# CHARACTERIZATIONS OF SOME PSEUDO-EINSTEIN RULED REAL HYPERSURFACES IN COMPLEX SPACE FORMS IN TERMS OF RICCI TENSOR 

Juan de Dios Pérez, Young Jin Suh and Hae Young Yang


#### Abstract

In this paper we define a new notion of pseudo-Einstein ruled real hypersurfaces, which are foliated by the leaves of pseudo-Einstein complex hypersurfaces in complex space forms $M_{n}(c), c \neq 0$. Also we want to give a new characterization of this kind of pseudo-Einstein ruled real hypersurfaces in terms of weakly $\eta$-parallel Ricci tensor and the certain commutative condition defined on the orthogonal distribution $T_{0}$ in $M_{n}(c)$.


## 1. Introduction

Let us denote by $M_{n}(c)$ a complex $n(\geq 2)$-dimensional Kaehler manifold of constant holomorphic sectional curvature $c$, which is said to be a complex space form. Then a complete and simply connected complex space form is isometric to a complex projective space $P_{n}(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^{n}$ or a complex hyperbolic space $H_{n}(\mathbb{C})$, according as $c>0, c=0$ or $c<0$. The induced almost contact metric structure of a real hypersurface $M$ of $M_{n}(c)$ is denoted by $(\phi, \xi, \eta, g)$.

Until now several kinds of real hypersurfaces have been investigated by many differential geometers from different view points ([2],[4],[6],[9],[12] and [13]). Among them in a complex projective space $P_{n}(\mathbb{C})$ [5] Cecil-Ryan and [9] Kimura proved that they are realized as the tubes of constant radius over Kaehler submanifolds if the structure vector field $\xi$ is principal. Also Berndt [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_{n}(\mathbb{C})$ are realized as the tubes of constant radius over certain submanifolds when the structure vector field $\xi$ is principal. Nowadays in $H_{n}(\mathbb{C})$ they are said to be of type $A_{0}, A_{1}, A_{2}$, and $B$.

[^0]When the structure vector field $\xi$ is not principal, Kimura [9] and Ahn, Lee and the second author [1] have constructed an example of ruled real hypersurfaces foliated by totally geodesic leaves, which are integrable submanifolds of the distribution $T_{0}$ defined by the subspace $T_{0}(x)=\left\{X \in T_{x} M: X \perp \xi\right\}, x \in M$, along the direction of $\xi$ and Einstein complex hypersurfaces in $P_{n}(\mathbb{C})$ and $H_{n}(\mathbb{C})$ respectively. The expression of the Weingarten map is given by

$$
A \xi=\alpha \xi+\beta U, A U=\beta \xi \text { and } A X=0
$$

where we have defined a unit vector $U$ orthogonal to $\xi$ in such a way that $\beta U=$ $A \xi-\alpha \xi$ and $\beta$ denotes the length of a vector field $A \xi-\alpha \xi$ and $\beta(x) \neq 0$ for any point $x$ in $M$, and for any $X$ in the distribution $T_{0}$ and orthogonal to $\xi$. Recently, several characterizations of such kind of ruled real hypersurfaces have been studied by the papers ( $[1],[6],[9],[11]$ and $[15])$.

Now as a general extension of this fact we introduce a new kind of ruled real hypersurfaces in $M_{n}(c)$ foliated by pseudo-Einstein leaves, which are integrable submanifolds of the distribution $T_{0}$ defined by the subspace $\left\{X \in T_{x} M: X \perp \xi\right\}$, along the direction of $\xi$ and pseudo-Einstein complex hypersurfaces in $M_{n}(c)$. Then such kind of ruled real hypersurfaces are said to be pseudo-Einstein, because its Ricci tensor of the integral submanifold $M(t)$ of the distribution $T_{0}$ is given by

$$
S^{t}=\left(\frac{n}{2} c-\mu\right) I+(\mu-\lambda)\left\{U \otimes U^{*}+\phi U \otimes(\phi U)^{*}\right\}
$$

Moreover, its expression of the Weingarten map is given by

$$
A U=\beta \xi+\gamma U+\delta \phi U \text { and } \quad A \phi U=\delta U-\gamma \phi U
$$

In Lemma 3.1 we know that the function $\lambda$ mentioned above is given by $\lambda=$ $2\left(\gamma^{2}+\delta^{2}\right)$.

When $\lambda=\mu$, ruled real hypersurfaces foliated by such kind of leaves are said to be Einstein. In particular, $\lambda=\mu=0$, this kind of Einstein ruled real hypersurfaces are congruent to ruled real hypersurfaces in $M_{n}(c)$ foliated by totally geodesic Einstein leaves $M_{n-1}(c)$, which are said to be totally geodesic ruled real hypersurfaces in the sense of Kimura [9] for $c>0$ and Ahn, Lee and the second author [1] for $c<0$. In such a situation the function $\gamma$ and $\delta$ both vanish identically.

From this point of view Kimura and Maeda [11] proved the following
Theorem A. Let $M$ be a real hypersurface of $P_{n} \mathbb{C}, n \geqq 3$. Then the second fundamental form is $\eta$-parallel and the holomorphic distribution $T_{0}$ is integrable if and only if $M$ is locally a ruled real hypersurface.

Moreover, Ahn, Lee and the second author [1] proved the following

Theorem B. Let $M$ be a real hypersurface of $M_{n}(c), c \neq 0, n \geqq 3$. Assume that $\xi$ is not principal. Then it satisfies

$$
g((A \phi-\phi A) X, Y)=0
$$

for any vector fields $X$ and $Y$ in $T_{0}$ and the second fundamental form is $\eta$-parallel if and only if $M$ is locally a ruled real hypersurface.

Even though the second fundamental form for ruled real hypersurfaces in above is $\eta$-parallel, but its Ricci tensor is not necessarilly $\eta$-parallel. In the previous paper [17] the second author introduced the new notion of pseudo-Einstein ruled real hypersurfaces in $M_{n}(c), c \neq 0$. The ruled real hypersurfaces of $M_{n}(c)$ defined by Kimura [9] and Ahn, Lee and the second author [1] respectively for $c>0$ and $c<0$ are Einstein ruled ones with zero Ricci curvatures. The purpose of this paper is to generalize such a notion of Einstein ruled ones into pseudo-Einstein ones.

We consider a distribution $T^{\prime \prime}$ defined by a subspace

$$
T^{\prime \prime}(x)=\left\{X \in T_{0}(x): g(X, U)(x)=0\right\}
$$

of the tangent subspace $T_{0}(x)$. Let $T^{\prime}$ be a distribution defined by a subspace

$$
T^{\prime}(x)=\left\{X \in T^{\prime \prime}(x): g\left(X, \phi U_{(x)}\right)=0\right\}
$$

of the tangent subspace $T^{\prime \prime}$.
Now let us consider much more generalized condition than that of $\eta$-paralle Ricci tensor. The Ricci tensor $S$ of the real hypersurface $M$ of $M_{n}(c)$ is said to be weakly $\eta$-parallel, if it satisfies

$$
\begin{equation*}
g\left(\left(\nabla_{X} S\right) Y, Z\right)=0, X \in T_{0}, Y, Z \in T^{\prime} \tag{I}
\end{equation*}
$$

Of course in section 3 it can be verified that the Ricci tensor of pseudo-Einstein ruled real hypersurfaces in $M_{n}(c)$ is weakly $\eta$-parallel.

Let us consider another geometric condition defined on the distribution $T_{0}$ in such a way that

$$
\begin{equation*}
g((A \phi+\phi A) X, Y)=0 \tag{II}
\end{equation*}
$$

for any $X, Y \in T_{0}$, which gives an integrability of the distribution $T_{0}$. Then the purpose of this paper is to give a new characterization of pseudo-Einstein ruled real hypersurface in terms of the Ricci tensor as follows:

Theorem 1. Let $M$ be a real hypersurface of $M_{n}(c), c \neq 0$ and $n \geqq 3$. If it satisfies (I) and (II) and its mean curvature is non-constant along the distribution $T^{\prime}$, then $M$ is locally congruent to a pseudo-Einstein ruled real hypersurface.

