

## CHARACTERIZATIONS OF THE EXPONENTIAL FUNCTION BY THE VALUE DISTRIBUTION

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**1. Introduction.** Baker [1] has shown the following characterization of the exponential function.

Let  $f(z)$  be a transcendental entire function. Assume that for every complex number  $w$  there is a straight line  $L_w$  of the complex plane on which all the solutions of  $f(z)=w$  lie. Then  $f(z)=a+b\cdot\exp(Az)$  with constants  $a, b, A$ .

Recently Kobayashi [6] has shown the following theorem.

**THEOREM A.** *Let  $f(z)$  be a transcendental entire function. Assume that there are three distinct finite complex numbers  $a_j$  and three straight lines  $L_j$  of the complex plane on which all the solutions of  $f(z)=a_j$  lie ( $j=1, 2, 3$ ). Assume further that these three values never lie on any straight line of the complex plane. Then  $f(z)=P(\exp Az)$  with a quadratic polynomial  $P(z)$  and a non-zero constant  $A$ .*

In this note we shall give a generalization of Baker's result. In what follows we shall mean a strip by the set  $\{az+b; |\operatorname{Re} z| \leq 1\}$ , where  $a (\neq 0)$  and  $b$  are constants.

**THEOREM 1.** *Let  $f(z)$  be a transcendental entire function and  $k$  a positive number. Assume that for every complex number  $w$  there is a strip  $S_w$  of width  $k$  of the complex plane in which all the solutions of  $f(z)=w$  lie. Then  $f(z)=a+b\cdot\exp(Az)$  with constants  $a, b, A, bA \neq 0$ .*

**THEOREM 2.** *Let  $f(z)$  be a transcendental real entire function of finite lower order and  $k$  a positive number. Assume that for every real number  $w$  there is a strip  $S_w$  of width  $k$  of the complex plane in which all the solutions of  $f(z)=w$  lie. Then  $f(z)=a+b\cdot\exp(Az)$  with real constants  $a, b, A, bA \neq 0$ .*

**THEOREM 3.** *Let  $f(z)$  be a transcendental entire function and  $G$  an open subset of the complex plane. Assume that for every  $w \in G$  there is a straight line  $L_w$  of the complex plane on which all but a finite number of the solutions of  $f(z)=w$  lie. Then  $f(z)=a+b\cdot\exp(Az)$  with constants  $a, b, A, bA \neq 0$ .*

**THEOREM 4.** *Let  $f(z)$  be a transcendental entire function,  $G$  an open subset*

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of the complex plane and  $n$  a positive integer. Assume that for every  $w \in G$  there are  $n$  straight lines  $L_{w,1}, \dots, L_{w,n}$  of the complex plane, which are parallel with one another, on which all the solutions of  $f(z)=w$  lie. Then  $f(z)=Q(\exp Az)$  with a rational function  $Q$  of order at most  $n$  and a non-zero constant  $A$ .

*Remarks.* The function  $e^z + e^{az}$  ( $a > 1$ ) shows that the assumption "for every complex number" in Theorem 1 cannot be improved. If  $G$  is a straight line of the complex plane, then the conclusion of Theorem 3 cannot hold generally. This is shown by the function  $f(z)=z \cos z$ . It is easily seen that for every real number  $w$  all but a finite number of the solutions of  $f(z)=w$  lie on the real axis.

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**2. Statement of known results.** We need the following theorems.

**THEOREM A [5].** Let  $f(z)$  be an entire function of finite genus  $q$  ( $\geq 1$ ). If its zeros  $\{a_n\}$  satisfy

$$\lim_{n \rightarrow \infty} \arg a_n = 0 \quad (|\arg a_n| \leq \pi),$$

then  $f(z)$  has zero as a deficient value.

**THEOREM B [5].** Let  $f(z)$  be an entire function of finite genus  $q$  ( $\geq 2$ ). If its zeros  $\{a_n\}$  lie in a strip of the complex plane, then  $f(z)$  has zero as a deficient value.

**THEOREM C [5].** Let  $f(z)$  be a non-constant entire function satisfying

$$\liminf_{r \rightarrow \infty} \frac{T(r, f)}{r} = 0.$$

Then the smallest convex set which contains the zeros of  $f(z)$  also contains the zeros of  $f'(z)$ .

**LEMMA A [5].** Let  $f(z)$  be an entire function of genus at most one. Assume that 0 is not a lacunary value of  $f(z)$  and all the zeros  $\{a_n\}$  of  $f(z)$  lie in the strip

$$\{z; |\operatorname{Re} z| \leq h\}.$$

Then

$$\operatorname{Re} \frac{f'(z)}{f(z)} = A + \sum_n \operatorname{Re} \frac{1}{z - a_n}$$

with a real constant  $A$ . Further if  $A$  is positive, zero, or negative, then for real number  $x$

$$\lim_{x \rightarrow +\infty} |f(x)| = +\infty,$$

$$\lim_{x \rightarrow +\infty} |f(x)| = \lim_{x \rightarrow -\infty} |f(x)| = +\infty, \quad \text{or} \quad \lim_{x \rightarrow -\infty} |f(x)| = +\infty$$

respectively.

LEMMA B [5]. *If  $f(z)$  is regular in the half plane  $\{z; \operatorname{Re} z > h > 0\}$  and fails to take there 0 and 1, then*

$$\log |f(z)| \leq \frac{A(1+h+|z|)^2}{(x-h)} \quad (\operatorname{Re} z = x > h),$$

where  $A$  is a positive constant.

**3. Lemmas.** In this section we shall prove the following lemmas.

LEMMA 1. *Let  $f(z)$  be an entire function. Assume that there exist four distinct finite complex numbers  $a_j$  and four strips  $S_j$  of the complex plane such that all the solutions of  $f(z) = a_j$  lie in  $S_j$  ( $j=1, \dots, 4$ ). Assume further that the four strips  $S_j$  are parallel with one another and  $S_i \cap S_j = \emptyset$  ( $i \neq j$ ). Then  $f(z)$  has at most order one.*

The proof is essentially the same as that of Theorem 4 in [5], hence omitted.

