Characterization on mixed super quasi-Einstein manifold

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Abstract

In this paper we study characterizations of odd and even dimensional mixed super quasi-Einstein manifold and we give three and four dimensional examples (both Riemannian and Lorentzian) of mixed super quasi-Einstein manifold to show the existence of such manifold. Also in the last section we give the examples of warped product on mixed super quasi-Einstein manifold.

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1 Introduction

A Riemannian manifold $(M,\ g)$ with dimension $(n \geq 2)$ is said to be an Einstein manifold if it satisfies the condition $S(X,\ Y) = \frac{r}{n}g(X,\ Y)$, holds on M, here S and r denote the Ricci tensor and the scalar curvature of $(M,\ g)$ respectivley. According to [2] the above equation is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity. The notion of quasi-Einstien manifold was defined in [7]. A non-flat Riemannian manifold $(M,\ g)$, $(n \geq 2)$ is said to be an quasi Einstein manifold if the condition $S(X,\ Y) = \alpha g(X,\ Y) + \beta\rho(X)\rho(Y)$, is fulfilled on M, where α and β are scalars of which $\beta \neq 0$ and ρ is non-zero 1-form such that $g(X,\ \xi) = \rho(X)$ for all vector field X and ξ is a unit vector field.

In [5], M.C.Chaki introduced super quasi-Einstein manifold, denoted by $S(QE)_n$ and gave an example of a 4-dimensional semi Riemannian super quasi-Einstein manifold, where the Ricci tensor S of type $(0,2)$ which is not identically zero satisfies the condition

$$S(X,\ Y) = \alpha g(X,\ Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X,Y), \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta$ are scalars such that β, γ, δ are nonzero and A, B are two nonzero 1-forms such that $g(X,\xi_1) = A(X)$ and $g(X,\xi_2) = B(X)$, $\xi_1,\ \xi_2$ being unit vectors which are orthogonal, i.e., $g(\xi_1,\xi_2) = 0$ and D is symmetric $(0,2)$ tensor with zero trace which satisfies the condition $D(X,\xi_1) = 0,\ \forall\ X \in \chi(M)$.

Here $\alpha, \beta, \gamma, \delta$ are called the associated scalars ,and A, B are called the associated main and auxiliary 1-forms respectively, $\xi_1,\ \xi_2$ are main and auxiliary generators and D is called the associated tensor of the manifold.

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In [3], A.Bhattacharyya, M.Tarafdar and D.Debnath introduced the notion of mixed super quasi-Einstein manifold, denoted by $MS(QE)_n$ with an example. A non-flat Riemannian manifold$(M,\ g)$, $(n \geq 3)$ is called mixed super quasi-Einstein manifold if its the Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) = \ & \alpha g(X,Y) + \beta A(X)A(Y) + \varrho B(X)B(Y) \\ & + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X,Y), \end{aligned} \tag{1.2}$$

where $\alpha, \beta, \varrho, \gamma, \delta$ are scalars such that $\beta, \varrho, \gamma, \delta$ are nonzero and A, B are two nonzero 1-forms such that

$$g(X,\xi_1) = A(X),\ g(X,\xi_2) = B(X),\ g(\xi_1,\xi_2) = 0, \forall X \tag{1.3}$$

$\xi_1,\ \xi_2$ being unit vectors which are orthogonal, D is symmetric $(0,2)$ tensor which satisfies the condition

$$D(X,\xi_1) = 0,\ traceD = 0\ \forall\ X \in \chi(M). \tag{1.4}$$

Here $\alpha, \beta, \varrho, \gamma, \delta$ are called the associated scalars, and A, B are called the associated main and auxiliary 1-forms respectively, ξ_1, ξ_2 are main and auxiliary generators and D is called the associated tensor of the manifold.

In [11], A.A.Shaikh, S.Jana introduced the notion of pseudo generalized quasi-Einstein manifold, denoted by $P(GQE)_n$ with examples. A non-flat Riemannian manifold$(M,\ g)$, $(n > 2)$ is called pseudo generalized quasi Einstein manifold if its the Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,\ Y) = \ \alpha g(X,\ Y) + \beta A(X)A(Y) + \varrho B(X)B(Y) + \delta D(X,Y), \tag{1.5}$$

where α, β, ϱ and δ are nonzero and A, B are two nonzero 1-forms such that

$$g(X,\xi_1) = A(X),\ g(X,\xi_2) = B(X),\ g(\xi_1,\xi_2) = 0, \forall X, \tag{1.6}$$

$\xi_1,\ \xi_2$ being unit vectors which are orthogonal, D is symmetric $(0,2)$ tensor which satisfies the condition

$$D(X,\xi_1) = 0,\ traceD = 0\ \forall\ X \in \chi(M). \tag{1.7}$$

Here $\alpha, \beta, \varrho, \delta$ are called the associated scalars, and A, B are called the associated 1-forms of the manifold and D is called the structure tensor of the manifold.

