CHARACTERIZATION OF WEIGHTED BESOV SPACES

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Abstract. We find conditions on the weight w in order characterize functions in weighted Besov spaces $B^{p,q}_{w,\phi}$ in terms of differences $\Delta_x f$.

Introduction.

There are many ways to define Besov spaces (see [1, 19, 24]). It is well known that Besov spaces can be defined, for instance in terms of convolutions $f * \phi_t$ with different kinds of smooth functions ϕ and that they can be also described by means of differences $\Delta_x f$ (see [10, 11, 22]).

Our objective will be to find weights (which extend the case t^α) where we can still get such a characteritation of weighted Besov spaces and to give a general procedure which works not only in the classical case but also in the weighted one. Our arguments will be based upon two main points: The Calderón's formula, a quite simple Schur Lemma.

We want to notice that this characterization can be used to get the atomic decomposition of the spaces.

The paper is divided into two sections. Section 1 has a preliminary character and it is devoted to introduce the notation and the main lemmas to be used later on. In Section 2 we prove the result about coincidence of seminorms in the spaces defined by differences and convolutions.

Throughout the paper a weight $w : \mathbb{R}^+ \to \mathbb{R}^+$ will be a measurable function $w > 0$ a.e., $1 \le p, q \le \infty$ and p', q' stand for the conjugate exponents. $\mathcal{S}$ denotes the Schwartz class of test functions on $\mathbb{R}^n$, $\mathcal{S}'$ the space of tempered distributions, $\mathcal{S}_0$ the set of functions in $\mathcal{S}$ with mean zero and ${\mathcal{S}'}_0$ its topological dual.

Given a weight w and $1 \le p, q \le \infty$ we shall denote by $\Lambda^{p,q}_w$ the space of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$||f||_{\Lambda^{p,q}_w} = \left(\int_{\mathbb{R}^n} \frac{||\Delta_x f||_p^q}{w(|x|)^q} \frac{dx}{|x|^n} \right)^{\frac{1}{q}} < \infty \qquad (1 \le q < \infty)$$

or

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$$||f||_{\Lambda^{p,\infty}_w} = \inf\{C > 0 : ||\Delta_x f||_p \leq Cw(|x|) \quad a.e. x \in \mathbb{R}^n\} < \infty \qquad (q = \infty)$$

where $\Delta_x f(y) = f(x + y) - f(y)$.

Given a weight w, $\phi \in \mathcal{S}_0$ and $1 \leq p, q \leq \infty$ we shall denote by $B^{p,q}_{w,\phi}$ the space functions $f : \mathbb{R}^n \to \mathbb{C}$ with $f \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$ such that

$$||f||_{B^{p,q}_{w,\phi}} = \left(\int_{\mathbb{R}^n} \frac{||\phi_t * f||_p^q}{w(t)^q} \frac{dt}{t}\right)^{\frac{1}{q}} < \infty \qquad (1 \leq q < \infty)$$

or

$$||f||_{B^{p,\infty}_{w,\phi}} = \inf\{C > 0 : ||\phi_t * f||_p \leq Cw(t) \quad \text{a.e.} t > 0\} < \infty \qquad (q = \infty)$$

where $\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{x}{t}\right)$.

To state the results of the paper, let us first recall the following notions.

A weight w is said to satisfy Dini condition if there exists $C > 0$ such that

$$\int_0^s \frac{w(t)}{t} dt \leq Cw(s) \quad \text{a.e. } s > 0.$$

A weight w is said to be a b_1-weight if there exists $C > 0$ such that

$$\int_s^\infty \frac{w(t)}{t^2} dt \leq C\frac{w(s)}{s} \quad \text{a.e. } s > 0.$$

We shall denote by $\mathcal{W}_{0,1}$ the space of b_1-weights which satisfy Dini condition.

Let us also use the notation $\mathcal{A}$ and $\mathcal{A}_1$ for the following classes

$$\mathcal{A} = \{\phi \in \mathcal{S}_0 : \int_0^\infty \left(\hat{\phi}(t\xi)\right)^2 \frac{dt}{t} = 1 \text{ for } \xi \in \mathbb{R}^n \setminus \{0\}\}.$$

$$\mathcal{A}_1 = \{\phi \in \mathcal{A} : \phi \text{ radial and real}, supp\ \phi \subseteq \{|x| \leq 1\}, \int_{\mathbb{R}^n} x_i\phi(x)dx = 0, i = 1, ..., n\}.$$

Section 2 is devoted to prove the following theorem.

