Characterizations of hermitian varieties by intersection numbers

J. Schillewaert, J.A. Thas
Department of Pure Mathematics and Computer Algebra, Ghent University
Krijgslaan 281 S 22, B-9000 Gent, Belgium
jschille@cage.ugent.be,jat@cage.ugent.be

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Abstract

In this paper, we give characterizations of the classical generalized quadrangles $H(3,q^2)$ and $H(4,q^2)$, embedded in $PG(3,q^2)$ and $PG(4,q^2)$, respectively. The intersection numbers with lines and planes characterize $H(3,q^2)$, and $H(4,q^2)$ is characterized by its intersection numbers with planes and solids.

This result is then extended to characterize all hermitian varieties in dimension at least 4 by their intersection numbers with planes and solids.

1 Introduction

In the early nineties Tallini and Ferri gave a characterization of the parabolic quadric $Q(4,q)$ by its intersection numbers with planes and solids [9]. Here we will characterize the hermitian varieties $H(3,q^2)$ and $H(4,q^2)$, embedded in $PG(3,q^2)$ and $PG(4,q^2)$, respectively. The intersection numbers with lines and planes will characterize $H(3,q^2)$, while the intersection numbers with planes and solids will characterize $H(4,q^2)$. Finally, we determine all point sets of $PG(n,q^2)$, $n \geq 4$, having only intersection numbers with planes and solids which are also intersection numbers of hermitian varieties in n dimensions with planes and solids.

2 Notation and background on polar spaces and generalized quadrangles

Polar spaces were first described axiomatically by Veldkamp [14]. Later on, Tits simplified Veldkamp's list of axioms and further completed the theory [13]. We recall Tits' definition of polar spaces.

A polar space of rank n, $n \geq 2$, is a point set P together with a family of subsets of P called subspaces, satisfying the following axioms.

(i) A subspace, together with the subspaces it contains, is a d-dimensional projective space with $-1 \leq d \leq n-1$ (d is called the dimension of the subspace).

(ii) The intersection of two subspaces is a subspace.

(iii) Given a subspace V of dimension $n-1$ and a point $p \in P \backslash V$, there is a unique subspace W of dimension $n-1$ such that $p \in W$ and $V \cap W$ has dimension $n-2$; W contains all points of V that are joined to p by a line (a line is a subspace of dimension 1).

(iv) There exist two disjoint subspaces of dimension $n-1$.

The finite classical polar spaces are the following structures.

(i) The non-singular quadrics in odd dimension, $Q^+(2n+1,q), n \geq 1$, and $Q^-(2n+1,q), n \geq 2$, together with the subspaces they contain, giving a polar space of rank $n+1$ and n, respectively. The non-singular parabolic quadrics $Q(2n,q), n \geq 2$, in even dimension, together with the subspaces they contain, giving a polar space of rank n.

(ii) The non-singular hermitian varieties in $PG(2n,q^2), n \geq 2$ (respectively, $PG(2n+1,q^2)$, $n \geq 1$), together with the subspaces they contain, giving a polar space of rank n (respectively, rank $n+1$).

(iii) The points of $PG(2n+1,q), n \geq 1$, together with the totally isotropic subspaces of a non-singular symplectic polarity of $PG(2n+1,q)$, giving a polar space of rank $n+1$.

By theorems of Veldkamp and Tits, all polar spaces with finite rank at least 3 are classified. In the finite case (i.e. the polar space has a finite number of points), we get the following theorem, which can be found in [13].

Theorem 2.1 *A finite polar space of rank at least 3 is classical.*

Buekenhout and Shult described polar spaces as point-line geometries, and it is this description we will use.

Definition A Shult space is a point-line geometry $S = (P, B, I)$, with B a non-empty set of subsets of P of cardinality at least 2, such that the incidence relation I (which is containment here) satisfies the following axiom. For each line $L \in B$ and for each point $p \in P \backslash L$, the point p is collinear with either one or all points of the line L.

A Shult space is non-degenerate if no point is collinear with all other points.
A subspace of a Shult space $S = (P, B, I)$ is a subset W of P such that any two points of W are on a common line and any line containing distinct points of W is completely contained in W. A Shult space is linear if two distinct lines have at most one common point. Buekenhout and Shult proved the following fundamental theorem [8].

Theorem 2.2 *(i) Every non-degenerate Shult space is linear.*

(ii) If S is a non-degenerate Shult space of finite rank at least 3, and if all lines contain at least three points, then the Shult space together with all its subspaces is a polar space.

A finite generalized quadrangle (GQ) of order (s, t) is an incidence structure $S = (P, B, I)$ in which P and B are disjoint non-empty sets of objects called points and lines respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms.

(GQ1) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(GQ2) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(GQ3) If p is a point and L is a line not incident with p, then there is a unique point-line pair (q, M) such that $pIMIqIL$.

A generalized quadrangle (GQ) of order (s, t) contains $(s + 1)(st + 1)$ points, see [12]. If $s = t$, then S is also said to be of order s.

If S has a finite number of points and if $s > 1$, it is easy to show one can replace axiom ($GQ1$) by the following axioms.

(GQ1') No point is collinear with all others.

(GQ1") There is a point on at least two lines.

It is this alternative definition which we will use in our proofs.

2.1 The classical generalized quadrangles

Consider a non-singular quadric of Witt index 2, that is of projective index 1, in $PG(3, q)$, $PG(4, q)$ and $PG(5, q)$ respectively. The points and lines of these quadrics form generalized quadrangles which are denoted by $Q^+(3, q)$, $Q(4, q)$ and $Q^-(5, q)$ respectively, and of order $(q, 1)$, (q, q) and (q, q^2) respectively. Next, let H be a non-singular hermitian variety in $PG(3, q^2)$, respectively $PG(4, q^2)$. The points and lines of H form a generalized quadrangle $H(3, q^2)$, respectively $H(4, q^2)$, which has order (q^2, q), respectively (q^2, q^3). The points of $PG(3, q)$ together with the totally isotropic lines with respect to a symplectic polarity form a GQ, denoted by $W(q)$, of order q. The generalized quadrangles defined here are the so-called classical generalized quadrangles.

Definition Let V be a vector space over some skew field (not necessarily finite-dimensional). A generalized quadrangle $S = (P, B, I)$ is fully embedded in the projective space $PG(V)$ if there is a map π from P (respectively B) to the set of points (respectively lines) of $PG(V)$ such that:

(i) π is injective on points,

(ii) if $x \in P$ and $L \in B$ with xIL, then $x^\pi \in L^\pi$,

(iii) the set of points x^π, where $x \in P$, generates $PG(V)$,

(iv) every point in $PG(V)$ on the image of a line of the quadrangle is also the image of a point of the quadrangle.

The following beautiful theorem is due to Buekenhout and Lefèvre [7].

Theorem 2.3 *Every finite generalized quadrangle fully embedded in projective space is classical.*

A lot of information on finite generalized quadrangles can be found in the reference work [12].

3 Notation and background on blocking sets

A blocking set B in $\Pi = PG(2, q)$ is a set of points of Π which meets every line. A line is an example of a blocking set, but a blocking set containing a line is called trivial.

A blocking set is called minimal if for every $p \in B$, $B - \{p\}$ is not a blocking set. It is easy to prove the following useful lemma.

Lemma 3.1 *A blocking set B is minimal if and only if for every point p of B, there is some line L such that $B \cap L = \{p\}$.*

A blocking set containing k points is called a blocking k-set. The following theorem gives an upper and a lower bound on the size of a non-trivial minimal blocking set.