Let M be an m-dimensional, $m \geq 3$, Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(U \wedge V)$ the sectional curvature of M associated with a plane section $\pi \subseteq T_pM$, where $\{U, V\}$ is an orthonormal basis of π. For a n-dimensional subspace $L \subseteq T_pM$, $2 \leq n \leq m$, its scalar cuvrvature $\tau(L)$ is denoted by $\tau(L) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$, where $\{e_1, e_2, ..., e_n\}$ is any orthonormal basis of L ([8]).
The notion of warped product generalizes that of a surface of revolution. It was introduced in [4], for studying manifolds of negative curvature. Let (B, g_B), (F, g_F) be two Riemannian manifolds with dim $B = m > 0$, dim$F = k > 0$ and $f : B \longrightarrow (0, \infty)$, $f \in C^\infty(B)$. The warped product $M = B \times_f F$ is the Riemannian manifold $B \times F$ furnished with the metric $g_M = g_B + f^2 g_F$. B is called the base of M, F is the fibre and the warped product is called a simply Riemannian product if f is a constant function. The function f is called the warping function of the warped product[10].

The well-known characterization of 4-dimensional Einstein spaces was given by Singer and Thorpe in [12]. Later we have seen that the generalization of 4-dimensional Einstein spaces was given by Chen in [6]. On the other hand, in [8] Dumitru obtained the result for odd dimensional Einstein spaces. Also these results (both odd and even dimensions) were generalized to quasi Einstein manifold by Bejan in [1]. Also characterization of super quasi-Einstein manifold for both of odd and even dimensions was studied in [9]. From above studies, we have given characterization of mixed super quasi-Einstein manifold for both of odd and even dimensions with three and four dimensional examples of mixed super quasi-Einstein manifold to ensure the existence of such manifold. Next we show that a mixed super quasi-Einstein manifold is pseudo generalized quasi Einstein manifold if either of generators is parallel vector field. In the last section we have given examples of warped product on mixed super quasi-Einstein manifold.

2. Characterization of mixed super quasi-Einstein manifold

In this section we establish the characterization of odd and even dimensional $MS(QE)_n$.

Theorem 2.1. *A Riemannian manifold of dimension (2n+1) with $n \geq 2$ is mixed super quasi Einstein manifold if and only if the Ricci operator Q has eigen vector fields ξ_1 and ξ_2 such that at any point $p \in M$, there exist three real numbers a, b and c satisfying*

$$\tau(P) + a = \tau(P^\perp); \quad \xi_1, \xi_2 \in T_pP^\perp,$$

$$\tau(N) + b = \tau(N^\perp); \quad \xi_1 \in T_pN, \xi_2 \in T_pN^\perp,$$

$$\tau(R) + c = \tau(R^\perp); \quad \xi_1 \in T_pR, \xi_2 \in T_pR^\perp,$$

for any n-plane sections P, N and $(n+1)$-plane section R where $P^\perp, N^\perp$ and $R^\perp$ denote the orthogonal complements of P, N and R in T_pM respectively and

$$a = \{\alpha + \beta + \varrho + \delta[\sum_{i=n+1}^{2n-1} D(e_i, e_i) + D(e_{2n+1}, e_{2n+1}) - \sum_{i=1}^{n} D(e_i, e_i)]\}/2,$$

$$b = \{\alpha - \beta + \varrho + \delta[\sum_{i=n+1}^{2n+1} D(e_i, e_i) - \sum_{i=1}^{n-1} D(e_i, e_i)]\}/2,$$

$$c = \{\varrho - \alpha - \beta + \delta[\sum_{i=n+2}^{2n+1} D(e_i, e_i) - \sum_{i=1}^{n} D(e_i, e_i)]\}/2,$$

where α, β, ϱ and δ are scalars.

Proof. First suppose that M is a $(2n+1)$ dimensional mixed super quasi Einstein manifold, so

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \varrho B(X)B(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X,Y), \quad (2.1)$$

where $\alpha, \beta, \varrho, \gamma, \delta$ are scalars such that $\beta, \varrho, \gamma, \delta$ are nonzero and A, B are two nonzero 1-forms such that $g(X, \xi_1) = A(X)$ and $g(X, \xi_2) = B(X)$, ξ_1, ξ_2 being unit vectors which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$ and D is symmetric $(0,2)$ tensor with zero trace which satisfies the condition $D(X, \xi_1) = 0, \forall X \in \chi(M)$.

Let $P \subseteq T_pM$ be an n-dimensional plane orthogonal to ξ_1, ξ_2 and let $\{e_1, e_2.......e_n\}$ be orthonormal basis of it. Since ξ_1 and ξ_2 are orthogonal to P, we can take orthonormal basis $\{e_{n+1}, e_{n+2}.......e_{2n+1}\}$ of $P^{\perp}$ such that $e_{2n} = \xi_1$ and $e_{2n+1} = \xi_2$. Thus $\{e_1, e_2.......e_n, e_{n+1}, e_{n+2}.......e_{2n+1}\}$ is an orthonormal basis of T_pM. Then we can take $X = Y = e_i$ in (2.1), we have

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} \alpha + \delta D(e_i, e_i), & \text{for } 1 \leq i \leq 2n-1 \\ \alpha + \beta, & \text{for } i = 2n \\ \alpha + \varrho + \delta D(\xi_2, \xi_2), & \text{for } i = 2n+1 \end{cases}$$