LEMMA 2. *Let  $f(z)$  be an entire function. Assume that there exist three distinct finite complex numbers  $a_j$  and three strips  $S_j$  of the complex plane such that all the solutions of  $f(z) = a_j$  lie in  $S_j$  ( $j=1, 2, 3$ ). Assume further that no two of the three strips  $S_j$  run parallel with each other. Then the order  $\rho$ , of  $f(z)$ , is finite and*

$$\rho \leq \max \left\{ \frac{\pi}{\omega_1 + \omega_2}, \frac{\pi}{\omega_2 + \omega_3}, \frac{\pi}{\omega_3 + \omega_1} \right\},$$

where  $\omega_j$  ( $j=1, 2, 3$ ) are apertures of those three angular sectors which are components of  $\mathbf{C} \setminus (S_1 \cup S_2 \cup S_3)$  and adjom.

*Proof.* We assume, without loss of generality, that

$$S_1 = \{z; |\operatorname{Im} z| \leq k\}, \quad S_2 = \{z; |\operatorname{Im}(ze^{-i\omega_2})| \leq k\},$$

$$S_3 = \{z; |\operatorname{Im}(ze^{i\omega_3})| \leq k\}, \quad \left(0 < \omega_2, \omega_3 \leq \frac{\pi}{2}, k > 0\right).$$

Let  $\varepsilon$  be an arbitrarily fixed positive number less than  $\min(\omega_2, \omega_3)$ . We choose real numbers  $\alpha, \beta$  such that

$$\{z^\beta e^{i\alpha}; \operatorname{Re} z > 0\} = \{z; \omega_2 - \varepsilon > \arg z > -\omega_3 + \varepsilon\}.$$

Let

$$F(z) = (f(z^3 e^{i\alpha}) - a_3) / (a_3 - a_2),$$

then for a suitable positive number  $h$   $F(z)$  fails to take 0 and 1 in  $\text{Re } z > h$ . Thus by Lemma B

$$(3.1) \quad \log^+ |F(z)| \leq A \frac{(1+h+|z|)^2}{(x-h)} \quad (\text{Re } z = x > h).$$

Let  $\delta$  be an arbitrarily fixed number in  $(0, \pi/2)$ . We choose a positive number  $\gamma$  satisfying  $\cos \delta - (h/r) > \gamma$  for every sufficiently large  $r$ . Then by (3.1)

$$\begin{aligned} \int_{-\delta}^{\delta} \log^+ |F(re^{i\theta})| d\theta &\leq A(1+h+r)^2 \frac{1}{r} \int_{-\delta}^{\delta} \frac{d\theta}{\cos \theta - h/r} \\ &\leq A \frac{2\delta}{\gamma} \frac{(1+h+r)^2}{r} = O(r) \end{aligned}$$

for every sufficiently large  $r$ . Thus

$$(3.2) \quad \int_{-\beta\delta+\alpha}^{\beta\delta+\alpha} \log^+ |f(re^{i\theta})| d\theta = O(r^{1/\beta}).$$

If  $\varepsilon$  is sufficiently small and  $\delta$  is sufficiently close to  $\pi/2$ , then

$$\{e^{i\theta}; \beta\delta + \alpha > \theta > -\beta\delta + \alpha\} \supset \left\{e^{i\theta}; \frac{\omega_2}{2} > \theta > \frac{-\omega_3}{2}\right\}.$$

Further if  $\varepsilon$  tends to 0 from above, then  $1/\beta$  tends to  $\pi/(\omega_2 + \omega_3)$  from above. Thus by (3.2)

$$\limsup_{r \rightarrow \infty} (\log r)^{-1} \log \left( \int_{-\omega_3/2}^{\omega_2/2} \log^+ |f(re^{i\theta})| d\theta \right) \leq \pi/(\omega_2 + \omega_3).$$

For other angular sectors we obtain similar results. Thus the order of  $f(z)$  is at most

$$\max \left\{ \frac{\pi}{\omega_1 + \omega_2}, \frac{\pi}{\omega_2 + \omega_3}, \frac{\pi}{\omega_3 + \omega_1} \right\}.$$

LEMMA 3. Let  $f(z)$  be an entire function. Assume that there exist three distinct finite complex numbers  $a_j$  and three straight lines  $L_j$  of the complex plane on which all but a finite number of the solutions of  $f(z) = a_j$  lie ( $j=1, 2, 3$ ). Then the order of  $f(z)$  is finite.

*Proof.* It is sufficient to consider only the case that the three lines  $L_j$  are distinct and parallel with one another. Indeed in other cases the assertion of Lemma 3 follows at once from Theorem 1 in [2].

We can assume, without loss of generality, that

$$L_i = \{z; \text{Re } z = h_i\} \quad (h_1 > 0, h_2 = 0, h_3 < 0),$$

and that  $f(z) \neq a_i$  for every  $z$  in  $C \setminus (\{z; |z| < 1\} \cup L_i)$  ( $i=1, 2, 3$ ). Let  $w = \phi(z) =$

$z-z^{-1}$ , then the function  $z=\phi^{-1}(w)$  maps the half plane  $\operatorname{Re} w > 0$  conformally onto the region  $\{z; \operatorname{Re} z > 0, |z| > 1\}$ . Let  $g(z)=(f(z)-a_2)/(a_3-a_2)$ , then  $g(\phi^{-1}(w))$  fails to take 0 and 1 in  $\operatorname{Re} w > 0$ . Thus by Lemma B

$$(3.3) \quad \log^+ |g(\phi^{-1}(w))| \leq \frac{A(1+|w|)^2}{\operatorname{Re} w} \quad (\operatorname{Re} w > 0).$$

If  $z$  ( $\operatorname{Re} z > 0$ ) is sufficiently large, then  $\operatorname{Re} w = \operatorname{Re}(z(1-|z|^{-2})) > \operatorname{Re}(z/2)$ . From (3.3) we thus obtain

$$\log^+ |g(z)| \leq \frac{8A|z|^2}{\operatorname{Re} z} \quad (\operatorname{Re} z > 0)$$

for sufficiently large  $z$ . Therefore

$$(3.4) \quad \log^+ |f(z)| \leq \frac{9A|z|^2}{\operatorname{Re} z} \quad (\operatorname{Re} z > 0)$$

for sufficiently large  $z$ . Similarly, we obtain

$$(3.5) \quad \log^+ |f(z)| \leq \frac{9A|z|^2}{-\operatorname{Re} z} \quad (\operatorname{Re} z < 0)$$

for sufficiently large  $z$ . Applying the essentially same method as in the proof of Theorem 4 in [5] to (3.4) and (3.5), we conclude that  $f(z)$  has at most order one. Lemma 3 is thus proved.