Theorem 3.2 *Let B be a non-trivial minimal blocking set in $PG(2, q)$. Then*

(i) [4] $|B| \geq q + \sqrt{q} + 1$ with equality if and only if q is a square and B is a Baer subplane.

(ii) [6] $|B| \leq q\sqrt{q} + 1$ with equality if and only if q is a square and B is a unital.

Next we introduce multiple blocking sets.

Definition An s-fold blocking set in $PG(2,q)$ is a set of points of $PG(2,q)$ that intersects every line in at least s points. It is called minimal if no proper subset is an s-fold blocking set.

A 1-fold blocking set is a blocking set. The following theorem shows that if $s > 1$, then in order to find s-fold blocking sets of small cardinality we must look for sets not containing a line.

Theorem 3.3 *Let B be an s-fold blocking set of $PG(2,q), s > 1$.*

(i) [5] If B contains a line, then $|B| \geq sq + q - s + 2$.

(ii) [1] If B does not contain a line, then $|B| \geq sq + \sqrt{sq} + 1$.

For small s this theorem can be improved and if s is small and q is a square the smallest minimal s-fold blocking sets are classified [2]. Finally we introduce blocking sets in higher dimensional spaces.

Definition A blocking set with respect to t-spaces in $PG(n,q)$ is a set B of points such that every t-dimensional subspace of $PG(n,q)$ meets B in at least one point.

The following result by Bose and Burton gives a nice characterization of the smallest ones [3].

Theorem 3.4 *If B is a blocking set with respect to t-spaces in $PG(n,q)$ then $|B| \geq |PG(n-t,q)|$ and equality holds if and only if B is an $(n-t)$-dimensional subspace.*

4 Characterizations of $H(3,q^2)$ and $H(4,q^2)$

4.1 The classical generalized quadrangle $H(3,q^2)$

We know that a non-singular hermitian variety $H(3,q^2)$ in $PG(3,q^2)$ intersects lines either in a point, a Baer subline or all q^2+1 points of the line. So the intersection numbers with lines are 1, $q+1$, and q^2+1. It intersects planes either in a non-singular hermitian variety $H(2,q^2)$ or in a cone with vertex a point and as base a Baer subline. So the intersection numbers with planes are q^3+1 and q^3+q^2+1.

We introduce the following terminology for the considered set K, which we will also use in the next section. We call lines intersecting K in one point tangent lines, lines intersecting K in $q+1$ points Baer lines, and lines intersecting K in q^2+1 points full lines.

We call planes that intersect our set K in q^3+1 points non-singular planes, and planes that intersect K in q^3+q^2+1 points singular planes. We denote the intersection of a plane α and the set K by K_α.

Theorem 4.1 *If the intersection numbers with lines and planes of the point set K of $PG(3,q^2)$ are intersection numbers with lines and planes of $H(3,q^2)$, then K is a hermitian variety $H(3,q^2)$.*

Proof Firstly, we show that every plane α contains a tangent line. Suppose the contrary; then every line of α intersects K in at least $q+1$ points. This means that K_α is a $(q+1)$-fold blocking set in α. By Theorem 3.3(ii) $(q+1)$-fold blocking sets have size at least $(q+1)q^2 + \sqrt{(q+1)q^2} + 1$, which yields a contradiction with the assumptions on the intersection numbers with planes.

Next we show that there exists a full line. Suppose this is not the case, then we distinguish two cases.

Either no singular plane exists in which case all planes are non-singular. We claim that in this case all lines must have the same intersection number with the set K. Suppose the contrary, so assume that there exists at least one tangent line, and at least one Baer line, and consider all planes in $PG(3,q^2)$ through a line L, respectively M, where we take L to be a tangent line and M to be a Baer line. This yields two different numbers for the size of K, a contradiction. Since all planes contain at least one tangent line, this means that all lines are tangent ones. Then consider a plane α and a point p of K inside α. Look at all lines through p inside α. Then α would contain only the point p, a contradiction.

The other case is that there exists a singular plane α. Take a point p of K_α lying on a tangent line L and consider all lines through p inside α. After counting we find at most $1 + (q^2)q = 1 + q^3$ points, again a contradiction. So we certainly have full lines.

We show that singular planes exist. To do this, consider a full line L and a point p of K which is not incident with L. Consider all lines through p in the plane α generated by p and L. All these lines contain at least 2 points hence they are Baer lines or full lines. So we have at least $1 + q(q^2+1) = q^3 + q + 1$ points of the set K in α. So α necessarily is a singular plane, and elementary counting yields that exactly one of the lines through p inside α is a full line, say M.

This line M intersects L in a point r. Consider a point p' in K_α not lying on L or M. Through this point p' there goes a full line N in α by the previous. The line N necessarily contains r, otherwise it intersects L and M in two different points s and t respectively, but then through $s \notin M$ there are two full lines in α that intersect M, namely L and N, a contradiction. Hence, all lines through r in α are either tangent lines or full lines, so counting yields that inside α there pass $q+1$ full lines through r.

We can also calculate the size of K. Consider all planes through a full line; since they are all singular by the previous arguments, we get

$$|K| = (q^2+1)q^3 + q^2 + 1 = (q^3+1)(q^2+1).$$

We define S to be the incidence structure with as points the points of the set K, as lines the full lines, and where a point p and a line L are said to be incident if L passes through p. We have already proved that axioms $(GQ2)$ and $(GQ3)$ for generalized quadrangles hold for S.

We show that it is impossible that one point p of S is collinear with all other points of S. If this were the case, then consider a plane π not through p. Since this plane intersects K in either q^3+1 or q^3+q^2+1 points, K would contain either $1+q^2(q^3+1)$ or $1+q^2(q^3+q^2+1)$ points, which yields a contradiction with the size of K.

Since we proved that there exists a point p in S such that there go at least 3 lines of S through p, and since no point of S is collinear with all other points of S, we have shown that the incidence structure S is a generalized quadrangle. Hence by the theorem of Buekenhout and Lefèvre it is a classical one, and by looking at the different classical generalized quadrangles, we see it has to be $H(3,q^2)$. $\square$

4.2 The classical generalized quadrangle $H(4,q^2)$

We consider a set of points K in $PG(4,q^2)$, such that all intersection numbers with planes and solids are also intersection numbers with planes and solids of the non-singular hermitian variety $H(4,q^2)$. We show that K has to be $H(4,q^2)$. Since $H(4,q^2)$ intersects a plane either in a line, a non-singular hermitian variety $H(2,q^2)$, or a cone with vertex a point p and base a Baer subline $H(1,q^2)$, we have as intersection numbers with planes

$$q^2+1,\ q^3+1,\ q^3+q^2+1.$$

We will call planes with intersection number either q^2+1, q^3+1 or q^3+q^2+1, small, medium and large, respectively.

The intersection of a non-singular hermitian variety $H(4,q^2)$ with a solid is either a non-singular hermitian variety $H(3,q^2)$ or a cone with vertex a point p and base a non-singular hermitian variety $H(2,q^2)$, so we get as intersection numbers with solids

$$(q^2+1)(q^3+1),\ q^2(q^3+1)+1.$$

We will call solids with intersection number $q^2(q^3+1)+1$ singular, the other ones non-singular. For a given plane α, the set of points belonging to $K\cap\alpha$ will be denoted by K_α. A line L intersecting K in q^2+1 points is called a full line.

Theorem 4.2 *If each intersection number with planes and solids of a point set K in $PG(4,q^2)$ is also an intersection number with planes and solids of $H(4,q^2)$, then K is a non-singular hermitian variety $H(4,q^2)$.*

To prove Theorem 4.2 we first need a series of lemmas.

