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Characterizations of Fuzzy Subalgebras in BCK/BCI-Algebras

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Abstract

The concepts of $(\in, \in\vee q_0^\delta)$-fuzzy subalgebras and $\in\vee q_0^\delta$-level sets

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are introduced, and related properties are investigated. Relations between an $(\in, \in)$-fuzzy subalgebra and an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra are discussed, and characterizations of $(\in, \in\vee q_0^\delta)$-fuzzy subalgebras are discussed. Homomorphic image and pre-image of an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra are considered.

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1 Introduction

Murali [7] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [8], played a vital role to generate some different types of fuzzy algebraic structures. It is worth pointing out that Bhakat and Das [1, 2] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in\vee q)$-fuzzy subgroup. In particular, $(\in, \in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Also, Jun [3, 4] considered the concepts of (α, β)-fuzzy subalgebras by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subalgebra, and introduced the concept of an $(\in, \in\vee q)$-fuzzy subalgebra. As a general form of "quasi-coincident with" relation (q), Jun et al. [5] introduced the concept of "δ-quasi-coincident with" relation (q_0^δ), and apply it to fuzzy subgroups.

In this paper, we apply this new notion to BCK/BCI-algebras. We introduce the notion of $(\in, \in\vee q_0^\delta)$-fuzzy subalgebras, which is a generalization of $(\in, \in\vee q)$-fuzzy subalgebras, and investigate related properties. We discuss relations between an $(\in, \in)$-fuzzy subalgebra and an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra. We give a condition for an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra to be an $(\in, \in)$-fuzzy subalgebra. We provide characterizations of an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra. We consider the homomorphic (pre) image of an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra.

2 Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(i) $(\forall x, y, z \in X)\,(((x * y) * (x * z)) * (z * y) = 0)$,

(ii) $(\forall x, y \in X)\,((x * (x * y)) * y = 0)$,

(iii) $(\forall x \in X)\,(x * x = 0)$,

(iv) $(\forall x, y \in X)\,(x * y = y * x = 0 \Rightarrow x = y)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. We refer the reader to the books [6] for further information regarding BCK/BCI-algebras.

A fuzzy set λ in a set X of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set λ in a set X, Pu and Liu [8] gave meaning to the symbol $x_t \alpha \lambda$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

To say that $x_t \in \lambda$ (resp. $x_t\, q\, \lambda$) means that $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set λ.

To say that $x_t \in\vee q\, \lambda$ (resp. $x_t \in\wedge q\; \lambda$) means that $x_t \in \lambda$ or $x_t\, q\, \lambda$ (resp. $x_t \in \lambda$ and $x_t\, q\, \lambda$).

Jun et al. [5] considered a general form of quasi-coincident fuzzy point. Let $\delta \in (0,1]$. For a fuzzy point x_t and a fuzzy set λ in a set X, we say that x_t is a *δ-quasi-coincident* with λ, written $x_t\, q_0^\delta\, \lambda$, (see [5]) if $\lambda(x) + t > \delta$.

Obviously, $x_t\, q\, \lambda$ implies $x_t\, q_0^\delta\, \lambda$. If $\delta = 1$, then the δ-quasi-coincident with λ is the quasi-coincident with λ, that is, $x_t\, q_0^1\, \lambda = x_t\, q\, \lambda$.

To say that $x_t \in\vee\, q_0^\delta\, \lambda$ (resp. $x_t \in\wedge\, q_0^\delta \lambda$) means that $x_t \in \lambda$ or $x_t\, q_0^\delta\, \lambda$ (resp. $x_t \in \lambda$ and $x_t\, q_0^\delta\, \lambda$).

3 Generalizations of $(\in, \in\vee q)$-fuzzy subalgebras

In what follows let δ and X denote an element of $(0,1]$ and a BCK/BCI-algebra, respectively, unless otherwise specified.