LEMMA 4. Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of non-zero complex numbers such that  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ , and  $n(r, a_n)$  the counting function of the sequence  $\{a_n\}$ . Assume that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty.$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{|z-a_n|^2} \geq \frac{1}{4|z|^2} n(|z|, a_n) \quad (z \neq 0).$$

*Proof.* Let  $z$  be an arbitrarily fixed non-zero complex number. Let  $|z|=r$ , then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{|z-a_n|^2} &\geq \sum_{n=1}^{\infty} \frac{1}{(r+|a_n|)^2} \\ &= \int_0^{\infty} \frac{1}{(t+r)^2} dn(t, a_n) = 2 \int_0^{\infty} \frac{1}{(t+r)^3} n(t, a_n) dt \\ &= \frac{2}{r^2} \int_0^{\infty} \frac{1}{(s+1)^3} n(rs, a_n) ds \geq \frac{2}{r^2} n(r, a_n) \int_1^{\infty} \frac{1}{(s+1)^3} ds = \frac{1}{4r^2} n(r, a_n). \end{aligned}$$

Lemma 4 is thus proved.

**4. Proof of Theorem 1.** Firstly by Lemma 1 and Lemma 2 the order of  $f(z)$  must be finite. Let us denote

$$C^* = \{a \in C; \delta(a, f) = 0\}.$$

Then by Theorem B the genus of  $f(z) - a$  is at most one for every  $a \in C^*$ . Further by the following lemma all the strips  $S_a$  ( $a \in C^*$ ) are parallel with one another.

LEMMA 5. *Let  $f(z)$  be a transcendental entire function of finite order, and  $A$  an infinite set of complex numbers containing no deficient value of  $f(z)$ . Assume that the genus of  $f(z) - a$  is at most one for every  $a \in A$ , and that for every  $a \in A$  there is a strip  $T_a$  of the complex plane in which all the solutions of  $f(z) = a$  lie. Then all the strips  $T_a$  ( $a \in A$ ) are parallel with one another.*

*Proof.* We consider the following two cases.

$$(1) \liminf_{r \rightarrow \infty} T(r, f)/r = 0. \quad (2) \liminf_{r \rightarrow \infty} T(r, f)/r \neq 0.$$

Firstly we consider the case (1). If there exist two values,  $a, b$  ( $\in A$ ) such that  $T_a$  and  $T_b$  are not parallel with each other, then by Theorem C  $f'(z)$  has at most a finite number of zeros. Therefore by (1)  $f(z)$  must be a polynomial. This is a contradiction. Thus all the strips  $T_a$  ( $a \in A$ ) are parallel with one another in this case.

Secondly we consider the case (2). Suppose that  $T_a$  and  $T_b$  are not parallel with each other for some  $a, b$  ( $\in A$ ). There exist infinitely many elements  $\{a_n\}_{n=1}^\infty$  of  $A$  and a straight line

$$L = \{te^{i\omega}; t \in R\} \quad (0 \leq \omega < \pi)$$

such that the direction of  $T_{a_n}$  approaches that of  $L$  as  $n \rightarrow \infty$ . Then using Lemma A and Lindelöf-Iversen-Gross' theorem [7] we deduce that

$$\lim_{r \rightarrow \infty} |f(re^{i\theta})| = +\infty$$

uniformly for  $|\theta - \omega + \pi/2| \leq \theta^* < \pi/2$  or for  $|\theta - \omega - \pi/2| \leq \theta^* < \pi/2$ , where  $\theta^*$  is an arbitrarily fixed number in  $(0, \pi/2)$ . Thus  $a$ -points or  $b$ -points of  $f(z)$  must lie in a half strip. Therefore by Theorem A and (2) we conclude that  $a$  or  $b$  is a deficient value of  $f(z)$ . This is a contradiction. Thus all the strips  $T_a$  ( $a \in A$ ) are parallel with one another in this case. Lemma 5 is thus proved.

By Lemma 5 we can assume, without loss of generality, that all the strips  $S_a$  ( $a \in C^*$ ) run parallel with the imaginary axis. We now consider a sequence  $\{x_n\}_{n=-\infty}^{+\infty}$  of complex numbers such that  $\text{Re}(x_{n+1}) - \text{Re}(x_n) \geq 3k$ , and that  $f(x_n) = w_n \in C^*$  ( $n = 0, \pm 1, \pm 2, \dots$ ). Then  $S_{w_i} \cap S_{w_j} = \emptyset$  ( $i \neq j$ ). From Lemma A

$$(4.1) \quad \operatorname{Re} \frac{f'(z)}{f(z)-w_n} = A_n + \sum_m \operatorname{Re} \frac{1}{z-a_m},$$

where  $A_n$  is a real constant and  $\{a_m\}$  the  $w_n$ -points of  $f(z)$  ( $n=0, \pm 1, \pm 2, \dots$ ).  
 Next we prove

LEMMA 6. *The following two cases do not occur.*

- (1)  $A_n > 0, A_m < 0$  ( $n > m$ ).
- (2)  $A_n = A_m = 0$  ( $n \neq m$ ).