Main Theorem. *Let $1 \leq p, q \leq \infty$, $\phi \in \mathcal{A}$ and w be a weight that can be factorized as $w(t) = \lambda^{\frac{1}{q'}}(t)\mu^{\frac{-1}{q}}(t^{-1})$ where $\lambda, \mu \in \mathcal{W}_{0,1}$. Then*

$$\Lambda^{p,q}_w = B^{p,q}_{w,\phi} \qquad \textit{(with equivalent seminorms).}$$

For particular cases $w(t) = t^\alpha$ the reader is referred to [10, 11, 14, 22] for similar results for special functions ϕ and their applications. In our weighted situation some closely related results for the unit disc are included in [3] and [5].

The reader should be aware that the case $1 < q < \infty$ in Main Theorem could have been shown by interpolation with the extreme cases, but a direct proof is presented in the paper.

§1. Preliminaries.

Let us recall some notions on weights we shall need later.

Definition 1.1. *Let $\varepsilon \geq 0$, $\delta \geq 0$ and w be a weight. w is said to be a d_ε-weight if exists $C > 0$ such that*

$$\int_0^s t^\varepsilon w(t)\frac{dt}{t} \leq C s^\varepsilon w(s) \quad \text{a.e. } s > 0. \tag{1.1}$$

w is said to be a b_δ-weight if there exists $C > 0$ such that

$$\int_s^\infty \frac{w(t)}{t^\delta}\frac{dt}{t} \leq C\frac{w(s)}{s^\delta} \quad \text{a.e. } s > 0. \tag{1.2}$$

If (d_ε)(respect. (b_δ)) denotes de class of d_ε-weights (respect. b_δ-weights) we write

$$\mathcal{W}_{\varepsilon,\delta} = (d_\varepsilon) \cap (b_\delta).$$

The following properties are elementary and left to the interested reader

$$w \in (d_\varepsilon) \Rightarrow w \in (d_{\varepsilon'}) \text{ for any } \varepsilon' > \varepsilon. \tag{1.3}$$

$$w \in (b_\delta) \Rightarrow w \in (b_{\delta'}) \text{ for any } \delta' > \delta. \tag{1.3$'$}$$

$$\text{Let } \overline{w}(t) = w(t^{-1}) \text{ then } w \in (b_\varepsilon) \Longleftrightarrow \overline{w} \in (d_\varepsilon). \tag{1.4}$$

$$w \in \mathcal{W}_{\varepsilon,\delta} \Rightarrow w(t) \geq C \min\left(t^{-\varepsilon}, t^\delta\right). \tag{1.5}$$

Let us now give some examples.

It is elementary to see that if $\alpha \in \mathbb{R}$ and $w_\alpha(t) = t^\alpha$ then $w_\alpha \in \mathcal{W}_{\varepsilon,\delta}$ for any $\delta > \alpha$ and $\varepsilon > -\alpha$.

Let us give a bit more general example. Let $\alpha, \beta \in \mathbb{R}$ and $w_{\alpha,\beta}(t) = t^\alpha(1 + |\log t|)^\beta$. Then $w_{\alpha,\beta} \in \mathcal{W}_{\varepsilon,\delta}$ for any $\delta > \alpha$ and $\varepsilon > -\alpha$.

