Multidim Syst Sign Process (2013) 24:51–64
DOI 10.1007/s11045-011-0154-3

Characterizations of global reachability of 2D structured systems

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Received: 9 February 2011 / Revised: 19 May 2011 / Accepted: 27 May 2011 /
Published online: 12 June 2011

Abstract The new concept of 2D structured system is defined and characterizations of global reachability are obtained. This paper extends well known results for the 1D case, according to which a structured system (A_λ, B_λ) is (generically) reachable if and only if its graph is spanned by a cactus, or, equivalently, if and only if the pair (A_λ, B_λ) is full generically row rank and irreducible.

Keywords Structured system · 2D discrete system · Directed graphs · Global reachability

1 Introduction

In 1974, Lin 1974 introduced the concept of structure for a linear time-invariant control system in order to model phenomena where the only available information is the existence or absence of relations between the relevant variables.

In a structured system, the system matrices are supposed to have entries that are either zero or then assume arbitrary values, and each nonzero entry is identified with a parameter. In this setting, several system theoretic properties have been defined in a generic way, i.e., as holding for almost all the concretizations of the values of the parameters.

This work was partially supported by *FCT–Fundação para a Ciência e Tecnologia* through *CIDMA–Centro de Investigação e Desenvolvimento em Matemática e Aplicações* of the University of Aveiro, Portugal.

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In Lin (1974) the graph associated to structured systems is defined and the study of the system theoretic properties from a graph theoretic point of view is carried out. Lin's results, originally derived for single-input systems, were extended to the multi-input case making both use of algebraic (Glover and Silverman 1976; Shields and Pearson 1976) and graph (Mayeda 1981) approaches.

This paper presents our first contribution on the extension of the theory of 1D structured systems to the two-dimensional (2D) case. Whereas 1D dynamical systems evolve only over time (a one-dimensional variable), 2D systems evolve over a two-dimensional domain (for instance (1D) space-time or 2D space) (Fornasini and Marchesini 1978). One of the most frequent representations of 2D systems is the Fornasini-Marchesini state space model (Fornasini and Marchesini 1978). In this model one distinguishes between local states (i.e., the values of the state at a certain point (i, j)) and global states (consisting of the collection of all local states along a separation set or "propagation front"). This has motivated the definition of system theoretic properties both at a local and at a global level. One of these properties is reachability, that is defined as the possibility of attaining an arbitrary state starting from the origin, by using a suitable control sequence. Note that Fornasini-Marchesini models include as a particular case the well-known Roesser models, where the local state is divided into two parts: one that is updated in the vertical direction (vertical state) an the other that is updated in the horizontal direction (horizontal state).

In this paper we focus on the study of the property of global reachability for 2D structured systems described by a Fornasini-Marchesini model. This property is characterized both in algebraic and graph theoretic terms.

2 Structured systems

A matrix $M \in \mathbb{R}^{n\times m}$ is said to be a *structured matrix* if its entries are either fixed zeros or independent parameters, in which case they are referred to as the nonzero entries. In this paper, we assume that the actual value of each of the nonzero entries is unknown, but can take any real value (including zero). Therefore a structured matrix M having r nonzero entries can be parameterized by means of a parameter vector $\lambda \in \mathbb{R}^r$ and is denoted by M_λ.

Example 2.1 Let $\lambda_i, \ i = 1, 2, 3$, be free parameters. The matrix

$$M_\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & 0 \end{bmatrix}$$

is a structured matrix where $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$.

However, neither

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & 0 \end{bmatrix} \text{nor} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 0 \end{bmatrix}$$

are structured matrices. □

Let us consider a discrete time-invariant system of the form

$$x(t+1) = Ax(t) + Bu(t), \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}, \ B \in \mathbb{R}^{n\times m}, \ x(\cdot) \in \mathbb{R}^n$ denotes the state of the system and $u(\cdot) \in \mathbb{R}^m$ the input.

If in the system (2.1) we assume that the matrices A and B are structured matrices having together r nonzero entries, the system can be parameterized by means of a parameter vector $\lambda \in \mathbb{R}^r$. The parameterized system thus obtained is called a *structured system* and is denoted by

$$x(t+1) = A_\lambda x(t) + B_\lambda u(t) \tag{2.2}$$

with $\lambda \in \mathbb{R}^r$, or simply by (A_λ, B_λ).

By choosing λ, system (2.2) becomes completely known and can be written as a system of the form (2.1). Thus, for each value of λ, its system theoretic properties can be studied in the usual way. It is clear that these properties may depend on the parameter values and hold for some of them while for others not. In this context, for structured systems, the relevant issue is not whether a property holds for some particular parameter values, but rather whether it is a *generic property*, in the sense that it holds "for almost all parameter values", i.e., it holds for all parameter values except for those in some proper algebraic variety in the parameter space (which is a set with Lebesgue measure zero) (Davison and Wang 1973). Hence we shall say that the structured system (2.2) has a certain property P if P is a generic property of the system.

In this paper we shall focus on the study of reachability. As is well-known, the system (2.1) is said to be *reachable* if for every $x^* \in \mathbb{R}^n$ there exist $t^* > 0$ and an input sequence $u(t),\ t = 0, 1, \ldots, t^* - 1$, that steers the state from $x(0) = 0$ to $x(t^*) = x^*$.