Moreover, as an application of Theorem 1 we obtain more specified result if we assume that the mean curvature of $M$ is non-vanishing. That is, we have the following

Theorem 2. Let $M$ be a real hypersurface of $M_{n}(c), c \neq 0$ and $n \geqq 3$ and its mean curvature is non-vanishing. If it satisfies (I) and (II) and its mean curvature is non-constant along the distribution $T^{\prime}$, then $M$ is locally congruent to a ruled real hypersurface.

In section 3 we will recall fundamental properties of pseudo-Einstein ruled real hypersurfaces of $M_{n}(c), c \neq 0$, and will show that its Ricci tensor is weakly $\eta$-parallel. In section 4 we shall prove the main theorem.

Kimura and Maeda ([11]) have constructed a ruled real hypersurface $M$ in complex projective space $P_{n}(\mathbb{C})$ which was foliated by totally geodesic submanifolds $P_{n-1}(\mathbb{C})$. In such a case its mean curvature $H$ is given by $H=\frac{\alpha}{n}$, where the function $\alpha$ is denoted by $\alpha=g(A \xi, \xi)$. In general its mean curvature $H$ is nonvanishing. After finishing the proof of Lemma 4.4, by using Theorem 1 we will prove Theorem 2 which gives another new characterization of such kind of ruled real hypersurfaces in complex space form $M_{n}(c)$.

The present authors would like to express their sincere gratitudes to the referee for his valuable comments to develop the first version of this manuscript.

## 2. Preliminaries

First of all, we recall fundamental properties of real hypersurfaces in a complex space form. Let $M$ be a real hypersurface in a complex $n$-dimensional complex space form ( $M_{n}(c), \bar{g}$ ) of constant holomorphic sectional curvature $c$, and let $C$ be a unit normal vector field on a neighborhood in $M$. We denote by $J$ the almost complex structure of $M_{n}(c)$. For a local vector field $X$ on the neighborhood in $M$, the images of $X$ and $C$ under the linear transformation $J$ can be represented as

$$
J X=\phi X+\eta(X) C, \quad J C=-\xi
$$

where $\phi$ defines a skew-symmetric transformation on the tangent bundle $T M$ of $M$, where $\eta$ and $\xi$ denote a 1 -form and a vector field on the neighborhood in $M$, respectively. Then it is seen that $g(\xi, X)=\eta(X)$, where $g$ denotes the Riemannian metric tensor on $M$ induced from the metric tensor $\bar{g}$ on $M_{n}(c)$. The set of tensors $(\phi, \xi, \eta, g)$ is called an almost contact metric structure on $M$. They satisfy the following properties :

$$
\phi^{2}=-I+\eta \otimes \xi, \quad \phi \xi=0, \quad \eta(\phi X)=0, \quad \eta(\xi)=1
$$

for any vector field $X$, where $I$ denotes the identity transformation. Furthermore the covariant derivatives of the structure tensors are given by

$$
\begin{equation*}
\nabla_{X} \xi=\phi A X, \quad\left(\nabla_{X} \phi\right) Y=\eta(Y) A X-g(A X, Y) \xi \tag{2.1}
\end{equation*}
$$

for any vector fields $X$ and $Y$ on $M$, where $\nabla$ is the Riemannian connection on $M$ and $A$ denotes the shape operator of $M$ in the direction of $C$.

Since the ambient space is of constant holomorphic sectional curvature $c$, the equations of Gauss and Codazzi are obtained by:

$$
\begin{align*}
R(X, Y) Z & =\frac{c}{4}\{g(Y, Z) X-g(X, Z) Y \\
& +g(\phi Y, Z) \phi X-g(\phi X, Z) \phi Y-2 g(\phi X, Y) \phi Z\}  \tag{2.2}\\
& +g(A Y, Z) A X-g(A X, Z) A Y
\end{align*}
$$

$$
\begin{aligned}
\left(\nabla_{X} A\right) Y- & \left(\nabla_{Y} A\right) X \\
& =\frac{c}{4}\{\eta(X) \phi Y-\eta(Y) \phi X-2 g(\phi X, Y) \xi\}
\end{aligned}
$$

where $R$ denotes the Riemannian curvature tensor of $M$ and $\nabla_{X} A$ denotes the covariant derivative of the shape operator $A$ with respect to $X$. The second fundamental form is said to be $\eta$-parallel if the shape operator $A$ satisfies $g\left(\left(\nabla_{X} A\right) Y, Z\right)=$ 0 for any vector fields $X, Y$ and $Z$ in $T_{0}$.

On the other hand, the Ricci tensor $S$ is given by

$$
\begin{equation*}
S=\frac{c}{4}\{(2 n+1) I-3 \eta \otimes \xi\}+h A-A^{2} \tag{2.4}
\end{equation*}
$$

where $I$ denotes the identity transformation and $h$ is the trace of $A$.
Next we assume the condition that

$$
\begin{equation*}
g((A \phi+\phi A) X, Y)=0 \tag{II}
\end{equation*}
$$

for any vector fields $X$ and $Y$ in $T_{0}$.
On the other hand, by (2.1) we know

$$
\nabla_{X} Y=\left(\nabla_{X} Y\right)_{0}-g(Y, \phi A X) \xi
$$

where $\left(\nabla_{X} Y\right)_{0}$ denotes the $T_{0}$ component of the vector field $\nabla_{X} Y$. Then by differentiating the condition (II) and using also (2.1) and the above formula, we have

$$
\begin{align*}
& g\left(\left(\nabla_{X} A\right) Y, \phi Z\right)-g\left(\left(\nabla_{X} A\right) Z, \phi Y\right) \\
& \quad=\beta[g(Y, U) g(A X, Z)-g(Z, U) g(A X, Y)  \tag{2.5}\\
& \quad+g(Y, \phi U) g(\phi A X, Z)-g(Z, \phi U) g(\phi A X, Y)]
\end{align*}
$$

for any vector fields $X, Y$ and $Z$ in $T_{0}$.
Now we here calculate the covariant derivative of the Ricci tensor $S$. By (2.4) we get for any $X, Y$ in $T_{0}$

$$
\begin{aligned}
\left(\nabla_{X} S\right) Y= & -\frac{3}{4} \operatorname{cg}(\phi A X, Y) \xi+d h(X) A Y \\
& +h\left(\nabla_{X} A\right) Y-\left(\nabla_{X} A\right) A Y-A\left(\nabla_{X} A\right) Y
\end{aligned}
$$

from which it turns out to be

$$
\begin{align*}
g\left(\left(\nabla_{X} S\right) Y, Z\right)= & d h(X) g(A Y, Z)+h g\left(\left(\nabla_{X} A\right) Y, Z\right) \\
& -g\left(\left(\nabla_{X} A\right) Y, A Z\right)-g\left(\left(\nabla_{X} A\right) Z, A Y\right) \tag{2.6}
\end{align*}
$$

for any vector fields $X, Y$ and $Z$ in $T_{0}$. Replacing $Z$ by $\phi Z$ in this equation, we get

$$
\begin{aligned}
g\left(\left(\nabla_{X} S\right) Y, \phi Z\right)= & d h(X) g(A Y, \phi Z)+h g\left(\left(\nabla_{X} A\right) Y, \phi Z\right) \\
& -g\left(\left(\nabla_{X} A\right) Y, A \phi Z\right)-g\left(\left(\nabla_{X} A\right) \phi Z, A Y\right) .
\end{aligned}
$$

