Journal of Statistical Theory and Applications, Vol. 13, No. 4 (December 2014), 311-316

Characterizations of distributions via record values with random exponential shifts

Mohammad Ahsanullah

College of Business Administration, Rider University
Lawrenceville, New Jersey 08648, U.S.A.
E-mail: ahsan@rider.edu

Valery B. Nevzorov

Department of Mathematics and Mechanics
St. Petersburg University, St. Petersburg, Russia 198904
E-mail: valnev@mail.ru

Ludmila Nevzorova

St. Petersburg Humanitarian University of Trade Unions
St. Petersburg, Russia 192238
E-mail: studnev@mail333.com

Received 12 March 2014
Accepted 11 July 2014

Abstract

Some new characterizations of probability distributions based on properties of record values shifted by independent exponential variables are obtained. Explicit expressions for the inverse functions of the corresponding variables are presented. Among others some characterizations of exponential distributions and limiting distributions of maximal and minimal extreme values are given.

Keywords and phrases: Characterizations; Order statistics; Record values; Exponential distribution; Weibull distribution.

1. Introduction

In some sense the given work is a continuation of the investigations presented in [1]. Among others the following problem was investigated there.

Let X, X_1 and X_2 be independent identically distributed random variables (r.v.'s) with an absolutely continuous distribution function (d.f.) F(x). Let also ξ_1 and ξ_2 be independent (and independent from X's) r.v.'s having the standard E(1) exponential distribution with the density function p(x)=exp(-x), x≥0. It is known (see, for example, [1]-[4],[6]) that the following relations

$$X+\xi_1 \overset{d}{=} \max\{X_1,X_2\} \tag{1}$$

and

$$X \overset{d}{=} \max\{X_1,X_2\} - \xi_2 \tag{2}$$

characterize correspondingly the exponential distributions with d.f.'s

$$F(x) = 1-\exp(-(x-C)/2),\ x\geq C, \tag{3}$$

where C is any arbitrary constant, and the logistic distributions with d.f.'s

Copyright: the authors

$$F(x) = C/(C+\exp(-x)),\ -\infty<x<\infty, \tag{4}$$

where C is any positive constant.

It is interesting to mention that very similar relations (1) and (2) characterize two rather different (exponential and logistic) distributions. Hence it is naturally to ask if it is possible to describe some general family of distributions which unites these two types (3) and (4). The natural way to find the necessary family is to consider more general than (1) and (2) relation

$$X_1+a\xi_1 \stackrel{d}{=} \max\{X_1,X_2\} - b\xi_2, \tag{5}$$

where $a\geq 0$, $b\geq 0$ and $a+b>0$.

Let Q(x) be the inverse (quantile) function of F(x), i.e.

$$Q(x)=\inf\{y: F(y) \geq x\},\ 0<x<1. \tag{6}$$

As a partial case of Theorem 3 from [1] it can be obtained that (5) holds if and only if

$$Q(x)=\log\{Cx^b(1-x)^{-d}\},\ 0<x<1, \tag{7}$$

where C is any positive constant and $d=2a+b$.

Indeed, if $a=1$ and $b=0$ one gets from (7) that

$$Q(x)=\log\{C(1-x)^{-2}\},\ 0<x<1, \tag{8}$$

and hence F(x) has the form given in (3).

If $a=0$ and $b=1$ we obtain from (7) that

$$Q(x)=\log\{Cx/(1-x)\},\ 0<x<1, \tag{9}$$

and then

$$F(x)=C_1/(C_1+\exp(-x)),\ -\infty<x<\infty,$$

where $C_1=1/C>0$.

Mention here one more (in some sense, intermediate between two previous) case when it is possible to get F(x) from (7) in the explicit form.

Let $a=1/2$ and $b=1$. Then one can obtain that F(x) has the following form:

$$F(x)=1+\exp(-(x-C))-((1+\exp(-(x-c)))^2-1)^{1/2},\ -\infty<x<\infty, \tag{10}$$

where C is an arbitrary constant.

Thus (7) describes the family of distributions which includes both types of distributions (exponential and logistic) mentioned above as partial cases.

Now we are going to solve the analogous problem for one more very popular in the probability theory class of random variables - the so-called record values.