Indeed, let us take $\delta > \alpha$. Then making the change of variable $t = su$ we have

$$\begin{aligned}\int_s^\infty \frac{w_{\alpha,\beta}(t)}{t^{\delta+1}}dt &= \int_s^\infty t^{\alpha-\delta}(1 + |\log t|)^\beta \frac{dt}{t} \\ &\leq s^{\alpha-\delta}\int_1^\infty u^{\alpha-\delta}(1 + |\log s| + \log u)^\beta \frac{du}{u}.\end{aligned}$$

For $\beta < 0$ then

$$\int_s^\infty \frac{w_{\alpha,\beta}(t)}{t^{\delta+1}}dt \leq \frac{1}{\delta-\alpha}s^{\alpha-\delta}(1+|\log s|)^\beta = \frac{C_\beta}{\delta-\alpha}\frac{w_{\alpha,\beta}(s)}{s^\delta}.$$

For $\beta > 0$, using $(a+b)^\beta \leq C_\beta(a^\beta + b^\beta)$, we have

$$\begin{aligned}\int_s^\infty \frac{w_{\alpha,\beta}(t)}{t^{\delta+1}}dt &\leq C_\beta s^{\alpha-\delta}\left((1+|\log s|)^\beta \int_1^\infty u^{\alpha-\delta}\frac{du}{u} + \int_1^\infty u^{\alpha-\delta}(\log u)^\beta \frac{du}{u}\right)\\ &\leq C(\alpha,\beta,\delta)\frac{w_{\alpha,\beta}(s)}{s^\delta}.\end{aligned}$$

Since $w_{\alpha,\beta}(t) = w_{-\alpha,\beta}(t^{-1})$ then also have $w_{\alpha,\beta}$ is a d_ε-weight for $\varepsilon > -\alpha$. □

Let us now establish the main lemma to be used later on. Observe that a net $\{\phi_i\}_{i\in\Lambda}$ converges to ϕ in $\mathcal{S}_0'$ if there exist $\{c_i\}_{i\in\Lambda} \subset \mathbb{C}$ such that $\phi_i - c_i$ converges to ϕ in $\mathcal{S}'$.

One of the main facts in our approach, which follows ideas from [6] and [14], is the use of the Calderón reproducing formula.

Let $\phi \in \mathcal{A}$ and $\psi \in \mathcal{S}$ then for $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\hat{\psi}(\xi) = \int_0^\infty (\phi_t * \phi_t * \psi)\hat{}(\xi)\frac{dt}{t}.$$

This shows that $\psi_{\varepsilon,\delta} = \int_\varepsilon^\delta \phi_t * \phi_t * \psi \frac{dt}{t}$ converges to ψ in $\mathcal{S}$.

Lemma A. *(see Appendix [14]). Let $f \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$ and $\phi \in \mathcal{A}$. For $0 < \varepsilon < \delta$ define*

$$f_{\varepsilon,\delta}(x) = \int_\varepsilon^\delta (\phi_t * \phi_t * f)(x)\frac{dt}{t}.$$

Then $f_{\varepsilon,\delta}$ converges to f in ${\mathcal{S}'}_0$ as $\varepsilon \to 0$ and $\delta \to \infty$.

To finish this preliminary section let us state a version of Schur lemma that will be useful for our purposes and whose elementary proof we include here for the sake of completeness.

Lemma B. *Let $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two σ-finite measure spaces and let $K : \Omega_1 \times \Omega_2 \to \mathbb{R}^+$ be a measurable function and write $T_K(f)$ for*

$$T_K(f)(w_2) = \int_{\Omega_1} K(w_1, w_2) f(w_1) d\mu_1(w_1).$$

If there exist $C > 0$ and measurable functions $h_i : \Omega_i \to \mathbb{R}^+$ $(i = 1, 2)$ such that

$$\int_{\Omega_1} K(w_1, w_2) h_1^{q'}(w_1) d\mu_1(w_1) \leq C h_2^{q'}(w_2) \quad \mu_2 - \text{a.e.} \tag{1.9}$$

$$\int_{\Omega_2} K(w_1, w_2) h_2^q(w_2) d\mu_2(w_2) \leq C h_1^q(w_1) \quad \mu_1 - \text{a.e.} \tag{1.10}$$

Then T_K defines a bounded operator from $L^q(\Omega_1, \mu_1)$ into $L^q(\Omega_2, \mu_2)$.