Characterizations of reachability for completely specified systems of type (2.1) are given by the following results (Kučera 1991).

Theorem 2.1 *The system* (2.1) *is reachable if and only if* $\operatorname{rank} \mathcal{R}^n = n$*, where* $\mathcal{R}^n$ *is the reachability matrix of the system, i.e.,*

$$\mathcal{R}^n := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

Theorem 2.2 (PBH test) *The system* (2.1) *is reachable if and only if*

$$\operatorname{rank} \begin{bmatrix} zI - A \mid B \end{bmatrix} = n, \quad \forall z \in \mathbb{C}.$$

By Theorem 2.1, system (2.2) is reachable if and only if the reachability matrix

$$\mathcal{R}^n = \begin{bmatrix} B_\lambda \ A_\lambda B_\lambda \cdots A_\lambda^{n-1} B_\lambda \end{bmatrix}$$

has rank n for almost all $\lambda \in \mathbb{R}^r$. But, noting that $\mathcal{R}^n$ is a polynomial matrix in r indeterminates, we can show that this is equivalent to say that $\operatorname{rank} \mathcal{R}^n = n$, for some $\lambda^* \in \mathbb{R}$. This means that the structured system (2.2) is reachable if and only if it is reachable for one choice of λ. However, this characterization or, equivalently, the study of the rank of the polynomial matrix $\mathcal{R}^n$ can only be used in experimental tests, where a value of λ that makes the system reachable has to be found.

Note that given two structured matrices A_λ and B_λ the reachability matrix $\mathcal{R}^n$ is not necessarily structured, as is illustrated by the next example.

Example 2.2 Let

$$A_\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 0 & \lambda_3 \end{bmatrix} \text{ and } B_\lambda = \begin{bmatrix} \lambda_4 \\ 0 \end{bmatrix}$$

be two structured matrices. Then

$$\mathcal{R}^2 = \begin{bmatrix} B_\lambda & A_\lambda B_\lambda \end{bmatrix} = \begin{bmatrix} \lambda_4 & \lambda_1\lambda_4 \\ 0 & 0 \end{bmatrix}$$

is not a structured matrix since its nonzero entries are not independent. □

Moreover, define a *structured polynomial matrix* $M_\lambda(z)$ as

$$M_\lambda(z) = M_k^\lambda z^k + \cdots + M_1^\lambda z + M_0^\lambda$$

for some nonnegative integer k, where the matrix $\begin{bmatrix} M_0^\lambda \mid \ldots \mid M_k^\lambda \end{bmatrix}$ is a structured matrix. Then the matrix

$$\begin{bmatrix} zI - A_\lambda \mid B_\lambda \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} z + \begin{bmatrix} -A_\lambda & B_\lambda \end{bmatrix}$$

associated to the pair (A_λ, B_λ) of structured matrices is also not structured.

Thus, if we wish to make a study of reachability of a structured system by analyzing structured matrices we must use different tools.

The next two concepts are fundamental for this study (Dion et al. 2003).

Let $A_\lambda \in \mathbb{R}^{n\times n}$ and $B_\lambda \in \mathbb{R}^{n\times m}$ be structured matrices. The pair (A_λ, B_λ) is said to be:

- *reducible*, or to ***be in form I***, if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P^{-1}A_\lambda P = \begin{bmatrix} A_{11}^\lambda & 0 \\ A_{21}^\lambda & A_{22}^\lambda \end{bmatrix} \quad \text{and} \quad PB_\lambda = \begin{bmatrix} 0 \\ B_2^\lambda \end{bmatrix}$$

  where A_{ij}^λ is an $n_i \times n_j$ structured matrix for $i, j = 1, 2$, with $0 < n_1 \leq n$ and $n_1 + n_2 = n$, and where B_2^λ is an $n_2 \times m$ structured matrix.
- *not of full generic row rank*, or to ***be in form II***, if the generic rank of $\begin{bmatrix} A_\lambda & B_\lambda \end{bmatrix}$ is less than n.

Recall that the *generic rank* of a structured matrix M_λ is ρ if it is equal to ρ for almost all $\lambda \in \mathbb{R}^r$. This coincides with the maximal rank that M_λ achieves as a function of the parameter λ.

A necessary and sufficient condition for a pair (A_λ, B_λ) to be in form II is that $\begin{bmatrix} A_\lambda & B_\lambda \end{bmatrix}$ has a zero submatrix of order $k \times l$ where $k + l \geq n + m + 1$ (Shields and Pearson 1976).

For structured systems of type (2.2), the following result has been proved (see Glover and Silverman 1976; Lin 1974; Shields and Pearson 1976).

Theorem 2.3 *The structured system* (2.2) *is (generically) reachable if and only if the pair* (A_λ, B_λ) *is neither in form I nor in form II.*

2.1 Graph of a structured system

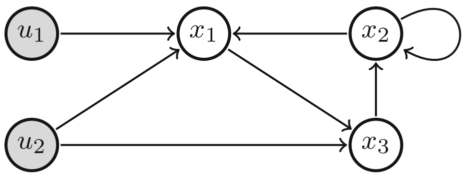
Structured systems can be represented by means of directed graphs, and this type of representation allows studying well-known system theoretic poperties from a graph theoretic point of view. This approach was first introduced by Lin 1974 for the single-input case and extended latter by Mayeda 1981 to the multi-input case.

