Characterizations of metric spaces by the use of their midsets: intervals*

by

Anthony D. Berard, Jr. (Ohio)

1. Introduction. In this article we study metric spaces with certain midset properties and find sufficient conditions to make them homeomorphic to intervals of real numbers. If (X, ϱ) is a nontrivial metric space and x and y are distinct points of X, then $\{q \in X|\ \varrho(x, q) = \varrho(y, q)\}$ will be called the *midset* of x and y in X and will be denoted by $A(x, y)$. If X is a nontrivial metric space for which each midset consists of a unique point, then we will say that X has *the unique midpoint property* (UMP).

The metric space (X, ϱ) is said to be *strongly convex* provided for each pair of distinct points in X there is a unique middle point. A *middle point* of x and y is a point m such that $\varrho(x, m) = \varrho(m, y) = \frac{1}{2}\varrho(x, y)$. Notice that any middle point for x and y is an element of $A(x, y)$. Menger [3] proved that a complete strongly convex metric space is isometric to a closed interval provided there exist points e_1 and e_2 in X such that for all $x \in X$, $\varrho(e_1, x) + \varrho(x, e_2) = \varrho(e_1, e_2)$; that is, there exist points e_1 and e_2 in X such that every point of X is between e_1 and e_2. Lelek and Nitka [2] showed that a compact strongly convex metric space which is one dimensional and which has the property that no middle point of x and y is a middle point of x' and y unless $x = x'$ is homeomorphic to the unit interval I. It is well known ([1], p. 54) that if X is a metric continuum with just two non-cut points then X is homeomorphic to the unit interval I.

We see from these results that convexity, the existence and number of cut points, compactness, and completeness are important in characterizing intervals. In this paper we weaken or remove some of these properties and prove that a connected metric space with the unique midpoint property is homeomorphic to an interval of the reals.

*Much of the work in this paper was accomplished in the authors' dissertation which was written at Case Western Reserve University under the direction of Dr. George M. Rosenstein, Jr.

The following lemma will be fundamental to many of the arguments of this paper.

LEMMA 1. *Let X be a connected metric space and x and y be distinct points of X; then $X-A(x,y)$ is disconnected.*

Proof. Indeed, $X-A(x,y) = A \cup B$ where $A = \{q \in X|\ \varrho(x,q) < \varrho(y,q)\}$, $B = \{q \in X|\ \varrho(x,q) > \varrho(y,q)\}$, and A and B are mutually separated. ■

To simplify the statements of some our proofs, when X is a connected metric space and x and y are distinct points of X, we will call the separation defined in Lemma 1 the *usual separation* for x and y. Notice that for any x and y in a connected metric space, $A(x,y)$ is a separating set.

Throughout the paper we will use the following conventions. For the sets $\{q \in X|\ \varrho(x,q) = \varepsilon\}$, $\{q \in X|\ \varrho(x,q) < \varepsilon\}$, $\{q \in X|\ \varrho(x,q) \leqslant \varepsilon\}$, and $\{q \in X|\ \varrho(x,q) > \varepsilon\}$ we will write, respectively, $D(x,\varepsilon)$, $B(x,\varepsilon)$, $\bar{B}(x,\varepsilon)$, and $c\bar{B}(x,\varepsilon)$. We will use R to denote the reals and R^+ to denote the non-negative reals.

2. Non-cut points. Much of the development which follows depends on the existence of non-cut points.

LEMMA 2. *If X is a connected metric space with* UMP, *then a necessary and sufficient condition that a point z of X be a non-cut point of X is that there do not exist distinct points x and y of X such that $\varrho(x,z) = \varrho(y,z)$.*

Proof. Let z be a cut point of X and let $X-z = A \cup B$ be a separation. Since A and B are nonvoid, there exist $a \in A$ and $b \in B$. Let $\varepsilon = \frac{1}{2}\min\{\varrho(a,z), \varrho(b,z)\}$. Assume that z satisfies the condition. There is at most one point x in X such that $\varrho(z,x) = \varepsilon$ and we may assume that $x \in A$. Then $B-\bar{B}(z,\varepsilon) = B-B(z,\varepsilon)$ and $X = [A \cup B(z,\varepsilon)] \cup [B-\bar{B}(z,\varepsilon)]$ is a separation of X contradicting its connectedness.