Proof. From (1.9) and Hölder's inequality we have

$$|T_K(f)(w_2)| \leq C h_2(w_2) \left(\int_{\Omega_1} K(w_1, w_2) h_1^{-q}(w_1) |f(w_1)|^q d\mu_1(w_1) \right)^{\frac{1}{q}}.$$

Apply now (1.10) and Fubini's theorem to get

$$\begin{aligned} ||T_K(f)||_q &\leq C \left(\int_{\Omega_1} \left(\int_{\Omega_2)} K(w_1, w_2) h_2^q(w_2) d\mu_2(w_2) \right) h_1(w_1)^{-q} |f(w_1)|^q d\mu_1(w_1) \right)^{\frac{1}{q}} \\ &\leq C^2 ||f||_q. \end{aligned}$$

§2. Characterization of Besov spaces

Let us first establish some general facts that can be used to relate properties about differences $\Delta_x f$ and convolutions $\phi_t * f$.

Lemma 2.1. *Let $1 \leq p \leq \infty$, $\rho \geq 0$ and $\phi \in \mathcal{A}$. Then there exists $C > 0$ such that if $f \in L^1 \left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}} \right)$ then we have:*

$$||\phi_t * f||_p \leq C \int_{\mathbb{R}^n} \min \left(\left(\frac{|x|}{t} \right)^n, \left(\frac{t}{|x|} \right)^\rho \right) ||\Delta_x f||_p \frac{dx}{|x|^n}. \tag{2.1}$$

$$||\Delta_x f||_p \leq C \int_0^\infty \min \left(1, \frac{|x|}{t} \right) ||\phi_t * f||_p \frac{dt}{t}. \tag{2.2}$$

Proof. Notice that, since $\int_{\mathbb{R}^n} \phi(x) dx = 0$, then

$$\phi_t * f(y) = \int_{\mathbb{R}^n} \phi_t(x) \Delta_{-x} f(y) dx.$$

From Minkowski's inequality one gets

$$||\phi_t * f||_p \leq \int_{\mathbb{R}^n} \frac{|x|^n}{t^n} \left| \phi \left(\frac{-x}{t} \right) \right| ||\Delta_x f||_p \frac{dx}{|x|^n}. \tag{2.1'}$$

Hence (2.1) follows from the trivial estimates

$$|y|^{n+\rho}|\phi(y)| \leq C \quad \text{if} \quad |y| \geq 1.$$
$$|\phi(y)| \leq C \quad \text{if} \quad |y| \leq 1.$$

To prove (2.2) observe first that for $0 < \varepsilon < \delta$

$$\Delta_x f_{\varepsilon,\delta}(y) = \int_\varepsilon^\delta (\Delta_{-x}\phi_t) * \phi_t * f(y)\frac{dt}{t}. \tag{2.2$'$}$$

Hence Minkowski's inequality and Young's inequality give

$$||\Delta_x f_{\varepsilon,\delta}||_p \leq \int_\varepsilon^\delta ||\Delta_{-x}\phi_t||_1||\phi_t * f||_p\frac{dt}{t}.$$

Note that

$$||\Delta_y\phi||_1 \leq 2||\phi||_1 \quad \text{if} \quad |y| \geq 1.$$
$$||\Delta_y\phi||_1 \leq |y|\int_{\mathbb{R}^n} \max_{|z-u|<1} |\bigtriangledown \phi(z)|du \quad \text{if} \quad |y| \leq 1.$$

Hence

$$\|\Delta_{-x}\phi_t\|_1 = \left\|\Delta_{\frac{-x}{t}}\phi\right\|_1 \leq C\min\left(1, \frac{|x|}{t}\right).$$

Therefore, using the previous estimate (2.2$'$) and Lemma A, a simple limiting argument shows (2.2). □

Although for the purposes of this paper only a particular case of next lemma will be used we state a general version of it that we find interesting in its own right.

Lemma 2.2. *Given* $0 \leq \varepsilon, \delta < \infty$, $1 < q < \infty$, *and* w *a weight, let us consider*

$$R_{\varepsilon,\delta}(s,t) = \frac{w(s)}{w(t)}\min\left(\left(\frac{s}{t}\right)^\varepsilon, \left(\frac{t}{s}\right)^\delta\right).$$

