Characterizations of Some Special Space-like Curves in
Minkowski Space-time

Melih Turgut and Suha Yilmaz

(Department of Mathematics of Dokuz Eylul University, 35160 Buca, Izmir, Turkey)

Email: melih.turgut@gmail.com

Abstract: In this work, a system of differential equation on Minkowski space-time E_1^4, a special case of Smarandache geometries ([4]), whose solution gives the components of a space-like curve on Frenet axis is constructed by means of Frenet equations. In view of some special solutions of this system, characterizations of some special space-like curves are presented.

Key words: Minkowski space-time, Frenet frame, Space-like curve.

AMS(2000): 53C50, 51B20.

§1. Introduction

It is safe to report that the many important results in the theory of the curves in E^3 were initiated by G. Monge; and G. Darboux pionnered the moving frame idea. Thereafter, F. Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry (for more details see [2]). At the beginning of the twentieth century, A.Einstein's theory opened a door of use of new geometries. One of them, Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold - a special case of Smarandache geometries ([4]), was introduced and some of classical differential geometry topics have been treated by the researchers.

In the case of a differentiable curve, at each point a tetrad of mutually orthogonal unit vectors (called tangent, normal, first binormal and second binormal) was defined and constructed, and the rates of change of these vectors along the curve define the curvatures of the curve in four dimensional space [1].

In the present paper, we write some characterizations of space-like curves by the components of the position vector according to Frenet frame. Moreover, we obtain important relations among curvatures of space-like curves.

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^4 are briefly presented (a more complete elementary treatment can be found in [1]).

Minkowski space-time E_1^4 is an Euclidean space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4.

Since g is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of the three causal characters; it can be space-like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and null (light-like) if $g(v, v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$. Therefore, v is a unit vector if $g(v, v) = \pm 1$. Next, vectors v, w in E_1^4 are said to be orthogonal if $g(v, w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. The hypersphere of center $m = (m_1, m_2, m_3, m_4)$ and radius $r \in R^+$ in the space E_1^4 defined by

$$H_0^3(m, r) = \left\{ \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in E_1^4 : g(\alpha - m, \alpha - m) = -r^2 \right\}. \tag{1}$$

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4. Then T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Space-like or time-like curve $\alpha(s)$ is said to be parameterized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. Let $\vartheta = \vartheta(s)$ be a curve in E_1^4. If tangent vector field of this curve is forming a constant angle with a constant vector field U, then this curve is called an inclined curve.

Let $\alpha(s)$ be a curve in the space-time E_1^4, parameterized by arclength function s. Then for the unit speed curve α with non-null frame vectors the following Frenet equations are given in [5] :

$$
\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \mu_1 \kappa & 0 & \mu_2 \tau & 0 \\ 0 & \mu_3 \tau & 0 & \mu_4 \sigma \\ 0 & 0 & \mu_5 \sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{2}
$$

Due to character of α, we write following subcases.

Case 1 α is a space-like vector. Thus T is a space-like vector. Now, we distinguish according to N.

Case 1.1 If N is space-like vector, then B_1 can have two causal characters.

Case 1.1.1 B_1 is space-like vector, then μ_i $(1 \leq i \leq 5)$ read

$$\mu_1 = \mu_3 = -1, \quad \mu_2 = \mu_4 = \mu_5 = 1,$$

and T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(N,N) = g(B_1, B_1) = 1, g(B_2, B_2) = -1.$$

Case 1.1.2 B_1 is time-like vector, then μ_i $(1 \leq i \leq 5)$ read

$$\mu_1 = -1, \mu_2 = \mu_3 = \mu_4 = \mu_5 = 1,$$

and T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(N,N) = g(B_2, B_2) = 1, g(B_1, B_1) = -1.$$

Case 1.2 N is time-like vector. Then μ_i $(1 \leq i \leq 5)$ read

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1, \mu_5 = -1,$$

and T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(B_1, B_1) = g(B_2, B_2) = 1, g(N,N) = -1.$$

Case 2 α is a time-like vector. Thus T is a time-like vector. Then μ_i $(1 \leq i \leq 5)$ read

$$\mu_1 = \mu_2 = \mu_4 = 1, \mu_3 = \mu_5 = -1,$$

and T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = -1, g(N,N) = g(B_1, B_1) = g(B_2, B_2) = 1.$$

Here κ, τ and σ are, respectively, first, second and third curvature of the curve α.

In another work [3], authors wrote a characterization of space-like curves whose image lies on H_0^3 with following statement.

Theorem 2.1 *Let $\alpha = \alpha(s)$ be an unit speed space-like curve with curvatures $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$ in E_1^4. Then α lies on H_0^3 if and only if*

$$\frac{\sigma}{\tau} \frac{d}{ds}(\frac{1}{\kappa}) - \frac{d}{ds}\left\{\frac{1}{\sigma}\left[\frac{\tau}{\kappa} + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d}{ds}(\frac{1}{\kappa})\right)\right]\right\} = 0. \tag{3}$$

In the same space, Yilmaz (see [6]) gave a formulation about inclined curves with the following theorem.

