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Characterizations of the $\bar{\partial}$-cohomology groups for a family of weakly pseudoconvex manifolds

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Introduction.

On strongly pseudoconvex manifolds one can know the finiteness or the vanishing of the $\bar{\partial}$-cohomology groups by the method of a priori estimate ([**1**, **5**, **9**]).

In the case of the $\bar{\partial}$-problem on weakly pseudoconvex manifolds, the method of a priori estimate seems to be less powerful (for instance, see [**6**, **7**, **10**, **13**, **14**]).

Let T^n be a complex n-dimensional torus and $\mathrm{Pic}^0(T^n)$ the Picard group, that is, the group of holomorphic line bundles on T^n with Chern class zero. Let $E\in\mathrm{Pic}^0(T^n)$. In [**4**] Grauert showed that there exists a C^∞ weakly plurisubharmonic exhaustion function on E. So we can regard $\mathrm{Pic}^0(T^n)$ as a family of weakly pseudoconvex manifolds.

In this paper we obtain a criterion for the $\bar{\partial}$-cohomology in this family $\mathrm{Pic}^0(T^n)$, using the theory of Diophantine approximation.

It was known that $\mathrm{Pic}^0(T^n)$ is again a complex n-dimensional torus. Concretely we give an isomorphism i: $\boldsymbol{C}^n/\Lambda\cong\mathrm{Pic}^0(T^n)$ in Lemma 1, where Λ is a discrete lattice of rank $2n$ in $\boldsymbol{C}^n$. We define on $\mathrm{Pic}^0(T^n)$ the invariant distance

$$d(E, F) := \min\{\|a-b+c\|\,;\ i(a+\Lambda)=E,\ i(b+\Lambda)=F,\ c\in\Lambda\}\,,$$

where $\|(z_1, \cdots, z_n)\| := \max|z_i|$. The unit element of the group $\mathrm{Pic}^0(T^n)$ is denoted by $\mathbf{1}$. We put

$$Q := \{E\in\mathrm{Pic}^0(T^n)\,;\ E^l=\mathbf{1} \text{ for some } l\geqq 1\}.$$

Using Diophantine approximation on $\{d(\mathbf{1}, E^l)\,;\ l\geqq 1\}$, we define the following subsets of $\mathrm{Pic}^0(T^n)$.

$$\mathcal{P} := \{E\in\mathrm{Pic}^0(T^n)\,;\ \inf_{l>1}\exp(al)d(\mathbf{1}, E^l)>0 \text{ for any } a>0\}\,.$$

$$\mathcal{R} := \{E\in\mathrm{Pic}^0(T^n)\setminus Q\,;\ \inf_{l>1}\exp(al)d(\mathbf{1}, E^l)=0 \text{ for some } a>0\}\,.$$

$$\mathcal{P}^* := \{E\in\mathrm{Pic}^0(T^n)\,;\ \inf_{l>1}\exp(al)d(\mathbf{1}, E^l)>0 \text{ for some } a>0\}.$$

$$\mathcal{R}^* := \{E\in\mathrm{Pic}^0(T^n)\setminus Q\,;\ \inf_{l>1}\exp(al)d(\mathbf{1}, E^l)=0 \text{ for any } a>0\}\,.$$

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Then

$$\mathrm{Pic}^0(T^n) = \mathcal{P} \cup Q \cup \mathcal{R} \text{ (disjoint)},$$

$$\mathrm{Pic}^0(T^n) = \mathcal{P}^* \cup Q \cup \mathcal{R}^* \text{ (disjoint)}, \ \mathcal{P} \subsetneqq \mathcal{P}^* \text{ and } \mathcal{R}^* \subsetneqq \mathcal{R}.$$

The purpose of this paper is to prove the following theorems (the result of Theorem 1 is announced without details in [7]).

THEOREM 1. *Let* $E \in \mathrm{Pic}^0(T^n)$ *and* $\mathcal{O}_E$ *the structure sheaf of* E. *Then*

(i) *if* $E \in Q$, $H^p(E, \mathcal{O}_E)$ *is an infinite-dimensional Hausdorff space* $(1 \leq p \leq n)$,

(ii) *if* $E \in \mathcal{P}^*$, $\dim H^p(E, \mathcal{O}_E) = \binom{n}{p}$ $(1 \leq p \leq n)$,

(iii) *if* $E \in \mathcal{R}^*$, $H^p(E, \mathcal{O}_E)$ *is not Hausdorff* $(1 \leq p \leq n)$.

For every $E \in \mathrm{Pic}^0(T^n)$ we get a C^∞ plurisubharmonic exhaustion function $\Phi: E \to [0, \infty)$ (see Lemma 3). We put

$$E_C := \{x \in E;\ \Phi(x) < c\} \quad (0 < c \leq \infty).$$

Since the zero section $\mathbf{0}$ of E is biholomorphic onto T^n, we write also $T^n \subset E$ for the zero section of E. Let

$$H^p(T^n, \mathcal{O}_E) = \mathrm{ind\,lim}\, H^p(U, \mathcal{O}_E),$$

where U runs through the set of open neighborhoods of T^n in E. Hence we can give $H^p(T^n, \mathcal{O}_E)$ the inductive limit topology.

THEOREM 2. *If* $E \in \mathcal{P}$, *then*

(i) $\dim H^p(E_C, \mathcal{O}_E) = \binom{n}{p}$ *for* $0 < c \leq \infty$ *and* $1 \leq p \leq n$,

(ii) $\dim H^p(T^n, \mathcal{O}_E) = \dim H^p(T^n, \mathcal{O}_{T^n}) = \binom{n}{p}$ *for* $1 \leq p \leq n$,

(iii) *the restriction mapping:* $H^p(E_{C_2}, \mathcal{O}_E) \to H^p(E_{C_1}, \mathcal{O}_E)$ *is bijective for* $0 < c_1 < c_2 \leq \infty$ *and* $1 \leq p \leq n$.

THEOREM 3. *If* $E \in \mathcal{R}$, *then* $H^p(T^n, \mathcal{O}_E)$ *is not Hausdorff for* $1 \leq p \leq n$.

REMARK. For a weakly pseudoconvex manifold $E \in \mathcal{P}$ and its zero section $T^n \subset E$, the assertions of Theorem 1 and Theorem 2 are similar to the case of strongly pseudoconvex manifolds and their exceptional sets. On the other hand, for the class $\mathcal{R}$ the statement of Theorem 3 is in marked contrast to the case of strongly pseudoconvex manifolds.

We can find an analogous phenomenon with the above in the theory of complex structures of neighborhoods of elliptic curves imbedded in complex surfaces with topologically trivial normal bundles ([2, 12]). For this analogy we propose a problem in § 4.

§1. Preliminaries and lemmata.