For any vector field $V$ we denote by $V_{0}$ the component in the distribution $T_{0}$. Since we see $A \phi Z=-\phi A Z-\beta g(Z, \phi U) \xi$ by (II), we have

$$
\begin{align*}
& g\left(\left(\nabla_{X} A\right) Y, A \phi Z\right)+g\left(\left(\nabla_{X} A\right) \phi Y, A Z\right) \\
& =-g\left(\left(\nabla_{X} A\right) Y, \phi A Z\right)-\beta g(Z, \phi U) g\left(\left(\nabla_{X} A\right) Y, \xi\right)  \tag{2.7}\\
& \quad+g\left(\left(\nabla_{X} A\right)(A Z)_{0}, \phi Y\right)+\beta g(Z, U) g\left(\left(\nabla_{X} A\right) \xi, \phi Y\right)
\end{align*}
$$

where we denote by $(A Z)_{0}$ the $T_{0}$-component of the vector field $A Z$. By replacing $Z$ by $(A Z)_{0}$ in (2.5), the above equation (2.7) is reformed as

$$
\begin{align*}
& g\left(\left(\nabla_{X} A\right) Y, A \phi Z\right)+g\left(\left(\nabla_{X} A\right) \phi Y, A Z\right) \\
& =-\beta[g(Y, U) g(A X, A Z)-g(A Z, U) g(A X, Y) \\
& \quad+g(Y, \phi U) g(\phi A X, A Z)-g(A Z, \phi U) g(\phi A X, Y)  \tag{2.8}\\
& \left.\quad-\beta^{2} g(X, U) g(Y, U) g(Z, U)\right] \\
& \quad+\beta g(Z, U) g\left(\left(\nabla_{X} A\right) \phi Y, \xi\right)-\beta g(Z, \phi U) g\left(\left(\nabla_{X} A\right) Y, \xi\right)
\end{align*}
$$

where we have used the formula that

$$
A Z=(A Z)_{0}+\beta g(Z, U) \xi
$$

Now by (II), (2.5), (2.6) and (2.8) we can assert an important formula which will be used to prove our results:

$$
\begin{align*}
& g\left(\left(\nabla_{X} S\right) Y, \phi Z\right)+g\left(\left(\nabla_{X} S\right) Z, \phi Y\right) \\
&=2 d h(X) g(A X, \phi Z)+2 h g\left(\left(\nabla_{X} A\right) Y, \phi Z\right) \\
&-\beta h {[g(Y, U) g(A X, Z)-g(Z, U) g(A X, Y)} \\
& \quad+g(Y, \phi U) g(\phi A X, Z)-g(Z, \phi U) g(\phi A X, Y)] \\
&+\beta {[g(Y, U) g(A X, A Z)-g(A Z, U) g(A X, Y)} \\
& \quad+g(Y, \phi U) g(\phi A X, A Z)-g(A Z, \phi U) g(\phi A X, Y) \\
& \quad\left.\quad \beta^{2} g(X, U) g(Y, U) g(Z, U)\right]  \tag{2.9}\\
&\left.-\beta g(Z, U) g\left(\nabla_{X} A\right) \phi Y, \xi\right)+\beta g(Z, \phi U) g\left(\left(\nabla_{X} A\right) Y, \xi\right) \\
&+\beta {[g(Z, U) g(A X, A Y)-g(A Y, U) g(A X, Z)} \\
& \quad+g(Z, \phi U) g(\phi A X, A Y)-g(A Y, \phi U) g(\phi A X, Z) \\
& \quad\left.\quad \beta^{2} g(X, U) g(Y, U) g(Z, U)\right] \\
&-\beta g(Y, U) g\left(\left(\nabla_{X} A\right) \phi Z, \xi\right)+\beta g(Y, \phi U) g\left(\left(\nabla_{X} A\right) Z, \xi\right) .
\end{align*}
$$

## 3. Pseudo-Einstein ruled real hypersurface

This section is concerned with necessary properties about ruled real hypersurfaces. First of all, we define a ruled real hypersurface $M$ of $M_{n}(c), c \neq 0$. Let $\mathfrak{D}$ be a $J$-invariant integrable ( $2 n-2$ )-dimensional distribution defined on $M_{n}(c)$ whose integral manifolds are holomorphic planes spanned by unit normals $C$ and $J C$ and let $\gamma: I \rightarrow M_{n}(c)$ be an integral curve for the vector $J C$. For any $t(\in I)$ let $M_{n-1}^{(t)}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_{n}(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma^{\prime}(t)$ and $J \gamma^{\prime}(t)$. Set $M=\left\{x \in M_{n-1}^{(t)}(c): t \in I\right\}$. Then the construction of $M$ asserts that $M$ is a real hypersurface of $M_{n}(c)$, which is called a ruled real hypersurface. This means that there exists a ruled real hypersurfaces of $M_{n}(c)$ with the given distribution $\mathfrak{D}$. We denote by $A_{C}$ and $A_{J C}$ the shape operator of any integral submanifold $M(t)$ of $\mathfrak{D}$ in $M_{n}(c)$ in the direction of $C$ and $J C$.

Under this construction the ruled real hypersurface $M$ of $M_{n}(c), c \neq 0$, has some fundamental properties. Let $M$ be a ruled real hypersurface with the given distribution $\mathfrak{D}$ of $M_{n}(c), c \neq 0$ and let $A$ be the its shape operator of the ruled real hypersurface $M$ in $M_{n}(c)$.

Now let us put $\xi=-J C$ and $A \xi=\alpha \xi+\beta U$, where $U$ is a unit vector orthogonal to $\xi$ and $\alpha$ and $\beta(\beta \neq 0)$ denote certain differentiable functions defined on $M$. For
any unit vector field $V$ along $\mathfrak{D}$, let $V^{*}$ be the corresponding 1 -form defined by $V^{*}(V)=g(V, V)=1$. If they satisfy

$$
A_{C}^{2}+A_{J C}^{2}=\lambda I+\mu\left(V \otimes V^{*}+J V \otimes(J V)^{*}\right)
$$

for a certain vector field $V$, where $\lambda$ and $\mu$ are smooth function on $M$, then the ruled real hypersurface $M$ with the given distribution $\mathfrak{D}$ of $M_{n}(c)$ is said to be pseudo-Einstein and if $\lambda=\mu=0$, then it is said to be totally geodesic and it is the ruled real hypersurface $M$ in the sense of Kimura [9]. If the ruled real hypersurface $M$ is pseudo-Einstein, Einstein or totally geodesic, then it can be easily seen that any integral submanifold of $\mathfrak{D}$ is pseudo-Einstein, Einstein or totally geodesic, respectively, because $\mathfrak{D}$ is $J$-invariant.

Since $T_{0}(=\mathfrak{D})$ is integrable, we see

$$
\begin{equation*}
g((A \phi+\phi A) X, Y)=0 \tag{II}
\end{equation*}
$$

for any vector fields $X$ and $Y$ in $T_{0}$.
On the other hand, $M(t)$ is a submanifold of codimension 2 and $\xi$ and $C$ are orthonormal normal vector fields on its leaf in $M_{n}(c)$. So we have

$$
\begin{align*}
\bar{\nabla}_{X} Y & =\nabla_{X} Y+g(A X, Y) C \\
& =\nabla_{X}^{t} Y+g\left(A_{C} X, Y\right) C+g\left(A_{\xi} X, Y\right) \xi \tag{3.1}
\end{align*}
$$

where $\bar{\nabla}$ and $\nabla^{t}$ are the covariant derivatives in the ambient space $M_{n}(c)$ and in the submanifold $M(t)$, respectively and moreover $A_{C}$ and $A_{\xi}$ are the shape operators in the direction of $C$ and $\xi$, respectively. Then we have

$$
g\left(\bar{\nabla}_{X} Y, \xi\right)=g\left(\nabla_{X} Y, \xi\right)=-g\left(\bar{\nabla}_{X} \xi, Y\right)=g\left(A_{\xi} X, Y\right)
$$

for any $X, Y \in T_{0}$, from which it implies that

$$
\begin{equation*}
A_{\xi} X=-\phi A X, X \in T_{0} \tag{3.2}
\end{equation*}
$$