2. Characterizations of probability distributions by properties of record statistics

Let X, X_1, X_2,… be a sequence of independent r.v.'s with a common absolutely continuous d.f. F(x). Define upper record times ***L(n)*** and upper record values ***X(n),*** n=1,2,…, as follows:

$$L(1)=1,\ L(n)=\min\{j>L(n-1):\ X_j>X_{L(n)}\},\ n=2,3,\dots$$

and

$$X(n)=X_{L(n)},\ n=1,2,\dots.$$

Analogously it is possible to determine the lower record times ***l(n)*** and the lower record values ***x(n).*** In this situation

$$l(1)=1,\ \ l(n)=\min\{j>l(n-1):\ X_j<X_{l(n)}\},\ n=2,3,\dots,$$

and

$$x(n)=X_{l(n)},\ n=1,2,\dots.$$

Indeed, $X(n)=\max\{X_1,X_2,\dots,X_{L(n)}\}$ and $x(n)=\min\{X_1,X_2,\dots,X_{l(n)}\}$, n=1,2,….

As above, for simplicity, we consider the case n=2 only. The first situation is connected with the equality

$$X+\xi_1 \overset{d}{=} X(2). \qquad (11)$$

As the partial case of the general theorem (*Theorem 1*), which will be presented below, it will be seen that (11) holds if and only if X has the exponential d.f.

$$F(x)=1-\exp(-(x-C)),\ x\geq C, \qquad (12)$$

where C is an arbitrary constant.

As another partial case of the Theorem 1 it can be obtained that the relation

$$X=X(2)-\xi_2 \qquad (13)$$

is valid if and only if

$$F(x)=1-\exp(-\exp(x-C)),\ \ -\infty<x<\infty, \qquad (14)$$

where C is any arbitrary constant. Note that d.f. (14) presents one of three possible families of limiting distributions for minimal order statistics.

Now our aim is to find the general family of d.f.'s which includes the both ((12) and (14)) given types of distributions. The natural way for it is to consider the relations which are analogous to equality (5):

$$X_1+a\xi_1 \overset{d}{=} X(2)-b\xi_2, \qquad (15)$$

where a≥0, b≥0 and a+b>0.

Let Q(x) as above be the inverse function of F(x). It appears that the following result is valid.

Theorem 1. *Let X, X_1, X_2,… be a sequence of independent r.v.'s with a common continuous d.f. F(x). Then relation (15) holds for any arbitrary a and b, such that a≥0, b≥0 and a+b>0, if and only if*

$$Q(x)=b\log(-\log(1-x))-a\log(1-x)+C,\ 0<x<1, \tag{16}$$

where C is any constant.

Corollary 1. *If a=1, b=0 , then Q(x)=C-log(1-x), 0<x<1, and hence*

$$F(x)=1-\exp(-(x-C)),\ \text{if } x\geq C, \text{ and } F(x)=0,\ \text{if } x<C.$$

Thus the equality $X+\xi_1 \stackrel{d}{=} X(2)$ characterizes the exponential distribution with d.f.

$$F(x)=\max\{0,\ 1-\exp(-(x-C)).$$

Corollary 2. *If a=0, b=1 , then Q(x)=log(-log(1-x)), 0<x<1, and hence*

$$F(x)=1-\exp(-\exp(x-C)),\ -\infty<x<\infty,$$

that is the equality X=X(2)- ξ_2 characterizes the limiting for minimal order statistics type of distributions.

As usual if we have some results for the upper record values X(n) it is easy to obtain the analogous results for the lower record values x(n). It is enough to consider r.v.'s $Y_1=-X_1$,

$Y_2=-X_2$,… instead of X's. Then the corresponding lower records y(1)>y(2)>… for the sequence Y_1,Y_2,… and the upper record values X(1)<X(2)<… for X's satisfy in distribution the following relation:

$$y(n) \stackrel{d}{=} -X(n),\ n=1,2,\dots .$$

Hence Theorem 1 can be rewritten for the lower records as follows.

Theorem 2. *Let X, X_1,X_2,… be a sequence of independent r.v.'s with a common continuous d.f. F(x). Then the relation*

$$X- a\xi_1 \stackrel{d}{=} x(2) + b\xi_2, \tag{17}$$

holds for any arbitrary a and b, such that a≥0, b≥0 and a+b>0, if and only if

$$Q(x)=C-b\log(-\log x)+a\log x,\ 0<x<1, \tag{18}$$

where C is any constant.