If $w(s) = \lambda^{\frac{1}{q'}}(s)\mu^{\frac{-1}{q}}(s^{-1})$ *for some pair of weights* $\lambda,\ \mu \in \mathcal{W}_{\varepsilon,\delta}$, *then there exist* $C > 0$ *and* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *measurable such that*

$$\int_0^\infty R_{\varepsilon,\delta}(s,t)g^{q'}(s)\frac{ds}{s} \leq Cg^{q'}(t). \tag{2.3}$$

$$\int_0^\infty R_{\varepsilon,\delta}(s,t)g^{q}(t)\frac{dt}{t} \leq Cg^{q}(s). \tag{2.4}$$

Proof. Let us take $g(t) = \lambda^{\frac{1}{qq'}}(t)\mu^{\frac{1}{qq'}}(t^{-1})$. Then $g^{q'}(s) = \dfrac{\lambda(s)}{w(s)}$ and $g^q(t) = w(t)\mu(t^{-1})$.

Therefore

$$\begin{aligned}\int_0^\infty R_{\varepsilon,\delta}(s,t)g^{q'}(s)\frac{ds}{s} &= \frac{1}{w(t)}\int_0^\infty \lambda(s)\min\left(\left(\frac{s}{t}\right)^\varepsilon,\left(\frac{t}{s}\right)^\delta\right)\frac{ds}{s}\\ &= \frac{1}{t^\varepsilon w(t)}\int_0^t s^\varepsilon\lambda(s)\frac{ds}{s}+\frac{t^\delta}{w(t)}\int_t^\infty\frac{\lambda(s)}{s^\delta}\frac{ds}{s}\\ &\le C\frac{\lambda(t)}{w(t)} = Cg^{q'}(t).\end{aligned}$$

On the other hand

$$\begin{aligned}\int_0^\infty R_{\varepsilon,\delta}(s,t)g^{q}(t)\frac{dt}{t} &= w(s)\int_0^\infty \mu(t^{-1})\min\left(\left(\frac{s}{t}\right)^\varepsilon,\left(\frac{t}{s}\right)^\delta\right)\frac{dt}{t}\\ &= \frac{w(s)}{s^\delta}\int_0^s t^\delta\mu(t^{-1})\frac{d}{t}+s^\varepsilon w(s)\int_s^\infty\frac{\mu(t^{-1})}{t^\varepsilon}\frac{dt}{t}\\ &= \frac{w(s)}{s^\delta}\int_{s^{-1}}^\infty\frac{\mu(t)}{t^\delta}\frac{d}{t}+s^\varepsilon w(s)\int_0^{s^{-1}} t^\varepsilon\mu(t)\frac{dt}{t}\\ &\le C\mu(s^{-1})w(s) = Cg^q(s). \quad \square\end{aligned}$$

Let us now state the following result in order to avoid repeating arguments in several of the remaining proofs.

Lemma 2.3. *Let $1 \le p \le \infty$ and let f be a measurable function.*
If $\|\Delta_x f\|_p \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$ *then* $f \in L^1\left(\mathbb{R}^n, (\frac{dx}{1+|x|)^{n+1}}\right)$.

Proof. Choose $\Psi \in L^{p'}(\mathbb{R}^n, dx)$ with $\Psi > 0$ a.e. Then Hölder's inequality and Fubini's theorem give

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|f(x+y)-f(y)|}{(1+|x|)^{n+1}}dx\right)\Psi(y)dy < \infty.$$

Therefore

$$\int_{\mathbb{R}^n}\frac{|f(x+y)-f(y)|}{(1+|x|)^{n+1}}dx < \infty \quad \text{for a.e. } y \in \mathbb{R}^n.$$

Since $(1+|x|)^{-(n+1)} \in L^1(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n}\frac{|f(x+y)|}{(1+|x|)^{n+1}}dx < \infty \quad \text{for a.e. } y \in \mathbb{R}^n.$$

Finally since there exists $C > 0$ such that $1+|x+y| \ge C(1+|x|)$ for all $y \in \mathbb{R}^n$, then one has

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty. \quad \square$$

Let us now start with the case $q = \infty$ in the Main Theorem which easily follows from Lemma 2.1.