By use of (2.1) for any $1 \leq i \leq 2n+1$, we can write

$$S(e_1, e_1) = K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + ... + K(e_1 \wedge e_{2n-1}) + K(e_1 \wedge \xi_1) + K(e_1 \wedge \xi_2) = \alpha + \delta D(e_1, e_1),$$

$$S(e_2, e_2) = K(e_2 \wedge e_1) + K(e_2 \wedge e_3) + ... + K(e_2 \wedge e_{2n-1}) + K(e_2 \wedge \xi_1) + K(e_2 \wedge \xi_2) = \alpha + \delta D(e_2, e_2),$$

....

$$S(e_{2n-1}, e_{2n-1}) = K(e_{2n-1} \wedge e_1) + K(e_{2n-1} \wedge e_2) + K(e_{2n-1} \wedge e_3)$$

$$+........... + K(e_{2n-1} \wedge \xi_2) = \alpha + \delta D(e_{2n-1}, e_{2n-1}),$$

$$S(\xi_1, \xi_1) = K(\xi_1 \wedge e_1) + K(\xi_1 \wedge e_2) + ... + K(\xi_1 \wedge e_{2n-1})$$
$$+K(\xi_1 \wedge \xi_2) = \alpha + \beta,$$

$$S(\xi_2, \xi_2) = K(\xi_2 \wedge e_1) + K(\xi_2 \wedge e_2) + ... + K(\xi_2 \wedge e_{2n-1})$$
$$+K(\xi_2 \wedge \xi_1) = \alpha + \varrho + \delta D(\xi_2, \xi_2).$$

Adding first n-equations, we get

$$2\tau(P) + \sum_{1 \leq i \leq n < j \leq 2n+1} K(e_i \wedge e_j) = n\alpha + \delta \sum_{i=1}^{n} D(e_i, e_i). \tag{2.2}$$

Then adding the last $(n+1)$ equations, we have

$$2\tau(P^{\perp}) + \sum_{1 \leq j \leq n < i \leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \beta + \varrho$$
$$+\delta \sum_{i=n+1}^{2n-1} D(e_i, e_i) + \delta D(\xi_2, \xi_2). \tag{2.3}$$

Then, by substracting the equation (2.2) and (2.3), we obtain

$$\tau(P^{\perp}) - \tau(P) = \{\alpha + \beta + \varrho + \delta[\sum_{i=n+1}^{2n-1} D(e_i, e_i) + D(e_{2n+1}, e_{2n+1}) - \Sigma_{i=1}^{n} D(e_i, e_i)]\}.$$

Therefore $\tau(P) + a = \tau(P^{\perp})$, where,

$$a = \{\alpha + \beta + \varrho + \delta[\sum_{i=n+1}^{2n-1} D(e_i, e_i) + D(e_{2n+1}, e_{2n+1}) - \sum_{i=1}^{n} D(e_i, e_i)]\}/2.$$

Similarly, Let $N \subseteq T_pM$ be an n-dimensional plane orthogonal to ξ_2 and let $\{e_1, e_2.......e_n\}$ be orthonormal basis of it. Since ξ_2 is orthogonal to N, we can take an orthonormal basis $\{e_{n+1}, e_{n+2}.......e_{2n+1}\}$ of $N^{\perp}$ orthogonal to ξ_1, such that $e_n = \xi_1$ and $e_{2n+1} = \xi_2$, respectively. Thus, $\{e_1, e_2.......e_n, e_{n+1}, e_{n+2}.......e_{2n+1}\}$ is an orthonormal basis of T_pM. Then we can take $X = Y = e_i$ in (2.1) to have

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} \alpha + \delta D(e_i, e_i), & 1 \leq i \leq n-1 \\ \alpha + \beta, & i = n \\ \alpha + \delta D(e_i, e_i), & n+1 \leq i \leq 2n \\ \alpha + \varrho + \delta D(e_{2n+1}, e_{2n+1}), & i = 2n+1 \end{cases}$$

Adding first n-equations, we get

$$2\tau(N) + \sum_{1\leq i\leq n<j\leq 2n+1} K(e_i \wedge e_j) = n\alpha + \beta + \delta \sum_{i=1}^{n-1} D(e_i, e_i), \tag{2.4}$$

and adding the last $(n+1)$ equations, we have

$$2\tau(N^{\perp}) + \sum_{1\leq j\leq n<i\leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \varrho + \delta \sum_{i=n+1}^{2n+1} D(e_i, e_i). \tag{2.5}$$

Then, by substracting the equation (2.4) and (2.5), we obtain

$$\tau(N^{\perp}) - \tau(N) = \{\alpha - \beta + \varrho + \delta[\sum_{i=n+1}^{2n+1} D(e_i, e_i) - \sum_{i=1}^{n-1} D(e_i, e_i)]\}/2.$$

Therefore $\tau(N) + b = \tau(N^{\perp})$, where,

$$b = \{\alpha - \beta + \varrho + \delta[\sum_{i=n+1}^{2n+1} D(e_i, e_i) - \sum_{i=1}^{n-1} D(e_i, e_i)]\}/2.$$