The directed graph of a structured system of type (2.2), denoted by $\mathcal{G}(A_\lambda, B_\lambda)$ or simply by $\mathcal{G}$, is defined by a vertex set V and an edge set E. The vertex set V is given by $U \cup X$ where $U = \{u_1, \ldots, u_m\}$ and $X = \{x_1, \ldots, x_n\}$ are, respectively, the input and state vertex sets. The edge set E is described by $E_{A_\lambda} \cup E_{B_\lambda}$, with $E_{A_\lambda} = \{(x_j, x_i) : a_{ij}^\lambda \neq 0\}$ and $E_{B_\lambda} = \{(u_j, x_i) : b_{ij}^\lambda \neq 0\}$. Here, (v, v') denotes a directed edge from the vertex $v \in V$ to the vertex $v' \in V$ and $m^{\lambda}_{ij} \neq 0$ means that the (i, j)th entry m^{λ}_{ij} of the matrix M_{λ} is a free parameter.

Example 2.3 Consider the structured system (2.2) described by the matrices

$$A_{\lambda} = \begin{bmatrix} 0 & \lambda_1 & 0 \\ 0 & \lambda_2 & \lambda_3 \\ \lambda_4 & 0 & 0 \end{bmatrix}, \; B_{\lambda} = \begin{bmatrix} \lambda_5 & \lambda_6 \\ 0 & 0 \\ 0 & \lambda_7 \end{bmatrix}.$$

The graph $\mathcal{G}$ is illustrated in the next figure.



A sequence of edges $(v_{i-1}, v_i) \in E,\ i = 1, \ldots, t$, is called a *path* with *initial* vertex v_0 and *terminal* vertex v_t, and will be denoted by $v_0 \to v_1 \to \cdots \to v_t$. The path is called *simple* if the vertices are all distinct. A *cycle* is a path $v_0 \to v_1 \to \cdots \to v_t \to v_0$ where $v_0 \to v_1 \to \cdots \to v_t$ is a simple path.

A *stem* is a simple path whose initial vertex is in the set of input vertices U. A vertex is said to be *accessible* if it is the terminal vertex of a stem.

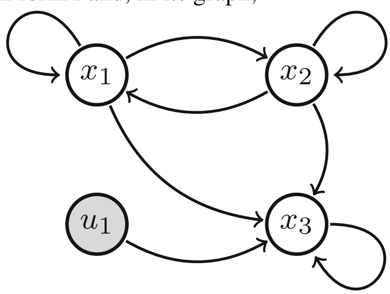
For instance, in the previous example, $x_1 \to x_3 \to x_2 \to x_1$ is a cycle and $u_1 \to x_1 \to x_3$ is a stem; moreover, all the state vertices are accessible.

Lemma 2.1 (Mayeda 1981) *The pair $(A_{\lambda}, B_{\lambda})$ is not in form I if and only if all the state vertices of $\mathcal{G}$ are accessible.*

Example 2.4 Consider the matrices

$$A_{\lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \text{ and } B_{\lambda} = \begin{bmatrix} 0 \\ 0 \\ \lambda_{34} \end{bmatrix}$$

The pair $(A_{\lambda}, B_{\lambda})$ is clearly in form I and, in its graph,



the vertices x_1 and x_2 are non-accessible. □

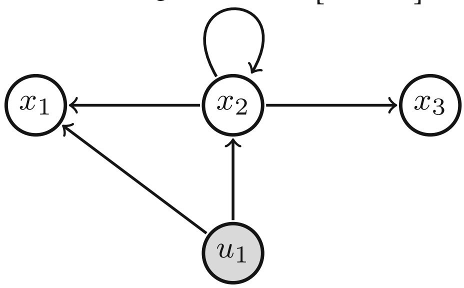
Two paths are *disjoint* if they consist of disjoint sets of vertices. A (possibly empty) set of mutually disjoint paths is called a *path family*. The notions of *cycle family* and *stem family* are defined analogously.

Lemma 2.2 (Mayeda 1981) *The pair (A_λ, B_λ) is not in form II if and only if there exists in $\mathcal{G}$ a disjoint union of a stem family and a cycle family that contains all the state vertices.*

Example 2.5 Let

$$C_\lambda = \begin{bmatrix} 0 & \lambda_{12} & 0 \\ 0 & \lambda_{22} & 0 \\ 0 & \lambda_{32} & 0 \end{bmatrix} \text{ and } D_\lambda = \begin{bmatrix} \lambda_{14} \\ \lambda_{24} \\ 0 \end{bmatrix}$$

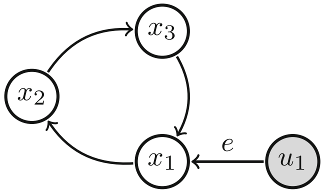
The pair (C_λ, D_λ) is in form II since the generic rank of $\begin{bmatrix} C_\lambda & D_\lambda \end{bmatrix}$ is 2. In fact, its graph

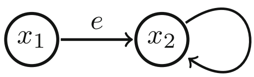


does not admit a disjoint union of a stem family and a cycle family that contains all the state vertices. Indeed, the only cycle families in this graph are $\mathcal{F}_1 = \emptyset$ and $\mathcal{F}_2 = \{x_2 \rightarrow x_2\}$. Clearly there is no stem family $\mathcal{S}$ satisfying $\mathcal{F}_i \cap \mathcal{S} = \emptyset$ and $X = \{x_1, x_2, x_3\} \subseteq \mathcal{F}_i \cup \mathcal{S}$ for $i = 1$ or $i = 2$.