Lemma 4.3 *The set K contains $|H(4,q^2)|$ points.*

Proof Call $H_1=q^2(q^3+1)+1$ the number of points of K contained in a singular solid, $H_2=(q^2+1)(q^3+1)$ the number of points of K contained in a non-singular solid, and x the total number of points contained in the set K. Call a the number of singular solids. Then counting the pairs (p,α) where p is a point and α a solid such that $p \in K \cap \alpha$, respectively the triples (p, r, α) with p, $r \in K \cap \alpha$, $p \neq r$, in two ways yields the following equations

$$aH_1 + (\frac{q^{10}-1}{q^2-1} - a)H_2 = x\frac{q^8-1}{q^2-1},$$

$$aH_1(H_1-1) + (\frac{q^{10}-1}{q^2-1} - a)H_2(H_2-1) = x(x-1)\frac{q^6-1}{q^2-1}.$$

From the first equation we obtain a in function of x. Substituting this in the second equation yields a quadratic equation in x. We get the following two solutions

$$x = q^7 + q^5 + q^2 + 1,\ \frac{q^9 + q^8 + q^7 + 2q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1}{q^2 + q + 1}.$$

The first solution is the desired one. The second one is not a natural number, hence we are done. □

Consider a small plane α and look at all solids through α inside $PG(4, q^2)$. Call the number of singular and non-singular ones a and b respectively, so $a + b = q^2 + 1$. An elementary counting yields the following equation

$$aq^5 + b(q^5 + q^3) + q^2 + 1 = (q^5 + 1)(q^2 + 1).$$

Solving these equations yields $a = q^2 + 1$ and $b = 0$. So we have shown that all solids containing a small plane are singular. Similar calculations learn that the other planes are contained in both kinds of solids.

Lemma 4.4 *There exist solids of both kinds.*

Proof Suppose the contrary. Then the calculations above show that all planes are small and all solids are singular. Take a line L containing c points of the set K. Consider all planes through L inside a solid. Then we get the following equation,

$$(q^2 + 1)(q^2 + 1 - c) + c = q^5 + q^2 + 1.$$

This yields a contradiction for all c. Hence, there exist solids of both kinds. □

Lemma 4.5 *For a large plane α, the set K_α is a blocking set in α.*

Proof Suppose that α contains a line L which is not blocked by K_α. Consider a large solid Π through α. Since through small planes, there only pass singular solids, all planes through L inside Π are medium or large ones, but then Π intersects the set K in more than $(q^2 + 1)(q^3 + 1)$ points, which yields a contradiction. □

Next we prove that also for small and medium planes α the sets K_α are blocking sets in α. We will need a separate proof for $q = 2$. We first prove the general case.

Lemma 4.6 *Consider a medium plane α. If $q > 2$, then K_α is a blocking set in α.*

Proof Suppose that K_α is not a blocking set in α, meaning that α contains a line L which is not blocked by K_α. Consider a singular solid Π through α, and look at all planes through L inside Π. By the previous lemma, all these planes are small or medium. Call a the number of small ones, and b the number of medium ones. We obtain the following equation

$$a(q^2 + 1) + b(q^3 + 1) = q^2(q^3 + 1) + 1.$$

If we use the fact that $a + b = q^2 + 1$, we find

$$b = q^2 - \frac{1}{q-1}.$$

Since b has to be a natural number, this yields a contradiction if $q > 2$. □

Lemma 4.7 *Consider a small plane α. If $q > 2$, then K_α is a blocking set in α and hence the points belonging to K_α form a line of α.*

Proof If α contains a line L which is not blocked by the set K_α, then by the previous lemmas all planes through L inside a solid Π on α must be small ones. But then Π intersects K in $(q^2 + 1)^2$ points, which yields a contradiction.

So K_α is a blocking set in α, and we know that blocking sets of size $q^2 + 1$ in a plane of order q^2 are lines by the Bose-Burton theorem. □

We prove the case $q = 2$ separately hereafter.

Lemma 4.8 *Let $q = 2$ and consider a plane α that is either small or medium. Then the set K_α is a blocking set for α. In particular, a small plane α intersects K in a line.*

Proof Suppose that a small plane α contains a line L which is not blocked by K_α. Consider a singular solid Π containing α, and call the number of small and medium planes through L inside Π, ϕ and ψ respectively. By Lemma 3.4 we know that $\phi + \psi = q^2 + 1$. Counting the number of intersection points of Π and K yields the following equation

$$\phi(q^2 + 1) + \psi(q^3 + 1) = q^2(q^3 + 1) + 1.$$

If we substitute $q = 2$ into these equations, we obtain that $\phi = 2$ and $\psi = 3$.

Next we show that the total number of small planes inside Π is 2. Call the number of small, medium and large planes inside Π, a, b and c respectively. Then by counting respectively the total number of planes in a solid, the incident pairs (p, α), where p is a point of K and α a plane of Π, and the triples (p, r, α) where p and r are distinct points of K and α is a plane in Π containing p and r, in two ways, we obtain the following equations, where $k = q^2(q^3+1)+1$,

$$a+b+c = (q^2+1)(q^4+1),$$

$$a(q^2+1)+b(q^3+1)+c(q^3+q^2+1) = k(q^4+q^2+1),$$
$$a(q^2+1)q^2+b(q^3+1)q^3+c(q^3+q^2+1)(q^3+q^2) = k(k-1)(q^2+1).$$

Solving these equations yields $a = q(q-1)$, so if $q = 2$ then $a = 2$. This implies that all other lines inside α are blocked, as if not we would have at least 3 small planes inside Π, since each non-blocked line has two small planes through it in Π and only α can be counted twice.

This implies that by adding an arbitrary point r on L to K_α we get a blocking set of size q^2+2 and so each of these sets has to contain a line M_r by Theorem 3.2. Since the point r on L was arbitrary this means all these sets contain the same line M already contained in K_α, but then L is blocked by K_α, a contradiction.

Consider a line L in a medium plane α. Consider a small solid Π containing α. The calculation in the beginning of the proof shows that L is contained in a small plane. Hence, L is blocked. $\square$

Remark: We have already shown that singular solids exist, and in this proof we have shown that all singular solids contain small planes. Since we know that small planes intersect K in a full line, we have shown that the set K contains full lines.

Lemma 4.9 *Every line intersecting K in c points, where $2 \leq c < q^2+1$, is contained in a medium plane.*

Proof Suppose a line L intersects K in c points, with $2 \leq c < q^2+1$, and has no medium planes through it. Consider a solid Π containing L and look at all planes through L inside Π. Since by assumption all these planes are large ones, Π would contain $(q^2+1)(q^3+q^2+1-c)+c$ points belonging to the set K. This yields a contradiction, even for a non-singular solid, unless $c = q^2+1$, a case which we exclude. $\square$

Lemma 4.10 *If α is a medium plane, it contains no full lines.*

Proof A medium plane α is always contained in a non-singular solid Π. Suppose that L is a full line lying in α and consider all planes through L inside Π. The proof of the foregoing lemma immediately shows that all these planes have to be non-singular ones, a contradiction. $\square$

Lemma 4.11 *If α is a medium plane, then K_α is a minimal blocking set in α.*

Proof Consider a line L in α containing c points of the set K, where $2 \leq c < q^2 + 1$. Note that such a line exists since we have shown that a medium plane does not contain full lines.