On the other hand, by (3.1) we have

$$
g(A X, Y)=g\left(A_{C} X, Y\right), X, Y \in T_{0}
$$

and therefore

$$
\begin{equation*}
A_{C} X=A X-\beta g(X, U) \xi, X \in T_{0} \tag{3.3}
\end{equation*}
$$

By (II) we have

$$
\begin{equation*}
A \phi X=-\phi A X-\beta g(X, \phi U) \xi, X \in T_{0} . \tag{3.4}
\end{equation*}
$$

It is easily seen that the traces of these shape operators are both equal to zero. Since the complex submanifold $M(t)$ of real codimension 2 in $M_{n}(c)$ is pseudo-Einstein, whose dimension is equal to $2 n-2(\geq 2)$, its Ricci tensor $S^{t}$ is given by

$$
S^{t}=\left(\frac{n}{2} c-\mu\right) I+(\mu-\lambda)\left(U \otimes U^{*}+\phi U \otimes(\phi U)^{*}\right)
$$

and we have

$$
\left\{\begin{array}{l}
\left(A_{\xi}^{2}+A_{C}^{2}\right) U=\lambda U  \tag{3.5}\\
\left(A_{\xi}^{2}+A_{C}^{2}\right) \phi U=\lambda \phi U \\
\left(A_{\xi}^{2}+A_{C}^{2}\right) X=\mu X, \quad X \in \mathfrak{D}, X \perp U, X \perp \phi U,
\end{array}\right.
$$

where $\lambda$ and $\mu$ are smooth functions on $M(t)$. By the direct calculation of the left hand side of the above relation and using the properties (3.2) and (3.3) we have
Lemma 3.1. (See [16] and [17]) Let $M$ be a proper pseudo-Einstein ruled real hypersurfaces in $M_{n}(c), c \neq 0, n \geq 3$. Then we have

$$
\begin{cases}A U= & \beta \xi+\gamma U+\delta \phi U  \tag{3.6}\\ A \phi U= & \delta U-\gamma \phi U, \quad \lambda=2\left(\gamma^{2}+\delta^{2}\right)\end{cases}
$$

In particular, if it is totally geodesic, we have $\gamma=\delta=0$.
On the other hand, we give the following for any $X$ orthogonal to $\xi, U$ and $\phi U$.

$$
\begin{equation*}
A^{2} X=\beta \epsilon \xi+\frac{\mu}{2} X \tag{3.7}
\end{equation*}
$$

because (3.2), (3.3), the third formula of (3.5) and the condition (II) imply that

$$
\begin{aligned}
\mu X & =-A_{\xi} \phi A X+A_{C}\{A X-\beta g(X, U) \xi\} \\
& =2\left(A^{2} X-\beta g(A X, U) \xi\right) .
\end{aligned}
$$

When the function $\mu$ in (3.5) vanishes, then (3.7) implies that

$$
\|A X\|^{2}=0
$$

for any $X \in T^{\prime}$, where $T^{\prime}=L(\xi, U, \phi U)^{\perp}$. Then the derivative of the Ricci tensor of pseudo-Einstein ruled real hypersurfaces in $M_{n}(c)$ satisfies

$$
\begin{equation*}
g\left(\left(\nabla_{X} S\right) Y, Z\right)=0 \tag{I}
\end{equation*}
$$

for any $Y, Z \in T^{\prime}$ and $X \in T_{0}$, because for such $X \in T_{0}$ and $Y, Z \in T^{\prime}$ we know that $A Y=A Z=0$, and naturally $g\left(\left(\nabla_{X} A\right) Y, Z\right)=0$ in (2.6). Of course, in totally geodesic Einstein ruled real hypersurfaces of $M_{n}(c)$ we know that $A X=0$ for
any $X$ orthogonal to $\xi, U$ and $\phi U$. In such a case the function $\mu$ also vanishes identically.

Now we introduce some examples of pseudo-Einstein ruled real hypersurfaces in complex projective space $P_{n}(\mathbb{C})$ which were given in [17].
Example 1. Let $M$ be a ruled real hypersurface in $P_{n}(\mathbb{C})$ foliated by complex hyperplane $P_{n-1}(\mathbb{C})$. Then the expression (3.1) implies that

$$
A_{\xi} X=0 \text { and } A_{C} X=0
$$

for any $X \in \mathfrak{D}$, where $\mathfrak{D}$ denotes the distribution of $P_{n-1}(\mathbb{C})$. This implies $A_{\xi}^{2}+A_{C}^{2}=$ 0 on the distribution $\mathfrak{D}$. Then its Ricci tensor is given by $S^{t}=\frac{n c}{2} I$. So we know that $M$ is a totally geodesic Einstein ruled real hypersurface in $P_{n}(\mathbb{C})$.
Example 2. Let $M$ be a real hypersurface in $P_{\boldsymbol{n}}(\mathbb{C})$ foliated by complex quadric $Q^{n-1}$. Then it can be easily seen in $[10]$ and $[17]$ that the shape operator $A_{C}$ defined on the distribution of the complex quadric $Q^{n-1}$ satisfies

$$
A_{C}^{2}=\lambda^{2} I .
$$

Moreover, we know that $A_{\xi} X=-\phi A X$ for $X \in \mathfrak{D}$. Then we know

$$
\begin{aligned}
A_{\xi}^{2} X & =\phi A \phi A X \\
& =\phi A \phi A_{C} X \\
& =-\phi^{2} A A_{C} X \\
& =-\phi^{2}\left\{A_{C}^{2} X+\beta g\left(A_{C} X, U\right) \xi\right\} \\
& =-\phi^{2}\left\{\lambda^{2} X\right\} \\
& =\lambda^{2} X
\end{aligned}
$$

where in the third equality we have used the integrability of the distribution $\mathfrak{D}$. So it follows that $\left(A_{\xi}^{2}+A_{C}^{2}\right) X=2 \lambda^{2} X$ for any $X \in \mathfrak{D}$. Then the Ricci tensor $S^{t}$ is given by $S^{t}=\left\{\frac{n}{2} c-2 \lambda\right\}$. From this we conclude that $M$ is not totally geodesic Einstein ruled real hypersurface.
Example 3. Let $\Gamma$ be a complex curve in $P_{n}(\mathbb{C})$. Now let us consider

$$
\phi_{\frac{\pi}{2}}(\Gamma)=\cup_{x \in \Gamma}\left\{\left.\exp _{x} \frac{\pi}{2} v \right\rvert\, v \text { is a unit normal vector of } \Gamma \text { at } x\right\} .
$$

Then $\phi_{\frac{\pi}{2}}(\Gamma)$ is an ( $n-1$ )-dimensional complex hypersurface in $P_{n}(\mathbb{C})$ (See [9], [10]), which is a submanifold of real codimension 2 in $P_{n}(\mathbb{C})$. Moreover, it is a pseudoEinstein complex hypersurface in $P_{n}(\mathbb{C})$. Then we construct a real hypersurface $M$ in $P_{n}(\mathbb{C})$ foliated by such kind of leaves along the integral curve of the normal vector field $\xi=-J C$.

For this, we consider a regular curve $\gamma: I \rightarrow M_{n}(c)$. Then we can construct a ruled real hypersurface $M$ foliated by pseudo-Einstein complex hypersurfaces in such a way that

$$
\begin{aligned}
M & =\cup_{t} \gamma(t) \times \phi_{\frac{\pi}{2}}(\Gamma) \\
& =\cup_{t} \phi_{\frac{\pi}{2}}^{(t)}(\Gamma) .
\end{aligned}
$$

Moreover, let us take a structure vector $\xi$ such that $\xi(\gamma(t))=\gamma^{\prime}(t)$ orthogonal to the tangent space of $\phi_{\frac{\pi}{2}}(\Gamma)$ at $\gamma(t)$. The vector $\xi(\gamma(t))$ can be smoothly extended to any point in $\phi_{\frac{\pi}{2}}^{(t)}(\Gamma)$ by parallel displacement $P$ in such a way that $P \xi\left((\gamma(t)) \perp T_{x} \phi_{\frac{\pi}{2}}^{(t)}(\Gamma)\right.$ for any $x$ in $\phi_{\frac{\pi}{2}}^{(t)}(\Gamma)$. Then in this case we call such a real hypersurface in $P_{n}(\mathbb{C})$ pseudo-Einstein ruled real hypersurface. Now let us show that its leaves are pseudoEinstein complex hypersufaces in $P_{n}(\mathbb{C})$.

