CHARACTERIZATIONS OF REAL MATRICES OF MONOTONE KIND

by

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An m by n real matrix A is said to be of <u>monotone kind</u> if

$$(1) \qquad Ax \geqq 0 \implies x \geqq 0 .$$

Collatz [2] treats square matrices of monotone kind and shows that for such matrices the above implication is equivalent to: A^{-1} exists and $A^{-1} \geqq 0$.[3)] Matrices of monotone kind have useful applications in numerical analysis [2, 7].

It is the purpose of this note to generalize Collatz's result to rectangular matrices, and also to show that, for the general rectangular case, a matrix of monotone kind can be further characterized as one for which the convex conical hull of the rows contains the nonnegative orthant.

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[3)] That is, each element of A^{-1} is nonnegative.

(For an m by n matrix A , the convex conical hull of the rows of A is defined as

$$K(A) = \{z \mid z = A^T u,\ u \geqq 0\}.$$

The nonnegative orthant E_+^n is defined by

$$E_+^n = \{x \mid x \in E^n,\ x \geqq 0\},$$

where E^n is the n-dimensional real Euclidean space.)

Theorem 1. Let A be an m by n real matrix. Then the following two statements are equivalent:

(2) A has a nonnegative left inverse. In other words, there exists an n by m matrix $Y \geqq 0$ such that $YA = I$.

(3) $$K(A) \supset E_+^n$$

Proof. Clearly (2) holds if and only if each row I_i of the identity matrix I of order n is a nonnegative linear combination of the rows of A . But this is equivalent to the statement that each unit vector is contained in K(A), which is the case if and only if (3) holds. Q.E.D.

Of course, if A is square, either (2) or (3) is equivalent to A being nonsingular and $Y = A^{-1}$ being nonnegative.

It can be shown by elementary arguments that (1) and (2) are equivalent for a square matrix A , and that (2) implies (1) for a general rectangular

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It can be shown by elementary arguments that (1) and (2) are equivalent for a square matrix A , and that (2) implies (1) for a general rectangular

matrix A . The proof that (1) implies (2) for a general rectangular A seems to require the use of either the duality theory of linear programming or a theorem of the alternative for linear inequalities, such as Motzkin's theorem [4, 5,8]. (Theorems of the alternative may be considered a consequence of the separation theorem for convex sets [1].)

Theorem 2. For any m by n real matrix A, (1) and (2) are equivalent.

Proof. If (2) holds, then $Ax \geqq 0$ implies that $x = YAx \geqq 0$, and (1) is established.

If (1) holds, then A must be of rank n. For, $Ax = 0$ implies that $Ax \geqq 0$ and $A(-x) \geqq 0$, and hence by (1), $x = 0$, and the rank of A is $n \leqq m$.

Thus if (1) holds and A is square ($m = n$), it is nonsingular, and (1) together with $AA^{-1} = I \geqq 0$ imply that $A^{-1} \geqq 0$.

For $m \geqq n$ a different argument is required. We note that $Ax \geqq 0$, $I_i x < 0$ has no solution for each $i = 1, \ldots, n$. By Motzkin's theorem [4, 5, 8] it follows that $yA = I_i$, $y \geqq 0$ has a solution for each i, and (2) follows. Q.E.D.

An alternate proof that (1) implies (2) may be based on the duality theory of linear programming [6] instead of on Motzkin's theorem. If (1) holds then

$$\underset{x}{\text{minimum}} \quad \{I_i x \mid Ax \geqq 0\} = 0 \quad \text{for each } i = 1, \ldots, n .$$

By the duality theory of linear programming [6]

$$\underset{y}{\text{maximum}} \quad \{0\,y \mid yA = I_i\,,\ \ y \geqq 0\} = 0 \ \text{ for each } i = 1, \ldots, n,$$

where the zero denotes an m vector of zeros. Hence for each $i = 1, \ldots, n$, $yA = I_i$, $y \geqq 0$, has a solution. This establishes (2) .

Remark. For square matrices, because $(A^{-1})^T = (A^T)^{-1}$, it follows from (2) above that any of the statements (1), (2) or (3) above is equivalent to any of the three statements below:

(1°) $A^T y \geqq 0 \implies y \geqq 0$.

(2°) $(A^T)^{-1}$ exists and $(A^T)^{-1} \geqq 0$.

(3°) $K(A^T) \supset E^n_+$.

Rectangular Matrices of Monotone Kind with Respect to Another Matrix: Let A be an m by n real matrix and let B be a k by n real matrix. Then the following are equivalent:

(1") $Ax \geqq 0 \implies Bx \geqq 0$

(2") $YA = B\,,\ \ Y \geqq 0$

(3") $K(A) \supset K(B)$

The equivalence of the above three statements is established by replacing I by B or B^T in the proofs of Theorems 1 and 2 (omitting in the latter case, the demonstration that A is of full rank and the special argument for non-singular A).

Finally it should be remarked that if we define the polar cone of the rows of a matrix A as

$$P(A) = \{x \mid Ax \geqq 0\},$$

then (1") above can be stated as

$$(1'') \qquad P(A) \subset P(B).$$

The equivalence of (1") and (3") follows then directly from the duality theorem for polyhedral convex cones of Goldman and Tucker [3, lemma 2].

Example. Consider the following m by 2 matrix $(m \geqq 2)$

$$A = \begin{bmatrix} r_1 \cos \theta_1 & r_1 \sin \theta_1 \\ \vdots & \vdots \\ r_m \cos \theta_m & r_m \sin \theta_m \end{bmatrix},$$

where $r_i \geqq 0$, $-\pi \leqq \theta_i \leqq \pi$, for $i = 1, \ldots, m$. Our necessary and sufficient condition (3) (that A be of monotone kind (1) or have a nonnegative left inverse (2)) becomes this: there exist i, j, $i \neq j$, such that for all $k \neq i$, $k \neq j$ $(1 \leqq i, j, k \leqq m)$ we have that

$$r_i > 0, \quad r_j > 0, \quad \theta_j \leqq \theta_k \leqq \theta_i$$

$$\frac{\pi}{2} \leqq \theta_i - \theta_j < \pi, \quad -\frac{\pi}{2} < \theta_j \leqq 0, \quad \frac{\pi}{2} \leqq \theta_i < \pi.$$

If A is a 2 by 2 matrix, then $i = 1$ or 2, $j = 1$ or 2, $i \neq j$, and the above condition is necessary and sufficient for A^{-1} to exist and $A^{-1} \geqq 0$. We have then

$$A^{-1} = \frac{1}{\sin(\theta_2 - \theta_1)} \begin{bmatrix} \dfrac{\sin\theta_2}{r_1} & \dfrac{-\sin\theta_1}{r_2} \\ \dfrac{-\cos\theta_2}{r_1} & \dfrac{\cos\theta_1}{r_2} \end{bmatrix} \geqq 0 .$$

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