Consider a singular solid Π through α, and look at all planes through L inside Π. We get at least

$$(q^3 + 1 - c)(q^2 + 1) + c$$

points belonging to $\Pi \cap K$ because we cannot have small planes through L since all small planes intersect K in a line. This yields $c \geq q$ otherwise Π would not be a singular solid. We show that $c = q$ cannot occur. Suppose the contrary. So take a line L such that L intersects K in q points. Consider a singular solid Π through L. The same counting as above yields that all planes through L inside Π must be medium planes. Hence, K would contain

$$(q^4 + q^2 + 1)(q^3 + 1 - q) + q = q^7 + q^4 + q^2 + 1$$

points, which yields a contradiction. So, $c \geq q + 1$. If K_α would be a non-minimal blocking set, then there would be a point p of K_α so that no line of α through p is a tangent one. Hence, all lines through p would contain at least $q + 1$ points. But then counting yields that K_α contains at least $1 + (q^2 + 1)q = q^3 + q + 1$ points, a contradiction. □

Using the foregoing lemma, we can easily determine the intersection numbers with lines.

Lemma 4.12 *All lines intersect K in 1, $q + 1$, or $q^2 + 1$ points.*

Proof Since every line intersecting K in c points, with $2 \leq c < q^2 + 1$, is contained in a medium plane and since medium planes are minimal non-trivial blocking sets, it follows from Theorem 3.2 that the blocking set is a unital and so $c = q + 1$. □

Proof of Theorem 4.2. A line L intersecting K in $q+1$ points is called a Baer line. For $q > 2$ the proof is easily finished by Theorem 23.5.19 in [11]. So we continue with $q = 2$. However, the following reasoning works for all q. Consider a point p in K and a line L in K which are not incident. By Lemma 4.10 the plane generated by them is a large one. The lines through p in that plane all are Baer lines or full lines. Elementary counting yields exactly one of them is a full one.

We define S to be an incidence structure with as points the points of the set K, as lines the full lines, and where a point p and a line L are said to be incident if L passes through p. We have already proved that axioms $(GQ2)$ and $(GQ3)$ for generalized quadrangles hold for S. We show that no point is collinear with all others. If there was a point p collinear with all others, then consider a solid Π not incident with p. Since this solid contains either $q^2(q^3 + 1) + 1$ or $(q^2 + 1)(q^3 + 1)$ points we would have

$$|K| \in \{1 + q^2(q^2(q^3 + 1) + 1), 1 + q^2((q^2 + 1)(q^3 + 1))\}$$

This yields a contradiction with the size of K. We prove that there is a point p with at least three lines through it. Let p be a point of K and let L be a full line of K with p not on L. As $(GQ2)$ and $(GQ3)$ are satisfied for the substructure S' induced by S in the plane α through p and L, the geometry S' consists of $q+1$ lines in α on p. So p is on at least 3 lines of S. We conclude that S is a generalized quadrangle. Theorem 2.3 implies that it is a classical one and by looking at the different classical ones, we see it has to be a non-singular hermitian variety $H(4, q^2)$. $\square$

5 Generalization

Consider a set of points K in $PG(n, q^2)$, $n \geq 4$, for which each intersection number with planes and solids is also an intersection number with planes and solids of a hermitian variety. First recall what the intersections of hermitian varieties with planes and solids look like.

A plane lies on the hermitian variety, or intersects it in either a non-singular hermitian variety $H(2, q^2)$, a cone with vertex a point and base a non-singular hermitian variety $H(1, q^2)$, or a line.

A solid lies on the hermitian variety, or intersects it in either a non-singular hermitian variety $H(3, q^2)$, a cone with vertex a point and base a non-singular hermitian variety $H(2, q^2)$, a cone with vertex a line and base a non-singular hermitian variety $H(1, q^2)$, or a plane.

So the intersection numbers of the set K with planes belong to

$$q^2 + 1, q^3 + 1, q^3 + q^2 + 1, q^4 + q^2 + 1,$$

and the intersection numbers of the set K with solids belong to

$$q^4 + q^2 + 1, q^5 + q^2 + 1, q^5 + q^3 + q^2 + 1, q^5 + q^4 + q^2 + 1, q^6 + q^4 + q^2 + 1.$$

We will call a plane (respectively a solid,a line) intersecting the set K in i points an i-plane (respectively i-solid, i-line). A plane intersecting the set K in q^4+q^2+1 points will be called a full plane, a solid intersecting the set K in $q^6 + q^4 + q^2 + 1$ points will be called a full solid, and a line intersecting the set K in $q^2 + 1$ points will be called a full line.

Lemma 5.1 *A $(q^4 + q^2 + 1)$-solid Π intersects the set K in a full plane.*

Proof Suppose some line L in Π does not intersect K. Consider all planes through L inside Π. Then we get at least $(q^2 + 1)^2$ points in $|\Pi \cap K|$, a contradiction. Hence all lines in Π are blocked by the set K. Hence, Theorem 3.4 implies that $\Pi \cap K$ is a plane. $\square$

Lemma 5.2 *A $(q^5+q^3+q^2+1)$-solid Π does not contain full planes nor (q^2+1)-planes.*

Proof Let $H1 = q^2 + 1$, $H2 = q^3 + 1$, $H3 = q^3 + q^2 + 1$, $H4 = q^4 + q^2 + 1$ and $x = q^5 + q^3 + q^2 + 1$. Call the number of $H1$-, $H2$-, $H3$- and $H4$-planes inside Π, a, b, c and t respectively. We use t as a parameter here and we will show it has to be zero.

Then by counting respectively the total number of planes in a solid, the incident pairs (p, α), where p is a point of K and α a plane of Π, and the triples (p, r, α) where p and r are distinct points of K and α is a plane in Π containing p and r, in two ways, we obtain the following equations

$$a + b + c = \frac{q^8 - 1}{q^2 - 1} - t,$$

$$aH1 + bH2 + cH3 = x\frac{q^6 - 1}{q^2 - 1} - tH4,$$

$$aH1(H1 - 1) + bH2(H2 - 1) + cH3(H3 - 1) = x(x - 1)\frac{q^4 - 1}{q^2 - 1} - tH4(H4 - 1).$$

Solving these equations yields $a = -t(q^2 - q + 1)$, hence $t = 0$. □

Lemma 5.3 *A $(q^5 + q^2 + 1)$-solid Π contains at most q full planes. If it contains q full planes, then there are no $(q^3 + q^2 + 1)$-planes contained in Π. Furthermore, a $(q^5 + q^2 + 1)$-solid always contains $(q^2 + 1)$-planes.*

Proof Call the number of full planes inside Π t as above. We apply the same counting technique and we find $a = q^3 - tq^2 + tq - t + 1$, $b = q^3(q^3 + t)$ and $c = q^4 - (t + 1)q^3 + (1 + t)q^2 - tq$. If we want c to be positive we see that $t \leq q$. □

Lemma 5.4 *If a $(q^5 + q^2 + 1)$-solid Π does not contain lines intersecting K in q points then it does not contain full planes.*

Proof Suppose Π does contain a full plane π. In the previous lemma we have shown that there exist $(q^3 + 1)$-planes in Π. Consider a $(q^3 + 1)$-plane α in Π and a point $p \in K \cap \alpha$ outside π and a line L in Π passing through p. If $|L \cap K| = x$ we get the following inequality

$$(q^2 + 1)(q^3 + 1 - x) + x \leq q^5 + q^2 + 1.$$

Hence $x \geq q$. Since we assume there are no q-lines in Π we have $x \geq q + 1$. Considering all lines through p inside α yields a contradiction. □

Lemma 5.5 *A $(q^5 + q^4 + q^2 + 1)$-solid contains $q + 1$ full planes.*

Proof We apply exactly the same method as in the proofs of the lemmas above. We get $a = q^3 + q^2 - q - 1 - t(q^2 - q + 1)$ and $b = q^3(-q + t - 1)$. The first equation implies $t \leq q + 1$, while the second implies $t \geq q + 1$, hence $t = q + 1$. □