In fact, if we consider the principal curvatures of the shape operator $A_{C}$ defined on the distribution of $\phi_{\frac{\pi}{2}}(\Gamma)$, it is given by

$$
\begin{gathered}
\cot \left(\frac{\pi}{2}+\theta\right) \text { with multiplicity } 1 \\
\cot \left(\frac{\pi}{2}-\theta\right) \text { with multiplicity } 1 \\
0 \text { with multiplicity } 2 n-4
\end{gathered}
$$

Then from this expression of the shape operator $A_{C}$ we can put

$$
A_{C} U=\cot \left(\frac{\pi}{2}+\theta\right) U, A_{C} \phi U=\cot \left(\frac{\pi}{2}-\theta\right) \phi U, \text { and } A_{C} X=0
$$

for a certain vector field $U \in \mathfrak{D}$ and any vector field $X \in \mathfrak{D}$ orthogonal to $U$ and $\phi U$, where $\mathfrak{D}$ denotes the distribution of $\phi_{\frac{\pi}{2}}(\Gamma)$ orthogonal to the structure vector $\xi$. Then it can be easily seen that

$$
\begin{aligned}
A_{C}^{2} U & =\cot ^{2}\left(\frac{\pi}{2}+\theta\right) U=\frac{\lambda}{2} U \\
A_{C}^{2} \phi U & =\cot ^{2}\left(\frac{\pi}{2}-\theta\right) \phi U=\frac{\lambda}{2} \phi U, \\
A_{C}^{2} X & =0
\end{aligned}
$$

for any $X$ orthogonal to $U$ and $\phi U$. Also if we apply the same method as in Example 2, the shape operator $A_{\xi}$ can be calculated. So naturally it follows that

$$
\begin{aligned}
\left(A_{\xi}^{2}+A_{C}^{2}\right) U & =\lambda U \\
\left(A_{\xi}^{2}+A_{C}^{2}\right) \phi U & =\lambda \phi U \\
\left(A_{\xi}^{2}+A_{C}^{2}\right) X & =0
\end{aligned}
$$

for any $X$ orthogonal to $U$ and $\phi U$. Accordingly, we have our assertion.

## 4. Proof of the Main Theorems

In this section we shall consider a characterization of certain kind of pseudoEinstein ruled real hypersurfaces in terms of the Ricci tensor $S$. Let $M$ be a real hypersurface of $M_{n}(c), c \neq 0$.

Let us first assume that the structure vector $\xi$ is not principal. We denote by $M_{0}$ an open subset in $M$ consisting of points $x$ at which $\beta \neq 0$. By the assumption the subset $M_{0}$ is not empty. So, we can put $A \xi=\alpha \xi+\beta U$ on $M_{0}$, where $U$ is a unit vector field in the holomorphic distribution $T_{0}$ and $\alpha$ and $\beta$ are smooth functions on $M$.

Let $L(\xi, U)$ or $L(\xi, U, \phi U)$ be a distribution spanned by $\xi, U$ or $\xi, U, \phi U$, respectively. We denote by $T^{\prime \prime}$ or $T^{\prime}$ an orthogonal complement of the distribution $L(\xi, U)$ or $L(\xi, U, \phi U)$, respectively. we also assume the following condition:

$$
\begin{gather*}
g\left(\left(\nabla_{X} S\right) Y, Z\right)=0, \quad X \in T_{0}, Y, Z \in T^{\prime}  \tag{I}\\
g((A \phi+\phi A) X, Y)=0, \quad X, Y \in T_{0} \tag{II}
\end{gather*}
$$

Now let $M_{0}$ be an open subset of $M$ consisting of points $x$ at which $\beta(x) \neq 0$. Since $\xi$ is not principal, $M_{0}$ can not be empty. By the assumption (II) it turns out to be

$$
\begin{equation*}
(A \phi+\phi A) X=-\beta g(X, \phi U) \xi, \quad X \in T_{0} \tag{4.1}
\end{equation*}
$$

Differentiating covariantly (II) with $X$ in $T_{0}$ and using (II) and the second equation of (2.1), we get directly

$$
\begin{align*}
& g\left(\left(\nabla_{X} A\right) Y, \phi Z\right)-g\left(\left(\nabla_{X} A\right) Z, \phi Y\right) \\
& =\beta[-g(A X, Y) g(Z, U)+g(A X, Z) g(Y, U)  \tag{4.2}\\
& \quad-g(\phi A X, Y) g(Z, \phi U)+g(\phi A X, Z) g(Y, \phi U)]
\end{align*}
$$

for any vector fields $X, Y$ and $Z$ in $T_{0}$.
Lemma 4.1. Let $M$ be a real hypersurface of $M_{n}(c), c \neq 0, n \geqq 3$ satisfying the conditions (I) and (II). Then it satisfies

$$
\begin{equation*}
A X \equiv 0(\bmod \xi, U, \phi U) \tag{4.3}
\end{equation*}
$$

on the open subset $M_{0}$ for any vector field $X$ in $T^{\prime}$.
Proof. By (2.9) and the assumption (I) we have for any vector fields $X$ in $T_{0}$ and $Y, Z$ in $T^{\prime}=\left\{X \in T^{\prime \prime}(x): g\left(X, \phi U_{(x)}\right)=0\right\}$

$$
\begin{align*}
& 2\left\{d h(X) g(A Y, \phi Z)+h g\left(\left(\nabla_{X} A\right) Y, \phi Z\right)\right\} \\
& \quad-\beta[g(A Y, U) g(A X, Z)+g(A Z, U) g(A X, Y)  \tag{4.4}\\
& \quad+g(A Y, \phi U) g(\phi A X, Z)+g(A Z, \phi U) g(\phi A X, Y)] \\
& =0 .
\end{align*}
$$

Hence we have

$$
\begin{aligned}
& 2\{d h(X) g(A Y, \phi Z)-d h(Y) g(A X, \phi Z)\} \\
& \quad+\beta[\{g(A X, U) g(A Y, Z)+g(A X, \phi U) g(\phi A Y, Z)\} \\
& \quad-\{g(A Y, U) g(A X, Z)+g(A Y, \phi U) g(\phi A X, Z)\}] \\
& \quad=0
\end{aligned}
$$

for any vector fields $X, Y$ and $Z$ in $T^{\prime}$. Accordingly, we have

$$
\begin{align*}
& \{2 d h(Y)-\beta g(A Y, \phi U)\} \phi A X-\{2 d h(X)-\beta g(A X, \phi U)\} \phi A Y \\
& \quad+\beta\{g(A X, U) A Y-g(A Y, U) A X\} \equiv 0(\bmod \xi, U, \phi U) \tag{4.5}
\end{align*}
$$

for any vector fields $X, Y$ and $Z$ in $T^{\prime}$. Under the assumption that the mean curvature $H$ is not constant along the distribution $T^{\prime}$ we know that $d h(Y) \neq 0$ for any $Y \in T^{\prime}$. From such a situation we do not have a case that all of coefficients are vanishing simultaneously. When one or two of the coefficients of $\phi A X$ and $\phi A Y$ are vanishing, we are able to assert our result by the same method given below.