Suppose that z does not satisfy the condition. Then there exist points x and y of X such that $\varrho(x,z) = \varrho(y,z)$. By Lemma 1 z is a cut point of X. ■

THEOREM 3. *If X is a connected metric space with* UMP, *then there exist at most two distinct non-cut points.*

Proof. Suppose that there exist three distinct non-cut points z_1, z_2, and z_3 in X. No one of these may be the midpoint of the other two. Assume without loss of generality that $\varrho(z_1,z_3) < \varrho(z_2,z_3) < \varrho(z_1,z_2)$. Let $\varrho(z_1,z_3) = \varepsilon_1$ and $\varrho(z_2,z_3) = \varepsilon_2$. Consider the set $M = B(z_2,\varepsilon_2) \cap cB(z_3,\varepsilon_1)$. $M \neq \emptyset$ since $z_2 \in M$ and $M \neq X$ since $z_3 \notin M$. Furthermore, M is both open and closed in X, since the boundary of M is empty. We have contradicted the connectedness of X. ■

LEMMA 4. *If X is a connected metric space with* UMP *and if X has two distinct non-cut points z_1 and z_2, then $X \subseteq [B(z_1,\varepsilon) \cap B(z_2,\varepsilon)] \cup \{z_1, z_2\}$ where $\varepsilon = \varrho(z_1,z_2)$.*

Proof. Choose any arbitrary x from X. Suppose that $x \notin B(z_1,\varepsilon) \cup B(z_2,\varepsilon)$. Then if $M = B(z_1,\varepsilon) \cup B(z_2,\varepsilon)$, we have $M \subseteq X$, $M \neq \emptyset$, $M \neq X$, and since the boundary of M is empty, M is both open and closed, contradicting the connectedness of X.

Now suppose that $x \in B(z_1,\varepsilon)$, $x \neq z_1$, and $x \notin B(z_2,\varepsilon)$. Let $\delta = \varrho(x,z_1)$. Then $N = B(z_2,\varepsilon) \cap cB(z_1,\delta)$ is a subset of X such that $N \neq \emptyset$, $N \neq X$, and since the boundary of N is empty, N is both open and closed. We have once again contradicted the connectedness of X. ■

Theorem 3 is quite important, as it says that if X is a connected metric space with unique midpoint property, then X contains at most two non-cut points. Lemma 4 gives us a way of bounding the space. These theorems give us the feeling that the non-cut points are at the "edge" of the space, or are the "ends" of the segment.

3. $I(a,b)$. We discuss certain well behaved subsets of our metric space X under the assumption that X has a non-cut point z.

DEFINITION. If X is a connected metric space with a non-cut point z, then for $x, y \in X$ we will say $x < y$ provided $\varrho(x,z) < \varrho(y,z)$.

LEMMA 5. *If X is a connected metric space with* UMP *and a non-cut point z, then $(X, <)$ is a simply ordered set.*

LEMMA 6. *If X is a connected metric space with* UMP *and a non-cut point z, then the order topology induced by $<$ is weaker than the metric topology.*

Proof. For any $a \in X$, $\{x \in X|\ x < a\} = B(z, \varrho(z,a))$ and $\{x \in X|\ x > a\} = c\bar{B}(z, \varrho(z,a))$. ■

Notice that under the hypotheses of Lemma 6, $x < y$ if and only if x separates z and y. Indeed, one can show by the techniques we employ that the ordering we have defined is the separation order (see [1]). The remainder of this section is devoted to showing that the metric topology is weaker than the order topology.

DEFINITION. Let X be a connected metric space with UMP and a non-cut point z. For $a, b \in X$, let $I(a,b) = \{x \in X|\ a \leqslant x \leqslant b\}$.

Notice that for all $a, b \in X$, $I(a,b)$ is closed.

LEMMA 7. *Let X be a connected metric space with* UMP *and a non-cut point z. Then for $a, b \in X$ with $a < b$, $I(a,b)$ is a connected metric space with* UMP *and exactly two non-cut points, a and b.*

Proof. $I(a,b)$ is connected. If not, then $I(a,b) = A \cup B$ where A and B are disjoint, nonempty closed sets. Let $\varepsilon_1 = \varrho(z,a)$ and $\varepsilon_2 = \varrho(z,b)$. Then $\varepsilon_1 < \varepsilon_2$. If $a \in A$ and $b \in B$, $X = [A \cup \bar{B}(z,\varepsilon_1)] \cup [B \cup cB(z,\varepsilon_2)]$ is a separation of the connected set X, while if a and b belong to A, then $X = [A \cup \bar{B}(z,\varepsilon_1) \cup cB(z,\varepsilon_2)] \cup B$ is a separation of X.