On the other hand, considering the matrices of Example 2.4, the generic rank of $\begin{bmatrix} A_\lambda & B_\lambda \end{bmatrix}$ is 3, thus (A_λ, B_λ) is not in form II, and in its graph the union of the stem $u_1 \rightarrow x_3$ and the cycle $x_1 \rightarrow x_2 \rightarrow x_1$ is disjoint and contains all the state vertices. □

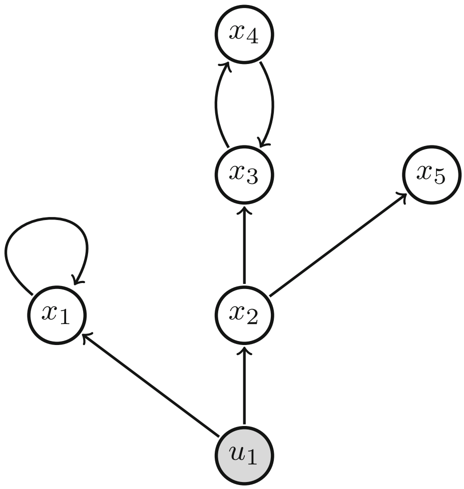
In the graph of a structured system, a *bud* is a cycle in X with an additional edge e that ends, but not begins in a vertex of the cycle. The edge e is called the *distinguished edge* of the bud. Two examples of buds (with distinguish edge e) are:





A graph $\mathcal{G}$ is called a *cactus* if there exist a stem $\mathcal{S}_0$ and buds $\mathcal{B}_1, \ldots, \mathcal{B}_l$, with $l \in \mathbb{N}$, such that $\mathcal{G} = \mathcal{S}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_l$ and for every $i = 1, \ldots, l$ the initial vertex of the distinguished edge of $\mathcal{B}_i$ is not the terminal vertex of $\mathcal{S}_0$ and is the only vertex belonging to $\mathcal{B}_i$ and $\mathcal{S}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{i-1}$ (where, in case $i = 1$, $\mathcal{S}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{i-1}$ is replaced by $\mathcal{S}_0$). A set of mutually disjoint cactus is called a *cacti*.

Example 2.6 The following graph is a cactus with stem $\mathcal{S}_0 : u_1 \rightarrow x_2 \rightarrow x_5$ and buds $\mathcal{B}_1 : u_1 \rightarrow x_1 \rightarrow x_1$ and $\mathcal{B}_2 : x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_3$.



We say that a subgraph $\mathcal{H}$ of $\mathcal{G}$ *spans* $\mathcal{G}$ if it contains all the vertices of $\mathcal{G}$. Theorem 2.3 can now be restated in terms of graphs.

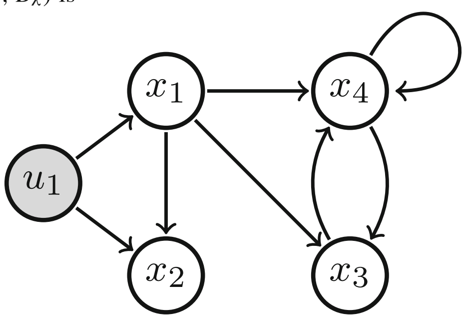
Theorem 2.4 (Mayeda 1981) *The structured system* (2.2) *is (generically) reachable if and only if there exists a cacti which spans* $\mathcal{G}(A_\lambda, B_\lambda)$.

Example 2.7 It is easy to see that the graphs of examples 2.4 and 2.5 are not spanned by a cacti and also that the structured systems represented by that graphs are not reachable.

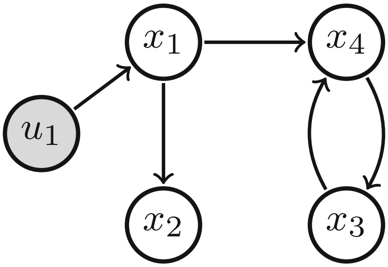
Consider now the structured matrices

$$A_\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & \lambda_3 \\ \lambda_4 & 0 & \lambda_5 & \lambda_6 \end{bmatrix} \text{ and } B_\lambda = \begin{bmatrix} \lambda_7 \\ \lambda_8 \\ 0 \\ 0 \end{bmatrix}$$

The graph of the pair (A_λ, B_λ) is



and is spanned by the cactus



with stem $\mathcal{S}_0 : u_1 \to x_1 \to x_2$ and bud $\mathcal{B}_1 : x_1 \to x_4 \to x_3 \to x_4$.

