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# CHARACTERIZATIONS OF CERTAIN WEAKLY PSEUDOCONVEX DOMAINS $E(k, \alpha)$ IN $C^n$

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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**Introduction.** Let $D$ be a domain in $C^n$ and $\mathrm{Aut}(D)$ the group of all biholomorphic transformations of $D$ onto itself. Let $p$ be a point of $\partial D$, the boundary of $D$. Throughout this paper, we say that *the condition* $(*)$ *is fulfilled for* $(D, p)$ if

$(*)$ *there exist a compact set $K$ in $D$, a sequence $\{k_\nu\}$ in $K$ and a sequence $\{\varphi_\nu\}$ in* $\mathrm{Aut}(D)$ *such that* $\lim_{\nu\to\infty} \varphi_\nu(k_\nu) = p$.

Moreover, a point $p \in \partial D$ is said to be *a strictly pseudoconvex boundary point of $D$* if there exist an open neighborhood $U$ of $p$ and a $C^2$-smooth strictly plurisubharmonic function $\rho: U \to \boldsymbol{R}$ such that $D \cap U = \{z \in U \mid \rho(z) < 0\}$ and $d\rho(z) \neq 0$ for all $z \in \partial D \cap U$.

In 1977, it was shown by Wong [14] that *if $D$ is a bounded strictly pseudoconvex domain in $C^n$ with $C^\infty$-smooth boundary and* $\mathrm{Aut}(D)$ *is non-compact, then $D$ is biholomorphically equivalent to the open unit ball $B^n$ in $C^n$*. It was later extended by Rosay to the following:

THEOREM R (Rosay [12]). *Let $D$ be a bounded domain in $C^n$ with a strictly pseudoconvex boundary point $p \in \partial D$. Assume that the condition $(*)$ is fulfilled for $(D, p)$. Then $D$ is biholomorphically equivalent to $B^n$.*

Here it seems natural to ask what happens when the point $p$ is a weakly pseudoconvex boundary point of $D$. In a recent work of Greene and Krantz [3] the weakly pseudoconvex domain

$$E(m) = \left\{ z \in C^n \mid -1 + \sum_{i=1}^{n-1} |z_i|^2 + |z_n|^{2m} < 0 \right\}, \quad 0 < m \in \boldsymbol{Z}$$

in $C^n$ is studied exclusively in connection with this problem and the following characterization of it is obtained as their main result:

THEOREM G-K (Greene and Krantz [3]). *Let $D$ be a bounded domain in $C^n$ with $C^{n+1}$-smooth boundary such that $p = (1, 0, \cdots, 0) \in \partial D$. Assume that there are neighborhoods $U$, $V$ of $p$ in $C^n$ such that, up to a local* *biholomorphism,* $U\cap\partial D$ *and* $V\cap\partial E(m)$ *coincide. Assume further that the condition* $(*)$ *is fulfilled for* $(D, p)$. *Then* $D$ *is biholomorphically equivalent to the domain* $E(m)$.

Their proof is very interesting, but contains a difficult and complicated lemma [3; Lemma 4.3], which was shown by the uniform estimates for the $\bar\partial$-equation on $D$. A glance at the proof of Theorem G-K tells us that the global $C^{n+1}$-smoothness assumption on $\partial D$ cannot be avoided with their technique. However, in view of Theorem R it would be naturally expected that the same conclusion is also true if only $D$ has a $C^2$-smooth boundary near the point $p$. The main purpose of this paper is to clear up this matter. In fact, employing the same technique as in our previous papers [6], [7] instead of using the $\bar\partial$-equation on $D$, we can avoid their hard part and obtain more general results without any smoothness assumption on $\partial D$.

In order to state our results, we here introduce the following notation: For every integer $k=1,\cdots,n$ and every real number $\alpha>0$, we set

$$\rho(k, \alpha; z) = -1 + \sum_{i=1}^{k}|z_i|^2 + \Big(\sum_{j=k+1}^{n}|z_j|^2\Big)^{\alpha}$$

and

$$E(k, \alpha) = \{z\in \boldsymbol{C}^n \mid \rho(k, \alpha; z) < 0\}\,.$$

So $E(m)=E(n-1, m)$; and if $k=n$ or $\alpha=1$, then $E(k,\alpha)$ is nothing but the open unit ball $B^n$. Moreover, note that $\partial E(k,\alpha)$ is not smooth in general. (Consider, for example, the domain $E(1, 1/4)=\{(z_1, z_2)\in \boldsymbol{C}^2 \mid -1+|z_1|^2+|z_2|^{1/2}<0\}$ in $\boldsymbol{C}^2$.) In this notation, we can prove the following:

THEOREM I. *Let* $D$ *be a bounded domain in* $\boldsymbol{C}^n$ *satisfying the following conditions*:

(i) $p=(1, 0,\cdots, 0)\in\partial D$;

(ii) *there is an open neighborhood* $U$ *of* $p$ *such that* $D\cap U = E(k,\alpha)\cap U$;

(iii) *the condition* $(*)$ *is fulfilled for* $(D, p)$.

*Then* $D$ *is biholomorphically equivalent to the domain* $E(k,\alpha)$.

In the theorem of Greene and Krantz [3], we may assume without loss of generality that there exists an open neighborhood $U$ of $p=(1, 0, \cdots, 0)$ such that $D\cap U = E(m)\cap U$ (see the proof of [3, Theorem 1.1]). Moreover, any smoothness of $\partial D$ is not assumed in our theorem. Therefore Theorem I is a natural generalization of Theorem G-K.

Clearly the condition (ii) of Theorem I imposes crucial restrictions on the boundary of $D$, and so we want to remove it. This cannot be achieved in full generality at this moment. But, under some additional condition on the convergence $\varphi_\nu(k_\nu) \to p$ we can prove the following theorem. (For the definition of R-lim, see Section 1.)

THEOREM II. *Let $D$ be a bounded domain in $C^n$ with $p = (1, 0, \cdots, 0) \in \partial D$. Assume that there exist an open neighborhood $U$ of $p$ and a continuous function $\rho: U \to \boldsymbol{R}$ such that:*

( i ) $D \cap U = \{z \in U \mid \rho(z) < 0\}$;

( ii ) $\rho(z) = \rho(k, \alpha; z) + R(z)$, $z \in U$ *with*

$$R(z) = o\Big(|z_1 - 1|^2 + \sum_{i=2}^{k} |z_i|^2 + \Big(\sum_{j=k+1}^{n} |z_j|^2\Big)^{\alpha}\Big)$$

*in a neighborhood of $p$; and assume further that:*

(iii) *There exist a compact set $K$ in $D$, a sequence $\{k_\nu\}$ in $K$ and a sequence $\{\varphi_\nu\}$ in* Aut($D$) *such that*

$$\text{R-}\lim_{\nu\to\infty} \varphi_\nu(k_\nu) = p\ ,$$

*Then $D$ is biholomorphically equivalent to the domain $E(k, \alpha)$.*

Taking account of the case of strictly pseudoconvex boundary points, it is reasonable that $R(z)$ has the estimate as in (ii). Moreover, it should be remarked that, in some sense, the assumption (iii) is not so strong. Indeed, in the model case $D = E(k, \alpha)$ with $\alpha \neq 1$, we have the following: For any convergent sequence $\varphi_\nu(k_\nu) \to p$, there exists a sequence $\{\tilde{\varphi}_\nu\}$ in Aut($D$) such that R-$\lim_{\nu\to\infty} \tilde{\varphi}_\nu(k_\nu) = p$ (see Example 2 in Section 1).

Next we assume that a complex manifold $M$ can be exhausted by biholomorphic images of a complex manifold $D$, that is, for any compact subset $K$ of $M$ there exists a biholomorphic mapping $f_K$ from $D$ into $M$ such that $K \subset f_K(D)$. Then, how can we describe $M$ using the data of $D$? In connection with this, Fridman [2] showed that *if a complete hyperbolic manifold $M$ of complex dimension $n$ in the sense of Kobayashi* [5] *can be exhausted by biholomorphic images of a bounded strictly pseudoconvex domain $D$ in $C^n$ with $C^3$-smooth boundary, then $M$ is biholomorphically equivalent either to $D$ or to the open unit ball $B^n$.* The following theorem tells us that the analogue is still valid for the weakly pseudoconvex domain $E(k, \alpha)$ with arbitrary $\alpha > 0$.

THEOREM III. *Let $M$ be a hyperbolic manifold of complex dimension $n$ in the sense of Kobayashi* [5]. *Assume that $M$ can be exhausted by biholomorphic images of the weakly pseudoconvex domain $E(k, \alpha)$. Then*

*$M$ is biholomorphically equivalent either to $E(k, \alpha)$ or to $B^n$.*

Our proofs of the theorems above are based on the normal family arguments developed in our previous papers [6], [7] and Pinčuk [10], [11]. Although there are some overlaps with those papers, we carry out the proofs in detail for the sake of completeness and self-containedness. After some preliminaries in Section 1, Theorems I, II and III will be proven in Sections 2, 3 and 4, respectively. In the final Section 5, we mention the analogues of Theorems I and II in the case where $D$ is a not necessarily bounded hyperbolic domain in $C^n$.

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**1. Preliminaries.** For later purpose, we shall recall some definitions and study the structure of the model space $E(k, \alpha)$ with arbitrary $\alpha > 0$.

Let $M$ and $N$ be complex manifolds and $\mathrm{Hol}(N, M)$ the family of all holomorphic mappings from $N$ into $M$. A sequence $\{f_\nu\}$ in $\mathrm{Hol}(N, M)$ is said to be *compactly divergent on* $N$ if, for any compact sets $L$, $K$ in $N$, $M$, respectively, there exists an integer $\nu_0$ such that $f_\nu(L) \cap K = \emptyset$ for all $\nu \geqq \nu_0$. After Wu [15], we shall define the tautness of complex manifolds as follows:

DEFINITION 1. A complex manifold $M$ is said to be *taut* if $\mathrm{Hol}(N, M)$ is a normal family for any complex manifold $N$, i.e., any sequence in $\mathrm{Hol}(N, M)$ contains a subsequence which is either uniformly convergent on every compact subset of $N$ or compactly divergent on $N$.