Lemma 5.6 *A $(q^5+q^4+q^2+1)$-solid Π is a union of $q+1$ full planes all passing through the same line L.*

Proof Suppose there does not pass a full plane lying in Π through p, with $p \in K \cap \Pi$. Consider an arbitrary line L in Π passing through p and intersecting K in x points. Consider all planes through L inside Π. Then we get at most

$$(q^2+1)(q^3+q^2+1-x)+x = q^5+q^4+q^2+1+q^3+(1-x)q^2$$

points in $\Pi \cap K$. This implies $x \leq q+1$. Consider all lines through p in a fixed plane β of Π through p; they all contain at most $q+1$ points of K, hence β contains at most $1+(q^2+1)q = q^3+q+1$, so at most q^3+1 points of the set K. Hence all planes in Π through p would contain at most q^3+1 points. Consider again an arbitrary line L in Π passing through p and repeat the argument above. We get the inequality

$$(q^2+1)(q^3+1-x)+x \geq q^5+q^4+q^2+1.$$

This yields a contradiction. Hence through all points $p \in \Pi \cap K$ there passes a full plane belonging to Π. Hence Π intersects K in a union of $q+1$ full planes. Since $|\Pi \cap K| = q^5+q^4+q^2+1$ these planes all intersect in the same line, otherwise we would have fewer points in $\Pi \cap K$. $\square$

Lemma 5.7 *If a 4-space Δ is such that $\Delta \cap K = (q^2+1)(q^5+1)$ then $\Delta \cap K$ is a non-singular hermitian variety $H(4,q^2)$.*

Proof Let $S1 = q^4+q^2+1$, $S2 = q^5+q^2+1$, $S3 = q^5+q^3+q^2+1$, $S4 = q^5+q^4+q^2+1$ $S5 = q^6+q^4+q^2+1$ and $x = q^7+q^5+q^2+1$. Call the number of $S1$-, $S2$-, $S3$-,$S4$- and $S5$-solids inside Π, a, b, c, t_1 and t_2 respectively. We use t_1 and t_2 as parameters here and we will show they have to be zero.

Then by counting respectively the total number of solids in a 4-space, the incident pairs (p,Π), where p is a point of K and Π a solid of Δ, and the triples (p,r,Π) where p and r are distinct points of K and Π is a solid in Δ containing p and r, in two ways, we obtain the following equations

$$a+b+c = \frac{q^{10}-1}{q^2-1} - t1 - t2,$$

$$aS1+bS2+cS3 = x\frac{q^8-1}{q^2-1} - t_1S4 - t_2S5,$$

$$aS1(S1-1)+bS2(S2-1)+cS3(S3-1) = x(x-1)\frac{q^6-1}{q^2-1} - t_1S4(S4-1) - t_2S5(S5-1).$$

Solving these equations yields $a = -\frac{t_2(q^4-q^3+2q^2-q+1)+t_1}{q^2-q+1}$. Hence we see that $t_1 = t_2 = a = 0$. Hence the only solids that occur in Δ are (q^5+q^2+1)- and $(q^5+q^3+q^2+1)$-solids. If Δ contains a full plane π then consider all solids in Δ through π. We get at most

$$(q^2+1)(q^5+q^3-q^4)+q^4+q^2+1$$

points, a contradiction. Hence we have the intersection numbers with planes and solids as in Theorem 4.2, so $\Delta \cap K$ is a non-singular hermitian variety $H(4,q^2)$. $\square$

Lemma 5.8 *If a 4-space Δ contains a line L intersecting K in q points, then this line cannot be contained in a (q^2+1)-plane of Δ. Furthermore, if L is only contained in (q^3+1)-planes of Δ, then $|K \cap \Delta| = q^7 + q^4 + q^2 + 1$.*

Proof Clearly L cannot be contained in full planes of Δ. First suppose that L is only contained in (q^3+1)-planes of Δ. Then $K \cap \Delta = (q^4+q^2+1)(q^3+1-q)+q = q^7+q^4+q^2+1$. Next suppose that L is contained in a (q^2+1)-plane α of Δ. By Lemma 5.2 α is not contained in a $(q^5+q^3+q^2+1)$-solid of Δ. Since (q^4+q^2+1)- and $(q^5+q^4+q^2+1)$-solids do not contain lines intersecting K in q points all solids through α in Δ are (q^5+q^2+1)-solids. Hence $|\Delta \cap K| = (q^2+1)q^5 + (q^2+1)$. This is impossible by the previous lemma. $\square$

Lemma 5.9 *In a 4-space Δ a (q^3+q^2+1)-plane β does not contain a q-line.*

Proof By the previous lemma a q-line can not be contained in a (q^2+1)-plane. If in some 4-space Δ, a q-line L lies in a (q^5+q^2+1)-solid Π then an easy counting learns that inside Π, L is only contained in (q^3+1)-planes. Hence if a (q^3+q^2+1)-plane β lying in Δ contains L, then β is not contained in (q^5+q^2+1)-solids of Δ. Since $(q^5+q^4+q^2+1)$-solids do not contain q-lines, in such a case we have

$$|K \cap \Delta| = (q^2+1)q^5 + q^3 + q^2 + 1 = q^7 + q^5 + q^3 + q^2 + 1.$$

Suppose Δ contains a (q^2+1)-plane α and look at all solids through α inside Δ. A $(q^5+q^3+q^2+1)$-solid and a full solid do not contain (q^2+1)-planes, hence we get

$$aq^4 + bq^5 + (q^2+1-a-b)(q^5+q^4) + q^2 + 1 = q^7 + q^5 + q^3 + q^2 + 1.$$

Solving this equation yields $a = \frac{q^3+(1-b)q-1}{q^2}$, but this is never an integer, contradiction. Hence Δ does not contain (q^2+1)-planes. Since $(q^4+q^2+1)-$, $(q^5+q^2+1)-$ and $(q^5+q^4+q^2+1)$-solids all contain (q^2+1)-planes, these solids do not occur in Δ.

Consider a $(q^5+q^3+q^2+1)$-solid in Δ. Such a solid contains (q^3+1)- and (q^3+q^2+1)-planes. Consider all solids through a (q^3+1)-plane (resp. a (q^3+q^2+1)-plane) inside Δ. By the previous all these solids are $(q^5+q^3+q^2+1)$-solids. This yields a contradiction since we get a different number of points in the respective cases. $\square$

Lemma 5.10 *A 4-space Δ intersecting K in $q^7+q^4+q^2+1$ points contains at most q full solids. Furthermore, if Δ contains q full solids, then it contains no $(q^5+q^4+q^2+1)$-solids.*

Proof Let $S1 = q^4+q^2+1$, $S2 = q^5+q^2+1$, $S3 = q^5+q^3+q^2+1$, $S4 = q^5+q^4+q^2+1$ $S5 = q^6+q^4+q^2+1$ and $x = q^7+q^4+q^2+1$. Call the number of $S1$-, $S2$-, $S3$-,$S4$- and $S5$-solids inside Π, a, b, c, t_1 and t_2 respectively. We use t_1 and t_2 as parameters here.