Analogously, Let $R \subseteq T_pM$ be an $(n{+}1)$-plane orthogonal to ξ_2 and let $\{e_1, e_2, ..., e_{n+1}\}$ be orthonormal basis of it. Since ξ_2 is orthogonal to R, we can take an orthonormal basis $\{e_{n+2}, e_{n+3}, ..., e_{2n}, e_{2n+1}\}$ of $R^{\perp}$ orthogonal to ξ_1, such that $e_{n+1} = \xi_1$ and $e_{2n+1} = \xi_2$. Thus,

$$\{e_1, e_2, ..., e_n, e_{n+1}, e_{n+2}, ..., e_{2n+1}\}$$

is an orthonormal basis of T_pM. Then we can take $X = Y = e_i$ in (2.1) to have

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} \alpha + \delta D(e_i, e_i), & 1 \leq i \leq n \\ \alpha + \beta, & i = n+1 \\ \alpha + \delta D(e_i, e_i), & n+2 \leq i \leq 2n \\ \alpha + \varrho + \delta D(e_{2n+1}, e_{2n+1}), & i = 2n+1 \end{cases}$$

Adding the first $n+1$-equations, we get

$$2\tau(R) + \sum_{1\leq i\leq n+1<j\leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \beta + \delta \sum_{i=1}^{n} D(e_i, e_i), \tag{2.6}$$

and adding the last n equations, we have

$$2\tau(R^{\perp})+\sum_{1\leq j\leq n+1<i\leq 2n+1}K(e_i\wedge e_j)=n\alpha+\varrho+\delta\sum_{i=n+2}^{2n+1}D(e_i,e_i). \tag{2.7}$$

Then, by substracting the equation (2.6) and (2.7), we obtain

$$\tau(R^{\perp})-\tau(R)=\{\varrho-\alpha-\beta+\delta[\sum_{i=n+2}^{2n+1}D(e_i,e_i)-\sum_{i=1}^{n}D(e_i,e_i)]\}/2.$$

Therefore $\tau(R)+c=\tau(R^{\perp})$, where,

$$c=\{\varrho-\alpha-\beta+\delta[\sum_{i=n+2}^{2n+1}D(e_i,e_i)-\sum_{i=1}^{n}D(e_i,e_i)]\}/2.$$

Conversely, let $g(lX,Y)=D(X,Y)$, where l is a symmetric endomorphism of the tangent space T_pM and let V be an arbitrary unit vector of T_pM, at $p\in M$, orthogonal to ξ_1 and ξ_2. Now D is symmetric as g is so, trace of D is zero and $D(X,\xi_1)=0$ for all $X\in\chi(M)$. We take an orthonormal basis $\{e_1,e_2.......e_n,e_{n+1},e_{n+2}.......e_{2n+1}\}$ of T_pM such that $V=e_1,e_{n+1}=\xi_1$ and $e_{2n+1}=\xi_2$. We consider n-plane section N and $(n+1)$-plane section R inT_pM as follows $N=$ span $\{e_2.......e_n,e_{n+1}\}$ and $R=$ span $\{e_1,e_2.......e_n,e_{n+1}\}$ respectively. Then we have $N^{\perp}=$ span $\{e_1,e_{n+2}.......e_{2n},e_{2n+1}\}$ and $R^{\perp}=$ span $\{e_{n+2}.......e_{2n}\}$ respectively. Now

$$\begin{aligned}
S(V,V)&=[K(e_1\wedge e_2)+K(e_1\wedge e_3)+......+K(e_1\wedge e_{n+1})]\\
&+[K(e_1\wedge e_{n+2})+..........+K(e_1\wedge e_{2n})+K(e_1\wedge e_{2n+1})]\\
&=[\tau(R)-\sum_{2\leq i<j\leq n+1}K(e_i\wedge e_j)]\\
&+[\tau(N^{\perp})-\sum_{n+2\leq i<j\leq 2n+1}K(e_i\wedge e_j)]\\
&=\tau(R)-\tau(N)+\tau(R^{\perp})-\tau(N^{\perp})\\
&=[\tau(R)+\tau(N)]+[b+\tau(N)-c-\tau(R)]\\
&=b-c.
\end{aligned}$$

Therefore, $S(V,V)=b-c$, for any unit vector $V\in T_pM$, orthogonal to ξ_1 and ξ_2. Then we can write for any $1\leq i\leq 2n+1$, $S(e_i,e_i)=b-c$, since $S(V,V)=(b-c)g(V,V)$. It follows that $S(X,X)=(b-c)g(X,X)+K_1A(X)A(X)+K_4D(X,X)$ and $S(Y,Y)=(b-c)g(Y,Y)+K_2B(Y)B(Y)+K_3[A(Y)B(Y)+B(Y)A(Y)]$ for any $X\in[\text{span}\{\xi_1\}]^{\perp}$ and $Y\in[\text{span}\{\xi_2\}]^{\perp}$, where A, B are the dual forms of ξ_1 and ξ_2 with respect to g, respectively and K_1,K_2,K_3,K_4 are scalars, such that $K_1\neq 0,K_2\neq 0,K_3\neq 0,K_4\neq 0$.