Let $d_M$, $d_N$ be the Kobayashi pseudodistances of $M$, $N$, respectively [5]. The following distance-decreasing property will play an important role in the proofs of our theorems: *Let $f: N \to M$ be a holomorphic mappping. Then*

$$d_M(f(p), f(q)) \leqq d_N(p, q) \quad \textit{for all } p, q \in N . \tag{1.1}$$

Consequently, every biholomorphic mapping $f$ from $N$ onto $M$ is an isometry with respect to $d_N$ and $d_M$; and if $N$ is a complex submanifold of $M$, then $d_M(p, q) \leqq d_N(p, q)$ for all $p, q \in N$.

Throughout this paper we use the following notation: For a point $z = (z_1, \cdots, z_n)$ of $C^n$ and a mapping $f = (f_1, \cdots, f_n)$ from a set $S$ into $C^n$, we set

$$z' = (z_1, \cdots, z_k) , \quad z'' = (z_{k+1}, \cdots, z_n) , \quad 'z = (z_1, \cdots, z_{n-1}) ,$$

$$'f = (f_1, \cdots, f_{n-1}) \quad \text{and} \quad |u|^2 = \sum_{i=1}^{l} |u_i|^2 \quad \text{for} \quad u = (u_1, \cdots, u_l) \in C^l .$$

Thus we can write the function $\rho(k, \alpha; z)$ and the domain $E(k, \alpha)$ in the form

$$\rho(k, \alpha; z) = -1 + |z'|^2 + |z''|^{2\alpha} ;$$
$$E(k, \alpha) = \{(z', z'') \in C^k \times C^{n-k} \,|\, |z'|^2 + |z''|^{2\alpha} < 1\} .$$

Recall that a domain $D$ in $C^n$ is called a Reinhardt domain if $((\exp\sqrt{-1}\theta_1)z_1, \cdots, (\exp\sqrt{-1}\theta_n)z_n) \in D$ whenever $(z_1, \cdots, z_n) \in D$ and $\theta_j \in R$, $j = 1, \cdots, n$. Moreover, we say that it is complete if $(z_1^\circ, \cdots, z_n^\circ) \in D$, $z = (z_1, \cdots, z_n) \in C^n$ and $|z_j| \leqq |z_j^\circ|$, $j = 1, \cdots, n$, implies $z \in D$. We now assert that $E(k, \alpha)$ *is a bounded pseudoconvex complete Reinhardt domain in* $C^n$ *containing the origin* $o$. Hence, by a result of Pflug [9] it is complete hyperbolic in the sense of Kobayashi [5]. Since $E(k, \alpha)$ is obviously a bounded complete Reinhardt domain in $C^n$ containing the origin, we have only to check that the domain

$$B = \{(x_1, \cdots, x_n) \in R^n \,|\, (\exp x_1, \cdots, \exp x_n) \in E(k, \alpha)\}$$

is geometrically convex in $R^n$ [8; p. 120]. To do so, let us take arbitrary points $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ of $B$ and arbitrary numbers $\lambda, \mu > 0$ such that $\lambda + \mu = 1$. Then, by using Hölder's inequality twice we obtain the following:

$$\begin{aligned}\sum_{i=1}^{k} \exp[2(\lambda x_i + \mu y_i)] &+ \Big(\sum_{j=k+1}^{n} \exp[2(\lambda x_j + \mu y_j)]\Big)^{\alpha} \\ &\leqq \Big(\sum_{i=1}^{k} \exp 2x_i\Big)^{\lambda} \cdot \Big(\sum_{i=1}^{k} \exp 2y_i\Big)^{\mu} + \Big[\Big(\sum_{j=k+1}^{n} \exp 2x_j\Big)^{\lambda} \cdot \Big(\sum_{j=k+1}^{n} \exp 2y_j\Big)^{\mu}\Big]^{\alpha} \\ &\leqq \Big[\sum_{i=1}^{k} \exp 2x_i + \Big(\sum_{j=k+1}^{n} \exp 2x_j\Big)^{\alpha}\Big]^{\lambda} \cdot \Big[\sum_{i=1}^{k} \exp 2y_i + \Big(\sum_{j=k+1}^{n} \exp 2y_j\Big)^{\alpha}\Big]^{\mu} < 1 ,\end{aligned}$$

which shows $\lambda x + \mu y \in B$. Thus $B$ is convex, as desired.

Next, setting $S = \{(0, z'') \in C^k \times C^{n-k} \,|\, |z''| = 1\} \subset \partial E(k, \alpha)$, we would like to show that $\partial E(k, \alpha)$ is real analytic and strictly pseudoconvex at every point contained in an open neighborhood $W$ of $S$. It is easy to see that there is an open neighborhood $W$ of $S$ on which $\rho(k, \alpha; z)$ is real analytic and $d\rho(k, \alpha; z) \neq 0$ for all $z \in W$. Once $\partial E(k, \alpha)$ is shown to be strictly pseudoconvex at every point $(0, z'') \in S$, one can obtain a desired neighborhood $W$ by the continuity of the Levi form. On the other hand, by direct calculation we obtain that

$$\begin{aligned}&\sum_{i,j=1}^{n} [\partial^2 \rho(k, \alpha; z)/\partial z_i \partial \bar{z}_j] \xi_i \bar{\xi}_j \\ &\qquad = |\xi'|^2 + \alpha |z''|^{2(\alpha-1)} |\xi''|^2 + \alpha(\alpha - 1) |z''|^{2(\alpha-2)} \Big| \sum_{j=k+1}^{n} \bar{z}_j \xi_j \Big|^2\end{aligned}$$

for every $\xi = (\xi', \xi'') \in C^k \times C^{n-k}$ and every $z \in W$; and

$$\left\{\xi \in C^n \,\middle|\, \sum_{i=1}^{n}[\partial\rho(k, \alpha; q)/\partial z_i]\xi_i = 0\right\} = \left\{\xi \in C^n \,\middle|\, \sum_{j=k+1}^{n} \bar{z}_j\xi_j = 0\right\}$$

for every $q = (0, z'') \in S$. Hence $\partial E(k, \alpha)$ is actually strictly pseudoconvex at every point of $S$, as desired.

We study the biholomorphic automorphism group $\mathrm{Aut}(E(k, \alpha))$ of $E(k, \alpha)$. Denoting by $M(r, s)$ the set of all $r\times s$ complex matrices for positive integers $r$, $s$, we consider the closed Lie subgroup $SU(k, 1)$ of $GL(k+1, C)$ consisting of all matrices

$$\gamma = \begin{pmatrix} A & \mathfrak{b} \\ \mathfrak{c} & d \end{pmatrix}; \quad \begin{matrix} A \in M(k, k)\,, & \mathfrak{b} \in M(k, 1) \\ \mathfrak{c} \in M(1, k)\,, & d \in M(1, 1) \end{matrix} \tag{1.2}$$

satisfying the relations

$$^t\bar{A}A - {}^t\bar{\mathfrak{c}}\mathfrak{c} = E_k\,, \quad {}^t\bar{\mathfrak{b}}\mathfrak{b} - |d|^2 = -1\,, \quad {}^t\bar{\mathfrak{b}}A = \bar{d}\mathfrak{c} \quad \text{and} \quad \det\gamma = 1\,,$$

where $E_k$ is the unit matrix of degree $k$. For each $\gamma \in SU(k, 1)$ represented as in (1.2) and each $U \in U(n-k)$, the unitary group of degree $n-k$, we define the transformation $\Psi(\gamma, U)$ by

$$\Psi(\gamma, U)\colon \begin{cases} z' \mapsto (Az' + \mathfrak{b})/(\mathfrak{c}z' + d) \\ z'' \mapsto U\cdot z''/(\mathfrak{c}z' + d)^{1/\alpha} \end{cases} \tag{1.3}$$

for $(z', z'') \in C^k\times C^{n-k}$ (think of $z'$, $z''$ as column vectors). Then, using the equality $|\mathfrak{c}z' + d|^2 - |Az' + \mathfrak{b}|^2 = 1 - |z'|^2$ for all $z' \in C^k$, one can check that each $\Psi(\gamma, U)$ gives rise to a biholomorphic automorphism of $E(k, \alpha)$. In fact, according to Sunada [13] the identity component $\mathrm{Aut}_o(E(k, \alpha))$ of the Lie group $\mathrm{Aut}(E(k, \alpha))$ coincides with the group

$$G(k, \alpha) = \{\Psi(\gamma, U) \,|\, \gamma \in SU(k, 1),\, U \in U(n-k)\}$$

provided that $\alpha \neq 1$. More precisely, we here assert that $\mathrm{Aut}(E(k, \alpha)) = G(k, \alpha)$ in our case. To verify this assertion, observe that the $G(k, \alpha)$-orbit passing through the origin $o \in E(k, \alpha)$ is of lowest dimension in the set of all $G(k, \alpha)$-orbits, i.e., $\dim(G(k, \alpha)\cdot o) < \dim(G(k, \alpha)\cdot z)$ for any point $z \in E(k, \alpha)\setminus G(k, \alpha)\cdot o$. Hence

$$g\cdot G(k, \alpha)\cdot o = G(k, \alpha)\cdot o = \{(z', 0) \in C^k\times C^{n-k} \,|\, |z'| < 1\}$$

for each $g \in \mathrm{Aut}(E(k, \alpha))$. This combined with a well-known theorem of H. Cartan [8; p. 67] assures that every element $g$ of $\mathrm{Aut}(E(k, \alpha))$ can be expressed as $g = \psi_g\cdot l_g$ for some $\psi_g \in G(k, \alpha)$ and $l_g \in GL(n; C)$. In particular, $l_g$ can be written in the form

$$l_g(z', z'') = (Az' + Bz'', Dz'')\,, \quad (z', z'') \in C^k\times C^{n-k}\,,$$

where $A \in SU(k) = SL(k; C)\cap U(k)$, $B \in M(k, n-k)$ and $D \in GL(n-k; C)$.