*Proof.* Firstly we assume that the case (1) occurs. Let us write  $A_n = A, A_m = -B, w_n = a, w_m = b, \operatorname{Re}(x_n) + k = r, \operatorname{Re}(x_m) - k = s$ . Let

$$L = \{a + t(a-b); t \in (0, \infty)\}.$$

By (4.1) and (1) we obtain

$$(4.2) \quad \begin{aligned} &\arg(f(x+iy)-a) - \arg(f(x)-a) \\ &= \int_0^y \operatorname{Re} \frac{f'(x+it)}{f(x+it)-a} dt \geq Ay \end{aligned}$$

for every  $(x, y) \in (r, \infty) \times \mathbf{R}$ . Thus from (4.2)

$$(4.3) \quad \{\arg(f(x+iy)-a); y \in [0, 2\pi/A]\} \supset [0, 2\pi]$$

for every  $x (> r)$ . On the other hand, Lemma A and Lindelöf-Iversen-Gross' theorem [7] imply

$$(4.4) \quad \lim_{x \rightarrow +\infty} |f(x+iy)| = +\infty$$

uniformly for  $y$  ( $0 \leq y \leq 2\pi/A$ ). By (4.3) and (4.4) we see that for some point  $z_1$  in  $\operatorname{Re} z > r, f(z_1)$  lies on  $L$ . Hence using the same argument as in the proof of Lemma 9 in [5], we can deduce that every sufficiently large value on  $L$  can be taken by  $f(z)$  in the half plane  $\operatorname{Re} z > r$ . For completeness we shall give a proof of this assertion.

Let  $v_1 = f(z_1)$ , and  $E(w, v_1)$  be the regular element of  $f^{-1}(w)$  with center  $v_1$  which satisfies  $E(v_1, v_1) = z_1$ . Put  $v_1 = a + e^{i\alpha}t_1$  with real constants  $\alpha, t_1$  ( $t_1 > 0$ ). We continue  $E(w, v_1)$  analytically along the segment  $\{a + te^{i\alpha}; t_1 \leq t < t_2 < \infty\}$ . Put

$$Z(t) = E(a + e^{i\alpha}(t_1+t), v_1), \quad 0 \leq t < (t_2 - t_1).$$

Then

$$(4.5) \quad f(Z(t)) = a + e^{i\alpha}(t_1+t).$$

If  $Z(t_*)$  ( $0 \leq t_* < t_2 - t_1$ ) is contained in the half plane  $\operatorname{Re} z > r$ , then  $Z(t)$  is differentiable at  $t_*$  and from (4.5)

$$(4.6) \quad f'(Z(t_*))Z'(t_*) = e^{i\alpha}.$$

From (4.1) and (1)

$$(4.7) \quad \operatorname{Re} \frac{f'(Z(t_*))}{f(Z(t_*))-a} \geq A > 0.$$

By (4.5), (4.6) and (4.7) we conclude

$$\operatorname{Re} Z'(t_*) > 0.$$

Therefore  $Z(t)$  must be contained in  $\operatorname{Re} z > r$  for every  $t (\in [0, t_2 - t_1])$ . If this analytic continuation defines a transcendental singularity at the point  $a + t_2 \exp(i\alpha)$ , then the path  $\Gamma = \{Z(t); 0 \leq t < (t_2 - t_1)\}$  must be an asymptotic path of  $f(z)$  and as  $z$  tends to infinity along this path  $\Gamma$ ,  $f(z)$  approaches the value  $a + t_2 \exp(i\alpha)$ . Thus, by Lindelöf-Iversen-Gross' theorem [7], we deduce that for real number  $x$

$$\lim_{x \rightarrow +\infty} f(x) = a + t_2 e^{i\alpha}.$$

This contradicts (4.4). Thus  $E(w, v_1)$  can be continued analytically along the half line  $L$  up to infinity. Hence we conclude that every sufficiently large value on  $L$  can be taken by  $f(z)$  in the half plane  $\operatorname{Re} z > r$ .

Similarly, every sufficiently large value on  $L$  can be taken by  $f(z)$  in the half plane  $\operatorname{Re} z < s$ . Some sufficiently large value on  $L$  is in  $C^*$ . Thus we have a contradiction, since every strip  $S_a (a \in C^*)$  is parallel with the imaginary axis. Thus (1) cannot occur.

We next show that the case (2) cannot occur. Indeed, if otherwise, then by (4.1) and (2)  $f'(z)$  fails to take 0 in  $C$ . Further by Lemma 1 the order of  $f(z)$  is at most one. Thus  $f(z) = a + b \cdot \exp(Az)$  with constants  $a, b, A$ . On the other hand by Lemma A and (2) we deduce that for real number  $x$

$$\lim_{x \rightarrow +\infty} |f(x)| = \lim_{x \rightarrow -\infty} |f(x)| = +\infty.$$

However the function  $f(z) = a + b \cdot \exp(Az)$  does not satisfy this asymptotic behavior. Thus (2) cannot occur. Lemma 6 is thus proved.

By Lemma 6 we have only the following two possibilities.

- 1) There exists an integer  $N$  such that  $A_n > 0$  for every  $n \leq N$ .
- 2) There exists an integer  $N$  such that  $A_n < 0$  for every  $n \geq N$ .

In each case by (4.1)  $f'(z)$  fails to take 0 in  $C$ . Further by Lemma 1 the order of  $f(z)$  is at most one. Thus  $f(z) = a + b \cdot \exp(Az)$  with constants  $a, b, A, bA \neq 0$ . The proof of Theorem 1 is now complete.

**5. Proof of Theorem 2.** Let  $L$  be the real axis. If  $a (\in \mathbf{R})$  is not a Picard exceptional value of  $f(z)$ , then  $S_a \supset L$  or  $S_a$  is at right angles to  $L$ . Thus by the theorem in [4] we see that  $S_a$  is at right angles to  $L$  for every sufficiently large  $a (\in \mathbf{R})$ . Further by Theorem 4 in [5] we see that the order of  $f(z)$  is at most one.

If there exist two real numbers  $a, b$  satisfying  $S_a \supset L, S_b \supset L$ , then by



Lemma A and Lindelöf-Iversen-Gross' theorem [7] we deduce that

$$\lim_{r \rightarrow +\infty} |f(re^{i\theta})| = +\infty$$

uniformly for  $|\theta - \pi/2| \leq \theta^* < \pi/2$  and for  $|\theta + \pi/2| \leq \theta^* < \pi/2$ , where  $\theta^*$  is an arbitrarily fixed number in  $(0, \pi/2)$ . Hence every sufficiently large real number must be a Picard exceptional value of  $f(z)$ . This is a contradiction. Thus  $S_a$  is at right angles to  $L$  for every  $a (\in \mathbf{R})$  with at most one exception.