Theorem 2.2 *Let $\alpha = \alpha(s)$ be a space-like curve in E_1^4 parameterized by arclength. The curve α is an inclined curve if and only if*

$$\frac{\kappa}{\tau} = A\cosh(\int_0^s \sigma ds) + B\sinh(\int_0^s \sigma ds), \tag{4}$$

where $\tau \neq 0$ and $\sigma \neq 0$, $A, B \in R$.

In this paper, we shall study these equations in Case 1.1.1.

§3. **Characterizations of Some Special Space-Like Curves in $\mathbf{E}_1^4$**

Let us consider an unit speed space-like curve $\xi = \xi(s)$ with Frenet equations in case 1.1.1 in Minkowski space-time. We can write this curve respect to Frenet frame $\{T, N, B_1, B_2\}$ as

$$\xi = \xi(s) = m_1 T + m_2 N + m_3 B_1 + m_4 B_2, \tag{5}$$

where m_i are arbitrary functions of s. Differentiating both sides of (5), and considering Frenet equations, we easily have a system of differential equation as follow:

$$\left\{ \begin{array}{c} \dfrac{dm_1}{ds} - m_2\kappa - 1 = 0 \\ \dfrac{dm_2}{ds} + m_1\kappa - m_3\tau = 0 \\ \dfrac{dm_3}{ds} + m_2\tau + m_4\sigma = 0 \\ \dfrac{dm_4}{ds} + m_3\sigma = 0 \end{array} \right\}. \tag{6}$$

This system's general solution have not been found. Owing to this, we give some special values to the components and curvatures. By this way, we write some characterizations.

Case 1 Let us suppose the curve $\xi = \xi(s)$ lies fully NB_1B_2 subspace. Thus, $m_1 = 0$. Using $(6)_1, (6)_2$ and $(6)_3$ we have other components, respectively,

$$\left\{ \begin{array}{c} m_2 = -\dfrac{1}{\kappa} \\ m_3 = -\dfrac{1}{\tau}\dfrac{d}{ds}(\dfrac{1}{\kappa}) \\ m_4 = \dfrac{1}{\sigma}\left[\dfrac{\tau}{\kappa} + \dfrac{d}{ds}\left(\dfrac{1}{\tau}\dfrac{d}{ds}(\dfrac{1}{\kappa})\right)\right] \end{array} \right\}. \tag{7}$$

These obtained components shall satisfy $(6)_4$. And therefore, we get following differential equation:

$$\frac{d}{ds}\left\{\frac{1}{\sigma}\left[\frac{\tau}{\kappa} + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d}{ds}(\frac{1}{\kappa})\right)\right]\right\} - \frac{\sigma}{\tau}\frac{d}{ds}(\frac{1}{\kappa}) = 0. \tag{8}$$

By the theorem (2.1), (8) follows that $\xi = \xi(s)$ lies on $H_0^3(r)$. Via this case, we write following results.

Corollary 3.1 *Let $\xi = \xi(s)$ be an unit speed space-like curve with curvatures $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$ in E_1^4.*

(i) If the first component of position vector of ξ on Frenet axis is zero, then ξ lies on H_0^3.
(ii) All space-like curves which lies fully NB_1B_2 subspace are spherical curves. And position vector of such curves can be written as

$$\xi = -\frac{1}{\kappa}N - \frac{1}{\tau}\frac{d}{ds}(\frac{1}{\kappa})B_1 + \frac{1}{\sigma}\left[\frac{\tau}{\kappa} + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d}{ds}(\frac{1}{\kappa})\right)\right]B_2. \tag{9}$$

Case 2 Let us suppose the curve $\xi = \xi(s)$ lies fully TB_1B_2 subspace. In this case $m_2 = 0$. Solution of (6) yields that

$$\begin{cases} m_1 = s + c \\ m_3 = -\dfrac{\kappa}{\tau}(s + c) \\ m_4 = \dfrac{1}{\sigma}\dfrac{d}{ds}\left(\dfrac{\kappa}{\tau}(s + c)\right) \end{cases}, \tag{10}$$

where c is a real number. Using $(6)_4$, we form a differential equation respect to $\dfrac{\kappa}{\tau}(s+c)$ as

$$\frac{d}{ds}\left\{\frac{1}{\sigma}\frac{d}{ds}\left(\frac{\kappa}{\tau}(s + c)\right)\right\} - \frac{\sigma\kappa}{\tau}(s + c) = 0. \tag{11}$$