Theorem 2.1. *Let $1 \le p \le \infty$, $\phi \in \mathcal{A}$ and $w \in \mathcal{W}_{0,1}$. Then*

$$\Lambda_w^{p,\infty} = B_{w,\phi}^{p,\infty} \qquad \text{(with equivalent seminorms).}$$

Proof. Assume $f \in \Lambda_w^{p,\infty}$. Note that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{(1+|x|)^{n+1}} dx &\le C \int_{\mathbb{R}^n} \frac{w(|x|)}{(1+|x|)^{n+1}} dx \\
&\le C \int_0^\infty \frac{w(t) t^{n-1}}{(1+t)^{n+1}} dt \\
&\le C \Big(\int_0^1 w(t) \frac{dt}{t} + \int_1^\infty w(t) \frac{dt}{t^2} \Big) < \infty
\end{aligned}$$

what combined with Lemma 2.3 gives

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty.$$

Let us prove that $||\phi_t * f||_p \le C w(t)$. From (2.1) in Lemma 2.1 for $\rho = 1$ we have

$$\begin{aligned}
||\phi_t * f||_p &\le C \left(\frac{1}{t^n} \int_{|x|<t} ||\Delta_x f||_p dx + t \int_{|x|>t} ||\Delta_x f||_p \frac{dx}{|x|^{n+1}} \right) \\
&\le C \left(\frac{1}{t^n} \int_{|x|<t} w(|x|) dx + t \int_{|x|>t} w(|x|) \frac{dx}{|x|^{n+1}} \right) \\
&\le C \left(\int_0^t \left(\frac{s}{t}\right)^n w(s) \frac{ds}{s} + t \int_t^\infty w(s) \frac{ds}{s^2} \right) \le C w(t).
\end{aligned}$$

Assume now $f \in B_{w,\phi}^{p,\infty}$. Then from (2.2) we have

$$\begin{aligned}
||\Delta_x f||_p &\le C \left(\int_0^{|x|} ||\phi * f||_p \frac{dt}{t} + |x| \int_{|x|}^\infty ||\phi * f||_p \frac{dt}{t^2} \right) \\
&\le C \left(\int_0^{|x|} \frac{w(t)}{t} dt + |x| \int_{|x|}^\infty \frac{w(t)}{t^2} dt \right) \le C w(|x|). \quad \square
\end{aligned}$$

We prove now the case $q = 1$ in the Main Theorem.

Theorem 2.2. *Let $1 \leq p \leq \infty$, $\phi \in \mathcal{A}$ and w such that $\mu(t) = w^{-1}(t^{-1}) \in \mathcal{W}_{0,1}$. Then*

$$\Lambda_w^{p,1} = B_{w,\phi}^{p,1} \qquad \textit{(with equivalent seminorms).}$$

Proof. Assume $f \in \Lambda_w^{p,1}$. Let us first prove that

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty.$$

From (1.5)

$$\frac{1}{|x|^n w(|x|)} \geq C \frac{1}{|x|^n} \min\left(1, \frac{1}{|x|}\right) \geq C \frac{1}{|x|^n} \min\left(|x|^n, \frac{1}{|x|}\right) \geq \frac{C}{(1+|x|)^{n+1}}.$$

Hence

$$\int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{(1+|x|)^{n+1}} dx \leq C \int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{w(|x|)} \frac{dx}{|x|^n} < \infty$$

and we apply Lemma 2.3 again.

We shall now prove that $||f||_{B_{w,\phi}^{p,1}} \leq C ||f||_{\Lambda_w^{p,1}}$.