We now choose a sequence  $\{x_n\}_{n=\pm\infty}$  of real numbers such that  $\text{Re}(x_{n+1} - x_n) \geq 3k$  and that  $S_{w_n} (w_n = f(x_n))$  is at right angles to  $L (n=0, \pm 1, \pm 2, \dots)$ . We define  $A_n$  by (4.1). Then as the proof of Theorem 1 we have only the following two possibilities.

- 1) There exists an integer  $N$  such that  $A_n > 0$  for every  $n \leq N$ .
- 2) There exists an integer  $N$  such that  $A_n < 0$  for every  $n \geq N$ .

In each case by (4.1)  $f'(z)$  fails to take 0 in  $\mathbf{C}$ . Thus  $f(z) = a + b \cdot \exp(Az)$  with real constants  $a, b, A, bA \neq 0$ . Theorem 2 is thus proved.

**6. Proof of Theorem 4.** Firstly we prove that the number of directions of the straight lines  $L_{w,1} (w \in G)$  is finite. Indeed, if otherwise, by Lemma 2 the order of  $f(z)$  must be finite. Therefore Theorem 1 in [3] implies that  $f(z)$  has at most a finite number of deficient values. Thus without loss of generality we can assume that  $G$  contains no deficient value of  $f(z)$ . Hence by Theorem B, the genus of  $f(z) - a$  is at most one for every  $a (\in G)$ . Thus by Lemma 5 we conclude that all the straight lines  $L_{w,1} (w \in G)$  are parallel with one another. This is a contradiction.

From the above result it is easily seen that there is an open subset  $G^*$  of  $G$  such that all the straight lines  $L_{w,1} (w \in G^*)$  are parallel with one another. We can assume, without loss of generality, that they are also parallel with the imaginary axis. Further we assume that there exists a points  $\alpha (\in G^*)$  such that  $\alpha$  is not a Picard exceptional value of  $f(z)$ , and that  $\{L_{\alpha,i}\}_{i=1}^n$  are  $n$  distinct straight lines each of which carries at least one  $\alpha$ -point of  $f(z)$ .

We choose  $n+1$   $\alpha$ -points  $\{z_i\}_{i=1}^{n+1}$  such that  $z_1$  and  $z_2$  lie on one of  $\{L_{\alpha,i}\}_{i=1}^n$ , say  $L_{\alpha,1}$ , and that  $z_i$  lies on  $L_{\alpha,i-1} (i=3, \dots, n+1)$ . By the assumption on  $\alpha$  it is easily seen that  $f'(z_i) \neq 0 (i=1, \dots, n+1)$ . Thus there exist neighborhoods  $U_i$  of  $z_i (i=1, \dots, n+1)$  and a neighborhood  $A$  of  $\alpha$  satisfying the following conditions.

- 1)  $f(z)$  is univalent in  $U_i \quad (i=1, 2, \dots, n+1)$ .
- 2)  $f(U_i) = A \quad (i=1, 2, \dots, n+1)$ .
- 3)  $A \subset G^*$ .
- 4)  $\{\text{Re } z; z \in U_1\} \cap \{\text{Re } z; z \in U_i\} = \emptyset \quad (i=3, \dots, n+1),$   
 $\{\text{Re } z; z \in U_2\} \cap \{\text{Re } z; z \in U_i\} = \emptyset \quad (i=3, \dots, n+1),$   
 $\{\text{Re } z; z \in U_i\} \cap \{\text{Re } z; z \in U_j\} = \emptyset \quad (i, j=3, \dots, n+1).$

If  $w \in A$ , then by 1) and 2) there exists one  $w$ -point of  $f(z)$  in each  $U_i$ , which is denoted by  $p_i$  ( $i=1, 2, \dots, n+1$ ). By 3) and 4) we deduce that  $\operatorname{Re} p_1 = \operatorname{Re} p_2$ . Put  $F=(f|_{U_2})^{-1} \circ (f|_{U_1})$ , then  $F$  is holomorphic in  $U_1$  and  $F(z)-z$  is purely imaginary for every  $z$  in  $U_1$ . Thus  $F(z) \equiv z+c$  in a neighborhood of  $z_1$  with a constant  $c$ . Hence we have

$$(6.1) \quad f(z+c) \equiv f(z).$$

Therefore  $f(z)=Q(\exp Bz)$ , where  $Q$  is a regular function in  $C \setminus \{0\}$  and  $B$  is a non-zero constant. From the assumption of Theorem 4, it is easily seen that  $Q$  is a rational function.

We may assume that  $Q(w)$  has no factorization of form  $Q(w)=P(w^N)$ , where  $P$  is a rational function and  $N$  is an integer ( $\geq 2$ ). Let  $m$  be the order of  $Q(w)$ . Then by the assumption of  $\alpha$  there are  $m$  distinct roots  $\{a_i\}$  of  $Q(w)=\alpha$ , which lie on  $n$  distinct circles whose center is at the origin. If  $m > n$ , then there exist two of  $\{a_i\}$ , say  $a_i, a_j$ , such that  $|a_i|=|a_j|$ . Then using essentially the same method which is used in showing (6.1), we easily deduce that

$$Q(w) \equiv Q(we^{i\theta}) \quad (\theta = \arg a_i - \arg a_j).$$

Thus  $Q(w)=P(w^N)$  with a rational function  $P$  and an integer  $N$  ( $\neq 0, \pm 1$ ). This is a contradiction. Thus  $m=n$ . Theorem 4 is thus proved.

**7. Proof of Theorem 3.** By Lemma 3 the order of  $f(z)$  must be finite. Hence Theorem 1 in [3] and the Denjoy-Carleman-Ahlfors Theorem imply that  $f(z)$  has at most a finite number of deficient values and asymptotic values. Thus we may assume that  $G$  contains neither deficient nor asymptotic values of  $f(z)$ . By Theorem B the genus of  $f(z)-a$  is at most one for every  $a$  ( $\in G$ ). By Lemma 5 all the straight lines  $L_a$  ( $a \in G$ ) are parallel with the imaginary axis.