For a complex n-dimensional torus T^n there exists a discrete lattice

$$\Gamma = \boldsymbol{Z}\{e_i,\ v_i=(v_{i1}, \cdots, v_{in});\ 1\leqq i\leqq n\}$$

such that $T^n \cong \boldsymbol{C}^n/\Gamma$, where e_i denotes the i-th unit vector of $\boldsymbol{C}^n$. We take

$$v^{n+1} = (v_{1\,n+1}, \cdots, v_{n\,n+1}) \in \boldsymbol{C}^n .$$

We put

$$v_i^* := (v_i,\ v_{i\,n+1}) \in \boldsymbol{C}^{n+1},$$

$$\Gamma^*(v^{n+1}) := \boldsymbol{Z}\{e_i^*,\ v_j^*;\ 1\leqq i\leqq n+1,\ 1\leqq j\leqq n\} \subset \boldsymbol{C}^{n+1},$$

where e_i^* denotes the i-th unit vector of $\boldsymbol{C}^{n+1}$. The projection $\pi: (z_1, \cdots, z_{n+1}) \mapsto (z_1, \cdots, z_n)$ induces the principal line bundle $\pi: \boldsymbol{C}^{n+1}/\Gamma^*(v^{n+1}) \to T^n$ with Chern class zero. $E(v^{n+1})$ denotes the line bundle on T^n associated with the above principal bundle. We get the map

$$i\ :\ \boldsymbol{C}^n \ni v^{n+1} \longmapsto E(v^{n+1}) \in \mathrm{Pic}^0(T^n)$$

and put

$$\Lambda := \boldsymbol{Z}\{e_i,\ v^i := (v_{1i}, \cdots, v_{ni});\ 1\leqq i\leqq n\} .$$

If $v^{n+1}=\sum_{1\leq i\leq n}(m_i e_i + m_{n+i} v^i) \in \Lambda$, we get the elliptic function $g(z)=\exp(2\pi\sqrt{-1}\sum_{1\leq i\leq n} m_{n+i} z_i)$ on T^n. $g(z)$ is regarded as a nowhere vanishing section of $E(v^{n+1})$. Hence $i(\Lambda)=\{\mathbf{1}\}$.

LEMMA 1. *The map i induces an isomorphism of $\boldsymbol{C}^n/\Lambda$ onto* $\mathrm{Pic}^0(T^n)$.

PROOF. To avoid confusion we only prove the lemma in the case of $n=1$. Then $\Lambda=\boldsymbol{Z}\{e_1, v^1\}$. Let $(\{U_i\}, \{f_{ij}\})$ be a defining 1-cocycle of $E\in \mathrm{Pic}^0(T^1)$. We can find $\{g_i : U_i \overset{C^\infty}{\to} \boldsymbol{C}^*\}$ satisfying $f_{ij}=g_j/g_i$ in $U_i\cap U_j$. We put $\varphi := 1/(2\pi\sqrt{-1})\,\partial\overline{\log g_i}$ and $\psi := \bar{\partial}\varphi$. There exists an $f\in C^\infty(T^1)$ such that $\psi=\bar{\partial}\partial f$ ([**11**]). Since $1/(2\pi\sqrt{-1})\partial\overline{\log g_i}-\partial f$ is a holomorphic (1, 0)-form on T^1 and $H^0(T^1, \Omega^1)=\boldsymbol{C}dz_1$, we have

$$\partial\left\{\frac{1}{2\pi\sqrt{-1}}\overline{\log g_i}-f\right\} = a_1 dz_1$$

for some $a_1\in \boldsymbol{C}$. We identify $\{U_i\}$ with an open covering of the fundamental region of T^1 in z_1-plane. Let $z_1^{(i)} := z_1|U_i$. We set

$$h_i := \bar{a}_1 z_1^{(i)} - \overline{a_1 z_1^{(i)}} - \bar{f} - \frac{1}{2\pi\sqrt{-1}}\log g_i ,$$

$$\tilde{f}_{ij} := \exp\{2\pi\sqrt{-1}(h_j - h_i) + \log f_{ij}\} ,$$

$$v_{12} := -2\sqrt{-1}\ \bar{a}_1 \operatorname{Im} v_{11},\quad v_1^* := (v_{11},\ v_{12}).$$

Then $(\{U_i\}, \{\tilde{f}_{ij}\})$ is also a defining 1-cocycle of E and $E=E(v_{12})$. Q. E. D.

We get on $\mathrm{Pic}^0(T^n)$ the invariant distance $d(E, F)$ defined in the introduction Let $v^{n+1}=(v_{1\,n+1}, \cdots, v_{n\,n+1})\in \boldsymbol{C}^n$ and $E=i(v^{n+1}+\Lambda)\in \mathrm{Pic}^0(T^n)$.

LEMMA 2. *Let* $m=(m_1, \cdots, m_{2n})\in \boldsymbol{Z}^{2n}$ *and* $l\in \boldsymbol{Z}_0^+ :=\{l\in \boldsymbol{Z}\,;\, l\geqq 0\}$. *Then*

$$d(\mathbf{1}, E^l)=\min\{\max_{1\leq i\leq n}|lv_{i\,n+1}-m_{n+i}+\sum_{1\leq j\leq n} v_{ij}m_j|\,;\, m\in \boldsymbol{Z}^{2n}\}.$$

PROOF. Let $a\in\Lambda$. Then there exists a multi-integer $m\in \boldsymbol{Z}^{2n}$ such that

$$a=\sum_{1\leq j\leq n}\{m_j(v_{1j}, \cdots, v_{nj})-m_{n+j}e_j\}.$$

Since $E^l=i(lv^{n+1}+\Lambda)$ and $\mathbf{1}=i(\Lambda)$,

$$\|a+lv^{n+1}\|=\max_{1\leq i\leq n}|lv_{i\,n+1}-m_{n+i}+\sum_{1\leq j\leq n}v_{ij}m_j|,$$

$$d(\mathbf{1}, E^l)=\min\{\|a+lv^{n+1}\|\,;\, a\in\Lambda\}. \qquad \text{Q. E. D.}$$

Let $v^{n+1}\in \boldsymbol{C}^n$ and $E=i(v^{n+1}+\Lambda)\in \mathrm{Pic}^0(T^n)$. Then the principal line bundle $E^*=E\setminus \mathbf{0}$ is identified with $\pi: \boldsymbol{C}^{n+1}/\Gamma^*(v^{n+1})\to T^n$. We put $v_{n+1}^*:=\sqrt{-1}\,e_{n+1}^*$. Then $\boldsymbol{C}^{n+1}=\boldsymbol{R}\{e_i^*, v_i^*\,;\, 1\leqq i\leqq n+1\}$. Let $z\in \boldsymbol{C}^{n+1}$. There exists a real vector $(t_1, \cdots, t_{2n+2})\in \boldsymbol{R}^{2n+2}$ such that

$$z=\sum_{1\leq i\leq n}(t_ie_i^*+t_{n+i}v_i^*)+t_{2n+1}e_{n+1}^*+t_{2n+2}v_{n+1}^*.$$

Then

$$\boldsymbol{C}^{n+1}\ni z\longmapsto (t_1, \cdots, t_{2n+2})\in \boldsymbol{R}^{2n+2}$$

induces a bireal-analytic isomorphism

$$\boldsymbol{C}^{n+1}/\Gamma^*(v^{n+1})\cong(\boldsymbol{R}/\boldsymbol{Z})^{2n+1}\times\boldsymbol{R}. \tag{1}$$

Since e_i, $v_i=(v_{i1}, \cdots, v_{in})$ are linearly independent over $\boldsymbol{R}$, the matrix $[\mathrm{Im}\, v_{ij}\,;\, 1\leqq i, j\leqq n]$ is invertible. We put

$$[\gamma_{ij}]:=[\mathrm{Im}\, v_{ij}]^{-1}, \quad x_i:=\mathrm{Re}\, z_i, \quad y_i:=\mathrm{Im}\, z_i. \tag{2}$$