Now from the above equations, we get from symmetry that S with tensors $(b-c)g+K_1(A\otimes A)+K_4D$

and $(b-c)+K_2(B\otimes B)+K_3[(A\otimes B)+(A\otimes B)]$ must coincide on the complement of ξ_1 and ξ_2, respectively, that is $S(X,Y)=(b-c)g(X,Y)+K_1A(X)A(Y)+K_2B(X)B(Y)+K_3[A(X)B(Y)+B(X)A(Y)]+K_4D(X,Y)$, for any $X,Y\in[\text{span}\{\xi_1,\xi_2\}]^{\perp}$. Since ξ_1 and ξ_2 are eigenvector fields of Q, we also have $S(X,\xi_1)=0$ and $S(Y,\xi_2)=0$ for any $X,Y\in T_pM$ orthogonal to ξ_1 and ξ_2. Thus, we can extend the above equation to

$$\begin{aligned} S(X,Z) &= (b-c)g(X,Z)+K_1A(X)A(Z)+K_2B(X)B(Z)) \\ &\quad +K_3[A(X)B(Z+A(Z)B(X)]+K_4D(X,Z), \end{aligned} \tag{2.8}$$

for any $X\in[\text{span}\{\xi_1,\xi_2\}]^{\perp}$ and $Z\in T_pM$, where K_1,K_2,K_3,K_4 are scalars and $K_1\neq 0,K_2\neq 0,K_3\neq 0,K_4\neq 0$. Now, let us consider the n-plane section P and $(n+1)$-plane section R in T_pM as follows $P=\text{span }\{e_1,e_2,...,e_n\}$ and $R=\text{span }\{e_1,e_2,...,e_n,\xi_1\}$. Then we have $P^{\perp}=\text{span }\{\xi_1,e_{n+2},...,e_{2n+1}\}$ and $R^{\perp}=\text{span }\{e_{n+2},...,e_{2n},e_{2n+1}\}$ respectively. Now

$$\begin{aligned} S(\xi_1,\xi_1) &= [K(\xi_1\wedge e_1)+K(\xi_1\wedge e_2)+......+K(\xi_1\wedge e_n)] \\ &+[K(\xi_1\wedge e_{n+2})+..........+K(\xi_1\wedge e_{2n})+K(e_1\wedge e_{2n+1})] \\ &=[\tau(R)-\sum_{1\leq i<j\leq n}K(e_i\wedge e_j)] \\ &+[\tau(P^{\perp})-\sum_{n+2\leq i<j\leq 2n+1}K(e_i\wedge e_j)] \\ &=\tau(R)-\tau(P)+\tau(P^{\perp})-\tau(R^{\perp}) \\ &=[\tau(R)+\tau(P)]+[a+\tau(P)-c-\tau(R)] \\ &=a-c \end{aligned}$$

Therefore we can write

$$S(\xi_1,\xi_1)=(b-c)g(\xi_1,\xi_1)+(a-b)A(\xi_1)A(\xi_1)+K_4D(\xi_1,\xi_1). \tag{2.9}$$

Analogously, let us consider the n-plane section P and $N\in T_pM$ as follows $P=\text{span }\{e_1,e_2,.......e_n\}$ and $N=\text{span }\{e_{n+1},e_{n+2}.......e_{2n}\}$ respectively. Then we have $P^{\perp}=\text{span }\{e_{n+1},e_{n+2}.......e_{2n},\xi_2\}$ and $N^{\perp}=\text{span }\{e_1,.......e_n,\xi_2\}$ respectively. Now, we have

$$\begin{aligned} S(\xi_2,\xi_2) &= [K(\xi_2\wedge e_1)+K(\xi_2\wedge e_2)+......+K(\xi_2\wedge e_n)] \\ &+[K(\xi_2\wedge e_{n+1})+K(\xi_2\wedge e_{n+2})+........+K(e_2\wedge e_{2n})] \\ &=[\tau(N^{\perp})-\sum_{1\leq i<j\leq n}K(e_i\wedge e_j)] \\ &+[\tau(P^{\perp})-\sum_{n+1\leq i<j\leq 2n}K(e_i\wedge e_j)] \\ &=\tau(N^{\perp})-\tau(P)+\tau(P^{\perp})-\tau(N) \\ &=[\tau(N)+b-\tau(P)]+[a+\tau(P)-\tau(N)] \end{aligned}$$

$$= a + b.$$

Then, we get

$$S(\xi_2, \xi_2) = (b-c)g(\xi_2, \xi_2) + (a+c)B(\xi_2)B(\xi_2) + K_3[A(\xi_2)B(\xi_2) + A(\xi_2)B(\xi_2)]. \tag{2.10}$$

Now from (2.8), (2.9) and (2.10) we can write the Ricci tensor by

$$\begin{aligned} S(X,\ Y) = \ & \mu_1 g(X,\ Y) + K_1 A(X)A(Y) + K_2 B(X)B(Y) \\ & + K_3[A(X)B(Y) + A(Y)B(X)] + K_4 D(X, Y), \end{aligned} \tag{2.11}$$

for any $X, Y \in T_pM$. From (2.11) it follows that M is a mixed super quasi Einstein manifold , where $\mu_1, K_1, K_2, K_3, K_4$ are scalars and $K_1 \neq 0, K_2 \neq 0, K_3 \neq 0, K_4 \neq 0$. Hence the theorem is proved.