Then by counting respectively the total number of solids in a 4-space, the incident pairs (p, Π), where p is a point of K and Π a solid of Δ, and the triples (p, r, Π) where

p and r are distinct points of K and Π is a solid in Δ containing p and r, in two ways, we obtain the following equations

$$a+b+c=\frac{q^{10}-1}{q^2-1}-t1-t2,$$

$$aS1+bS2+cS3=x\frac{q^8-1}{q^2-1}-t_1S4-t_2S5,$$

$$aS1(S1-1)+bS2(S2-1)+cS3(S3-1)=x(x-1)\frac{q^6-1}{q^2-1}-t_1S4(S4-1)-t_2S5(S5-1).$$

Solving this yields $c=\frac{(q^4-(q^3-q^2)(1+t_2)-qt_2-t_1)q^3}{q^2-q+1}$. Since c must be positive, we are done. $\square$

Lemma 5.11 *If all lines intersect K in 1, q, $q+1$ or q^2+1 points inside a (q^3+1)-plane α and if at least 2 lines in α intersect K in q^2+1 points, then α intersects K in a cone with vertex a point and base a line intersecting in K in q points. If there is no line intersecting K in q points in a (q^3+1)-plane, then this plane intersects K in a unital.*

Proof Assume that α contains more than one full line, so at least two, say L and M. These lines intersect in a point r. Take an arbitrary point p on L (respectively M) different from r. All lines through p inside α different from L (respectively M) intersect K in q points otherwise we get more than $1+q^2+q^2(q-1)=q^3+1$ points. Consider now a point s in $K\cap\alpha$ not lying on L or M. All lines through s inside α not through r intersect K in q points. Hence the line rs is a full line. We have proven that the point r is collinear with all other points, hence our claim is proved. The second fact is well-known. $\square$

Lemma 5.12 *If a 4-space Δ contains a line L intersecting K in q points, then it intersects K in $q^7+q^4+q^2+1$ points. Furthermore all lines in Δ intersect K in 1, q, $q+1$ or q^2+1 points and (q^3+q^2+1)-planes in Δ do not contain q-lines.*

Proof By Lemma 5.8 and Lemma 5.9 L is only contained in (q^3+1)-planes inside Δ. Then Lemma 5.8 shows that $|\Delta\cap K|=q^7+q^4+q^2+1$. Consider all solids through a (q^3+1)-plane of Δ inside Δ. Then we get at least

$$(q^2+1)(q^5+q^2-q^3)+q^3+1=q^7+q^4+q^2+1$$

points. Hence a (q^3+1)-plane of Δ is only contained in (q^5+q^2+1)-solids in Δ and not contained in a $(q^5+q^3+q^2+1)$-solid in Δ. This means there are no $(q^5+q^3+q^2+1)$-solids inside Δ, since by Lemma 5.2 such solids contain (q^3+1)-planes.

Consider a (q^2+1)-plane in Δ. If such a plane is not contained in a (q^4+q^2+1)-solid of Δ, then we get at least $(q^2+1)(q^5)+q^2+1$ points in $K\cap\Delta$, a contradiction. Hence (q^2+1)-planes contained in Δ intersect K in a line.

Consider a (q^3+q^2+1)-plane β in Δ and all solids through it inside Δ. Then counting yields there passes exactly one $(q^5+q^4+q^2+1)$-solid through β in Δ. Hence if a point p of K in Δ does belong to a (q^3+q^2+1)-plane of Δ, then there is a full plane through p in Δ by Lemma 5.6. Suppose there is a point p of K in Δ such that there does not pass a full plane through p in Δ. Then also no (q^3+q^2+1)-plane passes through p in Δ. Consider a line through p in Δ that contains x points of K. Hence we get the following inequality

$$(q^4+q^2+1)(q^3+1-x)+x \geq q^7+q^4+q^2+1.$$

Hence $x \leq q$. Consider all lines through p inside a fixed plane through p lying in Δ. Then we get at most $(q^2+1)(q-1)+1$ points, hence this plane should be a (q^2+1)-plane, but such a plane contains a full line, a contradiction. Hence through all points $p \in K \cap \Delta$, there passes a full plane. Hence it follows that a (q^5+q^2+1)-solid of Δ is the union of full lines.

In a (q^4+q^2+1)-, a $(q^5+q^4+q^2+1)$- and a full solid of Δ all lines intersect K in 1, $q+1$ or q^2+1 points. Hence if a line in Δ does not intersect K in 1, $q+1$ or q^2+1 points, then it lies in a (q^5+q^2+1)-solid Π inside Δ.

Consider a point $p \in K \cap \Pi$. Consider a line L through p in Π containing x points of K, where x is different from 1, $q+1$ and q^2+1. Consider all planes through L inside Π. Assume, by way of contradiction, that L is contained in a (q^3+q^2+1)-plane of Δ. As such a plane is contained in one $(q^5+q^4+q^2+1)$-solid of Δ, we have $x \in \{1, q+1, q^2+1\}$ by Lemma 5.6, a contradiction. Hence, all planes of Π containing L are (q^3+1)-planes. So we get the equality

$$(q^2+1)(q^3+1-x)+x = q^5+q^2+1.$$

Hence $x = q$, so all lines in Δ intersect K in 1, q, $q+1$ or q^2+1 points. $\square$

6 There is no line intersecting K in q points

In this section we assume that no line intersects the set K in q points. We define a point-line geometry $S=(P,B,I)$, where the points of P are the points of K, where the lines of B are the full lines and where incidence is containment.

Theorem 6.1 *The geometry S is a Shult space.*

Proof Consider a point p of S and a line L of S, such that p and L are not incident. We prove the main axiom for the incidence relation of a Shult space, and we refer to it as the 1-or-all axiom (see page 2 for the definition of a Shult space).

Consider the plane α generated by p and L and let Δ be a 4-dimensional subspace containing α. We distinguish several cases.

By Lemma 5.4 all (q^5+q^2+1)-solids in Δ do not contain full planes.

1)Suppose there is a solid Π containing α in Δ that contains a full plane β. Then Π intersects K either in β, a $(q^5+q^4+q^2+1)$-solid, which is the union of $q+1$ full planes through a line M, or a full solid. In all cases the 1-or-all axiom holds.

2)Suppose Δ does not contain full planes. Then Δ also does not contain (q^4+q^2+1)- or $(q^5+q^4+q^2+1)$-solids. By Theorem 4.2 Δ intersects K in a non-singular hermitian variety $H(4,q^2)$, hence the 1-or-all axiom holds.

3)Suppose Δ contains a full plane β, but no solid containing α inside Δ contains a full plane. Consider a solid Π in Δ containing β. If L belongs to Π then it is contained in a full plane lying in Π but then we are back in Case 1. So assume L and Π intersect in a point s. This point belongs to a full plane γ in Π, which might be equal to β. Consider the solid generated by p and γ. This solid is either a $(q^5+q^4+q^2+1)$-solid or a full solid. Inside this solid there is a full plane ψ that passes through p. If L intersects ψ we are back in Case 1, so assume $L \cap \psi = \emptyset$. Consider an arbitrary point $r \in L$ and the solid Π_r generated by r and ψ. This solid intersects K in a full solid or in a cone with vertex a line and base a line intersecting K in $q+1$ points. Hence the line rp intersects K in $q+1$ or q^2+1 points. Consider all lines through p inside α; then the assumptions on the intersection sizes of planes with the set K imply that the 1-or-all axiom holds. $\square$

Theorem 6.2 *If S is non-degenerate, then it is a non-singular hermitian variety in $PG(n,q^2)$, $n \geq 4$.*

Proof If there exists a full plane, then S is a non-degenerate Shult space of finite rank at least 3, and since all lines contain at least three points by definition, S with all its subspaces is a polar space. By Theorem 2.1 it is a finite classical polar space and by looking at the intersection numbers, we see that S is a non-singular hermitian variety.

If there exists no full plane, then the previous arguments show we have proved that axiom (GQ3) for generalized quadrangles is satisfied for S. Clearly, there is a point p through which there pass three lines of S. Hence, S is a generalized quadrangle.