Now we consider both of two coefficients of $\phi A X$ and $\phi A Y$ are non-vanishing. Then we can eliminate one of them as follows. In order to do this, let us replace $X$ with $\phi X$ in the above equation and taking account of the condition (II), we obtain again the linear combination of the vectors $A X, A Y, \phi A X$ and $\phi A Y$. From these two equations the vector $\phi A Y$ can be eliminated like the following equation for any $X, Y \in T^{\prime}$

$$
\begin{aligned}
& {[\{2 d h(\phi X)+\beta g(A X, U)\}\{2 d h(Y)-\beta g(A Y, \phi U)\}} \\
& \quad-\beta g(A Y, U)\{2 d h(X)-\beta g(A X, \phi U)\}] \phi A X \\
& +[\beta g(A X, U)\{2 d h(\phi X)+\beta g(A X, U)\} \\
& \quad-\beta g(A X, \phi U)\{2 d h(X)-\beta g(A X, \phi U)\}] A Y \\
& -[\beta g(A Y, U)\{2 d h(\phi X)+\beta g(A X, U)\} \\
& \quad \quad+\{2 d h(Y)-\beta g(A Y, \phi U)\}\{2 d h(X)-\beta g(A X, \phi U)\}] A X \equiv 0 \\
& (\bmod \xi, U, \phi U) .
\end{aligned}
$$

Let us write the above equation in such a way that

$$
\lambda(X, Y) \phi A X+\mu(X) A Y+\nu(X, Y) A X \equiv 0 \quad(\bmod \xi, U, \phi U)
$$

where $\lambda(X, Y)$,(resp. $\mu(X)$ and $\nu(X, Y)$ ) denotes the corresponding coefficients of $\phi A X$ (resp. $A Y$ and $A X$ ). Moreover, by virtue of our assumption we know that
these coefficients can not be simultaneously vanishing. So naturally let us consider the following equation for any $X, Y \in T^{\prime}$

$$
\begin{aligned}
& \phi A X \equiv \kappa(X, Y) A Y+\rho(X, Y) A X(\bmod \xi, U, \phi U) \\
& \phi A Y \equiv \kappa(Y, X) A X+\rho(Y, X) A Y(\bmod \xi, U, \phi U)
\end{aligned}
$$

where $\kappa(X, Y)=-\frac{\mu(X)}{\lambda(X, Y)}$ and $\rho(X, Y)=-\frac{\nu(X, Y)}{\lambda(X, Y)}$. The second equation mentioned above is just obtained by interchanging vector fields $X$ and $Y$ in $T^{\prime}$ in the first equation. Now applying $\phi$ to the first equation, then it follows that

$$
A X \equiv-\kappa(X, Y) \phi A Y-\rho(X, Y) \phi A X(\bmod \xi, U, \phi U) .
$$

From this, substituting the first and the second equation into the right side, and from our assumption we know that the coefficients of two vectors $A X$ and $A Y$ can not be vanishing simutaneously. So we are able to assert the following

$$
\begin{equation*}
A X \equiv a(X, Y) A Y \quad(\bmod \xi, U, \phi U) \tag{4.6}
\end{equation*}
$$

for any vector fields $X$ and $Y$ in $T^{\prime}$, where $a(X, Y)$ is a smooth function depends on $X$ and $Y$. Let $X_{1}, \ldots, X_{m}$ be an orthonormal basis of $T^{\prime}(x)$ at any point $x$ in $M_{0}$. Then we see by (4.6)

$$
A\left(X_{i}-a_{i} X_{1}\right) \equiv 0(\bmod \xi, U, \phi U), \quad i \geq 2 .
$$

Since $\left\{X_{i}-d_{i} X_{1} ; i \geq 2\right\}$ is a basis of an ( $n-1$ )-dimensional subspace in $T_{0}(x)$ with respect to $X_{1}$, we get $A X \equiv 0(\bmod \xi, U, \phi U)$ for any vector $X$ in this subspace. Accordingly, again by (4.6) we obtain (4.3). It completes the proof.

Now we want to show that the distribution $T^{\prime}$ is $A$-invariant and $\phi$-invariant, which is crucial and important to prove that our hypersurface is a pseudo-Einstein ruled one. Namely, we have the following
Lemma 4.2. Under the conditions (I) and (II) we have

$$
\left\{\begin{array}{l}
A \xi=\alpha \xi+\beta U  \tag{4.7}\\
A U=\beta \xi+\gamma U+\delta \phi U \\
A \phi U=\delta U-\gamma \phi U
\end{array}\right.
$$

on the open subset $M_{0}$. Namely, the subbundle $T^{\prime}$ is $A$-invariant and $\phi$-invariant on $M_{0}$.

Proof. On the non-empty open set $M_{0}$, we may suppose that we put

$$
A U=\beta \xi+\gamma U+\delta \phi U+\epsilon V
$$

where $\xi, U, \phi U$ and $V$ are orthonormal vector fields. So $V$ is contained in $T^{\prime}$. Then we have

$$
\begin{aligned}
g(A V, U) & =g(A U, V)=\epsilon \\
g(A V, \phi U) & =g(V, A \phi U)=-g(V, \phi A U)=0
\end{aligned}
$$

from which together with Lemma 4.1 it follows that

$$
\begin{equation*}
A V=\epsilon U, \quad A \phi V=-\epsilon \phi U \tag{4.8}
\end{equation*}
$$

On the other hand, in the covariant derivative of the Ricci tensor $S$, we replace $Y$ and $Z$ with $V$ and $\phi V$. By the assumption (I) we have

$$
\begin{equation*}
g\left(\left(\nabla_{X} S\right) V, \phi V\right)=0 \tag{4.9}
\end{equation*}
$$

for any vector field $X$ in $T_{0}$. Thus we have

$$
\begin{align*}
& d h(X) g(A V, \phi V)+h g\left(\left(\nabla_{X} A\right) V, \phi V\right)-g\left(\left(\nabla_{X} A\right) V, A \phi V\right)  \tag{4.10}\\
& \quad-g\left(\left(\nabla_{X} A\right) \phi V, A V\right)=0
\end{align*}
$$

for any vector field $X$ in $T_{0}$.
On the other hand, combining Lemma 4.1 with (4.4), we have

$$
\begin{equation*}
h g\left(\left(\nabla_{X} A\right) Y, \phi Z\right)=0 \tag{4.11}
\end{equation*}
$$

for any vector fields $X \in T_{0}, Y$ and $Z$ in $T^{\prime}$. Accordingly, the covariant derivative of the Ricci tensor $S$ implies that

$$
g\left(\left(\nabla_{X} A\right) Y, A Z\right)+g\left(\left(\nabla_{X} A\right) Z, A Y\right)=0
$$

for any vector fields $X, Y$ and $Z$ in $T^{\prime}$. This shows that the first term is skewsymmetric with respect to $Y$ and $Z$. Furthermore, let us take the skew-symmetric part of the above equation with respect to X and Y in $T^{\prime}$, we have

$$
g\left(\left(\nabla_{X} A\right) Z, A Y\right)=g\left(\left(\nabla_{Y} A\right) Z, A X\right)
$$

by the Codazzi equation (2.3). Hence it turns out to be symmetric with respect to $Y$ and $Z$. So we have

$$
g\left(\left(\nabla_{X} A\right) Y, A Z\right)=0
$$

for any vector fields $X, Y$ and $Z$ in $T^{\prime}$, from which together with (4.8) it follows that

$$
\begin{equation*}
g\left(\left(\nabla_{X} A\right) Y, A V\right)=\epsilon g\left(\left(\nabla_{X} A\right) Y, U\right)=0 \tag{4.12}
\end{equation*}
$$

for any vector fields $X$ and $Y$ in $T^{\prime}$. Thus the assumption (I) we see that

$$
\begin{align*}
g\left(\left(\nabla_{U} S\right) V, \phi V\right)= & d h(U) g(A V, \phi V)+h g\left(\left(\nabla_{U} A\right) V, \phi V\right) \\
& -g\left(\left(\nabla_{U} A\right) V, A \phi V\right)-g\left(\left(\nabla_{U} A\right) A V, \phi V\right) \\
= & h g\left(\left(\nabla_{U} A\right) V, \phi V\right)  \tag{4.13}\\
& +\epsilon\left\{g\left(\left(\nabla_{U} A\right) V, \phi U\right)-g\left(\left(\nabla_{U} A\right) U, \phi V\right)\right\} \\
= & 0 .
\end{align*}
$$

From this together with (4.2) and (4.8) it follows that for $U \in T_{0}$ and $V \in T^{\prime}$

$$
h g\left(\left(\nabla_{U} A\right) V, \phi V\right)-\beta \epsilon^{2}=0
$$

Then by the equation of Codazzi (2.3) and the second formula in (4.12), we get

$$
\epsilon=0
$$

on $M_{0}$, where we have used that $g\left(\left(\nabla_{X} A\right) Y, Z\right)$ is symmetric with respect to any vector fields $X, Y$ and $Z$ in $T_{0}$. Thus it completes the proof.