We obtain

$$t_i=x_i-\sum_{1\leq j,k\leq n}y_j\gamma_{jk}\,\mathrm{Re}\, v_{ki} \qquad (1\leqq i\leqq n),$$

$$t_{n+i}=\sum_{1\leq j\leq n}y_j\gamma_{ji} \qquad (1\leqq i\leqq n),$$

$$t_{2n+1}=x_{n+1}-\sum_{1\leq j,k\leq n}y_j\gamma_{jk}\,\mathrm{Re}\, v_{k\,n+1},$$

$$t_{2n+2}=y_{n+1}-\sum_{1\leq j,k\leq n}y_j\gamma_{jk}\operatorname{Im}v_{k\,n+1}. \tag{3}$$

We put

$$\Phi_1(t_1,\cdots,t_{2n+2}):=\exp\{2\pi\sqrt{-1}(t_{2n+1}+\sqrt{-1}\,t_{2n+2})\}.$$

From (1) and the identification $E^*\cong \boldsymbol{C}^{n+1}/\Gamma^*(v^{n+1})$, it follows that Φ_1 is a real analytic function on E^*. By (3) we can show

$$\Phi_1=\exp\{2\pi\sqrt{-1}\,(z_{n+1}-\sum_{1\leq j,k\leq n}y_j\gamma_{jk}v_{k\,n+1})\}.$$

We put $\zeta_{n+1}:=\exp(2\pi\sqrt{-1}\,z_{n+1})$ on E^* and $\zeta_{n+1}:=0$ on the zero section $\mathbf{0}$. Then ζ_{n+1} is a many-valued holomorphic function on E and each branch of ζ_{n+1} is a holomorphic fiber coordinate of E.

LEMMA 3. *Let*

$$\Phi(p):=\begin{cases}|\Phi_1(p)|^2 & p\in E^*=E\setminus\mathbf{0},\\ 0 & p\in\mathbf{0}.\end{cases}$$

Then Φ is a C^∞ plurisubharmonic exhausion function on E.

PROOF. Since

$$\Phi(p)=|\zeta_{n+1}(p)|^2\exp\{4\pi\sum_{1\leq j,k\leq n}y_j(p)\gamma_{jk}\operatorname{Im}v_{k\,n+1}\},$$

Φ is a C^∞ exhausion function on E and

$$\log\Phi(p)=\log|\zeta_{n+1}(p)|^2+4\pi\sum_{1\leq j,k\leq n}y_j(p)\gamma_{jk}\operatorname{Im}v_{k\,n+1}$$

for $p\in E^*$. This shows that $\log\Phi$ is plurisubharmonic on E^* and $\Phi=\exp(\log\Phi)$ is also plurisubharmonic on E^*. If $p\in\mathbf{0}$, then $\zeta_{n+1}(p)=0$ and the Levi form of Φ at p is given by

$$\exp\{4\pi\sum_{1\leq j,k\leq n}y_j\gamma_{jk}\operatorname{Im}v_{k\,n+1}\}d\zeta_{n+1}d\overline{\zeta_{n+1}}.$$

So Φ is plurisubharmonic on E. Q.E.D.

For an open set U in E we put

$$\mathcal{F}(U):=\{f\in C^\infty(U);\ f|\pi^{-1}(a)\cap U \text{ is holomorphic in } \pi^{-1}(a)\cap U \text{ for any } a\in T^n\},$$

where π is the projection of E onto T^n. Let $\mathcal{F}$ be the sheaf on E defined by the presheaf $\{\mathcal{F}(U)\}$. Let $\{V_i\}$ be a finite open covering of T^n such that $E|\pi^{-1}(V_i)$ is holomorphically trivial and V_i is simply connected and let D be an open set in E. We take the open covering $\mathfrak{U}_D:=\{\pi^{-1}(V_i)\cap D\}$ of D.

LEMMA 4. $H^p(\mathfrak{U}_D, \mathcal{F})=0$ $(p\geqq 1)$.

PROOF. Let $\{W_i\}$ be a refinement covering of $\{V_i\}$ with $\overline{W}_i\subset V_i$ and $\{\rho_i: V_i \overset{C^\infty}{\to}[0, 1]\}$ a partition of unity satisfying $\operatorname{supp}\rho_i\subset W_i$. We put $\tilde{\rho}_i := \rho_i\circ\pi\in C^\infty(\pi^{-1}(V_i))$. We prove the lemma only in the case $p=2$. Let $\{f_{ijk}\}\in \boldsymbol{Z}^2(\mathfrak{U}_D, \mathcal{F})$. Then $f_{ijk}\in\Gamma(\pi^{-1}(V_i\cap V_j\cap V_K)\cap D, \mathcal{F})$ and $f_{jkl}-f_{ikl}+f_{ijl}-f_{ijk}=0$. We put

$$g_{ijk}(p) := \begin{cases} \tilde{\rho}_i(p)f_{ijk}(p) & p\in\pi^{-1}(\overline{W}_i)\cap D, \\ 0 & p\in\{\pi^{-1}(V_j\cap V_k)\cap D\}\setminus\pi^{-1}(\overline{W}_i). \end{cases}$$

$$g_{jk} := \sum_i g_{ijk}.$$

Then $g_{jk}\in\Gamma(\pi^{-1}(V_j\cap V_k)\cap D, \mathcal{F})$ and $f_{ijk}=g_{jk}-g_{ik}+g_{ij}$. Q. E. D.

Let Φ be the weakly plurisubharmonic C^∞ exhaustion function obtained in Lemma 3. Let $E_C:=\{p\in E;\ \Phi(p)<c\}$. We take the open covering $\mathfrak{V}_{E_C}:= \{\pi^{-1}(V_i)\cap E_C\}$ of E_C.

LEMMA 5. $H^p(E_C, \mathcal{F}) = H^p(\mathfrak{V}_{E_C}, \mathcal{F})$ *for* $p\geqq 0$ *and* $0<c\leqq\infty$.

PROOF. We prove that $\mathfrak{V}_{E_C}$ is a Leray covering for the sheaf $\mathcal{F}\,|\,E_C$, that is, $H^p(\pi^{-1}(V_i)\cap E_C, \mathcal{F})=0$ $(p\geqq 1)$. We can assume that ζ_{n+1} is holomorphic and univalent on $\pi^{-1}(V_i)\cap E_C$. We get the resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C} \overset{\bar{\partial}_1}{\longrightarrow} \mathcal{C} \longrightarrow 0 \qquad \text{on } \pi^{-1}(V_i)\cap E_C,$$

where $\mathcal{C}$ is the sheaf of germs of C^∞ functions on E and $\bar{\partial}_1:=(\partial/\partial\overline{\zeta_{n+1}})$. We may regard $(z_1, \cdots, z_n)$ and $(z_1, \cdots, z_n, \zeta_{n+1})$ as holomorphic coordinate systems in V_i and $\pi^{-1}(V_i)\cap E_C$, respectively. Putting

$$\alpha(z_1, \cdots, z_n, \zeta_{n+1}) := (z_1, \cdots, z_n, \zeta_{n+1}\exp\{-2\pi\sqrt{-1}\sum_{1\leq j,k\leq n} y_j\gamma_{jk}v_{k\,n+1}\}),$$