Definition 3.1 *A fuzzy set λ in X is called an (α, β)-fuzzy subalgebra of X if for all $x, y \in X$ and $t, r \in (0, \delta]$,*

$$x_t\, \alpha\, \lambda,\; y_r\, \alpha\, \lambda \;\Rightarrow\; (x * y)_{\min\{t,r\}} \beta\, \lambda, \tag{1}$$

where $\alpha \in \{\in, q, q_0^\delta,\}$ and $\beta \in \{\in, q_0^\delta, \in\vee q_0^\delta,\}$.

We say that $x_t\,\overline{\alpha}\,\lambda$ if $x_t\,\alpha\,\lambda$ does not hold.

Example 3.2 *Let $X = \{0, a, b, c\}$ be a BCI-algebra in which the operation $*$ is described by Table 1.*

Table 1: Cayley table of the operation $*$

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a fuzzy set λ in X as follows:

$$\lambda : X \to [0,1],\ x \mapsto \begin{cases} 0.45 & \text{if } x = 0, \\ 0.74 & \text{if } x = b, \\ 0.35 & \text{if } x \in \{a, c\}. \end{cases}$$

Then λ is an $(\in, \in\vee\, q_0^{\delta})$-fuzzy subalgebra of X with $\delta \in (0, 0.9]$. If $\delta \in (0.9, 1]$, then λ is not an $(\in, \in\vee\, q_0^{\delta})$-fuzzy subalgebra of X.

It is obvious that if $\delta_1 \geq \delta_2$ in $(0, 1]$, then every $(\in, \in\vee\, q_0^{\delta_1})$-fuzzy subalgebra is an $(\in, \in\vee\, q_0^{\delta_2})$-fuzzy subalgebra, but the converse is not true as seen in Example 3.2.

Proposition 3.3 *If λ is a nonzero $(q_0^{\delta}, \in)$ (or $(q_0^{\delta}, q_0^{\delta})$)-fuzzy subalgebra of X, then the set*

$$X_0 := \{x \in X \mid \lambda(x) > 0\}$$

is a subalgebra of X.

Proof. Let $x, y \in X_0$. Then $\lambda(x) > 0$ and $\lambda(y) > 0$. Hence $\lambda(x) + \delta > \delta$ and $\lambda(y) + \delta > \delta$, that is, $x_\delta\, q_0^{\delta}\, \lambda$ and $y_\delta\, q_0^{\delta}\, \lambda$. It follows from (1) that $(x * y)_\delta \in \lambda$, i.e., $\lambda(x * y) \geq \delta > 0$. Thus $x * y \in X_0$. Thus X_0 is a subalgebra of X. For $(q_0^{\delta}, q_0^{\delta})$-fuzzy subalgebra case, we can prove similarly.

Proposition 3.4 *Let S be a subalgebra of X and let λ be a fuzzy set in X such that*

(i) $\lambda(x) \geq \frac{\delta}{2}$ *for all $x \in S$,*

(ii) $\lambda(x) = 0$ *for all $x \in X \setminus S$.*

Then λ is a $(q_0^\delta, \in\vee q_0^\delta)$-fuzzy subalgebra of X.

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, \delta]$ such that $x_{t_1}\, q_0^\delta\, \lambda$ and $y_{t_2}\, q_0^\delta\, \lambda$. Then $\lambda(x) + t_1 > \delta$ and $\lambda(y) + t_2 > \delta$, which imply that $x, y \in S$. Hence $x * y \in S$, and so $\lambda(x * y) \geq \frac{\delta}{2}$. If $\min\{t_1, t_2\} > \frac{\delta}{2}$, then $\lambda(x * y) + \min\{t_1, t_2\} > \delta$, i.e., $(x * y)_{\min\{t_1,t_2\}}\, q_0^\delta\, \lambda$. If $\min\{t_1, t_2\} \leq \frac{\delta}{2}$, then $\lambda(x * y) \geq \frac{\delta}{2} \geq \min\{t_1, t_2\}$ and so $(x * y)_{\min\{t_1,t_2\}} \in \lambda$. Therefore $(x * y)_{\min\{t_1,t_2\}} \in\vee\, q_0^\delta\, \lambda$, and λ is a $(q_0^\delta, \in\vee q_0^\delta)$-fuzzy subalgebra of X.