Using an exchange variable $t = \int\limits_0^s \sigma ds$ in (11), we easily have

$$\frac{d^2}{dt^2}\left(\frac{\kappa}{\tau}(u(t) + c)\right) - \frac{\kappa}{\tau}(u(t) + c) = 0, \tag{12}$$

where $u(t)$ is a real valued function. (12) has an elementary solution. It follows that

$$\frac{\kappa}{\tau}(u(t) + c) = k_1 e^t + k_2 e^{-t}, \tag{13}$$

where k_1, k_2 are real numbers. Using hyperbolic functions cosh and sinh, finally we write that

$$\frac{\kappa}{\tau}(s + c) = A_1 \cosh\int\limits_0^s \sigma ds + A_2 \sinh\int\limits_0^s \sigma ds, \tag{14}$$

where A_1 and A_2 real numbers. Moreover, integrating both sides of (11), we have

$$\left[\frac{\kappa}{\tau}(s + c)\right]^2 - \frac{1}{\sigma^2}\left[\frac{d}{ds}\left(\frac{\kappa}{\tau}(s + c)\right)\right]^2 = \text{constant}. \tag{15}$$

Now, we write following results by means of theorem (2.2) and above equations.

Corollary 3.2 *Let $\xi = \xi(s)$ be an unit speed space-like curve with curvatures $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$ in E_1^4 and second component of position vector of ξ on Frenet axis be zero. Then*

(i) there are relations among curvatures of ξ as $(11), (14)$ and (15);
(ii) there are no inclined curves in E_1^4 whose position vector lies fully in TB_1B_2 subspace;
(iii) position vector of ξ can be written as

$$\xi(s) = (s + c)T - \frac{\kappa}{\tau}(s + c)B_1 + \frac{1}{\sigma}\frac{d}{ds}\left(\frac{\kappa}{\tau}(s + c)\right)B_2. \tag{16}$$

Case 3 Let us suppose $m_3 = 0$ and $\kappa =$constant. Then, we arrive

$$\begin{cases} m_1 = \dfrac{c_4}{\kappa}\dfrac{d}{ds}\left(\dfrac{\sigma}{\tau}\right) \\ m_2 = -c_4\dfrac{\sigma}{\tau} \\ m_4 = c_4 \end{cases}. \tag{17}$$

Substituting $(17)_1$ and $(17)_2$ to $(6)_1$, we obtain following differential equation respect to $\dfrac{\sigma}{\tau}$

$$\frac{d^2}{ds^2}\left(\frac{\sigma}{\tau}\right) + \kappa^2 \frac{\sigma}{\tau} = \frac{\kappa}{c_4}. \tag{18}$$

(18) yields that

$$\frac{\sigma}{\tau} = l_1 \cos \kappa s + l_2 \sin \kappa s + \frac{1}{\kappa c_4}. \tag{19}$$

And therefore, we write following results.

Corollary 3.3 *Let $\xi = \xi(s)$ be an unit speed space-like curve with constant first curvature and $\tau \neq 0, \sigma \neq 0$ in E_1^4 and third component of position vector of ξ on Frenet axis be zero. Then*

(i)there is a relation among curvatures of ξ as (19);
(ii) position vector of ξ can be written as

$$\xi(s) = \frac{c_4}{\kappa} \frac{d}{ds}\left(\frac{\sigma}{\tau}\right) T - c_4 \frac{\sigma}{\tau} N + c_4 B_2. \tag{20}$$

Remark 3.4 Due to σ, m_4 can not be zero. Thus, the case $m_4 =$constant is similar to case 3.

And finally, considering system of equation (6), we write following characterizations.

Corollary 3.5 *Let $\xi = \xi(s)$ be an unit speed space-like curve with curvatures $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$ in E_1^4.*

(i) The components m_1 and m_2 can not be zero, together. This result implies that $\xi = \xi(s)$ never lies fully $B_1 B_2$ hyperplane. Similarly, the components m_2 and m_3 can not be zero, together. This result follows that $\xi = \xi(s)$ never lies fully in TB_2 hyperplane.
(ii) If the components $m_1 = m_2 = 0$, then, for the space-like curve $\xi = \xi(s)$, there holds $\kappa =$constant and $\frac{\sigma}{\tau} =$constant.
(iii) The components m_i, for $1 \leq i \leq 4$, can not be nonzero constants, together.

Remark 3.6 In the case when $\xi = \xi(s)$ is a space-like curve within other cases or when is a time-like curve, there holds corollaries which are analogous with corollary 3.1, 3.2, 3.3 and 3.5.

References

[1] B.O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York,1983.

[2] C.Boyer, *A History of Mathematics*, New York, Wiley,1968.

[3] C.Camci, K. Ilarslan and E. Sucurovic, On pseudohyperbolical curves in Minkowski space-time. *Turk J.Math.* **27** (2003) 315-328.

[4] L.F.Mao, Pseudo-manifold geometries with applications, *International J.Math. Combin.*, Vol.1(2007), No.1, 45-58.

[5] J.Walrave, Curves and surfaces in Minkowski space. Dissertation, K. U. Leuven, Fac. of Science, Leuven,1995.

[6] S. Yilmaz, Spherical Indicators of Curves and Characterizations of Some Special Curves in four dimensional Lorentzian Space L^4, Dissertation, Dokuz Eylul University, (2001).