By Theorem 2.4, the structured system (A_λ, B_λ) is reachable. This can alternatively be checked by computing the generic rank of the reachability matrix of this system:

$$\begin{aligned}\mathcal{R}^3 &= \begin{bmatrix} B_\lambda & A_\lambda B_\lambda & A_\lambda^2 B_\lambda & A_\lambda^3 B_\lambda \end{bmatrix} \\ &= \begin{bmatrix} \lambda_7 & 0 & 0 & 0 \\ \lambda_8 & \lambda_1\lambda_7 & 0 & 0 \\ 0 & \lambda_2\lambda_7 & \lambda_3\lambda_4\lambda_7 & (\lambda_2\lambda_5 + \lambda_4\lambda_6)\lambda_3\lambda_7 \\ 0 & \lambda_4\lambda_7 & (\lambda_2\lambda_5 + \lambda_4\lambda_6)\lambda_7 & (\lambda_3\lambda_4\lambda_5 + (\lambda_2\lambda_5 + \lambda_4\lambda_6)\lambda_6)\lambda_7 \end{bmatrix}.\end{aligned}$$

Considering $\lambda^* = (1, 1, 1, 1, 1, 1, 1, 0)$, the matrix

$$\mathcal{R}_{\lambda^*}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

has rank 4 and hence, by Theorem 2.1, the system $(A_{\lambda^*}, B_{\lambda^*})$ is globally reachable which, by definition, implies that the structured system (A_λ, B_λ) is globally reachable. □

The main goal of this paper is to generalize the previous theorem and Theorem 2.3 for 2D structured systems.

3 2D systems

One of the most frequent representations of 2D systems is the well-known Fornasini-Marchesini state space model (Fornasini and Marchesini 1978) which is described by the following 2D first order state updating equation

$$\begin{aligned} x(i+1, j+1) &= A_1 x(i, j+1) + A_2 x(i+1, j) \\ &\quad + B_1 u(i, j+1) + B_2 u(i+1, j), \end{aligned} \tag{3.1}$$

with *local states* $x(\cdot, \cdot) \in \mathbb{R}^n$, inputs $u(\cdot, \cdot) \in \mathbb{R}^m$, state matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ and input matrices $B_1, B_2 \in \mathbb{R}^{n\times m}$. In the sequel this 2D system will be denoted by (A_1, A_2, B_1, B_2).

Introducing the *shift operators*

$$\begin{aligned} \sigma_1 x(i, j) &:= x(i+1, j), \\ \sigma_2 x(i, j) &:= x(i, j+1), \end{aligned}$$

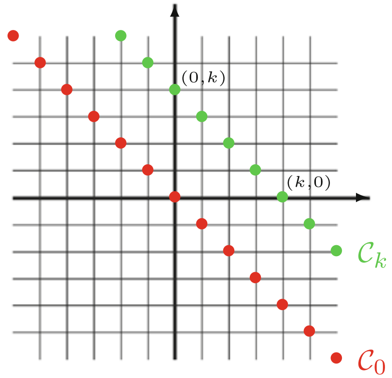
and defining a new operator $\sigma := \sigma_1\sigma_2^{-1}$ Eq. (3.1) can be written as

$$\sigma_1 x = (A_1 + A_2\sigma)\, x + (B_1 + B_2\sigma)\, u. \tag{3.2}$$

The initial conditions for this equation may be assigned by specifying the values of the state on the separation set $\mathcal{C}_0$, where

$$\mathcal{C}_k = \{(i, j) \in \mathbb{Z}^2 : i + j = k\},$$

see next figure.



Defining the *global state* on the separation set $\mathcal{C}_k$ as

$$\mathcal{X}_k(t) := (x(k + t, -t))_{t\in\mathbb{Z}}$$

and the global input as

$$\mathcal{U}_k(t) := (u(k + t, -t))_{t\in\mathbb{Z}}$$

by (3.2), the global state evolution is given by

$$\mathcal{X}_{k+1} = A(\sigma)\mathcal{X}_k + B(\sigma)\mathcal{U}_k, \tag{3.3}$$

where $A(\sigma) = A_1 + A_2\sigma,\ B(\sigma) = B_1 + B_2\sigma$, and the action of σ on $\mathcal{X}_k$ is given by

$$\begin{aligned}\sigma\mathcal{X}_k(t) &= (x(k + (t + 1), -(t + 1)))_{t\in\mathbb{Z}}\\ &= \mathcal{X}_k(t + 1).\end{aligned}$$

The action of σ on $\mathcal{U}_k$ is analogous.

Denote by $\mathbb{R}\left[\left[z, z^{-1}\right]\right]$ the set of bilateral Laurent formal power series in the indeterminate z with coefficients in $\mathbb{R}$ and define the $z-$transform $\mathcal{Z} : (\mathbb{R})^{\mathbb{Z}} \to \mathbb{R}\left[\left[z, z^{-1}\right]\right]$ by

$$\mathcal{Z}[\mathcal{W}_k] := \sum_{t=-\infty}^{+\infty} \mathcal{W}_k(t)z^{-t}$$

which will be denoted by $W_k(z)$, with $k \in \mathbb{Z}$. For vector signals in $\left(\mathbb{R}^l\right)^{\mathbb{Z}}$ the $z-$transform is defined componentwise.

It is easy to check that $\mathcal{Z}[\sigma\mathcal{X}_k] = zX_k(z)$ and hence, by (3.3), we obtain

$$X_{k+1}(z) = A(z)X_k(z) + B(z)U_k(z), \tag{3.4}$$

where $A(z) = A_1 + A_2z,\ B(z) = B_1 + B_2z$ and $U_k(z) := \mathcal{Z}[\mathcal{U}_k]$.