Theorem 3.5 *For any fuzzy set λ in X, the following are equivalent.*

(i) *λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X.*

(ii) $(\forall x, y \in X) \left(\lambda(x * y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}\right)$.

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X. For any $x, y \in X$, we consider two cases:

$$(1)\text{: } \min\{\lambda(x), \lambda(y)\} < \tfrac{\delta}{2} \text{ and } (2)\text{: } \min\{\lambda(x), \lambda(y)\} \geq \tfrac{\delta}{2}.$$

For the first case, suppose that $\lambda(x * y) < \min\{\lambda(x), \lambda(y)\}$ and take $t \in (0, \frac{\delta}{2}]$ such that $\lambda(x * y) < t \leq \min\{\lambda(x), \lambda(y)\}$. Then $x_t \in \lambda$ and $y_t \in \lambda$ but $(x * y)_{\min\{t,t\}} = (x * y)_t\, \overline{\in\vee q_0^\delta}\, \lambda$ since $(x * y)_t\, \overline{\in}\, \lambda$ and $\lambda(x * y) + t < 2t < \delta$, that is, $(x * y)_t\, \overline{q_0^\delta}\, \lambda$. This is a contradiction. Hence $\lambda(x * y) \geq \min\{\lambda(x), \lambda(y)\}$ whenever $\min\{\lambda(x), \lambda(y)\} < \frac{\delta}{2}$. Now assume that the case (2) is valid. Then $x_{\frac{\delta}{2}} \in \lambda$ and $y_{\frac{\delta}{2}} \in \lambda$, which imply that $(x * y)_{\frac{\delta}{2}} = (x * y)_{\min\left\{\frac{\delta}{2}, \frac{\delta}{2}\right\}} \in\vee\, q_0^\delta\, \lambda$. If $\lambda(x * y) < \frac{\delta}{2}$, then $\lambda(x * y) + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta$ and so $\lambda(x * y) < \frac{\delta}{2}$ which shows that $(x * y)_{\frac{\delta}{2}}\, \overline{\in}\, \lambda$ and $(x * y)_{\frac{\delta}{2}}\, \overline{q_0^\delta}\, \lambda$. This is a contradiction, and thus $\lambda(x * y) \geq \frac{\delta}{2}$. Therefore $\lambda(x * y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in X$.

Conversely assume that (ii) is valid. Let $x, y \in X$ and $t, r \in (0, \delta]$ such that $x_t \in \lambda$ and $y_r \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq r$. Suppose that $\lambda(x * y) < \min\{t, r\}$. If $\min\{\lambda(x), \lambda(y)\} < \frac{\delta}{2}$, then

$$\lambda(x * y) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, r\},$$

a contradiction. Hence $\min\{\lambda(x), \lambda(y)\} \geq \frac{\delta}{2}$, and so

$$\lambda(x * y) + \min\{t, r\} > 2\lambda(x * y) \geq 2\min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} = \delta.$$

This shows that $(x * y)_{\min\{t,r\}}\, q_0^\delta\, \lambda$. Therefore λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X.

Obviously, every $(\in, \in)$-fuzzy subalgebra is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra, but the converse is not true in general. In fact, the $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra λ with $\delta \in (0, 0.9]$ in Example 3.2 is not an $(\in, \in)$-fuzzy subalgebra of X since $b_{0.6} \in \lambda$ and $b_{0.7} \in \lambda$ but $(b * b)_{\min\{0.6,0.7\}} = 0_{0.6}\, \overline{\in}\, \lambda$.

We provide a condition for an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra to be an $(\in, \in)$-fuzzy subalgebra.