By Theorem 2.3, it is a classical one; going through the list of classical generalized quadrangles yields it is the non-singular hermitian variety $H(4,q^2)$. $\square$

Suppose now that S is degenerate, so there exist points collinear with all other points. We call such points singular points.

Lemma 6.3 *The singular points of S form a subspace Π_k of $PG(n,q^2)$.*

Proof Take two singular points p and r of S and consider a point t lying on the line $L = pr$. We want to show that t is a singular point. Since p and r are both singular, all the points of L belong to S, so $t \in S$. Let $M \neq L$ be an arbitrary line through t and consider the plane α generated by L and M. If α intersects S in L then M intersects S in 1 point. If α contains a point n of S not belonging to L, then through n there pass at least two full lines in α since p and r are singular points. Since we have the 1-or-all axiom all lines through n in α are full lines. Hence α is a full plane and M is a line of S. So every line through t intersects S either in 1 or in q^2+1 points. $\square$

Lemma 6.4 *If S contains singular points, then all lines not intersecting the subspace Π_k formed by the singular points, intersect S in 1, $q+1$, or q^2+1 points.*

Proof Consider a line L not intersecting Π_k. Take a singular point p and consider the plane generated by p and L. Since this plane contains either q^2+1, q^3+1, q^3+q^2+1 or q^4+q^2+1 points of S by assumption, we are done. □

Lemma 6.5 *If $n-k-1 \geq 4$, then S is a cone with vertex a k-dimensional space and base a non-singular hermitian variety.*

Proof If S is degenerate, then look at a complementary space $PG(n-k-1, q^2)$ of the space Π_k. By assumption, this space does not contain singular points of S. If $n-k-1 \geq 4$, then Theorem 6.2 shows that S intersects this space in a non-singular hermitian variety, hence S is a cone with vertex a k-dimensional space and base a non-singular hermitian variety. □

Now we consider all other cases one by one.

(a) If $n-k-1 = -1$, then S is the projective space $PG(n, q^2)$.

(b) If $n-k-1 = 0$, then S is a hyperplane of $PG(n, q^2)$.

(c) If $n-k-1 = 1$, then the complementary space is a line. If this line intersects K in one or in q^2+1 points we get a contradiction. If it intersects K in $q+1$ points, we have the union of $q+1$ hyperplanes.

(d) If $n-k-1 = 2$, then the complementary space is a plane π. By Lemma 6.4 all lines intersect in 1, $q+1$ or q^2+1 points.

(1)Suppose that π intersects S in q^2+1 points. Since all lines are blocked, this intersection has to be a line, but then we have singular points in the base, a contradiction.

(2)Suppose that π intersects S in q^3+1 points. Since we assume no lines intersect K in q points, all lines in π intersect S in 1 or in $q+1$ points, hence π intersects S in a unital. Indeed, if there would be a full line L in π then consider a point p in $\pi \cap S$ not belonging to L, and consider all lines through p in π. We would get at least $1 + (q^2+1)q = q^3+q+1$ points in $\pi \cap K$, a contradiction.

(3)Suppose that π intersects S in q^3+q^2+1 points. Since there is no line in π that intersects K in q points, the 1-axiom for generalized quadrangles is fulfilled. Hence some point $s \in S \cap \pi$ is collinear with all other points in $S \cap \pi$ otherwise we have a generalized quadrangle fully embedded in π, which yields a contradiction. But then s is a singular point, which is impossible.

(e) If $n-k-1 = 3$, then the complementary space is a solid Π.

By assumption, the solid Π does not contain a line intersecting S in q points. If Π contains a full plane then by Lemmas 5.2 and 5.4 it is either a (q^4+q^2+1)-solid, a $(q^5+q^4+q^2+1)$-solid or a full solid. In all cases we get singular points in the base, a contradiction. Hence Π does not contain full planes. Then Lemma 4.1 implies that $\Pi \cap K = H(3, q^2)$.

Theorem 6.6 *If a set of points K in $PG(n, q^2)$, $n \geq 4$, contains no lines intersecting K in q points, and if the intersection numbers of K with planes and solids are also intersection numbers of planes and solids with a hermitian variety in $PG(n, q^2)$, then K is either*

(i) The projective space $PG(n, q^2)$,

(ii) A hyperplane in $PG(n, q^2)$,

(iii) A hermitian variety in $PG(n, q^2)$.

(iv) A cone with vertex an $(n-2)$-dimensional space and base a line intersecting K in $q+1$ points.

(v) A cone with vertex an $(n-3)$-dimensional space and base a unital.

7 There is a line intersecting K in q points

In this section we assume that there is a line intersecting the set K in q points. By Lemma 5.12 we know that every line intersects the set K either in 1, q, $q+1$ or q^2+1 points, and that a line intersecting K in q points is only contained in (q^3+1)-planes. We introduce the following notations:

$$H_1(k) = \frac{q^{2k}-1}{q^2-1},$$

$$H_2(k) = q^{2k-1} + \frac{q^{2k-2}-1}{q^2-1},$$

$$H_3(k) = q^{2k-1} + \frac{q^{2k}-1}{q^2-1},$$

$$H_4(k) = \frac{q^{2k+2}-1}{q^2-1}.$$

Lemma 7.1 *Assume K is contained in $\Pi_n = PG(n, q^2)$.*

(1) Then $|K| = H_2(n)$.

(2) Next, consider any subspace Π_l of dimension l in Π_n, with $l \geq 1$. Then $|\Pi_l \cap K| \in \{H_1(l),\ H_2(l),\ H_3(l),\ H_4(l)\}$.

Proof (1) Since a q-line of Π_n is only contained in (q^3+1)-planes, we have

$$|\Pi_n \cap K| = (q^3+1-q)\frac{q^{2n-2}-1}{q^2-1} + q = H_2(n).$$

(2) The theorem clearly holds for $l = 1$ and $l = 2$. Now assume that $l \geq 3$. If Π_l contains a q-line, then $|\Pi_l \cap K| = H_2(l)$. So assume that Π_l does not contain a q-line. If M is a full line in Π_l and p is a point of K in Π_l not on M, then the plane α generated by M and p is not a (q^3+1)-plane by Lemma 5.11. If α is a (q^3+q^2+1)-plane, then an easy counting argument shows that α contains exactly one full line containing p. So the 1-or-all axiom holds in Π_l. First, let $l = 3$. If π is a (q^2+1)-plane in Π_3, then $\pi \cap K$ is a full line, and $\Pi_3 \cap K$ either is a plane, a cone with as vertex a point p and as base a unital, or a set of $q+1$ planes containing a common line. If Π does not contain a (q^2+1)-plane, but does contain a full plane, then $\Pi_3 \cap K = \Pi_3$. If Π_3 does not contain a (q^2+1)-plane, nor a full plane, then by Theorem 4.1 $\Pi_3 \cap K = H(3, q^2)$. In each of these three cases we have $|\Pi_3 \cap K| \in \{H_1(3),\ H_2(3),\ H_3(3), H_4(3)\}$.

Next, let $l \geq 4$. Then by Theorem 6.6, $|\Pi_l \cap K| \in \{H_1(l),\ H_2(l),\ H_3(l), H_4(l)\}$. □

Lemma 7.2 *(1) If $|K \cap \Pi_l| = H_1(l)$, $l \geq 1$, then $K \cap \Pi_l$ is an $(l-1)$-dimensional space.*

(2) If $|K \cap \Pi_l| = H_3(l)$, $l \geq 1$, then $K \cap \Pi_l$ is a union of $q+1$ $(l-1)$-dimensional spaces containing a common space Π_{l-2} of dimension $l-2$.