Then the fact $l_g(\partial E(k, \alpha)) = \partial E(k, \alpha)$ yields that

$$2\,\mathrm{Re}(Az', Bz'') + |Bz''|^2 + |Dz''|^{2\alpha} = |z''|^{2\alpha}\,, \quad (z', z'') \in \partial E(k, \alpha)\,,$$

where $(\cdot, \cdot)$ denotes the standard Hermitian inner product on $\boldsymbol{C}^k$. Consequently, $B = 0$, $D \in U(n-k)$ and $l_g(z', z'') = (Az', Dz'')$ for $A \in SU(k)$, $D \in U(n-k)$. Finally, noting that both groups $SU(k)$ and $U(n-k)$ are naturally imbedded in $G(k, \alpha)$, we conclude that $l_g \in G(k, \alpha)$ and so $\mathrm{Aut}(E(k, \alpha)) = G(k, \alpha)$, as desired.

Next we consider an arbitrary sequence $\{p^\nu\}_{\nu=1}^\infty$ in $E(k, \alpha)$ which converges to the point $p = (1, 0, \cdots, 0) \in \partial E(k, \alpha)$. Then there exists a sequence $\{\psi_\nu\}_{\nu=1}^\infty$ in $Aut(E(k, \alpha))$ such that

$$\psi_\nu(p^\nu) = (0, \cdots, 0, \tilde{t}_\nu) \quad \text{with} \quad 0 \leqq \tilde{t}_\nu < 1 \tag{1.4}$$

for all $\nu = 1, 2, \cdots$. Indeed, since the product group $SU(k) \times U(n-k)$ is naturally identified with a subgroup of $\mathrm{Aut}(E(k, \alpha))$, we may assume that

$$p^\nu = (x_\nu, 0, \cdots, 0, y_\nu) \quad \text{with} \quad 0 \leqq x_\nu,\ y_\nu < 1 \tag{1.5}$$

for $\nu = 1, 2, \cdots$. Consider the one-parameter subgroup

$$\gamma(t) = \left(\begin{array}{c|c|c} \cosh t & 0 & \sinh t \\ \hline 0 & E_{k-1} & 0 \\ \hline \sinh t & 0 & \cosh t \end{array}\right), \quad t \in \boldsymbol{R} \tag{1.6}$$

of $SU(k, 1)$ and set $\psi_\nu = \Psi(\gamma(t_\nu), E_{n-k})$, $t_\nu = \tanh^{-1}(-x_\nu)$ for $\nu = 1, 2, \cdots$. Then it is easily seen that each $\psi_\nu(p^\nu)$ has the desired form as in (1.4).

Summarizing the above, we obtain the following:

LEMMA. *The domain $E(k, \alpha)$ has the following properties*:

(1) *$E(k, \alpha)$ is complete hyperbolic in the sense of Kobayashi* [5]. *In particular, it is a taut domain* [4].

(2) *The boundary $\partial E(k, \alpha)$ of $E(k, \alpha)$ is real analytic and strictly pseudoconvex near the point $q = (0, \cdots, 0, 1) \in \partial E(k, \alpha)$.*

(3) *$\mathrm{Aut}(E(k, \alpha))$ is a connected Lie group consisting of all biholomorphic transformations of $E(k, \alpha)$ as defined in* (1.3).

(4) *Let $\{p^\nu\}_{\nu=1}^\infty$ be a sequence in $E(k, \alpha)$ which converges to the point $p = (1, 0, \cdots, 0) \in \partial E(k, \alpha)$. Then there is a sequence $\{\psi_\nu\}_{\nu=1}^\infty$ in $\mathrm{Aut}(E(k, \alpha))$ such that $\psi_\nu(p^\nu) = (0, \cdots, 0, t_\nu)$ with $0 \leqq t_\nu < 1$ for all $\nu = 1, 2, \cdots$.*

Finally we shall define the R-limit. Let us fix a domain $D$ in $\boldsymbol{C}^n$ such that $p = (1, 0, \cdots, 0) \in \partial D$ and the conditions (i), (ii) in Theorem II are satisfied for $D$. Without loss of generality, we may assume that the neighborhood $U$ of p is a small open Euclidean ball with center at $p$ satisfying the following inequalities:

$$2\,\mathrm{Re}(z_1-1)+A\Big[|z_1-1|^2+\sum_{i=2}^{k}|z_i|^2+\Big(\sum_{j=k+1}^{n}|z_j|^2\Big)^{\alpha}\Big]$$
$$\leqq \rho(z)\leqq 2\,\mathrm{Re}(z_1-1)+B\Big[|z_1-1|^2+\sum_{i=2}^{k}|z_i|^2+\Big(\sum_{j=k+1}^{n}|z_j|^2\Big)^{\alpha}\Big]$$

for every point $z\in U$, where $A$ and $B$ are arbitrarily given constants with $0<A<1<B$. Now, denoting by $N$ the unit vector $(1, 0, \cdots, 0)$, we consider the half line $L(z)=\{z+tN\,|\,t\geqq 0\}$ in $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$ for each point $z\in D\cap U$. Then $z$ has a unique farthest point $\zeta(z)$ in the set $\partial D\cap L(z)\cap U$, so that each point $z\in D\cap U$ can be written uniquely in the form $z=\zeta(z)+\lambda(z)N$, $\lambda(z)<0$. In particular, for a given sequence $\{p^\nu\}$ in $D$ converging to $p$ we have

$$(1.7)\qquad p^\nu=\zeta(p^\nu)+\lambda(p^\nu)N\,;\quad \zeta(p^\nu)=(\zeta_1(p^\nu),\cdots,\zeta_n(p^\nu))\in\partial D\cap U\,,\quad \lambda(p^\nu)<0$$

for all sufficiently large $\nu$. Clearly $\zeta(p^\nu)\to p$ and $\lambda(p^\nu)\to 0$ as $\nu\to\infty$.

DEFINITION 2. In the notation above, we say that $\{p^\nu\}$ *converges restrictedly to* $p$, and write $\text{R-lim}_{\nu\to\infty}p^\nu=p$, if the sequence $\{\mathrm{Re}(\zeta_1(p^\nu)-1)/\lambda(p^\nu)\}$ is a bounded sequence in $\boldsymbol{R}$.

We shall present two examples of sequences $\{p^\nu\}$ in $D$ which converge restrictedly to $p$. We set, for an arbitrary $\varepsilon>0$,

$$\Phi(z)=(\mathrm{Im}\,z_1)^2+\sum_{i=2}^{k}|z_i|^2+\Big(\sum_{j=k+1}^{n}|z_j|^2\Big)^{\alpha},\quad z\in\boldsymbol{C}^n\,;$$
$$C(\varepsilon)=\{z\in\boldsymbol{C}^n\,|\,\mathrm{Re}\,z_1\leqq 1-\varepsilon\cdot[\Phi(z)]^{1/2}\}\,.$$

So, if $\alpha=1$, the region $C(\varepsilon)$ is nothing but a cone with vertex at $p$ and axis in the direction of $-N$. The following example tells us that if $\{p^\nu\}$ converges to $p$ non-tangentially in the usual sense, then it converges restrictedly in our sense.

EXAMPLE 1. Assume that $\partial D$ is $C^1$-smooth near the point $p$ and $\{p^\nu\}$ converges to $p$ through the region $C(\varepsilon)$ for some $\varepsilon>0$. Then we have $\text{R-lim}_{\nu\to\infty}p^\nu=p$.

In fact, by our assumption, $\partial D$ is a $C^1$-smooth real hypersurface near $p$ and the vector $N$ is perpendicular to $\partial D$ at $p$ with respect to the Euclidean structure on $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$. Thus we can write uniquely $p^\nu=\zeta^\nu+\lambda^\nu N$ with some $\zeta^\nu\in\partial D$ and $\lambda^\nu<0$ for all sufficiently large $\nu$.

In order to check that the sequence $\{\mathrm{Re}(\zeta_1^\nu-1)/\lambda^\nu\}$ is bounded, we may assume (by passing to a subsequence if necessary) that $\mathrm{Re}(\zeta_1^\nu - 1) \neq 0$ for all $\nu = 1, 2, \cdots$. Since $\mathrm{R}(\zeta^\nu) = o((\mathrm{Re}(\zeta_1^\nu - 1))^2 + \Phi(\zeta^\nu))$ and

$$2\,\mathrm{Re}(\zeta_1^\nu - 1) + (\mathrm{Re}(\zeta_1^\nu - 1))^2 + \Phi(\zeta^\nu) + R(\zeta^\nu) = \rho(\zeta^\nu) = 0$$

for all large $\nu$, it follows that $\lim_{\nu\to\infty} \Phi(\zeta^\nu)/\mathrm{Re}(\zeta_1^\nu - 1) = -2$. On the other hand, we know by assumption that

$$\mathrm{Re}(p_1^\nu - 1) \leqq -\varepsilon\cdot[\Phi(p^\nu)]^{1/2} = -\varepsilon\cdot[\Phi(\zeta^\nu)]^{1/2} < 0$$

for all sufficiently large $\nu$. Thus

$$\begin{aligned}\lambda^\nu/\mathrm{Re}(\zeta_1^\nu - 1) &= [\mathrm{Re}(p_1^\nu - 1) - \mathrm{Re}(\zeta_1^\nu - 1)]/\mathrm{Re}(\zeta_1^\nu - 1) \\ &= |\mathrm{Re}(p_1^\nu - 1)/\mathrm{Re}(\zeta_1^\nu - 1)| - 1 \\ &\geqq \varepsilon\cdot[\Phi(\zeta^\nu)]^{1/2}/|\mathrm{Re}(\zeta_1^\nu - 1)| - 1 \to +\infty\ .\end{aligned}$$

Obviously this implies that $\text{R-}\lim_{\nu\to\infty} p^\nu = p$.

EXAMPLE 2. Let $\{k_\nu\}$ be a sequence of points contained in a compact subset of $E(k, \alpha)$, $\alpha \neq 1$, and let $\lim_{\nu\to\infty} \varphi_\nu(k_\nu) = (1, 0, \cdots, 0)$ for some sequence $\{\varphi_\nu\}$ in $\mathrm{Aut}(E(k, \alpha))$. Then there exists a new sequence $\{\tilde{\varphi}_\nu\}$ in $\mathrm{Aut}(E(k, \alpha))$ such that $\text{R-}\lim_{\nu\to\infty} \tilde{\varphi}_\nu(k_\nu) = (1, 0, \cdots, 0)$.