Proposition 3.6 *Let λ be an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X such that $\lambda(x) < \frac{\delta}{2}$ for all $x \in X$. Then λ is an $(\in, \in)$-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $t, r \in (0, \delta]$ be such that $x_t \in \lambda$ and $y_r \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq r$. It follows from the hypothesis and Theorem 3.5 that

$$\lambda(x * y) \geq \min\{\lambda(x),\, \lambda(y),\, \tfrac{\delta}{2}\} = \min\{\lambda(x),\, \lambda(y)\} \geq \min\{t,\, r\}$$

so that $(x * y)_{\min\{t,r\}} \in \lambda$. Hence λ is an $(\in, \in)$-fuzzy subalgebra of X.

For a subset S of X, a fuzzy set χ_S^δ in X defined by

$$\chi_S^\delta : X \to [0,1],\ x \mapsto \begin{cases} \delta & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

is called a δ-*characteristic fuzzy set* of S in X (see [5]).

Theorem 3.7 *For any subset S of X and the δ-characteristic fuzzy set χ_S^δ of S in X, the following are equivalent:*

(i) *S is a subalgebra of X.*

(ii) *χ_S^δ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X.*

Proof. (i) $\Rightarrow$ (ii) is straightforward.

Assume that χ_S^δ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X. Let $x, y \in S$. Then $\chi_S^\delta(x) = \delta = \chi_S^\delta(y)$, and so $x_\delta \in \chi_S^\delta$ and $y_\delta \in \chi_S^\delta$. It follows that $(x * y)_\delta = (x * y)_{\min\{\delta,\delta\}} \in\vee q_0^\delta\, \lambda$, which yields $\chi_S^\delta(x * y) > 0$. Hence $x * y \in S$ and S is a subalgebra of X.

Theorem 3.8 *A fuzzy set λ in X is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X if and only if the set*

$$U(\lambda; t) := \{x \in X \mid \lambda(x) \geq t\}$$

is a subalgebra of X for all $t \in (0, \frac{\delta}{2}]$.

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X. Let $t \in (0, \frac{\delta}{2}]$ and $x, y \in U(\lambda; t)$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$. It follows from Theorem 3.5 that

$$\lambda(x * y) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} \geq \min\{t, \tfrac{\delta}{2}\} = t$$

and so that $x * y \in U(\lambda; t)$. Therefore $U(\lambda; t)$ is a subalgebra of X.

Conversely, let λ be a fuzzy set in X such that $U(\lambda; t)$ is a subalgebra of X for all $t \in (0, \frac{\delta}{2}]$. Suppose that there are elements a and b of X such that

$$\lambda(a*b) < \min\{\lambda(a), \lambda(b), \tfrac{\delta}{2}\},$$

and take $t \in (0, \delta]$ such that $\lambda(a*b) < t \leq \min\{\lambda(a), \lambda(b), \frac{\delta}{2}\}$. Then $a, b \in U(\lambda; t)$ and $t \leq \frac{\delta}{2}$, which implies that $a*b \in U(\lambda; t)$ since $U(\lambda; t)$ is a subalgebra of X. This induces $\lambda(a*b) \geq t$, and this is a contradiction. Hence $\lambda(x*y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in X$, and therefore λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X by Theorem 3.5.

We say that the subalgebra $U(\lambda; t)$ in Theorem 3.8 is a *level subalgebra* of X.

Theorem 3.9 *Let $\{\lambda_i \mid i \in \Lambda\}$ be a family of $(\in, \in\vee q_0^\delta)$-fuzzy subalgebras of X. Then $\lambda := \bigcap_{i\in\Lambda} \lambda_i$ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X.*