Proof (1) Since all lines of Π_l are blocked by K and $|K \cap \Pi_l| = \frac{q^{2l}-1}{q^2-1}$, the result follows from Theorem 3.4.

(2)Follows easily from the proof of Lemma 7.1. □

Lemma 7.3 *Inside a space Π_l intersecting K in $H_2(l)$ points, an $(l-2)$-dimensional space Π_{l-2} intersecting K in $H_3(l-2)$ points is contained in exactly one $(l-1)$-dimensional space intersecting K in $H_3(l-1)$ points.*

Proof By Lemma 7.2, Π_{l-2} can only be contained in $(l-1)$-dimensional spaces of Π_l intersecting K in $H_2(l-1)$ or $H_3(l-1)$ points. If x of them intersect in $H_3(l-1)$ points, we get the following equation

$$(q^2+1-x)(H_2(l-1) - H_3(l-2)) + x(H_3(l-1) - H_3(l-2)) + H_3(l-2) = H_2(l).$$

Solving yields $x = 1$. □

Lemma 7.4 *For every space Π_l, $l \geq 2$, the set $K \cap \Pi_l$ is a union of $(l-2)$-dimensional spaces.*

Proof The result clearly holds for $l = 2$. So assume that $l \geq 3$. If $K \cap \Pi_l$ does not contain a line intersecting K in q points, the result is clear by Lemma 7.2 and the proof of Lemma 7.1.

So assume $K \cap \Pi_l$ does contain such a line. Consider a point $p \in K \cap \Pi_l$ and suppose p is not contained in an $(l-2)$-dimensional space of $\Pi_l \cap K$. Let Π_{l-1} be an $(l-1)$-dimensional subspace of Π_l containing p. By induction Π_{l-1} contains an $(l-3)$-dimensional space Π_{l-3} of $\Pi_{l-1} \cap K$ containing p. By Lemma 7.2 we necessarily have

$|K \cap \Pi_{l-1}| = H_2(l-1)$. Let Π_{l-2} be an $(l-2)$-dimensional subspace of Π_l containing Π_{l-3}. Counting points in the spaces $\Pi_{l-1} \subset \Pi_l$ containing Π_{l-2} and putting $|\Pi_{l-2} \cap K| = x$, we obtain

$$(q^2+1)(H_2(l-1)-x)+x = H_2(l).$$

So $x = H_2(l-2)$. Counting points in the spaces $\Pi_{l-2} \subset \Pi_l$ containing Π_{l-3} and putting $|\Pi_{l-3} \cap K| = y$, we obtain

$$(q^4+q^2+1)(H_2(l-2)-y)+y = H_2(l).$$

So $y = H_2(l-3)$. This contradicts $\Pi_{l-3} \subset K$. □

Lemma 7.5 *The set K is a cone with vertex an $(n-3)$-dimensional space and base a (q^3+1)-plane.*

Proof (1)Suppose K does not contain a hyperplane. There are only three types of hyperplane intersections by Lemma 7.1. The arguments sketched below can be worked out in detail exactly as in Lemma 5.10. One can count how many hyperplane intersections of each type there are. Furthermore, if a hyperplane intersects K in $H_3(n-1)$ points, one can count how many $(n-2)$-dimensional spaces inside the hyperplane intersect K in $H_3(n-2)$ points. By Lemma 7.3 above each $(n-2)$-dimensional space intersecting K in $H_3(n-2)$ points is contained in exactly one $(n-1)$-dimensional space intersecting K in $H_3(n-1)$ points. In this way one can count the number of $(n-2)$-dimensional spaces intersecting K in $H_3(n-2)$ points.

The number of $(n-2)$-dimensional spaces intersecting K in $H_3(n-2)$ points can also be counted directly in function of the number of $(n-2)$-dimensional spaces contained in $K \cap \Pi_n$. Comparing the two expressions yields a linear equation for the number of $(n-2)$-dimensional spaces contained in $K \cap \Pi_n$. So there can only be one solution. Since we know the example of a cone with vertex an $(n-3)$-dimensional space and base a unital the solution is q^3+1.

Furthermore, a standard counting argument yields there are also exactly q^3+1 hyperplanes that intersect K exactly in an $(n-2)$-dimensional space.

By Lemma 7.4 $\Pi_n \cap K$ also has to be the union of the q^3+1 $(n-2)$-dimensional spaces found above.

We show they all have to contain a common $(n-3)$-dimensional space. Counting the points of K in the q^2+1 hyperplanes containing a given $(n-2)$-dimensional space Π_{n-2} in K, we see that Π_{n-2} is contained in exactly one hyperplane Π_{n-1} for which $\Pi_{n-1} \cap K = \Pi_{n-2}$, and in no hyperplanes containing $H_2(n-1)$ points of K. Let Π'_{n-2} be a second $(n-2)$-dimensional space of K. Then $\Pi'_{n-2} \cap \Pi_{n-1}$ is an $(n-3)$-dimensional space contained in Π_{n-2}. We have proved that every two $(n-2)$-dimensional spaces completely contained in K intersect in an $(n-3)$-dimensional space. Dualizing, we see that all $(n-2)$-dimensional spaces contained in K either contain a common $(n-3)$-dimensional space, or are contained in a common hyperplane. Clearly, the latter cannot occur.

(2)Suppose K does contain a hyperplane Π_{n-1}. We know K is the union of $(n-2)$-dimensional spaces. Take a point $p \notin \Pi_{n-1}$ and consider the $(n-2)$-dimensional space through p contained in K. This space intersects Π_{n-1} in an $(n-3)$-dimensional space Π_{n-3}. Take an arbitrary point $r \in \Pi_{n-3}$, a point $s \in K \backslash (\Pi_{n-1} \cup \Pi_{n-2})$, and a point t in $\Pi_{n-2} \backslash \Pi_{n-3}$. Consider the plane π_{rst} generated by r, s and t. Due to Lemma 5.11 and Lemma 7.2 the line rs is a full line. Hence we are done. $\square$

Together with the results of Theorem 6.6 and Lemma 5.11 we have proved our main result.

Theorem 7.6 *If for a set of points K in $PG(n,q^2)$, $n \geq 4$ the intersection numbers with planes and solids are also intersection numbers of planes and solids with a hermitian variety, then K is either*

(i) The projective space $PG(n,q^2)$,

(ii) A hyperplane in $PG(n,q^2)$,

(iii) A hermitian variety in $PG(n,q^2)$,

(iv) A cone with vertex an $(n-2)$-dimensional space and base a line intersecting K in q or $q+1$ points,

(v) A cone with vertex an $(n-3)$-dimensional space and base a unital,

(vi) A cone with vertex an $(n-3)$-dimensional space and base a set $\tilde{K}$ of $PG(2,q^2)$ intersecting each line of $PG(2,q^2)$ in 1, q, $q+1$ or q^2+1 points and containing exactly one full line.

Remark. Let $\mathcal{M}$ be a maximal $\{q^3-q^2+q;q\}$-arc in $PG(2,q^2)$, that is, a point set $\mathcal{M}$ of size q^3-q^2+q in $PG(2,q^2)$ intersecting each line of $PG(2,q^2)$ in either 0 or q points; such a set is known to exist for any $q=2^h$, see [10]. Let M be a line of $PG(2,q^2)$ intersecting $\mathcal{M}$ in q points and let L be a line of $PG(2,q^2)$ containing no points of $\mathcal{M}$. Then $\tilde{K} = (\mathcal{M} \backslash M) \cup L$ can be taken as base of the cone described in Theorem 7.6 (vi).

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