Lemma 4.3. Under the conditions (I) and (II), we have $A Y=0$ for any vector field $Y$ in $T^{\prime}$ on $M_{0}$.

Proof. By Lemma 4.2 the subbundle $T^{\prime}$ is $A$-invariant on $M_{0}$. Furthermore Lemma 4.1 implies that we have $A Y \equiv 0(\bmod \xi, U, \phi U)$ for any vector field $Y$ in $T^{\prime}$. By the above facts it turns out to be $A Y=0$ for any vector field $Y$ in $T^{\prime}$.

Under these preparations of Lemmas 4.1, 4.2 and 4.3 we are in a position to prove our main theorems in the introduction.
Proof of Theorem 1. Suppose that the interior of $M-M_{0}$ is not empty. On the interior the function $\beta$ vanishes identically and therefore $\xi$ is prinicipal. Thus we have

$$
(A \phi+\phi A) \xi=0
$$

For any principal vector $X$ in $T_{0}$ with principal curvature $\lambda$, the condition (II) is reduced to $A \phi X=-\lambda \phi X+\theta(X) \xi$. From $A \xi=\alpha \xi$ the linear product of $A \phi X$ and $\xi$ gives us to $\theta(X)=0$. This means that

$$
\begin{equation*}
A \phi+\phi A=0 \tag{4.14}
\end{equation*}
$$

on the interior of $M-M_{0}$. It is seen in Ki and Suh [7] that (4.14) holds on $M$, then we have $c=0$. Since this property is local, we have $c=0$ on the interior of $M-M_{0}$, a contradiction. Thus the interior of $M-M_{0}$ must be empty and hence the open set $M_{0}$ is dense. Accordingly, under the condition of Theorem 1 we see
that $A X=0$ for any $X$ in $T^{\prime}$ on $M_{0}$. By the continuity of principal curvatures we see that the shape operator also satisfies such kind of properties on the whole $M$.

Since the distribution $T_{0}$ is integrable on $M$ by the definition, the integral manifold of $T_{0}$ can be regarded as the submanifold of codimension 2 in $M_{n}(c)$ whose normal vectors are $\xi$ and $C$. By the definition of the second fundamental form, we see

$$
\begin{aligned}
g\left(\bar{\nabla}_{X} Y, C\right) & =-g\left(\bar{\nabla}_{X} C, Y\right)=g\left(A_{C} X, Y\right)=g(A X, Y), \\
g\left(\bar{\nabla}_{X} Y, \xi\right) & =-g\left(\bar{\nabla}_{X} \xi, Y\right)=g\left(A_{\xi} X, Y\right)
\end{aligned}
$$

for any vector fields $X$ and $Y$ in $T_{0}$, where $\bar{\nabla}$ denotes the Riemannian connection of $M_{n}(c)$ and $A_{\xi}$ or $A_{C}$ denotes the shape operator of $M(t)$ in $M_{n}(c)$ in the direction of the normal $\xi$ or $C$ respectively. Namely, it is seen that these shape operators satisfy

$$
\begin{aligned}
\bar{\nabla}_{X} Y & =\nabla_{X} Y+g(A X, Y) C \\
& =\nabla_{X}^{t} Y+g\left(A_{\xi} X, Y\right) \xi+g\left(A_{C} X, Y\right) C
\end{aligned}
$$

where $\nabla^{t}$ denotes the Riemannian connection of the integral submanifold of $T_{0}$. Thus we see

$$
\begin{aligned}
A_{C} X & =A X+g\left(A_{C} X-A X, \xi\right) \xi=A X-\beta g(X, U) \xi, \quad X \in T_{0} \\
A_{\xi} X & =-\phi A X, \quad X \in T_{0}
\end{aligned}
$$

on $M_{0}$, because we have

$$
g\left(\bar{\nabla}_{X} Y, \xi\right)=-g\left(\bar{\nabla}_{X} \xi, Y\right)=-g(\phi A X, Y) X, Y \in T_{0}
$$

by (2.1). Since $T_{0}$ is $\phi$-invariant and therefore it is also $J$-invariant, its integral manifold is a complex hypersurface. Since the open subset $M_{0}$ is dense in $M$, by means of the continuity of principal curvatures, we have

$$
\begin{align*}
& A U=\beta \xi+\gamma U+\delta \phi U, \quad A \phi U=\delta U-\gamma \phi U \\
& A X=0, \quad X \in T^{\prime} \tag{4.15}
\end{align*}
$$

on $M$ and therefore it is seen that another shape operator $A_{\xi}$ of the integral submanifold $M(t)$ of $T_{0}$ satisfies

$$
A_{\xi} X= \begin{cases}\delta U-\gamma \phi U, & X=U  \tag{4.16}\\ -\gamma U-\delta \phi U, & X=\phi U \\ 0, & X \in T^{\prime}\end{cases}
$$

on $M(t)$ and it is also seen that another shape operator $A_{C}$ of the integral submanifold of $T_{0}$ satisfies

$$
A_{C} X= \begin{cases}\gamma U+\delta \phi U, & X=U  \tag{4.17}\\ \delta U-\gamma \phi U, & X=\phi U \\ 0, & X \in T^{\prime}\end{cases}
$$

By combining (4.16) with (4.17) and by the direct calculation, it is trivial that we have

$$
\left(A_{\xi}^{2}+A_{C}^{2}\right) X=2\left(\gamma^{2}+\delta^{2}\right) X, \quad X=U \text { and } \phi U
$$

In the case where $X$ are in $T^{\prime}$, we see

$$
\left(A_{\xi}^{2}+A_{C}^{2}\right) X=0
$$

This shows that an integral submanifold is pseudo-Einstein. Thus $M$ is a pseudoEinstein ruled real hypersurface.

Conversely, it is trivial by the fundamental properties discussed in section 3 that a pseudo-Einstein ruled real hypersurface $M$ of $M_{n}(c)$ satisfies the integrability condition (II) and the Ricci condition (I). So it completes the proof.

Proof of Theorem 2. Now we are going to prove Theorem 2. Let us denote by $M^{\prime}$ a subset of $M_{0}$ consisting of points $x$ in $M_{0}$ at which $\left(\gamma^{2}+\delta^{2}\right)(x) \neq 0$.
Lemma 4.4. Under the conditions (I) and (II) if the mean curvature is nonvanishing, then the subset $M^{\prime}$ is empty, i.e., the smooth functions $\gamma=g(A U, U)$ and $\delta=g(A U, \phi U)$ vanish identically on $M_{0}$.