Let us write for every  $a$  ( $\in G$ )

$$(7.1) \quad h(a) = \operatorname{Re} x \quad (x \in L_a),$$

$$(7.2) \quad \operatorname{Re} \frac{f'(z)}{f(z)-a} = A(a) + \sum_n \operatorname{Re} \frac{1}{z-a_n},$$

where  $A(a)$  is a real constant and  $\{a_n\}$  the  $a$ -points of  $f(z)$ . (7.2) follows from Lemma A.

Next we show the following Lemma 7 and Lemma 8. Which are modifications of Lemma 5 and Lemma 7 in [5].

LEMMA 7. *If  $h(a) < h(b)$  then  $A(a) < 0$  or  $A(b) > 0$  ( $a, b \in G$ ).*

*Proof.* Suppose that  $A(a) \geq 0, A(b) \leq 0$ . Let  $\{a_n\}$  be the  $a$ -points of  $f(z)$  which lie on  $L_a \setminus \{0\}$ . We choose a positive number  $R$  such that  $f(z) \neq a$  for every  $z$  ( $\operatorname{Re} z > R$ ). Put  $\varepsilon = (h(b) - h(a))/3, S = \{z; h(a) + \varepsilon \leq \operatorname{Re} z \leq R\}$ ,

$$\phi_a(z) = \operatorname{Re} \frac{f'(z)}{f(z) - a}.$$

Then by (7.2)

$$(7.3) \quad \phi_a(z) > 0 \quad (\operatorname{Re} z > R)$$

and

$$(7.4) \quad \phi_a(z) = A(a) + \sum_{n=1}^{\infty} \operatorname{Re} \frac{1}{z - a_n} + O\left(\frac{1}{z^2}\right) \quad (z \in S).$$

By (7.4) and Lemma 4 it is easily seen that

$$(7.5) \quad \phi_a(z) > 0$$

for every sufficiently large  $z (\in S)$ . From (7.3) and (7.5)  $f'(z)$  has at most a finite number of zeros in  $\operatorname{Re} z \geq h(a) + \varepsilon$ .

Similarly  $f'(z)$  has at most a finite number of zeros in  $\operatorname{Re} z \leq h(b) - \varepsilon$ . Thus  $f'(z)$  has at most a finite number of zeros in  $C$ . Therefore

$$(7.6) \quad f(z) = P(z) \cdot e^{\alpha z} + \beta$$

with constants  $\alpha, \beta$  and a polynomial  $P(z)$ . The assumption “ $A(a) \geq 0$  and  $A(b) \leq 0$ ”, Lemma A and Lindelöf-Iversen-Gross’ theorem [7] imply that

$$\lim_{r \rightarrow +\infty} |f(re^{i\theta})| = +\infty$$

uniformly for  $|\theta| \leq \theta^* < \pi/2$  and for  $|\theta - \pi| \leq \theta^* < \pi/2$ , where  $\theta^*$  is an arbitrarily fixed number in  $(0, \pi/2)$ . This asymptotic behavior contradicts (7.6). Lemma 7 is thus proved.

LEMMA 8. For every  $a (\in G)$  we have

$$\overline{(f(\bar{z} + h(a)) - a)} = (\cdot f(-z + h(a)) - a) \exp(2A(a)z + iB(a)) \overline{(P_a(\bar{z})/P_a(-z))}$$

with a polynomial  $P_a(z)$  and a real constant  $B(a)$ .

*Proof.* Let  $\{a_n\}$  be the  $a$ -points of  $f(z)$  which lie on  $L_a$  and  $\{a'_m\}$  the  $a$ -points of  $f(z)$  other than  $\{a_n\}$ . Let us write

$$(7.7) \quad f(z + h(a)) - a = z^p e^{Az + B} \prod_{a_n \neq h(a)} E\left(\frac{z}{a_n - h(a)}, 1\right) \prod_m E\left(\frac{z}{a'_m - h(a)}, 1\right),$$

$$(7.8) \quad P_a(z) = \begin{cases} \prod_m \left(1 - \frac{z}{a'_m - h(a)}\right) & \text{if } a'_m \text{ exists,} \\ 1 & \text{if } a'_m \text{ does not exist,} \end{cases}$$

$$(7.9) \quad g_a(z) = (f(z + h(a)) - a) / P_a(z),$$

where  $E(z, 1)$  is the Weierstrass primary factor of genus 1. Since all the zeros of  $g_a(z)$  are purely imaginary, by Lemma 5 in [5]

$$(7.10) \quad \overline{g_a(\bar{z})} = g_a(-z) \cdot \exp(\widetilde{A(a)}z + iB(a))$$

with a real constant  $B(a)$ , where

$$(7.11) \quad \widetilde{A(a)} = 2 \operatorname{Re} \left( A + \sum_m \frac{1}{a'_m - h(a)} \right).$$

From (7.7)

$$(7.12) \quad \operatorname{Re} \frac{f'(z)}{f(z) - a} = \widetilde{A(a)}/2 + \sum_n \operatorname{Re} \frac{1}{z - a_n} + \sum_m \operatorname{Re} \frac{1}{z - a'_m}.$$

By (7.2) and (7.12)

$$(7.13) \quad 2A(a) = \widetilde{A(a)}.$$

From (7.9), (7.10) and (7.13) we have the desired result.

Next we prove

LEMMA 9. *Let  $a, b \in G$ . Assume that  $h(a) > h(b)$ ,  $A(a)A(b) > 0$ . If  $A(a) > 0$ , then there exists a neighborhood  $E (\subset G)$  of  $a$ , such that  $h(p)$  is continuous in  $E$  and that  $A(p) > 0$  for every  $p$  in  $E$ . If  $A(a) < 0$ , then there exists a neighborhood  $E' (\subset G)$  of  $b$ , such that  $h(p)$  is continuous in  $E'$  and that  $A(p) < 0$  for every  $p$  in  $E'$ .*

*Proof.* In what follows we assume that  $A(a) > 0$ . When  $A(a) < 0$ , we only have to consider the function  $f(-z)$  instead of  $f(z)$ .