Using (2.1) in Lemma 2.1 with $\rho = 1$

$$\begin{aligned}
\int_0^\infty \frac{||\phi_t * f||_p}{w(t)} \frac{dt}{t} &\leq C \int_0^\infty \left[\int_{\mathbb{R}^n} \min\left(\left(\frac{|x|}{t}\right)^n, \left(\frac{t}{|x|}\right) \right) \frac{||\Delta_x f||_p}{w(t)} \frac{dx}{|x|^n} \right] \frac{dt}{t} \\
&= C \int_{\mathbb{R}^n} ||\Delta_x f||_p \left[\int_0^\infty \min\left(\left(\frac{|x|}{t}\right)^n, \left(\frac{t}{|x|}\right) \right) \mu\left(t^{-1}\right) \frac{dt}{t} \right] \frac{dx}{|x|^n} \\
&= C \int_{\mathbb{R}^n} ||\Delta_x f||_p \left[\int_0^{|x|} \frac{t\mu\left(t^{-1}\right)}{|x|} \frac{dt}{t} + \int_{|x|}^\infty \frac{|x|^n \mu\left(t^{-1}\right)}{t^n} \frac{dt}{t} \right] \frac{dx}{|x|^n} \\
&\leq C \int_{\mathbb{R}^n} ||\Delta_x f||_p \left[\frac{1}{|x|} \int_{|x|^{-1}}^\infty \mu(t) \frac{dt}{t^2} + \int_0^{|x|^{-1}} \mu(t) \frac{dt}{t} \right] \frac{dx}{|x|^n} \\
&\leq C \int_{\mathbb{R}^n} ||\Delta_x f||_p \mu\left(|x|^{-1}\right) \frac{dx}{|x|^n} \\
&= C \int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{w(|x|)} \frac{dx}{|x|^n}.
\end{aligned}$$

Take now $f \in B_{w,\phi}^{p,1}$. From (2.2) in Lemma 2.1 and Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{w(|x|)} \frac{dx}{|x|^n} &\leq C \int_0^\infty ||\phi_t * f||_p \left[\int_{\mathbb{R}^n} \mu\left(|x|^{-1}\right) \min\left(1, \frac{|x|}{t}\right) \frac{dx}{|x|^n} \right] \frac{dt}{t} \\
&= C \int_0^\infty ||\phi_t * f||_p \left[\int_0^\infty \mu(s) \min\left(1, \frac{1}{st}\right) \frac{ds}{s} \right] \frac{dt}{t} \\
&= C \int_0^\infty ||\phi_t * f||_p \left[\int_0^{t^{-1}} \frac{\mu(s)}{s} ds + \frac{1}{t} \int_{t^{-1}}^\infty \frac{\mu(s)}{s^2} ds \right] \frac{dt}{t} \\
&\leq C \int_0^\infty ||\phi_t * f||_p \mu\left(t^{-1}\right) \frac{dt}{t} \\
&= C \int_0^\infty \frac{||\phi_t * f||_p}{w(t)} \frac{dt}{t}. \quad \square
\end{aligned}$$

Theorem 1.3. *Let $1 \leq p \leq \infty$, $1 < q < \infty$, $\phi \in \mathcal{A}$ and w a weight such that*

$$w(t) = \lambda^{\frac{1}{q'}}(t) \mu^{\frac{-1}{q}}\left(t^{-1}\right)$$

for some pair of weights $\lambda, \mu \in \mathcal{W}_{0,1}$. Then

$$\Lambda_w^{p,q} = B_{w,\phi}^{p,q} \quad \textit{(with equivalent seminorms).}$$

Proof. Assume $f \in \Lambda_w^{p,q}$. Let us show first that

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty.$$

Let us denote

$$\Phi(x) = \frac{w(|x|)|x|^n}{(1+|x|)^{n+1}}.$$

We shall see that under the assumptions $\lambda, \mu \in \mathcal{W}_{0,1}$ one has that $\Phi \in L^{q'}\left(\mathbb{R}^n, \frac{dx}{|x|^n}\right)$. Indeed

$$\int_0^\infty \Phi^{q'}(t) \frac{dt}{t} = \int_0^\infty \lambda(t) \mu^{-q'/q}\left(t^{-1}\right) \frac{t^{nq'}}{(1+t)^{q'(n+1)}} \frac{dt}{t}.$$

Using (1.5) we have $\mu(s) \geq C \min(1, s)$. Therefore

$$\begin{aligned}
\int_0^\infty \Phi^{q'}(t) \frac{dt}{t} &\leq C \int_0^\infty \lambda(t) \max\left(1, t^{(q'-1)}\right) \frac{t^{nq'}}{(1+t)^{q'n+q'}} \frac{dt}{t} \\
&\leq C \left(\int_0^1 \lambda(t) \frac{dt}{t} + \int_1^\infty \frac{\lambda(t)}{t} \frac{dt}{t} \right) < \infty.
\end{aligned}$$

From Hölder's inequality one has

$$\int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{(1+|x|)^{n+1}} dx = \int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{w(|x|)} \Phi(x) \frac{dx}{|x|^n} < \infty$$

and we apply Lemma 2.3.