we get the map $\alpha:\ \pi^{-1}(V_i)\cap E_C\to V_i\times\{w\in\boldsymbol{C};\ |w|<\sqrt{c}\}$. Then α is diffeomorphic on $\pi^{-1}(V_i)\cap E_C$ and holomorphic in ζ_{n+1}. Let $f(z_1, \cdots, z_n, \zeta_{n+1})\in C^\infty(\pi^{-1}(V_i)\cap E_C)$. Putting

$$g_k(z_1, \cdots, z_n, w) := \frac{1}{2\pi\sqrt{-1}}\iint_{|\xi|<(2k-1)\sqrt{c}/2k} f\circ\alpha^{-1}(z_1, \cdots, z_n, \xi)\frac{d\xi\wedge d\bar{\xi}}{\xi-w}$$

on $V_i\times\{|w|<(2k-1)\sqrt{c}/2k\}$ for $k\in\boldsymbol{N}$, it follows from the Cauchy-Green formula that $\partial g_k/\partial\bar{w}=f\circ\alpha^{-1}$. From the Taylor expansion of $g_{k+1}-g_k$ with respect to w, $g_{k+1}-g_k$ can be approximated by C^∞ functions on $V_i\times\boldsymbol{C}$ which are holomorphic in $w\in\boldsymbol{C}$. Using this approximation and the standard argument for the Dolbeault lemma, we find a C^∞ function g such that $\partial g/\partial\bar{w}=f\circ\alpha^{-1}$ on $V_i\times\{|w|<\sqrt{c}\}$. We put

$$h := (g\circ\alpha)\exp\{2\pi\sqrt{-1}(-\sum_{1\leq j,k\leq n} y_j\gamma_{jk}\overline{v_{k\ n+1}})\}$$

on $\pi^{-1}(V_i)\cap E_C$. Then $\bar{\partial}_1 h=f$. This means $H^p(\pi^{-1}(V_i)\cap E_C, \mathcal{F})=0$ $(p\geqq 1)$.
Q.E.D.

Let $\mathcal{E}_{T^n}^{p,q}$ be the sheaf of germs of C^∞ forms of type (p, q) on T^n. We put

$$\mathcal{F}^{p,q} := \mathcal{F}\otimes\pi^*\mathcal{E}_{T^n}^{p,q}.$$

The following lemma is an immediate consequence of Lemmas 4 and 5 (Vogt [14] proved this lemma for $c=\infty$ implicitly).

LEMMA 6. *The sequence*

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{F}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{F}^{0,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{F}^{0,n} \longrightarrow 0$$

is exact on E and

$$H^p(E_C, \mathcal{O}_E) \cong \{\varphi\in H^0(E_C, \mathcal{F}^{0,p})\,;\ \bar{\partial}\varphi=0\}/\bar{\partial}H^0(E_C, \mathcal{F}^{0,p-1})$$

$(p\geqq 1,\ 0<c\leqq\infty)$.

For any open set D in E, we can define the natural locally convex topology of $H^0(D, \mathcal{F})$ which makes $H^0(D, \mathcal{F})$ an (F, S)-space. This topology makes $H^0(E_C, F^{0,p})$ and its closed subspace $\{\varphi\in H^0(E_C, \mathcal{F}^{0,p})\,;\ \bar{\partial}\varphi=0\}$ into (F, S)-spaces. Hence we can give $H^p(E_C, \mathcal{O}_E)$ the quotient locally convex topology by Lemma 6.

Using the matrix $[\gamma_{ij}]$ in (2), we put

$$(\zeta_1, \cdots, \zeta_n) := (z_1, \cdots, z_n)[\gamma_{ij}]. \tag{4}$$

Then $(\zeta_1, \cdots, \zeta_n)$ is regarded as a local coordinate system at each point of T^n. We have the global (0, 1)-forms

$$\pi^* d\overline{\zeta_i} = \sum_{1\leq k\leq n}\gamma_{ki}\pi^* d\overline{z_k} \quad (1\leqq i\leqq n)$$

on E. We put $\zeta_i^* := \zeta_i\circ\pi$ and $z_i^* := z_i\circ\pi (1\leqq i\leqq n)$. Then $\partial/\partial\overline{\zeta_i^*}=\sum_j \operatorname{Im} v_{ij}(\partial/\partial\overline{z_j^*})$ is a global vector field on E. Henceforth we write simply $d\overline{\zeta_i}$ and $\partial/\partial\overline{\zeta_i}$ instead of $\pi^* d\overline{\zeta_i}$ and $\partial/\partial\overline{\zeta_i^*}$, respectively. From the bireal-analytic isomorphism (1) it follows that

$$E_C^* = E_C\setminus 0 \cong (\boldsymbol{R}/\boldsymbol{Z})^{2n+1}\times\left\{t_{2n+2}\,;\ -\frac{\log c}{4\pi}<t_{2n+1}<\infty\right\}.$$

Let $\varphi\in H^0(E_C, \mathcal{F}^{0,p})$. We have the Fourier expansion of φ on $E_C^*\cong(\boldsymbol{R}/\boldsymbol{Z})^{2n+1}\times(-\log c/4\pi, \infty)$ in terms of $d\overline{\zeta_1}, \cdots, d\overline{\zeta_n}$;

$$\varphi = \frac{1}{p\,!}\sum_{1\leq i_1,\cdots,i_p\leq n,\ r\in Z^{2n+1}} b^r_{i_1\cdots i_p}(t_{2n+2})e_r(t')d\overline{\zeta_{i_1}}\wedge\cdots\wedge d\overline{\zeta_{i_p}},$$

where

$$b^{r}_{i_1\cdots i_p}(t_{2n+2}) \in C^{\infty}\Big(\Big(-\frac{\log c}{4\pi}, \infty\Big)\Big), \quad t'=(t_1, \cdots, t_{2n+1}) \in \boldsymbol{R}^{2n+1},$$

$$r=(r_1, \cdots, r_{2n+1}) \in \boldsymbol{Z}^{2n+1}, \quad e_r(t')=\exp\Big\{2\pi\sqrt{-1}\sum_{i=1}^{2n+1} r_i t_i\Big\}.$$

Since $\varphi|\pi^{-1}(a)$ is holomorphic in $\pi^{-1}(a)\cap E_C$ for any $a\in T^n$,

$$\frac{\partial\varphi}{\partial\bar{z}_{n+1}}=\frac{1}{2}\Big(\frac{\partial}{\partial t_{2n+1}}+\sqrt{-1}\ \frac{\partial}{\partial t_{2n+2}}\Big)\varphi=0.$$

Then $b^{r}_{i_1\cdots i_p}(t_{2n+2})=a^{r}_{i_1\cdots i_p}\exp(-2\pi r_{2n+1}t_{2n+2})$ for some constant $a^{r}_{i_1\cdots i_p}$. φ is of class C^{∞} in a neighborhood of the zero section $\mathbf{0}$ of E, so φ converges to $\varphi|\mathbf{0}$ as $t_{2n+2}\to+\infty$ and then $a^{r}_{i_1\cdots i_p}=0$ for $r_{2n+1}<0$. Hence, for any $\varphi\in H^0(E_C, \mathcal{F}^{0,p})$, we obtain the Fourier expansion:

$$\varphi=\frac{1}{p!}\sum_{1\le i_1,\cdots,i_p\le n,\, m\in Z^{2n},\, l\ge 0} a^{m,l}_{i_1\cdots i_p} e_m(t'')$$
$$\times\exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\}d\bar{\zeta}_{i_1}\wedge\cdots\wedge d\bar{\zeta}_{i_p}, \tag{5}$$

where $t''=(t_1, \cdots, t_{2n})\in\boldsymbol{R}^{2n}$, $t_{2n+1}\in\boldsymbol{R}$, $-(\log c)/4\pi<t_{2n+2}\leqq\infty$, $m=(m_1, \cdots, m_{2n})\in\boldsymbol{Z}^{2n}$ and $e_m(t'')=\exp\{2\pi\sqrt{-1}\ \sum_{i=1}^{2n} m_i t_i\}$.