Theorem 2.2. *A Riemannian manifold of dimension $2n$ with $n \geq 2$ is mixed super quasi Einstein manifold if and only if the Ricci operator Q has eigen vector fields ξ_1 and ξ_2 such that at any point $p \in M$, there exist three real numbers a, b and c satisfying*

$$\tau(P) + a = \tau(P^\perp); \quad \xi_1, \xi_2 \in T_pP^\perp,$$

$$\tau(N) + b = \tau(N^\perp); \quad \xi_1 \in T_pN, \xi_2 \in T_pN^\perp,$$

$$\tau(R) + c = \tau(R^\perp); \quad \xi_1 \in T_pR, \xi_2 \in T_pR^\perp,$$

for any n-plane section P, N and $(n+1)$-plane section R where $P^\perp, N^\perp$ and $R^\perp$ denote the orthogonal complements of P, N and R in T_pM respectively and

$$a = \{\beta + \varrho + \delta[\sum_{i=n+1}^{2n-2} D(e_i, e_i) + D(e_{2n}, e_{2n}) - \sum_{i=1}^{n} D(e_i, e_i)]\}/2,$$

$$b = \{2\alpha - \beta + \varrho + \delta[\sum_{i=n+1}^{2n} D(e_i, e_i) - \sum_{i=2}^{n} D(e_i, e_i)]\}/2,$$

$$c = \{\varrho - \beta + \delta[\sum_{i=1}^{n} D(e_i, e_i) - \sum_{i=n+1}^{2n} D(e_i, e_i)]\}/2,$$

where α, β, ϱ and δ are scalars.

Proof. Let P and R be n-plane sections and N be an $(n-1)$-plane section such that, $P =$

span$\{e_1, e_2, ...e_n\}$, R = span$\{e_{n+1}, e_{n+2}, ...e_{2n}\}$ and N = span$\{e_2, e_3, ...e_n\}$ respectively. Therefore the orthogonal complements of these sections can be written as $P^\perp$ = span$\{e_{n+1}, e_{n+2}, ...e_{2n}\}$, $R^\perp$ = span$\{e_1, e_2, ...e_n\}$ and $N^\perp$ = span$\{e_1, e_{n+1}, ...e_{2n}\}$.
Then rest of the proof is similar to the proof of Theorem 2.1.

Corollary 1. *For pseudo generalized quasi Einstein manifold the characterization will be same but the values of a, b, c will be different for both odd and even dimensional manifolds.*

3. $MS(QE)_n$ with the parallel vector field generators

Theorem 3.1. *A mixed super quasi Einstein manifold is pseudo generalized quasi Einstein manifold if either of generators is parallel vector field.*
Proof. By the definition of the Riemannian curvature tensor, if ξ_1 is parallel vector field, then we find that

$$R(X,Y)\xi_1 = \nabla_X\nabla_Y\xi_1 - \nabla_Y\nabla_X\xi_1 - \nabla_{[X,Y]}\xi_1 = 0,$$

and consequently we get

$$S(X, \xi_1) = 0. \tag{3.1}$$

Again, put Y = ξ_1 in the equation (1.2) and applying (1.3) and (1.4), we get

$$S(X, \xi_1) = (\alpha + \beta)g(X, \xi_1) + \gamma g(X, \xi_2).$$

So, if ξ_1 is a parallel vector field, by (3.1), we get

$$(\alpha + \beta)g(X, \xi_1) + \gamma g(X, \xi_2) = 0. \tag{3.2}$$

Now, putting X = ξ_2 in the equation (3.2) and using (1.3) we get $\gamma = 0$. So, if ξ_1 is parallel vector field in a mixed super quasi Einstein manifold, then the manifold is pseudo generalized quasi Einstein manifold.

Again, if ξ_2 is parallel vector field, then $R(X,Y)\xi_2 = 0$. Contracting, we get

$$S(Y, \xi_2) = 0. \tag{3.3}$$

Putting X = ξ_2 in the equation (1.2) and applying (1.3), we get

$$S(Y, \xi_2) = (\alpha + \varrho)g(Y, \xi_2) + \gamma g(Y, \xi_1) + \delta D(Y, \xi_2).$$

If, ξ_2 is a parallel vector field, by (3.3), we get

$$(\alpha+\varrho)g(Y,\xi_2)+\gamma g(Y,\xi_1)+\delta D(Y,\xi_2)=0. \tag{3.4}$$

Putting Y = ξ_1 and using (3.4), (1.3), (1.4), we get $\gamma = 0$, i.e., the manifold is pseudo generalized quasi Einstein manifold.