Let  $\varepsilon$  be an arbitrarily fixed positive number less than  $(h(a) - h(b))/3$  such that the set

$$C = \{z; 0 < |h(a) - (\operatorname{Re} z)| \leq 2\varepsilon\}$$

contains no  $a$ -point of  $f(z)$ . Using the same argument as in the proof of Lemma 7, we deduce that

$$(7.14) \quad \operatorname{Re} \frac{f'(z)}{f(z) - a} \geq A(a),$$

$$(7.15) \quad \operatorname{Re} \frac{f'(z)}{f(z) - b} \geq A(b)$$

for every  $z$  satisfying  $h(a) + \varepsilon \leq \operatorname{Re} z \leq h(a) + 2\varepsilon$  and  $|\operatorname{Im} z| \geq R_0$ , where  $R_0$  is a suitable positive number.

Let  $\gamma = |a - b|$  and  $\delta$  a positive number satisfying

$$(7.16) \quad \log \frac{\gamma + \delta}{\gamma - \delta} \leq \varepsilon A(b).$$

Put

$$(7.17) \quad \begin{aligned} L &= \{z; \operatorname{Re} z = h(a) + \varepsilon\}, \\ S &= \{z \in L; |a - f(z)| < \delta\}, \end{aligned}$$

$$(7.18) \quad T = \{z \in L; |a - f(z)| \geq \delta\}.$$

Let  $z \in S$  and  $|\operatorname{Im} z| \geq R_0$ , then by (7.15)

$$(7.19) \quad \begin{aligned} &\log |f(z + \varepsilon) - b| - \log |f(z) - b| \\ &= \int_0^\varepsilon \operatorname{Re} \frac{f'(x + t + iy)}{f(x + t + iy) - b} dt \geq A(b)\varepsilon \quad (z = x + iy). \end{aligned}$$

By (7.16), (7.17) and (7.19)

$$\begin{aligned} \log |f(z + \varepsilon) - b| &\geq \log |f(z) - b| + A(b)\varepsilon \\ &\geq \log(\gamma - \delta) + A(b)\varepsilon \geq \log(\gamma + \delta). \end{aligned}$$

Thus  $|f(z + \varepsilon) - b| \geq \gamma + \delta$ . Hence

$$(7.20) \quad |f(z + \varepsilon) - a| \geq \delta \quad (z \in S, |\operatorname{Im} z| \geq R_0),$$

Similarly, by (7.14) and (7.18)

$$(7.21) \quad |f(z + \varepsilon) - a| \geq \delta \quad (z \in T, |\operatorname{Im} z| \geq R_0).$$

Since  $f(z) - a$  does not vanish on  $\operatorname{Re} z = h(a) + 2\varepsilon$ , by (7.20) and (7.21) we see that for a suitable positive number  $\eta$

$$(7.22) \quad |f(z) - a| \geq \eta \quad (\operatorname{Re} z = h(a) + 2\varepsilon).$$

By Lemma 8 and (7.22)

$$(7.23) \quad |f(-z + h(a)) - a| \geq \eta e^{-4\varepsilon A(a)} (|P_a(-z)/P_a(\bar{z})|) \quad (\operatorname{Re} z = 2\varepsilon).$$

Since  $f(z) - a$  does not vanish on  $\operatorname{Re} z = h(a) - 2\varepsilon$ , by (7.22) and (7.23) we conclude that for a suitable positive number  $\zeta$

$$(7.24) \quad |f(z) - a| \geq \zeta \quad (\operatorname{Re} z = h(a) \pm 2\varepsilon).$$

Let  $\{a_i\}$  be the distinct  $a$ -points of  $f(z)$  outside  $\bar{C}$  and  $d$  a positive number satisfying the following conditions.

- 1)  $D_i \cap D_j = \emptyset$  ( $i \neq j$ ), where  $D_i = \{z; |z - a_i| \leq d\}$ .
- 2)  $D_i \cap \bar{C} = \emptyset$ .

Put

$$(7.25) \quad m = \min_{z \in \bigcup_i D_i} |f(z) - a|.$$

Let  $r$  be an arbitrarily fixed positive number less than  $\min(\zeta, m)$  and  $E = \{w; |w - a| < r\} \subset G$ . By (7.24) and (7.25)

$$(7.26) \quad f^{-1}(E) \cap (\partial \bar{C} \cup (\bigcup_i \partial D_i)) = \phi.$$

Since  $E$  contains no asymptotic value of  $f(z)$ , each component of  $f^{-1}(E)$  contains at least one  $a$ -point of  $f(z)$ . Thus by (7.26)

$$(7.27) \quad f^{-1}(E) \subset (\bar{C} \cup (\bigcup_i D_i)).$$

By (7.27) for every  $p$  in  $E$

$$(7.28) \quad |h(a) - h(p)| \leq 2\varepsilon.$$

Since  $\varepsilon$  can be taken arbitrarily small,  $h(p)$  is continuous at the point  $a$ . By (7.28)

$$(7.29) \quad h(p) > h(b) \quad (p \in E).$$

By Lemma 7 and (7.29)

$$(7.30) \quad A(p) > 0 \quad (p \in E).$$

Applying the same method to (7.29) and (7.30), we conclude that  $h(p)$  is continuous in  $E$ . Lemma 9 is thus proved.

Let  $H = \{p \in G; A(p) = 0\}$ . If  $p, q \in H$ , then by Lemma 7  $L_p = L_q$ . Thus  $H \subset f(L_p)$  ( $p \in H$ ), or  $H = \phi$ . Hence  $G \setminus H$  has infinitely many elements. Thus there exist two elements  $a, b$  ( $\in G$ ) satisfying  $A(a)A(b) > 0$ . There are the following two cases.

- 1)  $h(a) \neq h(b)$ .
- 2)  $h(a) = h(b)$ .

In the case 1), by Lemma 9, we easily see that for some two points  $\alpha, \beta$  in  $E$ , or in  $E'$ ,  $h(\alpha) = h(\beta)$  and  $A(\alpha)A(\beta) > 0$ . Thus in both cases there exist two elements  $\alpha, \beta \in G$  such that  $h(\alpha) = h(\beta)$  and  $A(\alpha)A(\beta) > 0$ .