Corollary 3. *If a=0, b=1 in (17) , then Q(x)=C-log(-logx), 0<x<1, and*

$$F(x)=\exp(-\exp(-(x-C))),\ -\infty<x<\infty,$$

where C is any arbitrary constant.
That is the equality

$$x(1) - \xi_1 \stackrel{d}{=} x(2) \tag{19}$$

characterizes the so-called Weibull (or doubly exponential) distribution, which belongs to one of three classical families of limit distributions for maximal order statistics.

3. Proof of Theorem 1.

Consider the situation with a>0 and b>0. The following result, which was proved in [1], can be useful.

Lemma. *Let Z_1 and Z_2 be two r.v.'s with continuous d.f.'s G(x) and H(x), respectively. Let also ξ_1 and ξ_2 be independent (and independent from Z's) r.v.'s having the common standard E(1) exponential distribution . Then the equality*

$$Z_1+a\xi_1 \overset{d}{=} Z_2 - b\xi_2 \tag{20}$$

holds for some a≥0, b≥0 and a+b>0, if and only if

$$bG'(x)+aH'(x)=G(x)-H(x). \tag{21}$$

In our case (let us recall equality (15)) we must take as G(x) and H(x) d.f.'s

$$G(x)=P(X<x)=F(x) \tag{22}$$

and

$$H(x)=P(X(2)<x)=1-(1-F(x))(1-\log(1-F(x))). \tag{23}$$

Expression (23) is the partial case of the general equality

$$P(X(n)<x)=1-(1-F(x))\sum_{k=0}^{n-1} (-\log(1-F(x)))^k/k!,\ n=1,2,\ldots, \tag{24}$$

which is valid for record values (see, for example, [5]).

In this case H'(x)=-F'(x)log(1-F(x)). We can get by substituting d.f.'s (22) and (23) to (21) that

$$b F'(x) -a F'(x)\log(1-F(x))=-(1-F(x))\log(1-F(x))$$

and hence

$$F'(x)(a\log(1-F(x))-b)=(1-F(x))\log(1-F(x)). \tag{25}$$

Denote R(x)=-log(1-F(x)). Then R'(x)=F'(x)/(1-F(x)).

We can rewrite (25) now as

$$bR'(x)/R(x) +aR'(x)=1. \tag{26}$$

It means that

$$C+b\log R(x)+aR(x)=x,$$

where C is some constant. Hence we see that d.f. F(x) satisfies the following relation

$$C+b\log(-\log(1-F(x)))-a\log(1-F(x))=x. \tag{27}$$

Substituting inverse function Q(x) instead of x to (27) one gets that

$$Q(x)=b\log(-\log(1-x)) -a\log(1-x) +C,$$

that is (16) holds for the case, when a>0 and b>0.

Copyright: the authors

Analogously it can be proved that (16) is valid also for situations when a=0, b>0 and a>0,b=0.

Thus Theorem 1 is proved.□

Acknowledgements
The work of the first author was supported by a summer 2014 grant from Rider University.
The work of the second author was partially supported by St. Petersburg State University (project no. 6.38.672.2013) and by the Russian Foundation for the Humanities (project no. 13_02_00338).

References

[1] M. Ahsanullah, V.B. Nevzorov and G.P. Yanev, Characterizations of distributions via order statistics with random exponential shifts, *J. Applied Statistical Science*, **18** (3) (2011) 1-8.
[2] M. Ahsanullah, A. Berred and V.B. Nevzorov, On characterizations of the exponential distributions, *J. Applied Statistical Science*, **19** (1) (2011) 37-42.
[3] E. O. George and G. S. Mudholkar, Some relationships between the logistic and the exponential distributions, in *Statistical Distributions in Scientific Work*, Vol.4, eds. C. Tallie et al. (Reidel, Dordrecht,1981), pp. 401-409.
[4] G. D. Lin and C.-Y. Hu, On characterizations of the logistic distribution, *J. Statistical Planning and Inference* **138** (2008) 1147-1156.
[5] V.B. Nevzorov, *Records: Mathematical Theory* (American Math. Soc., Providence, RI, 2001).
[6] V. O. Zykov and V. B. Nevzorov. On some characterizations of families of distributions, including logistic or exponential, by properties of order statistics, *Notes of Sci. Semin. POMI* **384** (2010) 176-181 (in Russian).