3.1 Global reachability

When dealing with 2D systems, the concept of reachability is naturally introduced in two different forms: a weak (local) and a strong (global) form which refer, respectively, to single local states and to global states. In this paper we shall focus on the global property which is defined next as in Fornasini and Marchesini (1978).

Definition 3.1 The 2D state space model (3.1) is ***globally reachable*** if, upon assuming $\mathcal{X}_0 \equiv 0$, for every global state sequence $\mathcal{X}^*$ with values in $\mathbb{R}^n$ there exists $k \in \mathbb{Z}_+$ and an input sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{k-1}$ such that the global state $\mathcal{X}_k$ coincides with $\mathcal{X}^*$. In this case, we say that $\mathcal{X}^*$ is reachable in k steps.

Note that the global state $\mathcal{X}^*$ is reachable in k steps if and only if there exist $U_0(z), \ldots, U_{k-1}(z) \in \mathbb{R}^m\left[\left[z, z^{-1}\right]\right]$ such that

$$X^*(z) := \mathcal{Z}[\mathcal{X}^*] = \mathcal{R}^k(z) \begin{bmatrix} U_{k-1}(z) \\ U_{k-2}(z) \\ \vdots \\ U_0(z) \end{bmatrix},$$

where $\mathcal{R}^k(z) := \left[B(z)\ A(z)B(z)\ \cdots\ A^{k-1}(z)B(z)\right]$.

The matrix

$$\mathcal{R}^n(z) = \left[B(z)\ A(z)B(z)\ \cdots\ A^{n-1}(z)B(z)\right],$$

where n is the dimension of the local state and the polynomial matrices $A(z)$ and $B(z)$ are defined as in (3.4), is called the *global reachability matrix* of the 2D system (A_1, A_2, B_1, B_2).

In the following theorem Fornasini and Marchesini (1978), global reachability is characterized in terms of the global reachability matrix.

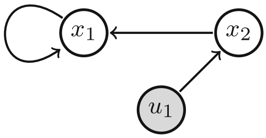
Theorem 3.1 *The 2D system (A_1, A_2, B_1, B_2) is globally reachable if and only if the polynomial matrix $\mathcal{R}^n(z)$ has rank n, i.e.,* $\operatorname{rank} \mathcal{R}^n(z) = n$.

4 2D structured systems

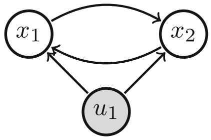
In the sequel we consider 2D systems of the form (3.1), where the matrices A_1, A_2, B_1 and B_2 are structured, i.e., their entries are either fixed zeros or independent free parameters. In this case, the polynomial matrices $A_\lambda(z) = A_1^\lambda + A_2^\lambda z$ and $B_\lambda(z) = B_1^\lambda + B_2^\lambda z$ are structured matrices too. Moreover, their evaluations for any $\nu^* \in \mathbb{C}$, yield matrices $A_\lambda(\nu^*)$ and $B_\lambda(\nu^*)$ that are also structured.

Definition 4.1 The *graph associated to the pair* $(A_\lambda(z), B_\lambda(z))$ is the superposition of the graphs associated to the pairs $(A_1^\lambda, B_1^\lambda)$ and $(A_2^\lambda, B_2^\lambda)$, eliminating the repeated edges that may appear.

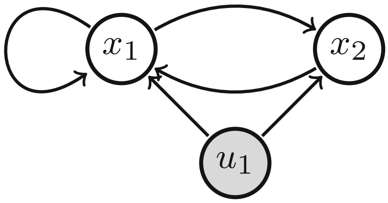
Example 4.1 Let $A_1^\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 0 & 0 \end{bmatrix}$, $B_1^\lambda = \begin{bmatrix} 0 \\ \lambda_5 \end{bmatrix}$, $A_2^\lambda = \begin{bmatrix} 0 & \lambda_3 \\ \lambda_4 & 0 \end{bmatrix}$ and $B_2^\lambda = \begin{bmatrix} \lambda_6 \\ \lambda_7 \end{bmatrix}$. The graphs associated to the pairs $(A_1^\lambda, B_1^\lambda)$ and $(A_2^\lambda, B_2^\lambda)$ are, respectively,



and



The graph associated to the pair $(A_\lambda(z), B_\lambda(z))$ is then



Similarly to the 1D case, we say that a 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is (*globally*) *reachable* if it is generically (globally) reachable, i.e., if it is reachable for almost all $\lambda \in \mathbb{R}^r$. Taking Definition 3.1 and Theorem 3.1 into account, this means that the reachability matrix $\mathcal{R}^n(z) = \left[B(z)\ A(z)B(z)\ \cdots\ A^{n-1}(z)B(z)\right]$, which is polynomial in λ and z, becomes a polynomial matrix in z of rank n for almost all $\lambda \in \mathbb{R}^r$. Due to the properties of polynomial matrices this means that $\mathcal{R}^n_{\lambda^*}(z)$ has rank n for at least one value of $\lambda^* \in \mathbb{R}^r$. Again this is equivalent to saying that $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$ is reachable for at least one value $\lambda^* \in \mathbb{R}^r$.

As in the 1D case, the notions of matrix pairs in *form I* and in *form II* play an important role in the characterization of 2D reachability, now applied to the polynomial matrix pair $(A_\lambda(z), B_\lambda(z))$.