In what follows we assume, without loss of generality, that  $\alpha = 0, \beta = 1, h(\alpha) = h(\beta) = 0$ . From Lemma 8 we have

$$\begin{aligned} \overline{f(\bar{z})} &= f(-z)e^{2A(0)z + iB(0)\overline{(P_0(\bar{z})/P_0(-z))}}, \\ \overline{f(\bar{z}) - 1} &= (f(-z) - 1)e^{2A(1)z + iB(1)\overline{(P_1(\bar{z})/P_1(-z))}}, \end{aligned}$$

Put

$$\begin{aligned} X(-z) &= e^{iB(0)\overline{(P_0(\bar{z})/P_0(-z))}}, \\ Y(-z) &= e^{iB(1)\overline{(P_1(\bar{z})/P_1(-z))}}, \\ A &= 2A(0), \quad B = 2A(1). \end{aligned}$$

Then

$$f(z)(X(z)e^{-Az} - Y(z)e^{-Bz}) = 1 - Y(z)e^{-Bz}.$$

Since  $B \neq 0$ , we easily have

$$(7.31) \quad f(z) = (e^{Bz} - Y(z)) / (X(z)e^{(B-A)z} - Y(z)).$$

We now consider the following two cases.

$$(1) \quad A=B. \quad (2) \quad A \neq B.$$

Firstly we consider the case (1). There are the following two subcases.

(1.1)  $X(z)$  and  $Y(z)$  are both constants.

(1.2)  $X(z)$  or  $Y(z)$  is not a constant.

Case (1.1). In this case, the assertion of Theorem 3 follows at once from (7.31) and (1).

Case (1.2). For instance, we assume that  $X(z)$  is not a constant. Another case, when  $Y(z)$  is not a constant, can be treated by the same method.

Let  $p \in G$ , and  $\{z_n\}$  be the  $p$ -points of  $f(z)$ . Put

$$F(z, p) = pX(z) + (1-p)Y(z).$$

Then by (7.31) and (1)

$$\exp(Bz_n) = F(z_n, p)$$

for sufficiently large  $n$ . Thus

$$(7.32) \quad |F(z_n, p)| = \exp(B \operatorname{Re} z_n) = \exp(B \cdot h(p))$$

for sufficiently large  $n$ . Since  $F(z, p)$  is regular at  $z = \infty$ , by (7.32)

$$(7.33) \quad |F(z, p)| = |F(\infty, p)| \quad (z \in L_p),$$

$$(7.34) \quad L_p = \{z; \operatorname{Re} z = (\log |F(\infty, p)|) / B\}.$$

Since  $X(z)$  and  $Y(z)$  have no common pole, any pole of  $X(z)$  must be also a pole of  $F(z, p)$  for every  $p$  in  $G \setminus \{0\}$ . Let  $t_0$  be a fixed pole of  $X(z)$ . Then (7.33) and Schwarz' reflection principle imply that  $F(z, p)$  vanishes at the point  $t_0 - 2((\operatorname{Re} t_0) - h(p))$ .

Put

$$c(p) = t_0 - 2((\operatorname{Re} t_0) - h(p)).$$

$F(\infty, p)$  is an analytic function of  $p$ . Hence there exists a point  $x$  in  $G \setminus \{0, 1\}$  satisfying

$$(7.35) \quad |F(\infty, x)| = |F(\infty, 1)|.$$

By (7.34) and (7.35) we have  $c(x) = c(1)$ . Thus

$$xX(c(1)) + (1-x)Y(c(1)) = xX(c(x)) + (1-x)Y(c(x)) = 0,$$

$$1 \cdot X(c(1)) + 0 \cdot Y(c(1)) = X(c(1)) = 0.$$

Hence  $(1-x)Y(c(1)) = 0$ . Since  $X(z)$  and  $Y(z)$  have no common pole,  $Y(c(1)) \neq 0$ . Thus  $x = 1$ . This is a contradiction. Thus the case (1.2) cannot occur.

Secondly, we consider the case (2). In this case by (7.31) we easily conclude

that  $B/(B-A)$  must be an integer. Put  $q=B/(B-A)$ . Then from (7.31)

$$(7.36) \quad f(z) = (e^{q(B-A)z} - Y(z)) / (X(z)e^{(B-A)z} - Y(z)).$$

Let  $\{a_n\}$  be the zeros of  $X(z) \cdot \exp((B-A)z) - Y(z)$ . From (7.36)

$$e^{(B-A)a_n} = Y(a_n) / X(a_n), \quad e^{q(B-A)a_n} = Y(a_n).$$

Thus

$$(Y(a_n) / X(a_n))^q = Y(a_n).$$

Therefore

$$(7.37) \quad Y(z)^{q-1} \equiv X(z)^q.$$

By (7.36) and (7.37),  $q$  cannot be 0 or 1. Therefore  $X(z)$  and  $Y(z)$  are both constants, since  $X(z)$  and  $Y(z)$  have no common pole. Let us write  $X(z) \equiv x$ ,  $Y(z) \equiv y$ .

From (7.36) and (7.37)

$$(7.38) \quad f(z) = (e^{q(B-A)z} - y) / (xe^{(B-A)z} - y),$$

$$(7.39) \quad y^{q-1} = x^q.$$

If  $q \neq 2, 1, 0, -1$ , then the order of the rational function

$$Q(w) = (w^q - y) / (xw - y)$$

is at least two. In this case, by the same method in the proof of Theorem 4, the function  $f(z) = Q(\exp(B-A)z)$  cannot fulfill the assumption of Theorem 3. Thus this case cannot occur. Hence  $q = 2$  or  $-1$ .

By (7.38) and (7.39) we have the following results.

(1) If  $q = 2$ , then

$$f(z) = (e^{(B-A)z} + x) / x.$$

(2) If  $q = -1$ , then

$$f(z) = (e^{-(B-A)z}) / (-y).$$

The proof of Theorem 3 is now complete.

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