$I(a, b)$ has UMP. For any pair of points in $I(a, b)$, their midpoint in X must be in $I(a, b)$; otherwise, $I(a, b)$ is not connected.

The only non-cut points of $I(a, b)$ are a and b. If a were a cut point of $I(a, b)$, then there would be $s, t \in I(a, b)$ with $s < t$ such that a separates s and t. But s separates a and t, contradicting ([4], Theorem 71). Similarly b is a non-cut point of $I(a, b)$. Since in connected metric spaces with UMP there are at most two such points, the assertion has been proven. ∎

Remark. Notice that if X is a connected metric space with UMP and two non-cut points, z and q, then $X = I(z, q)$, for, by Lemma 4, if $x \in X$, then $0 \leqslant \varrho(z, x) \leqslant \varrho(z, q)$.

LEMMA 8. *If X is a connected metric space with* UMP *and a non-cut point z, then if $m \in X$ and m is a cut point of X there exist $x, y \in B(m, \varepsilon)$ such that $x < m < y$ for any $\varepsilon > 0$.*

Proof. Since by the previous lemma, if for all $w \in X$, $w \leqslant m$, then m would be a non-cut point of X, there is a $w \in X$ such that $m < w$. Let $\delta_1 = \varrho(z, m)$ and $\delta_2 = \varrho(w, m)$. Let k be such that ε/k is less than the minimum of δ_1, δ_2, and ε. Then $D(m, \varepsilon/k)$ is not empty, for if it were, $B(m, \varepsilon/k)$ would be an open, closed, non-empty proper subset of X. Let $x \in D(m, \varepsilon/k)$. Either $x < m$ or $x > m$. Suppose that $x < m$. Then there is a point y in $D(m, \varepsilon/k) \cap cB(z, \delta_1)$ for otherwise $B(z, \delta_1) \cup B(m, \varepsilon/k)$ is an open, closed, non-empty proper subset of X. Since $\varrho(y, z) > \delta_1$, $x < m < y$. The argument is similar if $x > m$. ∎

THEOREM 9. *If X is a connected metric space with* UMP, *and a non-cut point z, then the order topology is the metric topology.*

Proof. Since we have already seen that the order topology is weaker than the metric topology, we need only show that for any $m \in X$ and any $\varepsilon > 0$, there exist x and y such that $m \in (x, y) \subseteq B(m, \varepsilon)$ where $x \in X$ or x is the symbol $-\infty$ and $y \in X$ or y is the symbol $+\infty$. If m is not a non-cut point, then by the previous theorem, there exist $x, y \in B(m, \varepsilon/4)$ with $x < m < y$. By Lemma 4, $I(x, y) \subseteq B(m, \varepsilon)$. If $m = z$, the result is obvious with y being the unique point which is a distance ε from z, and $x = -\infty$. If m is a non-cut point other than z, let x be the unique point a distance $\varrho(z, m) - \varepsilon/4$ from z and $y = +\infty$. ∎

4. Closed and half open intervals.

THEOREM 10. *If X is a connected metric space with* UMP *and a non-cut point z, then the function $f(x) = \varrho(z, x)$; $f\colon X \to f(X)$ is a homeomorphism and $f(X)$ is an interval of R.*

Proof. The function is clearly a continuous bijection, the injective property following from the fact that z is a non-cut point. Since X is

connected, $f(X)$ is also and is therefore an interval. We need only show that f is an open map.

Consider open sets of the form $\{q \in X |\ q < x\}$ and $\{q \in X |\ q > x\}$ for an arbitrary fixed point x of X. Now, $f(\{q \in X |\ q < x\}) = (-\infty, f(x)) \cap f(X)$ and $f(\{q \in X |\ q > x\}) = (f(x), \infty) \cap f(X)$ and these are appropriate sub-base elements for the topology induced on $f(X)$ by R.

THEOREM 11. *If X is a connected metric space with* UMP *and X has at least one non-cut point, then either:*

(i) *X has exactly one non-cut point and X is homeomorphic to R^+;*

or

(ii) *X has two distinct non-cut points and X is homeomorphic to* $[0, 1]$.

Proof. Let $f\colon X \to f(X) \subseteq R^+$ be the homeomorphism of Theorem 10.