Proof. Suppose that $x_t \in \lambda$ and $y_r \in \lambda$ for all $x, y \in X$ and $t, r \in (0, \delta]$. Assume that $(x*y)_{\min\{t,r\}} \overline{\in\vee q_0^\delta}\, \lambda$. Then $\lambda(x*y) < \min\{t, r\}$ and $\lambda(x*y) + \min\{t, r\} \leq \delta$, which imply that

$$\lambda(x*y) < \tfrac{\delta}{2} \tag{2}$$

Let $\Omega_1 := \{i \in \Lambda \mid (x*y)_{\min\{t,r\}} \in \lambda_i\}$ and

$$\Omega_2 := \{i \in \Lambda \mid (x*y)_{\min\{t,r\}}\, q_0^\delta\, \lambda_i\} \cap \{j \in \Lambda \mid (x*y)_{\min\{t,r\}} \overline{\in} \lambda_j\}.$$

Then $\Lambda = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. If $\Omega_2 = \emptyset$, then $(x*y)_{\min\{t,r\}} \in \lambda_i$ for all $i \in \Lambda$, that is, $\lambda_i(x*y) \geq \min\{t, r\}$ for all $i \in \Lambda$, which yields $\lambda(x*y) \geq \min\{t, r\}$. This is a contradiction. Hence $\Omega_2 \neq \emptyset$, and so for every $i \in \Omega_2$ we have $\lambda_i(x*y) < \min\{t, r\}$ and $\lambda_i(x*y) + \min\{t, r\} > \delta$. It follows that $\min\{t, r\} > \frac{\delta}{2}$. Now $x_t \in \lambda$ implies $\lambda(x) \geq t$ and thus $\lambda_i(x) \geq \lambda(x) \geq t \geq \min\{t, r\} > \frac{\delta}{2}$ for all $i \in \Lambda$. Similarly we get $\lambda_i(y) > \frac{\delta}{2}$ for all $i \in \Lambda$. Next suppose that $t := \lambda_i(x*y) < \frac{\delta}{2}$. Taking $t < r < \frac{\delta}{2}$, we get $x_r \in \lambda_i$ and $y_r \in \lambda_i$, but $(x*y)_{\min\{(r,r\}} = (x*y)_r \overline{\in\vee q_0^\delta}\, \lambda_i$. This contradicts that λ_i is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X. Hence $\lambda_i(x*y) \geq \frac{\delta}{2}$ for all $i \in \Lambda$, and so $\lambda(x*y) \geq \frac{\delta}{2}$ which contradicts (2). Therefore $(x*y)_{\min\{t,r\}} \in\vee q_0^\delta\, \lambda$ and consequently λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X.

The following example shows that the union of $(\in, \in\vee q_0^\delta)$-fuzzy subalgebras of X may not be an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X.

Example 3.10 *Let $X = \{0, 1, a, b\}$ be a BCI-algebra in which the operation $*$ is described by Table 2.*
Define fuzzy sets λ_1 and λ_2 in X_1 as follows:

$$\lambda_1 : X \to [0,1],\ x \mapsto \begin{cases} 0.48 & \text{if } x \in \{0, 1\}, \\ 0.20 & \text{if } x \in \{a, b\}, \end{cases}$$

Table 2: Cayley table of the operation $*$

$*$	0	1	a	b
0	0	0	a	a
1	1	0	b	a
a	a	a	0	0
b	b	a	1	0

and

$$\lambda_2 : X \to [0,1],\ x \mapsto \begin{cases} 0.5 & \text{if } x \in \{0,a\}, \\ 0.3 & \text{if } x \in \{1,b\}. \end{cases}$$

Then λ_1 and λ_2 are $(\in, \in\vee\, q_0^{0.9})$-fuzzy subalgebras of X. The union $\lambda_1 \cup \lambda_2$ of λ_1 and λ_2 is described as follows:

$$\lambda_1 \cup \lambda_2 : X \to [0,1],\ x \mapsto \begin{cases} 0.5 & \text{if } x \in \{0,a\}, \\ 0.48 & \text{if } x = 1, \\ 0.3 & \text{if } x = b, \end{cases}$$

and it is not an $(\in, \in\vee\, q_0^{0.9})$-fuzzy subalgebra of X since

$$\begin{aligned}(\lambda_1 \cup \lambda_2)(1 * a) &= (\lambda_1 \cup \lambda_2)(b) = 0.3 \not\geq 0.45 \\ &= \min\{(\lambda_1 \cup \lambda_2)(1), (\lambda_1 \cup \lambda_2)(a), 0.45\}.\end{aligned}$$