4. Examples of 3-dimensional and 4- dimensional mixed super quasi Einstein manifold

Example 4.1. Let us consider a Riemannian metric g on R^3 by

$$ds^2 = g_{ij}dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2)] + (dx^3)^2,$$

$(i, j = 1, 2, 3)$ and $x^3 \neq 0$. Then the only non-vanishing components of Christofell symbols, the curvature tensors and the Ricci tensors are

$$\Gamma^1_{13} = \Gamma^2_{23} = \frac{2}{3x^3}, \qquad \Gamma^3_{11} = \Gamma^3_{22} = -\frac{2}{3}(x^3)^{\frac{1}{3}}$$

$$R_{1331} = R_{2332} = -\frac{2}{9(x^3)^{\frac{2}{3}}}, \qquad R_{1221} = \frac{4}{9}(x^3)^{\frac{2}{3}}$$

$$R_{11} = R_{22} = \frac{2}{9(x^3)^{\frac{2}{3}}}, \qquad R_{33} = -\frac{4}{9(x^3)^2}$$

Let us consider the associated scalars $\alpha, \beta, \varrho, \gamma$ and δ and the associated tensor D as follows:

$$\alpha = -\frac{4}{9(x^3)^2}, \quad \beta = \frac{5(x^3)^{\frac{4}{3}}}{9}, \quad \varrho = \frac{7}{9(x^3)^2}, \quad \gamma = -\frac{1}{(x^3)^{\frac{5}{3}}}, \quad \delta = \frac{1}{15(x^3)^{\frac{5}{3}}},$$

and

$$D_{ij} = \begin{cases} \frac{5}{3}x^3 & \text{for} \quad i=j=1 \\ -\frac{5}{3}x^3 & \text{for} \quad i=j=2 \\ \frac{15}{(x^3)^{\frac{1}{3}}} & \text{for} \quad i=1, j=2 \\ \frac{15}{(x^3)^{\frac{1}{3}}} & \text{for} \quad i=2, j=1 \\ 0 & \text{for} \quad otherwise \end{cases}$$

the 1-form

$$A_i(x) = \begin{cases} \frac{1}{x^3} & \text{for} \quad i=1 \\ 0 & \text{for} \quad otherwise \end{cases} \qquad \text{and} \qquad B_i(x) = \begin{cases} (x^3)^{\frac{2}{3}} & \text{for} \quad i=2 \\ 0 & \text{for} \quad otherwise \end{cases}$$

Then we have

$$(i)R_{11} = \alpha g_{11} + \beta A_1 A_1 + \varrho B_1 B_1 + \gamma[A_1 B_1 + A_1 B_1] + \delta D_{11}$$

$$(ii) R_{22} = \alpha g_{22} + \beta A_2 A_2 + \varrho B_2 B_2 + \gamma[A_2 B_2 + A_2 B_2] + \delta D_{22}$$

$$(iii) R_{33} = \alpha g_{33} + \beta A_3 A_3 + \varrho B_3 B_3 + \gamma[A_3 B_3 + A_3 B_3] + \delta D_{33}$$

Since all the cases other than $(i) - (iii)$ are trivial, we can say that

$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \varrho B_i B_j + \gamma[A_i B_j + A_j B_i] + \delta D_{ij}$ for $i, j = 1, 2, 3$. Thus if (R^3, g) is a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij} dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2)] + (dx^3)^2,$$

$(i, j = 1, 2, 3)$ and $x^3 \neq 0$, then (R^3, g) is an $MS(QE)_3$.

Next we consider the Lorentzian metric g on R^3 by

$$ds^2 = g_{ij} dx^i dx^j = -(x^3)^{4/3}(dx^1)^2 + (x^3)^{4/3}(dx^2)^2) + (dx^3)^2, \quad (i, j = 1, 2, 3) \text{ and } x^3 \neq 0.$$

Now, by similar way after some construction of associated scalars and associated 1-forms, we can say that the manifold is a mixed super quasi-Einstein manifold. Therefore we get another example of $MS(QE)_3$.

Example 4.2. (R^3, g) is a Lorentzian manifold endowed with the metric given by

$$ds^2 = g_{ij} dx^i dx^j = -(x^3)^{4/3}(dx^1)^2 + (x^3)^{4/3}(dx^2)^2) + (dx^3)^2,$$

$(i, j = 1, 2, 3)$ and $x^3 \neq 0$, then (R^3, g) is an $MS(QE)_3$.

Example 4.3. Let us consider a Riemannian metric g on R^4 by

$$ds^2 = g_{ij} dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

$(i, j = 1, 2, 3, 4)$ and $p = \frac{e^{x^1}}{k^2}$, k is constant, then the only non-vanishing components of Christofell symbols, the curvature tensors and the Ricci tensors are

$$\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{p}{1 + 2p}$$

,

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{p}{1 + 2p}$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{p}{1 + 2p}$$

$$R_{11} = \frac{3p}{(1 + 2p)^2}, \quad R_{22} = R_{33} = R_{44} = \frac{p}{(1 + 2p)^2}$$

It can be easily seen that the scalar curvature r of the given manifold (R^4, g) is $r = \frac{6p}{(1+2p)^3}$, which is non-vanishing and non-constant.