Let us now prove

$$||f||_{B^{p,q}_{w,\phi}} \leq C||f||_{\Lambda^{p,q}_w}.$$

From (2.1) in Lemma 2.1 with $\rho = 1$

$$\frac{||\phi_t * f||_p}{w(t)} \leq C \int_{\mathbb{R}^n} K(x,t) \frac{||\Delta_x f||_p}{w(|x|)} \frac{dx}{|x|^n}$$

where

$$K(x,t) = \frac{w(|x|)}{w(t)} \min\left(1, \frac{t}{|x|}\right).$$

Take

$$(\Omega_1, \Sigma_1, \mu_1) = \left(\mathbb{R}^n, \mathcal{B}\left(\mathbb{R}^n\right), \frac{dx}{|x|^n}\right)$$

and

$$(\Omega_2, \Sigma_2, \mu_2) = \left((0,\infty), \mathcal{B}\left((0,\infty)\right), \frac{dx}{|x|^n}\right).$$

Since $K(x,t) = R_{0,1}(|x|,t)$ we can apply Lemma 2.2 with $\varepsilon = 0$ and $\delta = 1$ to get a measurable function g satisfying (2.3) and (2.4).

Now write $h_1(x) = g(|x|)$ and $h_2(t) = g(t)$. Obviously, using polar coordinates, (2.3) and (2.4) give (1.3) and (1.4) in Lemma B, what shows that T_K is bounded from $L^q\left(\mathbb{R}^n, \frac{dx}{|x|^n}\right)$ into $L^q\left((0,\infty), \frac{dt}{t}\right)$. Therefore

$$\begin{aligned} ||f||_{B^{p,q}_{w,\phi}} &\leq C \left|\left| T_K\left(\frac{||\Delta_x f||_p}{w(|x|)}\right)\right|\right|_{L^q\left((0,\infty), \frac{dt}{t}\right)} \\ &\leq C \left|\left| \frac{||\Delta_x f||_p}{w(|x|)} \right|\right|_{L^q\left(\mathbb{R}^n, \frac{dx}{|x|^n}\right)} \\ &\leq C||f||_{\Lambda^{p,q}_w}. \end{aligned}$$

(ii) Let us take $f \in B^{p,q}_{w,\phi}$. From (2.2) in Lemma 2.1

$$\frac{||\Delta_x f||_p}{w(|x|)} \leq C \int_0^\infty R(x,t) \frac{||\phi_t * f||_p}{w(t)} \frac{dt}{t}.$$

where

$$R(x,t) = \frac{w(t)}{w(|x|)} \min\left(1, \frac{|x|}{t}\right).$$

Now take

$$(\Omega_1, \Sigma_1, \mu_1) = \left((0,\infty), \mathcal{B}\left((0,\infty)\right), \frac{dx}{|x|^n}\right)$$

and

$$(\Omega_2, \Sigma_2, \mu_2) = \left(\mathbb{R}^n, \mathcal{B}\left(\mathbb{R}^n\right), \frac{dx}{|x|^n}\right).$$

Combine now again Lemma 2.2 and Lemma B to get the boundedness of T_R from $L^q\left((0,\infty), \frac{dt}{t}\right)$ into $L^q\left(\mathbb{R}^n, \frac{dx}{|x|^n}\right)$. Therefore

$$||f||_{\Lambda_w^{p,q}} \leq C \left|\left| T_R\left(\frac{||\phi_t * f||_p}{w(t)}\right)\right|\right|_{L^q\left(\frac{dx}{|x|^n}\right)} \leq C \left|\left|\frac{||\phi_t * f||_p}{w(t)}\right|\right|_{L^q\left(\frac{dt}{t}\right)} \leq C||f||_{B^{p,q}_{w,\phi}}. \quad \square$$

Remark. Note that in the previous theorem one of the embedding could have been proved under weaker assumptions. In fact, if $\lambda, \mu \in \mathcal{W}_{n,1}$ then $\Lambda_w^{p,q} \subseteq B^{p,q}_{w,\phi}$.

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