LEMMA 7. *Let* $\{a^{m,l}_{i_1\cdots i_p};\ m\in\boldsymbol{Z}^{2n}, l\geqq 0, 1\leqq i_1, \cdots, i_p\leqq n\}$ *be a sequence of complex numbers, where* $a^{m,l}_{i_1\cdots i_p}$ *are skew-symmetric in all indices* $i_1, \cdots, i_p$, *and let* φ *be the formal Fourier series defined by the right hand side in* (5). *Suppose* $0<c\leqq\infty$. *Then* φ *converges to a form in* $H^0(E_C, \mathcal{F}^{0,p})$ *if and only if, for any* $R\in(0, \sqrt{c})$ *and any* $k>0$,

$$C(k, R):=\sup\{|a^{m,l}_{i_1\cdots i_p}|\,\|m\|^k R^l;\ m\in\boldsymbol{Z}^{2n}, l\geqq 0, 1\leqq i_1<\cdots<i_p\leqq n\}<\infty.$$

PROOF. We can assume without loss of generality $p=0$. We put $\Phi_1:=\sum a^{m,l}e_m(t'')\xi^l$ for $(t'', \xi)\in\boldsymbol{R}^{2n}\times\{\xi\in\boldsymbol{C};\ |\xi|<\sqrt{c}\}$. From the well-known result for Fourier coefficients of C^{∞} functions, Φ_1 is of C^{∞} in $\boldsymbol{R}^{2n}\times\{|\xi|<\sqrt{c}\}$ and holomorphic in $\xi\in\{|\xi|<\sqrt{c}\}$ if and only if $C(k, R)<\infty$ for any $R\in(0, \sqrt{c})$ and any $k\geqq 0$. Q. E. D.

§ 2. Proof of Theorem 1.

We will accomplish the proof in the four steps (a), (b), (c) and (d).

(a) Let $E\in\mathrm{Pic}^0(T^n)$ and let Γ, $v_i=(v_{i1}, \cdots, v_{in})$, $v^{n+1}=(v_{1\,n+1}, \cdots, v_{n\,n+1})$, $\Gamma^*(v^{n+1})$, $v_i^*=(v_i, v_{i\,n+1})$ and Λ be as in § 1. We put

$$K_i^{m,l}:=lv_{i\,n+1}-m_{n+i}+\sum_{1\le j\le n} v_{ij}m_j \tag{6}$$

for $m=(m_1, \cdots, m_{2n})\in \boldsymbol{Z}^{2n}$, $l\in \boldsymbol{Z}_0^+$ and $1\leqq i\leqq n$. Then, by Lemma 2,

$$d(\mathbf{1}, E^l) = \min\{\max_{1\leq i\leq n} |K_i^{m,l}|\ ;\ m\in \boldsymbol{Z}^{2n}\}. \tag{7}$$

If follows from (3), (4) and (6) that

$$\frac{\partial}{\partial\bar{\zeta}_i}[e_m(t'')\exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\}]$$
$$= \pi K_i^{m,l} e_m(t'')\exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\}. \tag{8}$$

We put

$$j(m, l) := \min\{j\ ;\ 1\leqq j\leqq n,\ |K_j^{m,l}| = \max_{1\leq i\leq n} |K_i^{m,l}|\}$$

for $(m, l)\in \boldsymbol{Z}^{2n}\times \boldsymbol{Z}_0^+$. We take a form

$$\varphi = \frac{1}{p!}\sum_{1\leq i_1,\cdots,i_p\leq n,\, m\in Z^{2n},\, l\geq 0} a_{i_1\cdots i_p}^{m,l} e_m(t'')$$
$$\times\exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\}\overline{d\zeta_{i_1}}\wedge\cdots\wedge\overline{d\zeta_{i_p}} \tag{9}$$

in $H^0(E, \mathcal{F}^{0,p})$ and a form

$$\psi = \frac{1}{(p-1)!}\sum_{1\leq i_1,\cdots,i_{p-1}\leq n,\, m\in Z^{2n},\, l\geq 0} b_{i_1\cdots i_{p-1}}^{m,l} e_m(t'')$$
$$\times\exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\}\overline{d\zeta_{i_1}}\wedge\cdots\wedge\overline{d\zeta_{i_{p-1}}}$$

in $H^0(E, \mathcal{F}^{0,p-1})$ satisfying $\varphi=\bar{\partial}\psi$. Then, by (8) we obtain

$$a_{i_1\cdots i_p}^{m,l} = \sum_{1\leq k\leq p}(-1)^{k+1}\pi K_{i_k}^{m,l} b_{i_1\cdots\hat{i}_k\cdots i_p}^{m,l}. \tag{10}$$

The equation (10) implies that φ is $\bar{\partial}$-closed if and only if

$$\sum_{1\leq k\leq p+1}(-1)^{k+1}\pi K_{i_k}^{m,l} a_{i_1\cdots\hat{i}_k\cdots i_{p+1}}^{m,l} = 0. \tag{11}$$

Applying (11) to the $p+1$-tuple $(j(m, l), i_1, \cdots, i_p)$ of indices in place of $(i_1, \cdots, i_{p+1})$, we see that

$$K_{j(m,l)}^{m,l} a_{i_1\cdots i_p}^{m,l} = \sum_{1\leq k\leq p}(-1)^{k+1}\pi K_{i_k}^{m,l} a_{j(m,l)i_1\cdots\hat{i}_k\cdots i_p}^{m,l}. \tag{12}$$

It follows from (10) and (12) that

$$\bar{\partial}\Big[\frac{1}{(p-1)!}\sum_{1\leq i_1,\cdots,i_{p-1}\leq n} a_{j(m,l)i_1\cdots i_{p-1}}^{m,l} e_m(t'')$$
$$\times\exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\}\overline{d\zeta_{i_1}}\wedge\cdots\wedge\overline{d\zeta_{i_{p-1}}}\Big]$$

$$= \frac{\pi}{p!} K^{m,l}_{j(m,l)} \sum_{1 \leq j_1, \cdots, j_p \leq n} a^{m,l}_{j_1 \cdots j_p} e_m(t'')$$
$$\times \exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\} d\overline{\zeta_{j_1}} \wedge \cdots \wedge d\overline{\zeta_{j_p}}. \tag{13}$$

(b) We suppose $E \in Q$. We put

$$M := \{(m, l) \in \boldsymbol{Z}^{2n} \times \boldsymbol{Z}^+_0 ;\ K^{m,l}_i = 0 \text{ for } 1 \leqq i \leqq n\}.$$