Proof. Suppose that $M^{\prime}$ is not empty. From Lemma 4.3 it follows that $A Y=0$ for any vector field $Y$ in $T^{\prime}$. Consequently, we see

$$
g\left(\nabla_{X} U, Y\right)=0, \quad X, Y \in T^{\prime}
$$

on $M^{\prime}$.
In fact, differentiating $A Y=0$ with respect to $X \in T^{\prime}$ covariantly, we get

$$
\left(\nabla_{X} A\right) Y+A \nabla_{X} Y=0
$$

which yields that

$$
g\left(\left(\nabla_{X} A\right) Y, Z\right)+g\left(\nabla_{X} Y, A Z\right)=0
$$

for any vector fields $X$ and $Y$ in $T^{\prime}$ and $Z$ in $T_{0}$.
For any $X, Y \in T^{\prime}$ Lemma 4.3 gives $A X=A Y=0$. Moreover, by the condition (I) and the equation of Codazzi, we have for any $X, Y \in T^{\prime}$ and $Z \in T_{0}$ in (2.6)

$$
0=h g\left(\left(\nabla_{Z} A\right) X, Y\right)=h g\left(\left(\nabla_{X} A\right) Y, Z\right)
$$

From this it follows that

$$
\left(\nabla_{X} A\right) Y=0(\bmod \xi)
$$

for any $X, Y$ in $T^{\prime}$ when the mean curvature is non-vanishing. So the first term in above equation vanishes identically, we have $g\left(\nabla_{X} Y, A Z\right)=0$ for any vector fields $X$ and $Y$ in $T^{\prime}$ and $Z$ in $T_{0}$. By (4.7), according as $Z=U$ or $\phi U$, we have

$$
\gamma g\left(\nabla_{X} Y, U\right)+\delta g\left(\nabla_{X} Y, \phi U\right)=0
$$

or

$$
\delta g\left(\nabla_{X} Y, U\right)-\gamma g\left(\nabla_{X} Y, \phi U\right)=0
$$

for any vector fields $X$ and $Y$ in $T^{\prime}$ on $M^{\prime}$. Since the determinant of the coefficients in the above system of linear equations is given by $\gamma^{2}+\delta^{2} \neq 0$, we have

$$
g\left(\nabla_{X} Y, U\right)=0, \quad g\left(\nabla_{X} Y, \phi U\right)=0
$$

for any vector fields $X$ and $Y$ in $T^{\prime}$ on $M^{\prime}$. Thus it means that the vector field $\nabla_{X} U$ is expressed as the linear combination of $\xi, U$ and $\phi U$ on $M^{\prime}$.

On the other hand, the equation of Codazzi (2.3) gives us that

$$
\left(\nabla_{X} A\right) \xi-\left(\nabla_{\xi} A\right) X=-\frac{c}{4} \phi X
$$

for any $X$ in $T_{0}$. Then by the direct calculation of the left side of the above relation, we have

$$
\begin{aligned}
& d \alpha(X) \xi+d \beta(X) U+\frac{1}{4} c \phi X+\alpha \phi A X+\beta \nabla_{X} U \\
& \quad-A \phi A X-\nabla_{\xi}(A X)+A \nabla_{\xi} X=0
\end{aligned}
$$

for any vector field $X$ in $T_{0}$. Accordingly, we have

$$
\begin{equation*}
d \alpha(X) \xi+d \beta(X) U+\frac{1}{4} c \phi X+\beta \nabla_{X} U+A \nabla_{\xi} X=0 \tag{4.18}
\end{equation*}
$$

for any vector field $X$ in $T^{\prime}$. By Lemma 4.2 and the property that $A X=0, X \in T^{\prime}$ the last term in (4.18) is expressed as the linear combination of $\xi, U$ and $\phi U$, because $g\left(A \nabla_{\xi} X, Y\right)=0$ for any vector field $Y \in T^{\prime}$. This shows that $c \phi X=0$, a contradiction. It means that the subset $M^{\prime}$ becomes empty and the smooth functions $\gamma$ and $\delta$ vanish identically on the open subset $M_{0}$. It completes the proof.

Now, from Lemmas 4.2 and 4.4, we get

$$
\left\{\begin{array}{l}
A \xi=\alpha \xi+\beta U  \tag{4.19}\\
A U=\beta \xi \\
A \phi U=0
\end{array}\right.
$$

On the other hand, Theorem 1 implies that $M$ is a pseudo-Einstein ruled real hypersurface of $M_{n}(c), c \neq 0$. Furthermore, its integral submanifold is a pseudoEinstein codimension 2 submanifold in $M_{n}(c)$ with orthonormals $C$ and $\xi$. Moreover, from (4.19) together with (4.16) and (4.17) we know $A_{\xi}=0$ and $A_{C}=0$. So it becomes totally geodesic ruled ones.

Conversely, it is evident that a ruled real hypersurface $M$ in $M_{n}(c)$ in the sense of Kimura [9] and in Ahn, Lee and the second author [1] satisfies the Ricci condition (I) and the condition (II). From this we complete the proof of Theorem 2.

## References

1. S. S. Ahn, S. B. Lee and Y. J. Suh, On ruled real hypersurfaces in a complex space form, Tsukuba J. Math. 17 (1993), 311-322.
2. J. Berndt, Real hypersurfaces with constant principal curvatures in a complex hyperbolic space, J. Reine Angew. Math. 395 (1989), 132-141.
3. J. Berndt, Real hypersurfaces with constant principal curvatures in complex space forms, Geometry and Topology of Submanifolds II, Avignon, 1988, 1019, World Scientific, 1990.
4. J. Berndt and Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians, Monatshefte für Mathematik 127 (1999), 1-14.
5. T. E. Cecil and P. J. Ryan, Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. 269 (1982), 481-499.
6. R. Niebergal and P. J. Ryan, Real hypersurfaces in complex space forms,, Tight and Taut submanifolds, MSRI publications, Edited by T.E. Cecil and S.S. Chern 32 (1997), 233-339.
7. U-H. Ki and Y. J. Suh, On real hypersurfaces of a complex space form, Math. J. Okayama 32 (1990), 207-221.
8. U-H. Ki and Y. J. Suh, On a characterization of real hypersurfaces of type A in a complex space form, Canadian Math. Bull. 37 (1994), 238-244.
9. M. Kimura, Sectional curvatures of real hypersurfaces in complex projective space, Math. Ann. 296 (1986), 137-149.
10. M. Kimura, Some non-homogeneous real hypersurfaces in a complex projective space II (Characterization), The Bull. of the Faculty of Edu. Ibaraki Univ. 44 (1995), 17-31.
11. M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. 202 (1989), 299-311.
12. S. Montiel and A. Romero, On some real hypersurfaces of a complex hyperbolic space, Geometriae Dedicata 20 (1986), 245-261.
13. J. D. Pérez and Y. J. Suh, Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_{i}} R=0$, Differential Geom. and its Appl. 7 (1997), 211-217.
14. Y. J. Suh, On real hypersurfaces of a complex space form with $\eta$-parallel Ricci tensor, Tsukuba J. Math. 14-1 (1990), 27-37.
15. Y. J. Suh, Characterizations of real hypersurfaces in complex space forms in terms of Weingarten map, Nihonkai Math. J., 6 (1995), 63-79.
16. Y. J. Suh, On non-proper pseudo-Einstein ruled real hypersurfaces in complex space forms, Bull. of Korean Math. Soc. 36 (1999), 315-336.
17. Y. J. Suh, On pseudo-Einstein ruled real hypersurfaces in complex space forms, Note di Matematica . 19 (1999), 71-86.

Departamento de Geometría y Topología;
Facultad de Ciencias,
Universidad de Granada,
18071-Granada, Spain
E-mail: jdperez@goliat.ugr.es

Department of Mathematics,
Kyungpook National University,
Taegu 702-701, KOREA
E-mail: yjsuh@bh.knu.ac.kr

Received November 19, 2001 Revised December 3, 2001


[^0]:    Mathematical Subject Classifications (1991): 53C40, 53C15, 53B25.
    Key words: Einstein, Pseudo-Einstein ruled real hypersurface, Complex space form, Ricci tensor, Totally geodesic, Distribution, Weingarten map.

    * The second and the third authors were supported by the grant from BSRI Program, Korea Research Foundation, Korea 2001 Project No. KRF-2001-015-DP0034.