Theorem 3.11 *Let $\{\lambda_i \mid i \in \Lambda\}$ be a family of $(\in, \in\vee\, q_0^{\delta})$-fuzzy subalgebras of X such that $\lambda_i \subseteq \lambda_j$ or $\lambda_j \subseteq \lambda_i$ for all $i, j \in \Lambda$ Then $\lambda := \bigcup_{i\in\Lambda} \lambda_i$ is an $(\in, \in\vee\, q_0^{\delta})$-fuzzy subalgebra of X.*

Proof. For all $x, y \in X$, we have

$$\begin{aligned}\lambda(x * y) &= \left(\bigcup_{i\in\Lambda} \lambda_i\right)(x * y) = \bigvee_{i\in\Lambda} \lambda_i(x * y) \geq \bigvee_{i\in\Lambda} \min\{\lambda_i(x), \lambda_i(y), \tfrac{\delta}{2}\} \\ &= \min\left\{\bigvee_{i\in\Lambda} \lambda_i(x), \bigvee_{i\in\Lambda} \lambda_i(y), \tfrac{\delta}{2}\right\} = \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\}.\end{aligned}$$

Therefore λ is an $(\in, \in\vee\, q_0^{\delta})$-fuzzy subalgebra of X.

Definition 3.12 ([5]) *Let λ be a fuzzy set in X and $t \in (0,1]$. Then the set*

$$\Omega(\lambda; t) := \{x \in X \mid x_t \in\vee\, q_0^{\delta}\, \lambda\}$$

is called an $\in\vee\, q_0^{\delta}$-level set in X determined by λ and t.

Using Theorem 3.11, we can obtain the following theorem.

Theorem 3.13 *A fuzzy set* λ *in* X *is an* $(\in, \in\vee q_0^\delta)$*-fuzzy subalgebra of* X *if and only if the set* $\Omega(\lambda; t)$ *is a subalgebra of* X *for all* $t \in (0, \delta]$.

Proof. We omit the proof.

Proposition 3.14 *Let* S *be a subalgebra of* X. *For any* $t \in (0, \frac{\delta}{2}]$, *there exists an* $(\in, \in\vee q_0^\delta)$*-fuzzy subalgebra* λ *of* X *such that* $U(\lambda; t) = S$.

Proof. Let λ be a fuzzy set in X defined by

$$\lambda(x) = \begin{cases} t & \text{if} \quad x \in S, \\ 0 & \text{otherwise}, \end{cases}$$

for all $x \in X$ where $t \in (0, \frac{\delta}{2}]$. Obviously, $U(\lambda; t) = S$. Assume that there exist $a, b \in X$ such that $\lambda(a * b) < \min\{\lambda(a), \lambda(b), \frac{\delta}{2}\}$. Since $|\mathrm{Im}(\lambda)| = 2$, it follows that $\lambda(a * b) = 0$ and $\min\{\lambda(a), \lambda(b), \frac{\delta}{2}\} = t$, and so $\lambda(a) = t = \lambda(b)$, so that $a, b \in S$ but $a * b \notin S$. This is a contradiction, and so $\lambda(x * y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in X$. Using Theorem 3.5, we know that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X.

Theorem 3.15 *Let* $f : X \to Y$ *be a homomorphism of* BCK/BCI*-algebras and let* λ *and* ν *be* $(\in, \in\vee q_0^\delta)$*-fuzzy subalgebras of* X *and* Y, *respectively. Then*

(i) $f^{-1}(\nu)$ *is an* $(\in, \in\vee q_0^\delta)$*-fuzzy subalgebra of* X.