Then, by (7) there exists a positive constant c such that

$$|K^{m,l}_{j(m,l)}| = \max_{1 \leq i \leq n} |K^{m,l}_i| \geqq d(\mathbf{1}, E^l) \geqq c \tag{14}$$

for any $(m, l) \in \boldsymbol{Z}^{2n} \times \boldsymbol{Z}^+_0 \setminus M$. Suppose that

$$\varphi = \frac{1}{p!} \sum_{1 \leq i_1, \cdots, i_p \leq n,\ m \in Z^{2n},\ l \geq 0} a^{m,l}_{i_1 \cdots i_p} e_m(t'')$$
$$\times \exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\} d\overline{\zeta_{i_1}} \wedge \cdots \wedge d\overline{\zeta_{i_p}}$$

belongs the closure of $\bar{\partial} H^0(E, \mathcal{F}^{0,p-1})$ in $H^0(E, \mathcal{F}^{0,p})$. It follows from (8) that

$$\bar{\partial}[e_m(t'') \exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\}] = 0$$

for all $(m, l) \in M$. Then $a^{m,l}_{i_1 \cdots i_p} = 0$ for any $(m, l) \in M$. Putting

$$h^{m,l}_j := \frac{1}{(p-1)!} \sum_{1 \leq i_1, \cdots, i_{p-1} \leq n} a^{m,l}_{j i_1 \cdots i_{p-1}} e_m(t'')$$
$$\times \exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\} d\overline{\zeta_{i_1}} \wedge \cdots \wedge d\overline{\zeta_{i_{p-1}}}$$

and using (8) and (13), we obtain

$$\bar{\partial} h^{m,l}_j = \frac{1}{p!} \pi K^{m,l}_j \sum_{1 \leq i_1, \cdots, i_p \leq n} a^{m,l}_{i_1 \cdots i_p} e_m(t'')$$
$$\times \exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\} d\overline{\zeta_{i_1}} \wedge \cdots \wedge d\overline{\zeta_{i_p}}.$$

We set

$$\phi_1 := \sum_{(m,l) \in Z^{2n} \times Z^+_0 \setminus M} \{h^{m,l}_{j(m,l)} / \pi K^{m,l}_{j(m,l)}\}.$$

From (14) we can show that $\phi_1 \in H^0(E, \mathcal{F}^{0,p-1})$ and $\varphi = \bar{\partial}\phi_1$. This means that $\bar{\partial} H^0(E, \mathcal{F}^{0,p-1})$ is closed in $H^0(E, \mathcal{F}^{0,p})$. Since

$$\{e_m(t'') \exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\} d\overline{\zeta_{i_1}} \wedge \cdots \wedge d\overline{\zeta_{i_p}} ;$$
$$(m, l) \in M,\ 1 \leqq i_1 < \cdots < i_p \leqq n\}$$

is a set of linearly independent forms in $H^p(E, \mathcal{O}_E) \cong \{\varphi \in H^0(E, \mathcal{F}^{0,p});\ \bar{\partial}\varphi = 0\} / \bar{\partial} H^0(E, \mathcal{F}^{0,p-1})$, $\dim H^p(E, \mathcal{O}_E) = \infty$. Q. E. D.

(c) Let $E\in\mathcal{P}^*$. Suppose that $\varphi\in H^0(E, \mathcal{F}^{0,p})$ is as in (9) and that $\bar{\partial}\varphi=0$. Since e_i, v^i are linearly independent over $\boldsymbol{R}$ $(1\leqq i\leqq n)$, there exists a positive constant $b\in(0, 1)$ such that

$$\begin{aligned}\min\{\max_{1\leq i\leq n}|K_i^{m,l}|\,;\ m\in\boldsymbol{Z}^{2n}\setminus\{0\},\ l=0\}\\ &= \min\{\|\sum_{1\leq i\leq n}(m_i v^i+m_{n+i}e_i)\|\,;\ m\in\boldsymbol{Z}^{2n}\setminus\{0\}\}\\ &\geqq b.\end{aligned}$$

And since $E\in\mathcal{P}^*$, there exists a positive constant a such that

$$|K_{j(m,l)}^{m,l}|\geqq d(\mathbf{1}, E^l)\geqq b\exp(-al)$$

for any $(m, l)\in\boldsymbol{Z}^{2n}\times\boldsymbol{Z}_0^+\setminus\{(0, 0)\}$. We put

$$\lambda := \frac{1}{(p-1)!}\sum_{1\leq i_1,\cdots,i_{p-1}\leq n,\,(m,l)\in\boldsymbol{Z}^{2n}\times\boldsymbol{Z}_0^+\setminus\{(0,0)\}}[a_{j(m,l)i_1\cdots i_{p-1}}^{m,l}e_m(t'')/\pi K_{j(m,l)}^{m,l}]$$
$$\times\exp\{2\pi\sqrt{-1}\ l(t_{2n+1}+\sqrt{-1}\ t_{2n+2})\}d\overline{\zeta_i}\wedge\cdots\wedge d\overline{\zeta_{i_{p-1}}}.$$

Lemma 7 shows that

$$C(k, R) := \sup\{|a_{i_1\cdots i_p}^{m,l}|\,\|m\|^k R^l\,;\ m\in\boldsymbol{Z}^{2n},\ l\geqq 0,\ 1\leqq i_1<\cdots<i_p\leqq n\}<\infty$$

for $0<R<\infty$ and $k>0$. Then

$$\begin{aligned}|a_{j(m,l)i_1\cdots i_{p-1}}^{m,l}/\pi K_{j(m,l)}^{m,l}|\,\|m\|^k R^l\\ &\leqq |a_{j(m,l)i_1\cdots i_{p-1}}^{m,l}|\,\|m\|^k R^l\exp(al)\\ &\leqq C(k, R\exp a)<\infty\end{aligned}$$

for $0<R<\infty$, $k>0$, $(m, l)\in\boldsymbol{Z}^{2n}\times\boldsymbol{Z}_0^+\setminus\{(0, 0)\}$. Hence by Lemma 7 $\lambda\in H^0(E, \mathcal{F}^{0,p-1})$. It follows from (13) that

$$\varphi=\frac{1}{p!}\sum_{1\leq i_1,\cdots,i_p\leq n}a_{i_1\cdots i_p}^{0,0}d\overline{\zeta_{i_1}}\wedge\cdots\wedge d\overline{\zeta_{i_p}}+\bar{\partial}\lambda.$$

This shows $H^p(E, \mathcal{O}_E)\cong\boldsymbol{C}\{d\overline{\zeta_{i_1}}\wedge\cdots\wedge d\overline{\zeta_{i_p}}\,;\ 1\leqq i_1<\cdots<i_p\leqq n\}$.