Indeed, changing $\varphi_\nu$ into a suitable biholomorphic automorphism $\tilde{\varphi}_\nu = f_\nu \circ \varphi_\nu$, $f_\nu \in SU(k) \times U(n-k) \subset \mathrm{Aut}(E(k, \alpha))$ if necessary, we may assume as in (1.5) that

$$\varphi_\nu(k_\nu) = (x_\nu, 0, \cdots, 0, y_\nu) = \zeta^\nu + \lambda^\nu N$$

with $0 \leqq x_\nu, y_\nu < 1$, $\zeta^\nu = (\zeta_1^\nu, 0, \cdots, 0, \zeta_n^\nu) \in \partial E(k, \alpha)$, $\lambda^\nu < 0$ and $N = (1, 0, \cdots, 0)$. Here it can be seen that $\zeta^\nu$ and $\lambda^\nu$ are uniquely determined by $\varphi_\nu(k_\nu)$. Now, we claim that $\text{R-}\lim_{\nu\to\infty} \varphi_\nu(k_\nu) = (1, 0, \cdots, 0)$. To this end, note that $\{k_\nu\}$ lies in a compact subset of $E(k, \alpha)$ and recall the structure of $\mathrm{Aut}(E(k, \alpha))$. Then one can choose an $r$, $0 < r < 1$, in such a way that $\varphi_\nu(k_\nu) \in D(r)$ for all $\nu = 1, 2, \cdots$, where we have set

$$D(r) = \{(x, 0, \cdots, 0, y) \in \boldsymbol{R}^n \,|\, x^2 + (y/r)^{2\alpha} \leqq 1,\ 0 \leqq x, y\}\ .$$

Let us choose a unique point $q^\nu = \zeta^\nu + \mu^\nu N$, $\lambda^\nu \leqq \mu^\nu < 0$, such that

$$(\zeta_1^\nu + \mu^\nu)^2 + (\zeta_n^\nu/r)^{2\alpha} = 1 \quad \text{for each } \nu\ . \tag{1.8}$$

Then, substituting $(\zeta_n^\nu)^{2\alpha} = 1 - (\zeta_1^\nu)^2$ into (1.8) and rearranging the result, we obtain

$$(1 - \zeta_1^\nu)(1 + \zeta_1^\nu)/r^{2\alpha} = (1 - \zeta_1^\nu - \mu^\nu)(1 + \zeta_1^\nu + \mu^\nu)$$

for all $\nu$. Consequently

$$\begin{aligned}(1 - \zeta_1^\nu)/\mu^\nu &= (1 + \zeta_1^\nu + \mu^\nu)/[1 + \zeta_1^\nu + \mu^\nu - (1 + \zeta_1^\nu)/r^{2\alpha}] \\ &\to r^{2\alpha}/(r^{2\alpha} - 1) \quad \text{as} \quad \nu \to \infty\ .\end{aligned}$$

Since $|(\zeta_1^\nu - 1)/\lambda^\nu| \leqq |(\zeta_1^\nu - 1)/\mu^\nu|$ for all $\nu$, we conclude that $\{(\zeta_1^\nu - 1)/\lambda^\nu\}$ is a bounded sequence.

**2. Proof of Theorem I.** Passing to a subsequence if necessary, we may assume that $\{k_\nu\}$ converges to some point $k_o \in K$ and $\{\varphi_\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $\varphi: D \to \bar{D} \subset \boldsymbol{C}^n$. Let us define the holomorphic function $\Psi_p$ on $\boldsymbol{C}^n$ by

$$\Psi_p(z) = \exp(z_1 - 1)\ , \quad z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n\ ,$$

where $p = (1, 0, \cdots, 0) \in \partial E(k, \alpha) \cap \partial D$. Then obviously $\Psi_p$ is a holomorphic function for $E(k, \alpha) \cap U = D \cap U$ peaking at $p$ in the sense that

$$\Psi_p(p) = 1 \quad \text{and} \quad |\Psi_p(z)| < 1 \quad \text{for all} \quad z \in \overline{D \cap U} \setminus \{p\}\ .$$

This combined with the maximum principle for the holomorphic function $\Psi_p \circ \varphi$ defined on an open neighborhood of $k_o$ yields at once that $\varphi(z) = p$ for all $z \in D$. We can therefore assume that

$$\lim_{\nu \to \infty} \varphi_\nu(k_o) = p \quad \text{and} \quad p^\nu := \varphi_\nu(k_o) \in D \cap U = E(k, \alpha) \cap U$$

for $\nu = 1, 2, \cdots$. As in Greene and Krantz [3], we choose a sequence $\{\psi_\nu\}_{\nu=1}^\infty$ in $\mathrm{Aut}(E(k, \alpha))$ such that

$$q^\nu := \psi_\nu(p^\nu) = (0, \cdots, 0, t_\nu) \quad \text{with} \quad 0 \leqq t_\nu < 1 \tag{2.1}$$

for all $\nu = 1, 2, \cdots$. The existence of such a sequence of autmorphisms was already shown in Section 1. We have now two cases to consider.

*Case* 1. *$\{q^\nu\}_{\nu=1}^\infty$ has an accumulation point $q$ in $E(k, \alpha)$.* We shall prove that $D$ is biholomorphically equivalent to $E(k, \alpha)$ in this case. We may assume without loss of generality that

$$\lim_{\nu \to \infty} q^\nu = q \in E(k, \alpha)\ .$$

Now let us fix a family of relatively compact subdomains $D_j$ of $D$ such that

$$D = \bigcup_{j=1}^{\infty} D_j \supset \cdots \supset D_{j+1} \supset D_j \supset \cdots \supset D_1 \ni k_o \tag{2.2}$$

and choose an integer $j \geqq 1$ arbitrarily. Since $\varphi_\nu(z) \to p$ uniformly on $D_j$, there exists an integer $\nu(j)$ such that

$$\varphi_\nu(D_j) \subset D \cap U = E(k, \alpha) \cap U \quad \text{for all} \quad \nu \geqq \nu(j)\ .$$

So we can define biholomorphic mappings $f^\nu: D_j \to E(k, \alpha)$ by setting

$$f^\nu(z) = \psi_\nu(\varphi_\nu(z))\ , \quad z \in D_j \quad \text{for} \quad \nu \geqq \nu(j)\ . \tag{2.3}$$

Since $E(k, \alpha)$ is taut and $f^\nu(k_o) \to q \in E(k, \alpha)$, we can assume by taking a subsequence if necessary that $\{f^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $f(j)\colon D_j \to E(k, \alpha)$. By the usual diagonal argument, we may further assume that $\{f^\nu\}$ converges uniformly on $D_j$ to the holomorphic mapping $f(j)$ for all $j = 1, 2, \cdots$. Accordingly, we can define a holomorphic mapping $f\colon D \to E(k, \alpha)$ by $f(z) = f(j)(z)$, $z \in D_j$ for $j = 1, 2, \cdots$.

Setting $E_\nu = \psi_\nu(E(k, \alpha) \cap U) = \psi_\nu(D \cap U)$ for $\nu = 1, 2, \cdots$, we consider the biholomorphic mappings $g^\nu\colon E_\nu \to D$ defined by

$$g^\nu(z) = \varphi_\nu^{-1}(\psi_\nu^{-1}(z)) \,, \quad z \in E_\nu \quad \text{for} \quad \nu = 1, 2, \cdots .$$

Then it is clear that

$$g^\nu \circ f^\nu = \mathrm{id}_{D_j} \quad \text{and} \quad f^\nu \circ g^\nu = \mathrm{id}_{f^\nu(D_j)} \tag{2.4}$$

for all $\nu \geqq \nu(j)$, $j = 1, 2, \cdots$. Let $E'$ be an arbitrary subdomain of $E(k, \alpha)$ with compact closure. Then $\psi_\nu^{-1}(E') \subset E(k, \alpha) \cap U$ for all sufficiently large $\nu$. Passing to a subsequence if necessary, we can therefore assume that $\{g^\nu\}$ converges uniformly on every compact subset of $E(k, \alpha)$ to a holomorphic mapping $g\colon E(k, \alpha) \to \bar{D} \subset \boldsymbol{C}^n$. Once $g(E(k, \alpha)) \subset D$ is shown, the equations (2.4) imply that $g \circ f = \mathrm{id}_D$ and $f \circ g = \mathrm{id}_{E(k,\alpha)}$; consequently, $f$ gives a biholomorphic mapping from $D$ onto $E(k, \alpha)$. Thus we have only to show that $g(E(k, \alpha)) \subset D$. To this end, take a subdomain $E'$ of $E(k, \alpha)$ with compact closure such that $f(\bar{D}_1)$, $f^\nu(\bar{D}_1) \subset E'$ for all $\nu \geqq \nu_0$, where $D_1$ is the domain appearing in (2.2) and $\nu_0$ is a large integer. Then, for any point $z \in D_1$ there is a sequence $\{z_i\}_{i=1}^\infty$ in $E'$ such that $g^{\nu_i}(z_i) = z$ for all $i$ and $z_i \to z_0$ for some point $z_0 \in \bar{E}'$. Hence $z = \lim_{i\to\infty} g^{\nu_i}(z_i) = g(z_0) \in g(E(k, \alpha))$, and accordingly, $D_1 \subset g(E(k, \alpha))$. On the other hand, being the local uniform limit of regular holomorphic mappings $\{g^\nu\}$, the mapping $g$ is either regular on $E(k, \alpha)$ or the Jacobian determinant of $g$ vanishes identically on $E(k, \alpha)$. But, $g(E(k, \alpha))$ contains a non-empty open set in $\boldsymbol{C}^n$, as we have already seen above. Hence we conclude that $g\colon E(k, \alpha) \to \boldsymbol{C}^n$ is regular on $E(k, \alpha)$ and so $g(E(k, \alpha)) \subset D$ by [1; Lemma 0] or [8; p. 79], completing the proof in Case 1.