Definition 4.2 Let $A_\lambda(z) \in \mathbb{R}^{n\times n}[z]$ and $B_\lambda(z) \in \mathbb{R}^{n\times m}[z]$ be structured polynomial matrices. The pair $(A_\lambda(z), B_\lambda(z))$ is said to be:

- *reducible*, or to ***be in form I***, if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that
$$P^{-1}A_\lambda(z)P = \begin{bmatrix} A_{11}^\lambda(z) & 0 \\ A_{21}^\lambda(z) & A_{22}^\lambda(z)\end{bmatrix} \text{ and } PB_\lambda(z) = \begin{bmatrix} 0 \\ B_2^\lambda(z)\end{bmatrix}$$
where $A_{ij}^\lambda(z)$ is an $n_i \times n_j$ structured polynomial matrix for $i, j = 1, 2$, with $0 < n_1 \le n$ and $n_1 + n_2 = n$, and where $B_2^\lambda(z)$ is an $n_2 \times m$ structured polynomial matrix.
- *not of full generic row rank*, or to ***be in form II***, if the generic rank of $\left[A_\lambda(z)\ B_\lambda(z)\right]$ is less than n.

Note that the (generic) rank of $\left[A_\lambda(z)\ B_\lambda(z)\right]$ is to be understood as $\max\limits_{\lambda^*\in\mathbb{R}^r}$ (rank $[A_{\lambda^*}(z)$ $B_{\lambda^*}(z)]$), where rank $\left[A_{\lambda^*}(z)\ B_{\lambda^*}(z)\right]$ denotes the rank of this polynomial matrix in z.

Lemma 4.1 *Let* $\nu^* \in \mathbb{C}\setminus\{0\}$. *Then the pair of structured matrices* $(A_\lambda(z), B_\lambda(z))$ *is neither in form I nor in form II if and only if the pair of structured matrices* $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ *is not in form I nor in form II, where* $A_\lambda(z) = A_1^\lambda + A_2^\lambda z$ *and* $B_\lambda(z) = B_1^\lambda + B_2^\lambda z$.

Proof Since $\nu^* \in \mathbb{C}\setminus\{0\}$ both implications are obvious because the graphs associated to the pairs of structured matrices $(A_\lambda(z), B_\lambda(z))$ and $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ are equal. □

Remark 4.1 The "if" part also holds for $\nu^* = 0$. In fact, if $\nu^* = 0$ then $A_\lambda(0) = A_1^\lambda$ and $B_\lambda(0) = B_1^\lambda$. Therefore the graph $\mathcal{G}(A_\lambda(0), B_\lambda(0)) = \mathcal{G}(A_1^\lambda, B_1^\lambda)$ spans (and, in particular, has the same vertices as) $\mathcal{G}(A_\lambda(z), B_\lambda(z))$. If the pair $(A_\lambda(0), B_\lambda(0))$ is not in form I nor in form II, by Lemma 2.1 all the state vertices of the graph $\mathcal{G}(A_\lambda(0), B_\lambda(0))$ are accessible and by Lemma 2.2 there exists a disjoint union of a stem family and a cycle family that contains all the state vertices. Since $\mathcal{G}(A_\lambda(0), B_\lambda(0))$ spans $\mathcal{G}(A_\lambda(z), B_\lambda(z))$ the same happens for $\mathcal{G}(A_\lambda(z), B_\lambda(z))$.

The following example shows that the "only if" part does not hold for $\nu^* = 0$.

Example 4.2 Let $A_1^\lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2^\lambda = \begin{bmatrix} 0 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}$, $B_1^\lambda = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $B_2^\lambda = \begin{bmatrix} \lambda_5 \\ \lambda_6 \end{bmatrix}$.

Since all the entries of $(A_1^\lambda + A_2^\lambda, B_1^\lambda + B_2^\lambda)$ are free, this pair is neither in form I nor in form II and by Lemma 4.1 the same holds for the pair $(A_\lambda(z), B_\lambda(z))$. However, its clear that the pair $(A_\lambda(0), B_\lambda(0)) = (A_1^\lambda, B_1^\lambda)$ is in form I and II.

Based on Lemma 4.1 and Theorems 2.1, 2.3 and 3.1 we obtain the main result of this paper that characterizes the global reachability of 2D structured systems

Theorem 4.1 *A 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable if and only if the pair of structured matrices $(A_\lambda(z), B_\lambda(z))$ is neither in form I nor in form II, where $A_\lambda(z) = A_1^\lambda + A_2^\lambda z$ and $B_\lambda(z) = B_1^\lambda + B_2^\lambda z$.*

Proof By definition, the 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable if there exists $\lambda^* \in \mathbb{R}^r$ such that the 2D system $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$ is globally reachable.