(d) Finally suppose $E\in\mathcal{R}^*$. Let $l\in\boldsymbol{N}$. We take $m(l)=(m_1(l), \cdots, m_{2n}(l))\in\boldsymbol{Z}^{2n}$ satisfying

$$\max_{1\leq i\leq n}|K_i^{m(l),l}|=d(\mathbf{1}, E^l).$$

We put

$$u := lv^{n+1}+\sum_{1\leq j\leq n}\{m_j(l)v^j-m_{n+j}(l)e_j\},$$

then $u=(K_1^{m(l),l}, \cdots, K_n^{m(l),l})$. Since

$$\|u\|\leqq\max\{d(F_1, F_2)\,;\ F_1, F_2\in\mathrm{Pic}^0(T^n)\}<\infty,$$

$$\sum_j m_j(l)\operatorname{Im} v^j = \operatorname{Im} u - l \operatorname{Im} v^{n+1},$$

$$\sum_j m_{n+j}(l)e_j = \operatorname{Re} u - \sum_j m_j(l)\operatorname{Re} v^j + l \operatorname{Re} v^{n+1}$$

and $[\operatorname{Im} v_{ij}]$ is invertible, there exists a positive constant K such that $\|m(l)\| \leqq Kl$ for any $l \in \boldsymbol{N}$. From this fact and the assumption $E \in \mathfrak{R}^*$, there exist $l_k \in \boldsymbol{N}$ such that

$$\exp(-kl_k - k\|m(l_k)\|)/d(1, E^{l_k}) \geqq k$$

for all $k \in \boldsymbol{N}$. We put

$$b^{m,l} := \begin{cases} \exp(-kl_k - k\|m(l_k)\|)/K^{m(l_k),l_k}_{j(m(l_k),l_k)} \\ \qquad \text{if} \quad (m, l) = (m(l_k), l_k) \quad (k \in \boldsymbol{N}) \\ 0 \qquad \text{otherwise.} \end{cases}$$

Then $|b^{m(l_k),l_k}| \geqq k$. We take i_0 satisfying $1 \leqq i_0 \leqq n$ and $\sup\{k\,;\, i_0 = j(m(l_k), l_k)\} = \infty$. We can assume without loss of generality that $i_0 = n$. We put

$$\phi_2^{m,l} := b^{m,l} e_m(t'') \exp\{2\pi\sqrt{-1}\, l(t_{2n+1} + \sqrt{-1}\, t_{2n+2})\} d\overline{\zeta_1} \wedge \cdots \wedge d\overline{\zeta_{p-1}}.$$

It follows from (8) that

$$\bar\partial\phi_2^{m,l} = \sum_i \pi K_i^{m,l} b^{m,l} e_m(t'') \exp\{2\pi\sqrt{-1}\, l(t_{2n+1} + \sqrt{-1}\, t_{2n+2})\} d\overline{\zeta_i} \wedge d\overline{\zeta_1} \wedge \cdots \wedge d\overline{\zeta_{p-1}}.$$

Since

$$|K_i^{m(l_k),l_k} b^{m(l_k),l_k}| \leqq \exp(-kl_k - k\|m(l_k)\|)$$

for any $k \in \boldsymbol{N}$, we can prove, in virtue of Lemma 7, that

$$\sum_{(m,l)\in \boldsymbol{Z}^{2n}\times \boldsymbol{N}} \bar\partial\phi_2^{m,l} \in H^0(E, \mathcal{F}^{0,p}).$$

We write $\phi_2 = \sum_{(m,l)\in \boldsymbol{Z}^{2n}\times \boldsymbol{N}} \bar\partial\phi_2^{m,l}$. Then ϕ_2 belongs to the closure of $\bar\partial H^0(E, \mathcal{F}^{0,p-1})$. We assume, to reach a contradiction, that $\phi_2 = \bar\partial\lambda$ for some

$$\lambda = \sum \lambda^{m,l}_{i_1\cdots i_{p-1}} e_m(t'') \exp\{2\pi\sqrt{-1}\, l(t_{2n+1} + \sqrt{-1}\, t_{2n+2})\} d\overline{\zeta_{i_1}} \wedge \cdots \wedge d\overline{\zeta_{i_{p-1}}}.$$

We compare the term of $\sum \bar\partial\phi_2^{m,l}$ with that of $\bar\partial\lambda$ imvolving only the exterior differential form $d\overline{\zeta_n} \wedge d\overline{\zeta_1} \wedge \cdots \wedge d\overline{\zeta_{p-1}}$. Then

$$\pi K_n^{m,l} b^{m,l} = \pi K_n^{m,l} \lambda^{m,l}_{1\cdots p-1} + \sum_{1\leqq i\leqq p-1} (-1)^{p+i} \pi K_i^{m,l} \lambda^{m,l}_{1\cdots\hat{i}\cdots p-1\, n}.$$

And then

$$b^{m,l} = \lambda^{m,l}_{1\cdots p-1} + \sum_{1\leqq i\leqq p-1} (-1)^{p+i} (K_i^{m,l}/K_n^{m,l}) \lambda^{m,l}_{1\cdots\hat{i}\cdots p-1\, n}.$$

Since $\sup\{k\,;\, n = j(m(l_k), l_k)\} = \infty$, we can choose a subsequence $\{l_k^*\}$ of $\{l_k\}$ so that

$$|K_i^{m(l_k^*),l_k^*}/K_n^{m(l_k^*),l_k^*}| \leqq 1$$

for $1 \leqq i \leqq n$ and

$$\lim_{k\to\infty}\{\lambda_{1\cdots p-1}^{m(l_k^*),l_k^*}+\sum_{1\leqq i\leqq p-1}(-1)^{p+i}(K_i^{m(l_k^*),l_k^*}/K_n^{m(l_k^*),l_k^*})\lambda_{1\cdots\hat{i}\cdots p-1\,n}\}=0.$$

This contradicts that $|b^{m(l_k),l_k}| \geqq k$. Hence ϕ_2 belongs not to $\bar{\partial}H^0(E, \mathcal{F}^{0,p-1})$ but to the closure of $\bar{\partial}H^0(E, \mathcal{F}^{0,p-1})$.

§3. Proofs of Theorem 2 and Theorem 3.

First we prove Theorem 2(i). We suppose $E \in \mathcal{P}$. Since $\mathcal{P} \subset \mathcal{P}^*$, the statement (i) of Theorem 2 in the case of $c=\infty$ follows from Theorem 1. Then we can assume $0<c<\infty$. We recall the proof of Theorem 1 in §2. Let

$$\varphi=\frac{1}{p!}\sum_{1\leqq i_1,\cdots,i_p\leqq n,\, m\in Z^{2n},\, l\geqq 0} a_{i_1\cdots i_p}^{m,l} e_m(t'')$$
$$\times\exp\{2\pi\sqrt{-1}\,l(t_{2n+1}+\sqrt{-1}\,t_{2n+2})\}d\overline{\zeta_{i_1}}\wedge\cdots\wedge d\overline{\zeta_{i_p}}$$
$$\in H^0(E_C, \mathcal{F}^{0,p}) \qquad \left(-\frac{\log c}{4\pi}<t_{2n+2}\leqq\infty\right).$$

We put

$$\lambda := \frac{1}{(p-1)!}\sum_{1\leqq i_1,\cdots,i_{p-1}\leqq n,\,(m,l)\in Z^{2n}\times Z_0^+\setminus\{(0,0)\}}[a_{j(m,l)i_1\cdots i_{p-1}}^{m,l}e_m(t'')/\pi K_{j(m,l)}^{m,l}]$$
$$\times\exp\{2\pi\sqrt{-1}\,l(t_{2n+1}+\sqrt{-1}\,t_{2n+2})\}d\overline{\zeta_i}\wedge\cdots\wedge d\overline{\zeta_{i_{p-1}}}.$$

Since $E \in \mathcal{P}$, for any $a>0$ we can find a positive constant $C(a)$ satisfying

$$1/d(1, E^{l'}) \leqq C(a)\exp(al') \quad (l'>0),$$
$$1/|K_{j(m,l'')}^{m,l''}| \leqq C(a) \quad (m\in \boldsymbol{Z}^{2n}\setminus\{0\},\ l''=0).$$

Let $0<r<\sqrt{c}$ and $0<a_0<\log(\sqrt{c}/r)$. We obtain

$$|a_{j(m,l)i_1\cdots i_{p-1}}^{m,l}/K_{j(m,l)}^{m,l}|\,\|m\|^k r^l$$
$$\leqq C(a_0)|a_{j(m,l)i_1\cdots i_{p-1}}^{m,l}|\,\|m\|^k(r\exp a_0)^l.$$

Since $\varphi\in H^0(E_C, \mathcal{F}^{0,p})$ and $0<r\exp a_0<\sqrt{c}$, Lemma 7 shows that λ converges to a form in $H^0(E_C, \mathcal{F}^{0,p-1})$. Similarly to the step (c) in §2, we can show $\dim H^p(E_C, \mathcal{O}_E)=\binom{n}{p}$.