*Case* 2. *$\{q^\nu\}_{\nu=1}^\infty$ has no accumulation point in $E(k, \alpha)$.* In this case we show that both domains $D$ and $E(k, \alpha)$ are biholomorphically equivalent to the open unit ball $B^n$. We may assume that

$$\lim_{\nu\to\infty} q^\nu = (0, \cdots, 0, 1) =: q \in \partial E(k, \alpha) .$$

Since $q$ is a strictly pseudoconvex boundary point of $E(k, \alpha)$ by the lemma in Section 1, there exist a small open neighborhood $W$ of $q$ and a $C^2$-strictly plurisubharmonic function $\rho\colon W \to \boldsymbol{R}$ such that

(2.6) $$W \subset \{(z', z'') \in C^k \times C^{n-k} \mid |z'| \leqq 1/2\} ;$$

(2.7) $$E(k, \alpha) \cap W = \{z \in W \mid \rho(z) < 0\} \quad \text{and} \quad d\rho(z) \neq 0,\ z \in W ;$$

(2.8) $$(\partial\rho(q)/\partial z_1, \cdots, \partial\rho(q)/\partial z_{n-1}, \partial\rho(q)/\partial z_n) = (0, \cdots, 0, 1) .$$

To simplify the notation, we set

$$a_{ij} = (1/2)\cdot\partial^2\rho(q)/\partial z_i\partial z_j\ , \quad b_{i\bar{j}} = \partial^2\rho(q)/\partial z_i\partial\bar{z}_j$$

for $1 \leqq i,\ j \leqq n$ and consider the coordinate changes as follows:

$$H_1: u_j = z_j \quad (1 \leqq j \leqq n-1)\ , \quad u_n = z_n - 1\ ;$$

$$H_2: v_j = u_j \quad (1 \leqq j \leqq n-1)\ , \quad v_n = u_n + \sum_{i,j=1}^{n} a_{ij}u_iu_j\ .$$

Clearly, $H_1$ is a globally defined change of coordinates and $H_2$ is a well-defined change of coordinates in a sufficiently small neighborhood of $u = o$. In the new coordinates $v = (v_1, \cdots, v_n)$, we have by Taylor's formula

$$\rho(v) = 2\,\mathrm{Re}\, v_n + \sum_{i,j=1}^{n} b_{i\bar{j}}v_i\bar{v}_j + o(|v|^2)$$

in a neighborhood of the origin,

$$q = (0, \cdots, 0) \quad \text{and} \quad q^\nu = (0, \cdots, 0, \delta_\nu)$$

with $\delta_\nu = (t_\nu - 1)[1 + a_{nn}(t_\nu - 1)]$ for $\nu = 1, 2, \cdots$, where $t_\nu$ are the numbers given by (2.1). Hence

(2.9) $$\lim_{\nu\to\infty}(\delta_\nu, \delta_\nu/|\delta_\nu|) = (0, -1)\ .$$

In particular, we may assume that $0 < |\delta_\nu| < 1$ for all $\nu = 1, 2, \cdots$. Since $(b_{i\bar{j}})_{1\leq i,j\leq n-1}$ is a positive definite Hermitian matrix of degree $n-1$, it is diagonalizable. Thus, after a suitable change of coordinates $(v_1, \cdots, v_{n-1})$ in $C^{n-1}$, we can obtain a new coordinate system $w = (w_1, \cdots, w_n)$, $w_n = v_n$, with respect to which $\rho$ can be written in the form

(2.10) $$\rho(w) = 2\,\mathrm{Re}\, w_n + |'w|^2 + A(w)$$

in a small neighborhood of the origin, where $'w = (w_1, \cdots, w_{n-1})$ as in Section 1 and

$$A(w) = 2\,\mathrm{Re}\Big(\sum_{j=1}^{n} c_j w_j \bar{w}_n\Big) + o(|w|^2)$$

with some constants $c_1, \cdots, c_n \in C$. In particular, there are a continuous function $r(x)$ and a constant $C > 0$ such that

(2.11) $$r(x) \to 0 \quad \text{as} \quad x \to 0\ ;$$

$$|A(w)| \leqq C|w||w_n| + r(|w|^2)|w|^2 \quad \text{near} \quad w = o . \tag{2.12}$$

Let $\{D_j\}_{j=1}^{\infty}$ be the increasing family of relatively compact subdomains of $D$ defined in (2.2). Then, as in (2.3) we can define a family of biholomorphic mappings $f^\nu = \psi_\nu \circ (\varphi_{\nu|D_j})$ for $\nu \geqq \nu(j)$, $j = 1, 2, \cdots$ which converges uniformly on compact subsets to a holomorphic mapping $f: D \to \overline{E(k, \alpha)} \subset \boldsymbol{C}^n$ with $f(k_o) = q \in \partial E(k, \alpha)$. Taking now the plurisubharmonic function $\rho \circ f$ defined on an open neighborhood of $k_o$ instead of the holomorphic function $\Psi_p \circ \varphi$ in Case 1, we can see that $f(z) = q$ for all $z \in D$. Let us fix an integer $j \geqq 1$ arbitrarily. Then, since $f^\nu(z) \to q$ uniformly on $D_j$, there exists an integer $\nu_j$ such that

$$f^\nu(D_j) \subset E(k, \alpha) \cap W \quad \text{for all} \quad \nu \geqq \nu_j .$$

We define mappings $L_\nu: \boldsymbol{C}^n \to \boldsymbol{C}^n$ and $F^\nu: D_j \to \boldsymbol{C}^n$ by setting

$$L_\nu(w) = ('w/\sqrt{|\delta_\nu|}, -w_n/\delta_\nu) , \quad w = ('w, w_n) \in \boldsymbol{C}^n ; \tag{2.13}$$

$$F^\nu(z) = L_\nu(f^\nu(z)) , \quad z \in D_j \tag{2.14}$$

for all $\nu \geqq \nu_j$, where $\delta_\nu$ are the numbers appearing in (2.9). Then $L_\nu$ are non-singular linear transformations of $\boldsymbol{C}^n$ and $F^\nu$ are biholomorphic mappings $D_j$ into $\boldsymbol{C}^n$. Moreover, it is easily seen by construction that

$$F^\nu(k_o) = (0, \cdots, 0, -1) \quad \text{and} \quad F^\nu(D_j) \subset W_\nu \tag{2.15}$$

for all $\nu \geqq \nu_j$, where

$$W_\nu = L_\nu(E(k, \alpha) \cap W) = \{w \in \boldsymbol{C}^n \,|\, L_\nu^{-1}(w) \in W, \ \rho \circ L_\nu^{-1}(w) < 0\} \tag{2.16}$$

for $\nu = 1, 2, \cdots$. Now we would like to show that some subsequence of $\{F^\nu\}$ converges uniformly on every compact set in $D$ to a holomorphic mapping $F: D \to \boldsymbol{C}^n$. To see this, we set

$$\rho^\nu(w) = [\rho \circ L_\nu^{-1}(w)]/|\delta_\nu| \quad \text{and} \quad A^\nu(w) = [A \circ L_\nu^{-1}(w)]/|\delta_\nu|$$

for $\nu = 1, 2, \cdots$. It follows then from (2.10), (2.12) that

$$\rho^\nu(w) = 2\,\mathrm{Re}(-\delta_\nu w_n/|\delta_\nu|) + |'w|^2 + A^\nu(w) ; \tag{2.17}$$

$$|A^\nu(w)| \leqq [C\sqrt{|\delta_\nu|} + r(|L_\nu^{-1}(w)|^2)] \cdot |w|^2 \tag{2.18}$$

in a neighborhood of the origin. Now, for the sake of simplicity we put

$$w^\nu = F^\nu(z) \quad \text{for each point} \quad z \in D_j .$$

Since $L_\nu^{-1}(w^\nu) = f^\nu(z) \to q = o$ uniformly on $D_j$, it follows from (2.11) and (2.18) that $|A^\nu(w^\nu)|/|w^\nu|^2 \to 0$ uniformly on $D_j$. This combined with the inequality $\rho^\nu(w^\nu) < 0$ for $\nu \geqq \nu_j$ yields that

$$|w_n^\nu|^2 + 2\,\mathrm{Re}(\delta_\nu w_n^\nu/|\delta_\nu|) > |w^\nu|^2 + A^\nu(w^\nu) \geqq |w^\nu|^2/2 \geqq 0 \tag{2.19}$$

for all $\nu \geqq \nu_0$ and all $z \in D_j$, where $\nu_0$ is a large integer depending on $D_j$. Here we may assume by (2.9) that $|1 + (\delta_\nu/|\delta_\nu|)| < 1/3$ for $\nu \geqq \nu_0$. Thus $\{F_n^\nu\}_{\nu \geqq \nu_0}$ forms a normal family, because $F_n^\nu$ for every $\nu \geqq \nu_0$ can now be regarded as a holomorphic mapping from $D_j$ into the taut domain $\boldsymbol{C} \setminus \{1/2, 1\}$. Moreover $F_n^\nu(\{k_0\}) \cap \{-1\} \neq \varnothing$ for all $\nu$ by (2.15). Hence we may assume that $\{F_n^\nu\}_{\nu \geqq \nu_0}$ converges uniformly on compact subsets to a holomorphic function on $D_j$. By (2.19) this means that $\{F^\nu\}_{\nu \geqq \nu_0}$ is uniformly bounded on every compact subset of $D_j$, and consequently some subsequence of $\{F^\nu\}_{\nu \geqq \nu_0}$ converges uniformly on compact subsets to a holomorphic mapping from $D_j$ into $\boldsymbol{C}^n$. Hence, passing again to a subsequence if necessary, we may assume that $\{F^\nu\}$ itself converges uniformly on every compact set in $D$ to a holomorphic mapping $F: D \to \boldsymbol{C}^n$.