Suppose that $f(X)$ is unbounded. Then X has exactly one non-cut point. If X had two non-cut points, then from the remark following Lemma 7, we could conclude that for all $x \in X$ $f(x)$ is no greater than the distance between the non-cut points, contradicting the unboundness of f. Since $f(X)$ is an interval in R^+ and since the only unbounded interval in R^+ which contains 0 is R^+, $f(X) = R^+$.

Now suppose that $f(X)$ is bounded. Let $M = \sup f(X)$. Then X has a second non-cut point if and only if there is a $q \in X$ such that $f(q) = M$, in which case q is the non-cut point. To see this suppose $q \in X$ and is such that $f(q) = M$. Then $X = I(z, q)$ and q is a non-cut point by Lemma 7. On the other hand, if w is a second non-cut point and $f(w) < M$, then there is a $y \in X$ such that $f(w) < f(y)$. But as $\varrho(z, w) < \varrho(z, y)$, w separates z and y, a contradiction. Thus if X has only one non-cut point, $f(X) = [0, M)$ and X is homeomorphic to R^+, while if X has two non-cut points, $f(X) = [0, M]$ and X is homeomorphic to $[0, 1]$.

5. Open intervals.

THEOREM 12. *If X is a connected metric space with* UMP *and no non-cut points, then X is homeomorphic to R.*

Proof. Choose arbitrary distinct points $x, y \in X$. Let z be the unique midpoint for x and y, and let $X - z = A \cup B$ be the usual separation. We want to show that $A \cup \{z\}$ and $B \cup \{z\}$ satisfy the hypotheses of Theorem 11 (i). By [4], Theorem 60, $A \cup \{z\}$ is connected.

ASSERTION 12.1. $A \cup \{z\}$ has UMP.

Choose arbitrary distinct points $r, s \in A \cup \{z\}$. Then r and s are in X and there exists a unique point q in X such that $\varrho(r, q) = \varrho(s, q)$. Suppose that $q \in B$. Let $X - q = C \cup D$ be the usual separation. We have that

$$A \cup \{z\} = [(A \cup \{z\}) \cap C] \cup [(A \cup \{z\}) \cap D] \quad \text{and} \quad A \cup \{z\}$$

is disconnected.

ASSERTION 12.2. z is a non-cut point of $A \cup \{z\}$.

Assume z is a cut point of $A \cup \{z\}$. Let $A = A \cup \{z\} - z = G \cup H$ be a separation. There is an $\varepsilon > 0$ such that if $0 < \delta < \varepsilon$, there exist $g \in G$, $h \in H$, and $b \in B$ such that g, h, and b are in $D(z, \delta)$. For, choosing arbitrary points $g_1 \in G$, $h_1 \in H$, and $b_1 \in B$, let $\varepsilon = \min\{\varrho(z, g_1), \varrho(z, h_1), \varrho(z, b_1)\}$. Suppose that $\delta < \varepsilon$ and there exists no $g \in G$ such that $g \in D(z, \delta)$. Then

$$G \cup \{z\} = [G \cap c\bar{B}(z, \delta)] \cup [(G \cup \{z\}) \cap B(z, \delta)]$$

is a separation of $G \cup \{z\}$, contradicting the connectedness of $G \cup \{z\}$. Similarly, there exist appropriate h and b.

Choose $\gamma = \varepsilon/4$. Let $g_2, b_2, h_2 \in D(z, \gamma)$ where $g_2 \in G$, $b_2 \in B$, and $h_2 \in H$. Let $\sigma = \max\{\varrho(g_2, b_2), \varrho(h_2, b_2)\} < \varepsilon/2$. We may assume that $\sigma = \varrho(g_2, b_2) \leqslant \varepsilon/2$. Clearly $g_2 \in D(b_2, \sigma)$. Now there exists an $h_3 \in H$ such that $h_3 \in D(b_2, \sigma)$; otherwise,

$$H \cup \{z\} = [B(b_2, \sigma) \cap (H \cup \{z\})] \cup [c\bar{B}(b_2, \sigma) \cap (H \cup \{z\})]$$

is a separation contradicting the connectedness of $H \cup \{z\}$. As $h_3, g_2 \in A \cup \{z\}$, they have a midpoint a in $A \cup \{z\}$ which is clearly a midpoint for h_3 and g_2 in X. But b_2 is distinct from a and also a midpoint for h_3 and g_2 in X, which contradicts the unique midpoint property in X.