The assertion (ii) is an immediate consequence of (i).

Let $0<c_1<c_2\leqq\infty$ and $1\leqq p\leqq n$. (i) shows that

$$H^p(E_{C_i}, \mathcal{O}_E)\cong \boldsymbol{C}\{d\overline{\zeta_{i_1}}\wedge\cdots\wedge d\overline{\zeta_{i_p}};\ 1\leqq i_1<\cdots<i_p\leqq n\} \quad (i=1, 2).$$

Then the restriction mapping $H^p(E_{c_2}, \mathcal{O}_E)\to H^p(E_{c_1}, \mathcal{O}_E)$ is given by $\sum a_{i_1\cdots i_p}d\overline{\zeta_{i_1}}\wedge\cdots\wedge d\overline{\zeta_{i_p}}\mapsto\sum a_{i_1\cdots i_p}d\overline{\zeta_{i_1}}\wedge\cdots\wedge d\overline{\zeta_{i_p}}|E_{c_1}$. This proves the statement (iii).

Now we begin to prove Theorem 3. Let $E\in\mathcal{R}$. We put

$$a^* := \sup\{a>0;\ \inf_{l>0}\exp(al)d(\mathbf{1}, E^l)=0\},$$

$$c_0 := \exp(2a^*).$$

Then $\inf_{l>0}\exp(a'l)d(\mathbf{1}, E^l)=0$ for $0<a'<a^*$. Let $0<c<c_0$, $a:=\log\sqrt{c}$, $c_1:=(c+c_0)/2$, $a_1:=\log\sqrt{c_1}$ and $a_2:=(a_0+a_1)/2$. We use the notation $m(l)\in\boldsymbol{Z}^{2n}$ which is defined in §2. Then we can find a positive number K so that $\|m(l)\|\leqq Kl$ for any $l\in\boldsymbol{N}$. There exists a sequence $\{l_k;\ l_k>0\}$ such that $\exp(a_2l_k)\times d(\mathbf{1}, E^{l_k})\leqq 1/k$ for any $k\in\boldsymbol{N}$. Then we obtain

$$\exp\left(a_1l_k+\frac{a_2-a_1}{K}\|m(l_k)\|\right)d(\mathbf{1}, E^{l_k})\leqq 1/k \quad (k\in\boldsymbol{N}).$$

We put

$$b_*^{m,l} := \begin{cases} \exp\left(-a_1l_k-\dfrac{a_2-a_1}{K}\|m(l_k)\|\right)/K^{m(l_k),l_k}_{j(m(l_k),l_k)} \\ \qquad\qquad \text{if} \quad (m, l)=(m(l_k), l_k) \qquad (k\in\boldsymbol{N}) \\ 0 \qquad\qquad \text{otherwise}. \end{cases}$$

Replacing $\{b^{m,l}\}$ by the above $\{b_*^{m,l}\}$ in the argument of the step (d) in §2, we can show that the form

$$\psi_2^* := \sum\bar\partial[b_*^{m,l}e_m(t'')\exp\{2\pi\sqrt{-1}\,l(t_{2n+1}+\sqrt{-1}\,t_{2n+2})\}d\overline{\zeta_1}\wedge\cdots\wedge d\overline{\zeta_{p-1}}]$$

belongs not to $\bar\partial H^0(E_c, \mathcal{F}^{0,p-1})$ but to the closure of $\bar\partial H^0(E_c, \mathcal{F}^{0,p-1})$. Observing the argument in the step (d) in §2, we can prove that $\psi_2^*|E_{c^*}$ doesn't belong to $\bar\partial H^0(E_{c^*}, \mathcal{F}^{0,p-1})$ for any $c^*\in(0, c]$. Since $H^p(T^n, \mathcal{O}_E)$ is endowed with the inductive limit topology, the above fact proves that $H^p(T^n, \mathcal{O}_E)$ is not Hausdorff.

§4. Problem concerning complex structures of neighborhoods of elliptic curves.

Let C be an elliptic curve and E a holomorphic line bundle over C. E is said to be rigid if, for every imbedding with the normal bundle E of the base C into a complex analytic surface S, a sufficiently small neighborhood of the imbedded base is biholomorphically equivalent to a neighborhood of the zero section of the bundle E (see [2]).

Grauert [3] showed that if E is negative, then E is rigid.

Henceforth we assume that E is of Chern class zero, that is, $E\in\mathrm{Pic}^0(C)$.

Since C is biholomorphic onto a complex 1-dimensional torus, we obtain the classification $\mathrm{Pic}^0(C)=\mathcal{P}\cup Q\cup\mathcal{R}$ in the introduction. Theorem 2 and Theorem 3 show that

(i) if $E\in\mathcal{P}$, then $\dim H^1(C, \mathcal{O}_E)=1$,

(ii) if $E\in\mathcal{R}$, then $H^1(C, \mathcal{O}_E)$ is not Hausdorff.

We put

$$\mathcal{E} := \{E\in\mathrm{Pic}^0(C);\ \text{there exists a positive constant}\ a=a(E)\ \text{such that}\ d(\mathbf{1}, E^l)\geqq(2l)^{-a}\ \text{for any}\ l\in N\}.$$

Arnold [2] and Ueda [12] proved the following theorem.

THEOREM A. *Every element belonging to the class $\mathcal{E}$ is rigid.*

Ueda [12] gave the following

THEOREM B. *There exists a subset $\mathcal{B}$ of* $\mathrm{Pic}^0(C)$ *satisfying the following properties* (i) *and* (ii).

(i) *If $E\in\mathcal{B}$, then there exist $A=A(E)>1$ and $k=k(E)\geqq 2$ such that*

$$\liminf_{l\to\infty} A^l[d(\mathbf{1}, E^l)]^{1/(k^l-1)}=0.$$

(ii) *Every element belonging to the class $\mathcal{B}$ is not rigid.*

It follows from the statement (i) of Theorem B that $\mathcal{B}\subsetneqq\mathcal{R}$. Moreover we can show, by the definition of the subset $\mathcal{E}$, that $\mathcal{E}\subsetneqq\mathcal{P}$.

Here, with respect to the above theorems, we propose the following

PROBLEM. Is every element of the class $\mathcal{P}$ rigid? And is every element of the class $\mathcal{R}$ not rigid?

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