Here we consider the following domain $\mathscr{B}$ and the mapping $C$:

$$\mathscr{B} = \{w \in \boldsymbol{C}^n \,|\, 2 \operatorname{Re} w_n + |'w|^2 < 0\} ; \tag{2.20}$$

$$C: ('w, w_n) \mapsto (\sqrt{2}\, 'w/(w_n - 1), (w_n + 1)/(w_n - 1)) . \tag{2.21}$$

It is easily seen that there is an open neighborhood $X$ of $\bar{\mathscr{B}}$ such that $C$ gives rise to a biholomorphic mapping from $X$ into $\boldsymbol{C}^n$ and $C(\mathscr{B}) = B^n$. In particular, $\mathscr{B}$ is a strictly pseudoconvex domain with real analytic boundary. Now we wish to show that $F(D) \subset \mathscr{B}$. For this let us fix a point $z \in D$ arbitrarily. Then, since $w^\nu = F^\nu(z) \to F(z)$ and $L_\nu^{-1}(w^\nu) = f^\nu(z) \to q = o$ as $\nu \to \infty$, we obtain from (2.9), (2.17) and (2.18) that

$$2 \operatorname{Re} F_n(z) + |'F(z)|^2 = \lim_{\nu \to \infty} \rho^\nu(w^\nu) \leqq 0 ,$$

which says that $F(D) \subset \bar{\mathscr{B}}$. But, thanks to the strict pseudoconvexity of $\mathscr{B}$, the image $F(D)$ can meet the boundary $\partial\mathscr{B}$ only when $F$ is a constant mapping from $D$ into $\partial\mathscr{B}$. Consequently, $F(D) \subset \mathscr{B}$, since by (2.15) $F(D)$ contains the point $(0, \cdots, 0, -1)$ of $\mathscr{B}$.

Next we prove that $F: D \to \mathscr{B}$ is, in fact, a biholomorphic mapping from $D$ onto $\mathscr{B}$. Observe first that $L_\nu^{-1}(W_\nu) = E(k, \alpha) \cap W$ for all $\nu$ and $\psi_\nu^{-1}(E(k, \alpha) \cap W) \to \{p\}$ by the choice of $W$ as in (2.6). Hence there is an integer $\nu_0$ such that

$$\psi_\nu^{-1}(L_\nu^{-1}(W_\nu)) \subset E(k, \alpha) \cap U = D \cap U \quad \text{for all} \quad \nu \geqq \nu_0$$

and so we can define holomorphic mappings $G^\nu: W_\nu \to D$ by setting

$$G^\nu = \varphi_\nu^{-1} \circ \psi_\nu^{-1} \circ L_\nu^{-1} \quad \text{for} \quad \nu \geqq \nu_0 .$$

Clearly we have $G^\nu \circ F^\nu = \mathrm{id}_{D_j}$ and $F^\nu \circ G^\nu = \mathrm{id}_{F^\nu(D_j)}$ for all $\nu \geqq \max(\nu(j), \nu_0)$, $j = 1, 2, \cdots$. On the other hand, for an arbitrarily given subdomain $\mathscr{B}'$ of $\mathscr{B}$ with compact closure in $\mathscr{B}$ one can choose an integer $\nu(\mathscr{B}')$ in such a way that $\mathscr{B}'\subset W_\nu$ for all $\nu \geqq \nu(\mathscr{B}')$, because $\rho^\nu(w) \to 2\operatorname{Re} w_n + |'w|^2 < 0$ uniformly on $\mathscr{B}'$ by (2.9), (2.17) and (2.18). Therefore, passing to a subsequence if necessary, we may assume that $\{G^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $G: \mathscr{B} \to \bar{D}\subset C^n$. With exactly the same method as in Case 1 one can now check that $G(\mathscr{B})\subset D$ and $F$ defines a biholomorphic mapping from $D$ onto the domain $\mathscr{B} \cong B^n$.

Finally, assuming the correctness of Theorem II, we shall complete the proof by showing that $D$ and $E(k, \alpha)$ are both biholomorphically equivalent to $B^n$. For this purpose, let us choose a sequence of positive numbers $x_\nu$ in such a way that

$$x_\nu \uparrow 1 \quad \text{and} \quad p^\nu := (x_\nu, 0, \cdots, 0) \in D\cap U$$

for $\nu = 0, 1, 2, \cdots$. Since $D$ is now biholomorphically equivalent to $B^n$, there exists a sequence $\{\sigma_\nu\}_{\nu=1}^\infty$ in $\operatorname{Aut}(D)$ such that $\sigma_\nu(p^0) = p^\nu$ for $\nu = 1, 2, \cdots$. In particular, we have $\text{R-lim}_{\nu\to\infty} \sigma_\nu(p^0) = (1, 0, \cdots, 0)$. Moreover $D\cap U = E(k, \alpha)\cap U = \{z \in U\,|\,\rho(k, \alpha; z) < 0\}$ by assumption. As an immediate consequence of Theorem II, $D$ is biholomorphically equivalent to $E(k, \alpha)$. q.e.d.

**3. Proof of Theorem II.** By the change of coordinates $u_1 = z_1 - 1$, $u_j = z_j$ $(2 \leqq j \leqq n)$, we have $p = (0, \cdots, 0)$ and $\rho$ can be written in the form

$$\rho(u) = 2\operatorname{Re} u_1 + |u'|^2 + |u''|^{2\alpha} + R(u)\,, \quad R(u) = o(|u'|^2 + |u''|^{2\alpha})$$

in a neighborhood of the origin $u = o$. For any given constants $A$, $B$ with $0 < A < 1 < B$, we can therefore assume that

$$(3.1) \qquad 2\operatorname{Re} u_1 + A(|u'|^2 + |u''|^{2\alpha}) \leqq \rho(u) \leqq 2\operatorname{Re} u_1 + B(|u'|^2 + |u''|^{2\alpha})$$

on $U$ by shrinking $U$ if necessary. So the holomorphic function $\Psi_p(u) = \exp u_1$ on $C^n$ is peaking for $D\cap U$ at $p = o$. Hence, by the same reasoning as in Case 1 of the proof of Theorem I we may assume without loss of generality that

$$(3.2) \qquad \varphi_\nu(z) \to p \quad \text{uniformly on compact subsets of } D\,;$$

$$(3.3) \qquad \text{R-}\lim_{\nu\to\infty} \varphi_\nu(k_\nu) = p \quad \text{and} \quad p^\nu := \varphi_\nu(k_\nu) \in D\cap U\,, \quad \nu = 1, 2, \cdots.$$

Therefore, writing

$$(3.4) \qquad p^\nu = \zeta^\nu + \lambda^\nu N \quad \text{with some} \quad \zeta^\nu \in \partial D\cap U\,, \quad \lambda^\nu < 0$$

uniquely as in (1.7) and taking a subsequence if necessary, we obtain by the assumption (iii) that

$$\lim_{\nu\to\infty} \operatorname{Re} \zeta_1^\nu/|\lambda^\nu| = d_o \quad \text{for some finite number} \quad d_o \leqq 0 .$$

For the sake of simplicity, we set

$$r_\nu = |\lambda^\nu|^{1/2} , \quad s_\nu = |\lambda^\nu|^{1/(2\alpha)} \quad \text{for} \quad \nu = 1, 2, \cdots .$$

The proof is now divided into two cases as follows:

*Case* 1. $d_o = 0$. In this case, it follows at once from (3.1) that

$$(3.5) \qquad (\operatorname{Re} \zeta_1^\nu/|\lambda^\nu|, \zeta_i^\nu/r_\nu, \zeta_j^\nu/s_\nu, R(\zeta^\nu)/|\lambda^\nu|) \to (0, 0, 0, 0)$$

as $\nu \to \infty$ for each $i$, $j$ with $1 \leqq i \leqq k < j \leqq n$. Let us choose a sequence of relatively compact subdomains $D_j$ of $D$ such that

$$D = \bigcup_{j=1}^{\infty} D_j \supset \cdots \supset D_{j+1} \supset D_j \supset \cdots \supset D_1 \supset K ,$$

where $K$ is the compact subset of $D$ as in the theorem, and fix an integer $j \geqq 1$ arbitrarily. Since $\varphi_\nu(u) \to p$ uniformly on $D_j$, there exists an integer $\nu(j)$ such that

$$(3.6) \qquad \varphi_\nu(D_j) \subset D \cap U \quad \text{for all} \quad \nu \geqq \nu(j) .$$

Now define mappings $h_\nu$, $L_\nu$ and $F^\nu$ by

$$\begin{aligned} &h_\nu(u) = (u_1 - \zeta_1^\nu, \cdots, u_n - \zeta_n^\nu) , \quad u \in \boldsymbol{C}^n ; \\ &L_\nu(w) = (-w_1/\lambda^\nu, w_2/r_\nu, \cdots, w_k/r_\nu, w_{k+1}/s_\nu, \cdots, w_n/s_\nu) , \quad w \in \boldsymbol{C}^n ; \\ &F^\nu(u) = L_\nu \circ h_\nu \circ \varphi_\nu(u) , \quad u \in D_j \end{aligned}$$

for all $\nu \geqq \nu(j)$. Then both $h_\nu$ and $L_\nu$ are biholomorphic transformations of $\boldsymbol{C}^n$, while $F^\nu$ are biholomorphic mapping from $D_j$ into $\boldsymbol{C}^n$. It is clear that

$$(3.7) \qquad F^\nu(k_\nu) = (-1, 0, \cdots, 0) \quad \text{and} \quad F^\nu(D_j) \subset W_\nu$$

for all $\nu \geqq \nu(j)$, where

$$(3.8) \qquad W_\nu = \{w \in \boldsymbol{C}^n \,|\, (L_\nu \circ h_\nu)^{-1}(w) \in U, \ \rho \circ (L_\nu \circ h_\nu)^{-1}(w) < 0\}$$

for $\nu = 1, 2, \cdots$. Now we claim that some subsequence of $\{F^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $F: D \to \boldsymbol{C}^n$. For this, we set

$$\rho^\nu(w) = \rho \circ (L_\nu \circ h_\nu)^{-1}(w) , \quad R^\nu(w) = R \circ (L_\nu \circ h_\nu)^{-1}(w)$$

for $\nu = 1, 2, \cdots$ and

$$w^\nu = F^\nu(u) , \quad u \in D_j \quad \text{for} \quad \nu \geqq \nu(j) .$$

Then, since $(L_\nu \circ h_\nu)^{-1}(F^\nu(D_j)) = \varphi_\nu(D_j) \subset D \cap U$ for $\nu \geqq \nu(j)$, we obtain by (3.1), (3.7) and (3.8) that

$$0 > \rho^\nu(w^\nu) \geqq 2\,\mathrm{Re}(-\lambda^\nu w_1^\nu + \zeta_1^\nu) + A\cdot\left[|-\lambda^\nu w_1^\nu + \zeta_1^\nu|^2 + \sum_{i=2}^{k}|r_\nu w_i^\nu + \zeta_i^\nu|^2 + \left(\sum_{j=k+1}^{n}|s_\nu w_j^\nu + \zeta_j^\nu|^2\right)^\alpha\right]$$