ASSERTION 13.3. There is no non-cut point of $A \cup \{z\}$ except z.

Suppose that there exists a non-cut point $q \in A \cup \{z\}$ such that $q \neq z$. Since q is not a non-cut point of X there exist points r and s in X such that $\varrho(r, q) = \varrho(s, q)$. Let $X - q = E \cup F$ be the usual separation. Either $A \cup \{z\} - q \subseteq E$ or $A \cup \{z\} - q \subseteq F$. We may assume that $A \cup \{z\} - q \subseteq E$. Then we have that

$$B \cup \{z\} = [E \cap (B \cup \{z\})] \cup [F \cap (B \cup \{z\})]$$

is a separation contradicting the connectedness of $B \cup \{z\}$.

We have shown that $A \cup \{z\}$ satisfies the hypotheses of Theorem 11 (i) and is thus homeomorphic to R^+. Call the appropriate homeomorphism α. By similar reasoning $B \cup \{z\}$ is homeomorphic to R^+ and we will call the appropriate homeomorphism β. Now we may define the homeomorphism $\gamma\colon X \to R$ by

$$\gamma(x) = \begin{cases} \alpha(x) & \text{for } x \in A \cup \{z\}, \\ -\beta(x) & \text{for } x \in B \cup \{z\}. \ \blacksquare \end{cases}$$

COROLLARY 13. *If X is a connected metric space with* UMP, *then X is an interval and:*

(i) *if X has two distinct non-cut points X is a closed interval.*

(ii) *if X has exactly one non-cut point X is a half open interval.*

(iii) *if X has no non-cut points, X is an open interval.*

6. Examples. We would like to know more about the relationships between completeness, connectedness, convexity, and the unique midpoint property. We give an example of a complete connected metric space with the unique midpoint property for which the metric is not convex and an example of a complete metric space with the unique midpoint property which is not connected (indeed it is totally disconnected).

EXAMPLE 1. A complete connected metric space with UMP for which the metric is not convex. For $x, y \in [-1, 1] = X$ let $\sigma(x, y) = |x-y|/(1+|x-y|)$. Since σ is equivalent to the usual metric on X, (X, σ) is a compact, and thus complete, metric space. Notice that if $a, b \in X$, then $(a+b)/2$ is the unique midpoint. On the other hand, consider the points 1 and -1 in X. As $\sigma(-1, 1) = 2/3$, we would like to find some $x \in X$, $x \neq 1$, $x \neq -1$, such that $\sigma(-1, x) + \sigma(1, x) = 2/3$. Some elementary algebra shows that this is impossible. Thus X is not convex.

EXAMPLE 2. A metric space X which is complete and possesses UMP but which is totally disconnected. Let Y be the set consisting of all points of $[0, 1]$, which when expressed to the base 4 possess no ones or twos in their expansion. Notice that Y is homeomorphic to the Cantor ternary set. Let h and k be linear homeomorphisms of Y into $[0, 1/4] \times \{0\}$ and $[1, 5/4] \times \{0\}$, $h\colon Y \to [0, 1/4] \times \{0\}$ and $k\colon Y \to [1, 5/4] \times \{0\}$. Let q be the point $(5/8, 1)$ and set $X = \{q\} \cup h(Y) \cup k(Y)$. Let $\sigma\colon X \times X \to R^+$ be defined by $\sigma(q, q) = 0$, $\sigma(x, q) = \sigma(q, x) = 1$ whenever $x \in h(Y) \cup k(Y)$, and $\sigma(x, y) = \varrho(x, y)$ where ϱ is the usual metric on R, whenever x and y belong to $h(Y) \cup k(Y)$. (X, σ) is a complete metric space with the unique midpoint property, but X is totally disconnected.

References

[1] J. G. Hocking and G. S. Young, *Topology*, Reading, Mass: Addison-Wesley Publishing Co., Inc., 1961.

[2] A. Lelek and W. Nitka, *On convex metric spaces* I, Fund. Math. 49 (1961), pp. 183–204.

[3] K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. 100 (1928), pp. 75–163.

[4] R. L. Moore, *Foundations of Point Set Theory*, Revised Edition. American Mathematical Society Colloquium Publications, Vol. XIII. Providence: American Mathematical Society, 1962.

[5] G. T. Whyburn, *Analytic Topology*, American Mathematical Society Colloquium Publications, Vol. XXVIII. Providence: American Mathematical Society, 1942.

Reçu par la Rédaction le 17. 11. 1969