Let us consider the associated scalars $\alpha, \beta, \varrho, \gamma$ and δ and the associated tensor D as follows:

$$\alpha = \frac{p}{(1+2p)^3}, \quad \beta = 2p, \quad \varrho = -\frac{p}{(1+2p)^2}, \quad \gamma = -\frac{2\sqrt{p}}{1+2p}, \quad \delta = -\frac{2}{(1+2p)^2},$$

and

$$D_{ij} = \begin{cases} p & \text{for} \quad i = j = 1 \\ -p & \text{for} \quad i = j = 3 \\ 0 & \text{for} \quad otherwise \end{cases}$$

the 1-form

$$A_i(x) = \begin{cases} \frac{1}{1+2p} & \text{for} \quad i = 1 \\ 0 & \text{for} \quad otherwise \end{cases} \qquad \text{and} \qquad B_i(x) = \begin{cases} \sqrt{p} & \text{for} \quad i = 1 \\ -\sqrt{p} & \text{for} \quad i = 3 \\ 0 & \text{for} \quad otherwise \end{cases}$$

Then we have

$$(i) R_{11} = \alpha g_{11} + \beta A_1 A_1 + \varrho B_1 B_1 + \gamma[A_1 B_1 + A_1 B_1] + \delta D_{11}$$
$$(ii) R_{22} = \alpha g_{22} + \beta A_2 A_2 + \varrho B_2 B_2 + \gamma[A_2 B_2 + A_2 B_2] + \delta D_{22}$$
$$(iii) R_{33} = \alpha g_{33} + \beta A_3 A_3 + \varrho B_3 B_3 + \gamma[A_3 B_3 + A_3 B_3] + \delta D_{44}$$
$$(iv) R_{44} = \alpha g_{44} + \beta A_4 A_4 + \varrho B_4 B_4 + \gamma[A_4 B_4 + A_4 B_4] + \delta D_{44}$$

Since all the cases other than $(i) - (iv)$ are trivial, we can say that

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \varrho B_i B_j + \gamma[A_i B_j + A_j B_i] + \delta D_{ij}, \ \text{ for } i, j = 1, 2, 3, 4.$$

so if (R^4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij} dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

$(i, j = 1, 2, 3, 4)$. and $p = \frac{e^{x^1}}{k^2}$, k is constant, then (R^4, g) is an $MS(QE)_4$ with non-zero and non-constant scalar curvature.

If we consider the Lorentzian metric g on R^3 by

$$ds^2 = g_{ij} dx^i dx^j = -(1 + 2p)(dx^1)^2 + (1 + 2p)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

$(i, j = 1, 2, 3, 4)$ and $p = \frac{e^{x^1}}{k^2}$, k is constant. Now, by similar way after some construction of associated scalars and associated 1-forms, we can say that the manifold is a mixed super quasi-Einstein manifold. Therefore we get another example of $MS(QE)_4$.

Example 4.4. Let (R^4, g) be a Lorentzian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = -(1+2p)(dx^1)^2 + (1+2p)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

$(i, j = 1, 2, 3, 4)$ and $p = \frac{e^{x^1}}{k^2}$, k is constant. Then (R^4, g) is an $MS(QE)_4$ with non-zero and non-constant scalar curvature.

5. Examples of warped product on mixed super quasi-Einstein manifold

Example 5.1. Here we consider the example 4.1, a 3-dimensional example of mixed super quasi-Einstein manifold. Let (R^3, g) be a Riemannian manifold endowed with the metric given by

$ds^2 = g_{ij}dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2)] + (dx^3)^2$, where $(i, j = 1, 2, 3)$ and $x^3 \neq 0$.

To define warped product on $MS(QE)_3$, we consider the warping function $f : R \setminus 0 \longrightarrow (0, \infty)$ by $f(x^3) = (x^3)^{\frac{2}{3}}$ and observe that $f = (x^3)^{\frac{2}{3}} > 0$ is a smooth function. The line element defined on $R \setminus \{0\} \times R^2$ which is of the form $B \times_f F$, where $B = R \setminus \{0\}$ is the base and $F = R^2$ is the fibre.

Therefore the metric ds_M^2 can be expressed as $ds_B^2 + f^2 ds_F^2$ i.e.,

$$ds^2 = g_{ij}dx^i dx^j = (dx^3)^2 + \{(x^3)^{2/3}\}^2[(dx^1)^2 + (dx^2)^2],$$

which is the example of Riemannian warped product on $MS(QE)_3$.

Example 5.2. We consider the example 4.3, a 4-dimensional example of mixed super quasi-Einstein manifold. Let (R^4, g) be a Riemannian manifold endowed with the metric given by

$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2]$, where $(i, j = 1, 2, 3, 4)$, $p = \frac{e^{x^1}}{k^2}$, k is constant.

To define warped product on $MS(QE)_4$, we consider the warping function $f : R^3 \longrightarrow (0, \infty)$ by $f(x^1, x^2, x^3) = \surd(1+2p)$ and we observe that $f > 0$ is a smooth function. The line element defined on $R^3 \times R$ which is of the form $B \times_f F$, where $B = R^3$ is the base and $F = R$ is the fibre.

Therefore the metric ds_M^2 can be expressed as $ds_B^2 + f^2 ds_F^2$

i.e., $ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + [\surd(1+2p)]^2(dx^4)^2,$

which is the example of Riemannian warped product on $MS(QE)_4$.

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