(ii) *If, for any subset* T *of* X, *there exists* $x_0 \in T$ *such that*

$$\lambda(x_0) = \bigvee\{\lambda(x) \mid x \in T\},$$

then $f(\lambda)$ *is an* $(\in, \in\vee q_0^\delta)$*-fuzzy subalgebra of* Y *when* f *is onto.*

Proof. (i) Let $x, y \in X$ and $t, r \in (0, \delta]$ be such that $x_t \in f^{-1}(\nu)$ and $y_r \in f^{-1}(\nu)$. Then $(f(x))_t \in \nu$ and $(f(y))_r \in \nu$. Since ν is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of Y, it follows that

$$(f(x * y))_{\min\{t,r\}} = (f(x) * f(y))_{\min\{t,r\}} \in\vee\, q_0^\delta\, \nu$$

so that $(x * y)_{\min\{t,r\}} \in\vee\, q_0^\delta\, f^{-1}(\nu)$. Therefore $f^{-1}(\nu)$ is an $(\in, \in\vee q_0^\delta)$-fuzzy subalgebra of X.

(ii) Let $a, b \in Y$ and $t, r \in (0, \delta]$ be such that $a_t \in f(\lambda)$ and $b_r \in f(\lambda)$. Then $(f(\lambda))(a) \geq t$ and $(f(\lambda))(b) \geq r$. By assumption, there exists $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ such that

$$\lambda(x) = \bigvee\{\lambda(z) \mid z \in f^{-1}(a)\}$$

and

$$\lambda(y) = \bigvee\{\lambda(w) \mid w \in f^{-1}(b)\}.$$

Then $x_t \in \lambda$ and $y_r \in \lambda$. Since λ is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subalgebra of X, we have $(x*y)_{\min\{t,\,r\}} \in\vee\, q_0^\delta\, \lambda$. Now $x*y \in f^{-1}(a*b)$ and so $(f(\lambda))(a*b) \geq \lambda(x*y)$. Thus

$$(f(\lambda))(a*b) \geq \min\{t,\, r\} \text{ or } (f(\lambda))(a*b) + \min\{t,\, r\} > \delta$$

which means that $(a*b)_{\min\{t,\,r\}} \in\vee\, q_0^\delta\, f(\lambda)$. Consequently, $f(\lambda)$ is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subalgebra of Y.

References

[1] S. K. Bhakat, P. Das, On the definition of a fuzzy subgroup, *Fuzzy Sets and Systems*, **51** (1992), 235-241. http://dx.doi.org/10.1016/0165-0114(92)90196-b

[2] S. K. Bhakat, P. Das, $(\in, \in\vee\, q)$-fuzzy subgroup, *Fuzzy Sets and Systems*, **80** (1996), 359-368. http://dx.doi.org/10.1016/0165-0114(95)00157-3

[3] Y. B. Jun, On (α, β)-fuzzy subalgebras of BCK/BCI-algebras, *Bulletin of the Korean Math. Soc.*, **42** (2005), 703-711. http://dx.doi.org/10.4134/bkms.2005.42.4.703

[4] Y. B. Jun, Fuzzy subalgebras of type (α, β) in BCK/BCI-algebras, *Kyungpook mathematical journal*, 47 (2007), no.3, 403-410.

[5] Y. B. Jun, M. A. Öztürk and G. Muhiuddin, A generalization of $(\in, \in\vee q)$-fuzzy subgroups, *Int. J. Comput. Math.* (submitted).

[6] J. Meng and Y. B. Jun, *BCK-Algebras*, Kyung Moon Sa Co., Seoul, Korea, 1994.

[7] V. Murali, Fuzzy points of equivalent fuzzy subsets, *Inform. Sci.*, **158** (2004), 277-288. http://dx.doi.org/10.1016/j.ins.2003.07.008

[8] P. M. Pu and Y. M. Liu, Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal. Appl.*, **76** (1980), 571-599. http://dx.doi.org/10.1016/0022-247x(80)90048-7

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