and so

$$0 > 2\,\mathrm{Re}(w_1^\nu + \zeta_1^\nu/|\lambda^\nu|) + A\cdot\left[\sum_{i=2}^{k}|w_i^\nu + \zeta_i^\nu/r_\nu|^2 + \left(\sum_{j=k+1}^{n}|w_j^\nu + \zeta_j^\nu/s_\nu|^2\right)^\alpha\right]$$

for all $\nu \geqq \nu(j)$. Hence, if we define a domain $W(k, \alpha, A)$ in $\boldsymbol{C}^n$ and holomorphic mappings $\Phi^\nu\colon D_j \to \boldsymbol{C}^n$, $\nu \geqq \nu(j)$, by setting

$$(3.9) \qquad W(k, \alpha, A) = \left\{w \in \boldsymbol{C}^n \,|\, 2\,\mathrm{Re}\, w_1 + A\cdot\left[\sum_{i=2}^{k}|w_i|^2 + \left(\sum_{j=k+1}^{n}|w_j|^2\right)^\alpha\right] < 0\right\};$$

$$(3.10) \qquad \Phi^\nu = (F_1^\nu + \mathrm{Re}\,\zeta_1^\nu/|\lambda^\nu|,\ F_2^\nu + \zeta_2^\nu/r_\nu,\ \cdots,\ F_k^\nu + \zeta_k^\nu/r_\nu,\ F_{k+1}^\nu + \zeta_{k+1}^\nu/s_\nu,\ \cdots,\ F_n^\nu + \zeta_n^\nu/s_\nu)\,,$$

then every $\Phi^\nu$ gives rise to a holomorphic mapping from $D_j$ into $W(k, \alpha, A)$. On the other hand, it is easily seen that $W(k, \alpha, A)$ is biholomorphically equivalent to the domain $E(k, \alpha)$ via the correspondence $C_A\colon (w_1, \cdots, w_n) \mapsto (z_1, \cdots, z_n)$ given by

$$(3.11) \qquad C_A\colon \begin{cases} z_1 = (w_1 + 1)/(w_1 - 1) \\ z_i = (2A)^{1/2}\cdot w_i/(w_1 - 1)\,, & i = 2, \cdots, k \\ z_j = (2A)^{1/(2\alpha)}\cdot w_j/(w_1 - 1)^{1/\alpha}\,, & j = k+1, \cdots, n\,. \end{cases}$$

Hence $W(k, \alpha, A)$ is taut by the lemma in Section 1 and $\{\Phi^\nu\}$ forms a normal family. Moreover, it follows from (3.5) and (3.7) that

$$\Phi^\nu(k_\nu) = (-1 + \mathrm{Re}\,\zeta_1^\nu/|\lambda^\nu|,\ \zeta_2^\nu/r_\nu,\ \cdots,\ \zeta_k^\nu/r_\nu,\ \zeta_{k+1}^\nu/s_\nu,\ \cdots,\ \zeta_n^\nu/s_\nu) \to (-1, 0, \cdots, 0) \in W(k, \alpha, A) \quad \text{as} \quad \nu \to \infty\,,$$

that is, $\{\Phi^\nu\}$ is not compactly divergent on $D_j$. Therefore we may assume that $\{\Phi^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $\Phi\colon D_j \to W(k, \alpha, A)$. Here it is obvious from (3.5) and (3.10) that $\lim_{\nu\to\infty} F^\nu = \Phi$ uniformly on compact subsets of $D_j$. By the usual diagonal argument, we may further assume that $\{F^\nu\}$ itself converges uniformly on every compact subset of $D$ to a holomorphic mapping $F\colon D \to \boldsymbol{C}^n$.

We wish to prove that the image $F(D)$ is contained in the domain $W(k, \alpha) := W(k, \alpha, 1)$ defined in (3.9) with $A = 1$. To this end, recall that $R(u) = o(|u'|^2 + |u''|^{2\alpha})$. So there is a continuous function $r(x)$ such that

$$(3.12) \qquad r(x) \to 0 \quad \text{as} \quad x \to 0\,;$$

(3.13) $|R(u)| \leqq r(|u'|^2 + |u''|^{2\alpha})\cdot[|u'|^2 + |u''|^{2\alpha}]$ near the origin.

Since $(L_\nu \circ h_\nu)^{-1}(w) \to o$ uniformly on compact sets, these combined with (3.5) yield that

$$|R^\nu(w)/\lambda^\nu| \leqq r(x_\nu)\cdot y_\nu \to 0 \quad \text{as} \quad \nu \to \infty$$

uniformly on every compact subset of $C^n$, where we have set

$$x_\nu = |[(L_\nu \circ h_\nu)^{-1}(w)]'|^2 + |[(L_\nu \circ h_\nu)^{-1}(w)]''|^{2\alpha} ;$$

$$y_\nu = |r_\nu w_1 + \zeta_1^\nu/r_\nu|^2 + \sum_{i=2}^{k}|w_i + \zeta_i^\nu/r_\nu|^2 + |w'' + (\zeta^\nu)''/s_\nu|^{2\alpha} .$$

Now take a point $u \in D$ arbitrarily and set again $w^\nu = F^\nu(u)$. Then $w^\nu \to F(u)$ as $\nu \to \infty$ and it follows from (3.7), (3.8) that

$$(3.14) \qquad 0 > \rho^\nu(w^\nu)/|\lambda^\nu| = 2\,\mathrm{Re}(w_1^\nu + \zeta_1^\nu/|\lambda^\nu|) + |r_\nu w_1^\nu + \zeta_1^\nu/r_\nu|^2 + \sum_{i=2}^{k}|w_i^\nu + \zeta_i^\nu/r_\nu|^2 + \Big(\sum_{j=k+1}^{n}|w_j^\nu + \zeta_j^\nu/s_\nu|^2\Big)^\alpha + R^\nu(w^\nu)/|\lambda^\nu|$$

for all sufficiently large $\nu$, and so letting $\nu$ tend to infinity, we have

$$0 \geqq 2\,\mathrm{Re}\,F_1(u) + \sum_{i=2}^{k}|F_i(u)|^2 + \Big(\sum_{j=k+1}^{n}|F_j(u)|^2\Big)^\alpha .$$

Clearly this means $F(u) \in \overline{W(k, \alpha)}$ and accordingly $F(D) \subset \overline{W(k, \alpha)}$.

Next step is to show that $F(D) \subset W(k, \alpha)$. Observe first that the interior of the closure $\overline{W(k, \alpha)}$ coincides with $W(k, \alpha)$ in our case. Hence the problem reduces to showing that $F\colon D \to C^n$ is an open mapping. We define biholomorphic mappings $G^\nu\colon W_\nu \to D$, $\nu = 1, 2, \cdots$, by

$$G^\nu(w) = \varphi_\nu^{-1} \circ h_\nu^{-1} \circ L_\nu^{-1}(w) , \quad w \in W_\nu ,$$

where $W_\nu$ are the domains given by (3.8). Clearly we have

$$(3.15) \qquad G^\nu \circ F^\nu{}_{|D_j} = \mathrm{id}_{D_j} \quad \text{and} \quad F^\nu \circ G^\nu{}_{|F^\nu(D_j)} = \mathrm{id}_{F^\nu(D_j)}$$

for all $\nu \geqq \nu(j)$, $j = 1, 2, \cdots$. Let $W'$ be an arbitrary subdomain of $W(k, \alpha)$ with compact closure. Then we obtain by (3.5) and (3.14) that

$$\rho^\nu(w)/|\lambda^\nu| \to 2\,\mathrm{Re}\,w_1 + \sum_{i=2}^{k}|w_i|^2 + \Big(\sum_{j=k+1}^{n}|w_j|^2\Big)^\alpha < 0$$

uniformly on $W'$. Thus there exists an integer $\nu(W')$ such that

$$(3.16) \qquad W' \subset W_\nu \quad \text{for all} \quad \nu \geqq \nu(W') .$$

Now, by the compactness of $K$ we may assume that $k_\nu \to k_o \in K$. Then $F(k_o) = \lim_{\nu\to\infty} F^\nu(k_\nu) = (-1, 0, \cdots, 0) \in W(k, \alpha)$. Choose open neighborhoods $W'$, $D'$ of the points $(-1, 0, \cdots, 0)$, $k_o$ with compact closures in $W(k, \alpha)$,

$D$, respectively, in such a way that $F(\bar{D}')\subset W'$. There exists an integer $\nu(D', W')$ so large that

$$F^{\nu}(D')\subset W' \quad \text{for all} \quad \nu \geqq \nu(D', W') . \tag{3.17}$$

Once it is shown that $F: D \to C^n$ is injective on $D'$, $F(D)$ contains the non-empty open set $F(D')$, accordingly, we may conclude by the same reasoning as in the proof of Theorem I that $F(D)\subset W(k, \alpha)$. Now assume that $F(u_1) = F(u_2) = w$ for some $u_1, u_2 \in D'$. It follows then from (1.1) and (3.15) $\sim$ (3.17) that

$$\begin{aligned} d_{W'}(F^{\nu}(u_1), F^{\nu}(u_2)) &= d_{G^{\nu}(W')}(G^{\nu}(F^{\nu}(u_1)), G^{\nu}(F^{\nu}(u_2))) \\ &= d_{G^{\nu}(W')}(u_1, u_2) \geqq d_D(u_1, u_2) \end{aligned}$$

for all $\nu \geqq \max(\nu(W'), \nu(D', W'))$, and so letting $\nu \to \infty$ we have $u_1 = u_2$, as desired.

Finally we assert that $F: D \to W(k, \alpha)$ is a biholomorphic mapping from $D$ onto $W(k, \alpha)$. Indeed, thanks to the fact (3.16) we may assume without loss of generality that $\{G^{\nu}\}$ converges uniformly on every compact set in $W(k, \alpha)$ to a holomorphic mapping $G: W(k, \alpha) \to \bar{D}\subset C^n$. Then, repeating exactly the same argument as in the proof of Theorem I, we can verify that $G(W(k, \alpha))\subset D$ and $F$ defines a biholomorphic mapping from $D$ onto $W(k, \alpha)$. Since the domain $W(k, \alpha)$ is biholomorphically equivalent to $E(k, \alpha)$ via the correspondence $C_1$ defined by (3.11), we have completed the proof in the first case.