Then, by Theorem 3.1, rank $\mathcal{R}_{\lambda^*}^n(z) = n$, where $\mathcal{R}_{\lambda^*}^n(z)$ is the global reachability matrix of the 2D system $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$. Note that, in this case, the set

$$\mathcal{L} := \{\eta \in \mathbb{C} : \operatorname{rank} \mathcal{R}_{\lambda^*}^n(\eta) < \operatorname{rank} \mathcal{R}_{\lambda^*}^n(z)\}$$

corresponds to the common zeros of the $n \times n$ minors of $\mathcal{R}_{\lambda^*}^n(z)$, and is hence a finite set. Thus rank $\mathcal{R}_{\lambda^*}^n(z) = n$ means that there exist $\nu^* \in \mathbb{C} \setminus \mathcal{L}$ such that

$$\operatorname{rank} \mathcal{R}_{\lambda^*}^n(\nu^*) = n.$$

By Theorem 2.1, the system corresponding to the pair $(A_{\lambda^*}(\nu^*), B_{\lambda^*}(\nu^*))$ is reachable, for all $\nu^* \in \mathbb{C} \setminus \mathcal{L}$.

Thus, by definition, $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ is a structured system which is reachable.

By Theorem 2.3 we have that the pair of structured matrices $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ is neither in form I nor in form II and, by Lemma 4.1, $(A_\lambda(z), B_\lambda(z))$ is neither in form I nor in form II. The converse implication is analogous. □

It follows from the proof of previous theorem that the 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable if and only if the pair of structured matrices $(A_\lambda(\nu^*), B_\lambda(\nu^*))$ is neither in form I nor in form II, for all $\nu^* \in \mathbb{C} \setminus \mathcal{L}$. By Theorem 2.4 this means that the graph $\mathcal{G}(A_\lambda(\nu^*), B_\lambda(\nu^*))$ is spanned by a cacti. Since, for every $\nu^* \in \mathbb{C} \setminus \{0\}$, the graphs $\mathcal{G}(A_\lambda(z), B_\lambda(z))$ and $\mathcal{G}(A_\lambda(\nu^*), B_\lambda(\nu^*))$ are equal, the following graph theoretical characterization of global reachability holds.

Corollary 4.1 *A 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable if and only if there exists a cacti which spans the graph $\mathcal{G}(A_\lambda(z), B_\lambda(z))$.*

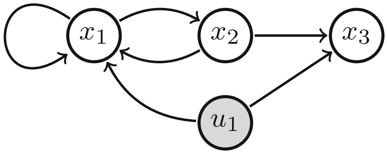
Example 4.3 Let

$$A_1^\lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{bmatrix}, A_2^\lambda = \begin{bmatrix} 0 & \lambda_3 & 0 \\ \lambda_4 & 0 & 0 \\ 0 & \lambda_5 & 0 \end{bmatrix}, B_1^\lambda = \begin{bmatrix} \lambda_6 \\ 0 \\ 0 \end{bmatrix} \text{ and } B_2^\lambda = \begin{bmatrix} 0 \\ 0 \\ \lambda_7 \end{bmatrix},$$

and define

$$A_\lambda(z) = A_1^\lambda + A_2^\lambda z = \begin{bmatrix} \lambda_1 & \lambda_3 z & 0 \\ \lambda_4 z & 0 & 0 \\ 0 & \lambda_2 + \lambda_5 z & 0 \end{bmatrix} \text{ and } B_\lambda(z) = B_1^\lambda + B_2^\lambda z = \begin{bmatrix} \lambda_6 \\ 0 \\ \lambda_7 z \end{bmatrix}.$$

The graph associated to the pair $(A_\lambda(z), B_\lambda(z))$ is



and it is easy to check that this graph is spanned by the cactus with stem $\mathcal{S}_0 : u_1 \to x_3$ and bud $\mathcal{B}_1 : u_1 \to x_1 \to x_2 \to x_1$, or alternatively by the cactus $\mathcal{S}_0 : u_1 \to x_1 \to x_2 \to x_3$. By the previous corollary, the 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable. This can alternatively be checked from the global reachability matrix:

$$\mathcal{R}^2(z) = \begin{bmatrix} B_\lambda(z) & A_\lambda(z)B_\lambda(z) & A_\lambda^2(z)B_\lambda(z) \end{bmatrix} = \begin{bmatrix} \lambda_6 & \lambda_1\lambda_6 & \lambda_1^2\lambda_6 + \lambda_3\lambda_4\lambda_6 z^2 \\ 0 & \lambda_4\lambda_6 z & \lambda_1\lambda_4\lambda_6 z \\ \lambda_7 z & 0 & (\lambda_2 + \lambda_5 z)\lambda_4\lambda_6 z \end{bmatrix}.$$

Considering $\lambda^* = (0, 1, 0, 1, 0, 1, 0)$, the polynomial matrix

$$\mathcal{R}_{\lambda^*}^2(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix}$$

has rank 3 and hence, by Theorem 3.1, the 2D system $(A_1^{\lambda^*}, A_2^{\lambda^*}, B_1^{\lambda^*}, B_2^{\lambda^*})$ is globally reachable which, by definition, implies that the 2D structured system $(A_1^\lambda, A_2^\lambda, B_1^\lambda, B_2^\lambda)$ is globally reachable.

5 Conclusions

In this paper we extended the study of structured dynamical systems to the 2D case, considering 2D Fornasini-Marchesini state space models. Necessary and sufficient conditions for structured global reachability were obtained that generalize well known results on the reachability of 1D structured systems. Such conditions are stated in terms of the generic rank and the irreducibility of suitably defined matrices, and in graph theoretic terms.

The characterization of other system properties such local reachability and (global and local) observability is under current investigation.

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