*Case* 2. $d_o \neq 0$. The following proof will be presented in outline, since the details of the steps can be filled in by consulting the corresponding passages in Case 1.

Passing to a subsequence if necessary, we may assume by (3.1) together with the estimate $R(u) = o(|u'|^2 + |u''|^{2\alpha})$ that

$$(\operatorname{Re} \zeta_1^{\nu}/|\lambda^{\nu}|,\ \zeta_i^{\nu}/r_{\nu},\ \zeta_j^{\nu}/s_{\nu},\ R(\zeta^{\nu})/|\lambda^{\nu}|) \to (d_o, d_i, d_j, 0) \tag{3.18}$$

for each $i$, $j$ with $1 \leqq i \leqq k < j \leqq n$, where $d_i$, $d_j$ are some finite complex numbers. Let us define holomorphic mappings $F^{\nu}$ and $\Phi^{\nu}$ in the same manner as in Case 1. Then, repeating exactly the same arguments as in Case 1, we can show that some subsequence of $\{\Phi^{\nu}\}$ converges uniformly on compact subsets of $D$ to a holomorphic mapping $\Phi: D \to W(k, \alpha, A)$, where $W(k, \alpha, A)$ is the domain in $C^n$ defined by (3.9). Clearly this combined with (3.10), (3.18) guarantees that some subsequence of $\{F^{\nu}\}$ also converges uniformly on compact subsets to a holomorphic mapping $F$: $D \to C^n$. In exactly the same way as in Case 1, it can be shown that $F$ defines a biholomorphic mapping from $D$ onto the domain

$$W'(k, \alpha) = \left\{ w \in \boldsymbol{C}^n \,|\, 2\,\mathrm{Re}(w_1 + d_0 + |d_1|^2/2) + \sum_{i=2}^{k} |w_i + d_i|^2 + \left( \sum_{j=k+1}^{n} |w_j + d_j|^2 \right)^{\alpha} < 0 \right\} ,$$

which is obviously biholomorphically equivalent to $W(k, \alpha)$ via a parallel translation in $\boldsymbol{C}^n$. Therefore, we have shown that $D$ is also biholomorphically equivalent to $E(k, \alpha)$ in Case 2. q.e.d.

4. **Proof of Theorem III.** To begin with, we fix a family $\{M_j\}_{j=1}^{\infty}$ of relatively compact subdomains of $M$ such that

$$(4.1) \qquad M = \bigcup_{j=1}^{\infty} M_j \supset \cdots \supset M_{j+1} \supset M_j \supset \cdots \supset M_1 \ni k_0 ,$$

where $k_0$ is an arbitrarily fixed point of $M$. Since $M$ can be exhausted by biholomorphic images of $E(k, \alpha)$, there exists a sequence $\{\psi_\nu\}_{\nu=1}^{\infty}$ of biholomorphic mappings from $E(k, \alpha)$ into $M$ such that

$$M_\nu \subset \psi_\nu(E(k, \alpha)) , \quad \nu = 1, 2, \cdots .$$

We set

$$\varphi_\nu = \psi_\nu^{-1} \colon \psi_\nu(E(k, \alpha)) \to E(k, \alpha) , \quad \nu = 1, 2, \cdots .$$

Without loss of generality, we may assume that $\{\varphi_\nu\}$ converges uniformly on every compact set in $M$ to a holomorphic mapping $\varphi\colon M \to \overline{E(k, \alpha)} \subset \boldsymbol{C}^n$. Replacing $\psi_\nu$, $\varphi_\nu$ by suitable holomorphic mappings of the form $\psi_\nu \circ \sigma_\nu^{-1}$, $\sigma_\nu \circ \varphi_\nu$ with some $\sigma_\nu \in \mathrm{Aut}(E(k, \alpha))$, if necessary, we may further assume that

$$q^\nu := \varphi_\nu(k_0) = (0, \cdots, 0, t_\nu) \quad \text{with} \quad 0 \leqq t_\nu < 1$$

for all $\nu = 1, 2, \cdots$. Again we have two cases to consider.

*Case* 1. *$\{q^\nu\}$ has an accumulation point $q$ in $E(k, \alpha)$.* We claim that $M$ is biholomorphically equivalent to $E(k, \alpha)$. We may assume that $q^\nu \to q$ and $\{\varphi^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $\varphi\colon M \to E(k, \alpha)$, since $E(k, \alpha)$ is taut and $\{\varphi_\nu(k_0)\}$ lies in a compact subset of $E(k, \alpha)$. Here we assert that $\varphi\colon M \to E(k, \alpha)$ is injective. Indeed, suppose that $\varphi(x_1) = \varphi(x_2) = z$ for $x_1, x_2 \in M$. It follows then from (1.1) that

$$\begin{aligned} d_{E(k,\alpha)}(\varphi_\nu(x_1), \varphi_\nu(x_2)) &= d_{\psi_\nu(E(k,\alpha))}(\psi_\nu(\varphi_\nu(x_1)), \psi_\nu(\varphi_\nu(x_2))) \\ &= d_{\psi_\nu(E(k,\alpha))}(x_1, x_2) \geqq d_M(x_1, x_2) \end{aligned}$$

for all sufficiently large $\nu$. Consequently, we have $x_1 = x_2$, because $M$ is hyperbolic and $d_{E(k,\alpha)}(\varphi_\nu(x_1), \varphi_\nu(x_2)) \to d_{E(k,\alpha)}(z, z) = 0$ as $\nu \to \infty$. Therefore, identifying $M$ with the bounded domain $\varphi(M)\subset E(k, \alpha)$ and replacing the system $(\{f^\nu\}, \{g^\nu\}, D, \{D_j\})$ by $(\{\varphi_\nu\}, \{\psi_\nu\}, M, \{M_j\})$ in Case 1 of the proof of Theorem I, we can show that $M$ is biholomorphically equivalent to $E(k, \alpha)$.

*Case* 2. *$\{q^\nu\}_{\nu=1}^\infty$ has no accumulation point in $E(k, \alpha)$.* In this case, we shall prove that $M$ is biholomorphically equivalent to the open unit ball $B^n$. Without loss of generality, we may assume that:

$$\lim_{\nu\to\infty} q^\nu = (0, \cdots, 0, 1) =: q \in \partial E(k, \alpha) ; \tag{4.2}$$

$$\varphi_\nu(x) \to q \quad \text{uniformly on compact subsets of } M . \tag{4.3}$$

Hence there exists an integer $\nu_j$ such that

$$\varphi_\nu(M_j)\subset E(k, \alpha)\cap W \quad \text{for all} \quad \nu \geqq \nu_j ,$$

where $M_j$ is an arbitrary subdomain of $M$ appearing in the sequence (4.1) and $W$ is the same neighborhood of $q$ as that defined in Case 2 of the proof of Theorem I. Introducing a new coordinate system $w = (w_1, \cdots, w_n)$ in $C^n$ as in Case 2 of the proof of Theorem I, we define biholomorphic mappings $L_\nu: C^n \to C^n$ and $F^\nu: M_j \to C^n$ for $\nu \geqq \nu_j$ by

$$L_\nu(w) = ('w/\sqrt{|\delta_\nu|}, -w_n/\delta_\nu) , \quad w = ('w, w_n) \in C^n ;$$
$$F^\nu(x) = L_\nu(\varphi_\nu(x)) , \quad x \in M_j$$

as in (2.13) and (2.14). Then it can be shown that some subsequence of $\{F^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $F: M \to \mathscr{B}$, where $\mathscr{B}$ is the domain in $C^n$ defined in (2.20). Indeed, considering the biholomorphic mappings

$$G^\nu(w) = \psi_\nu(L_\nu^{-1}(w)) , \quad w \in L_\nu(E(k, \alpha)\cap W) = W_\nu$$

for $\nu = 1, 2, \cdots$, one can check that $F$ is a biholomorphic mapping from $M$ into $\mathscr{B} \cong B^n$. In particular, $M$ can be regarded as a bounded domain in $C^n$. Therefore, repeating the same argument as in Case 2 of the proof of Theorem I, we conclude that $M$ is biholomorphically equivalent to the domain $\mathscr{B} \cong B^n$. q.e.d.

**5. Concluding remarks.** Let $D$ be a domain in $C^n$ and $p$ a point of $\bar{D}$. Then we say that *$D$ is hyperbolically imbedded at $p$* if, for any neighborhood $W$ of $p$ in $C^n$, there exists a neighborhood $V$ of $p$ in $C^n$ such that

$$\bar{V}\subset W \quad \text{and} \quad d_D(D\cap(C^n \setminus W), D\cap V) > 0 .$$

Note that, if $D$ is a bounded domain in $C^n$, then $D$ is hyperbolically imbedded at every point $p$ of $\bar{D}$.

REMARK 1. In Theorems I and II, the boundedness assumption on $D$ can be replaced by the following weaker one: *$D$ is a not necessarily bounded hyperbolic domain in $C^n$ which is hyperbolically imbedded at $p=(1, 0, \cdots, 0)\in\partial D$.*

Indeed, by the existence of a local peaking function for $D$ at $p$, one can extract in the same manner as in [7; Lemma 2] a subsequence of $\{\varphi_\nu\}\subset\mathrm{Aut}(D)$ which converges uniformly on compact subsets of $D$ to the constant mapping $C_p(z)=p$, $z\in D$. Hence, the rests of the proofs of Theorems I and II will go through without any change.

REMARK 2. By a simple modification of the proof of Theorem II, one can see that the analogue of Theorem II is also valid for more general domains

$$E=\left\{(z_1, \cdots, z_s)\in \boldsymbol{C}^{n_1}\times\cdots\times\boldsymbol{C}^{n_s}\,\middle|\,|z_1|^2+\sum_{i=2}^{s}|z_i|^{2\alpha_i}<1\right\},$$

where $0\leqq n_i\in\boldsymbol{Z}$, $0<\alpha_i\in\boldsymbol{R}$ for $i=2, \cdots, s$ and $1\leqq n_1\in\boldsymbol{Z}$.

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