

CHARACTERIZING A CLASS OF TOTALLY REAL SUBMANIFOLDS OF S^6 BY THEIR SECTIONAL CURVATURES

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Abstract. The first author introduced in a previous paper an important Riemannian invariant for a Riemannian manifold, namely take the scalar curvature function and subtract at each point the smallest sectional curvature at that point. He also proved a sharp inequality for this invariant for submanifolds of real space forms. In this paper we study totally real submanifolds in the nearly Kähler six-sphere that realize the equality in that inequality. In this way we characterize a class of totally real warped product immersions by one equality involving their sectional curvatures.

1. Introduction. In [C], the first author gives a general best possible inequality between the main intrinsic invariants of a submanifold M^n in a Riemannian space form $\tilde{M}^m(c)$, namely its sectional curvature function K and its scalar curvature function τ , and the main extrinsic invariant, namely its mean curvature function $\|H\|$, H being the mean curvature vector field of M in \tilde{M} . It is convenient to define a Riemannian invariant δ_M of M^n by

$$\delta_M(p) = \tau(p) - \inf K(p),$$

where $\inf K$ is the function assigning to each $p \in M^n$ the infimum of $K(\pi)$, where π runs over all planes in $T_p M$ and τ is defined by $\tau = \sum_{i < j} K(e_i \wedge e_j)$. The inequality can be written as follows.

$$(1.1) \quad \delta_M \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2} (n+1)(n-2)c.$$

He then started to investigate those submanifolds, with dimension $n \geq 3$, for which the above inequality actually becomes an equality, i.e. submanifolds which satisfy

$$(1.2) \quad \delta_M = \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2} (n+1)(n-2)c.$$

For such submanifolds, a distribution can be defined by

$$\mathcal{D}(p) = \{X \in T_p M \mid (n-1)h(X, Y) = n\langle X, Y \rangle H, \forall Y \in T_p M\}.$$

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If the dimension of $\mathcal{D}(p)$ is constant, it is shown in [C] that the distribution \mathcal{D} is completely integrable.

In this paper, we investigate 3-dimensional totally real submanifolds in the nearly Kähler 6-sphere $S^6(1)$. Since such a submanifold is always minimal (cf. [E3]), we get

$$(1.3) \quad \delta_M \leq 2.$$

When M has constant scalar curvature (τ is constant), a complete classification of submanifolds satisfying the equality in (1.3) has been obtained in [CDVV1]. Here, we will investigate those totally real 3-dimensional submanifolds in $S^6(1)$ which satisfy:

- (1) $\delta_M = 2$,
- (2) the dimension of the distribution \mathcal{D} is constant (and hence it is a completely integrable distribution),
- (3) the distribution \mathcal{D}^\perp is also integrable.

We will relate submanifolds satisfying the above conditions to minimal (non-totally geodesic) totally real immersions of surfaces N^2 into $S^6(1)$ whose ellipse of curvature is a circle. The ellipse of curvature of a surface at a point p is the set $\{h(u, u) \mid u \in T_p M, \|u\| = 1\}$ in the normal space, where h is the second fundamental form. It is shown in [BVW] that every such immersion is linearly full in a totally geodesic S^5 . An alternative proof of this will be given in Section 5. Other characterizations will be given below. The Main Theorem we prove here is:

MAIN THEOREM. *Let $f : M^2 \rightarrow S^6(1)$ be a minimal (non-totally geodesic) totally real immersion in $S^6(1)$ whose ellipse of curvature is a circle. Then M^2 is linearly full in a totally geodesic S^5 . Let N be a unit vector perpendicular to this S^5 . Then*

$$(1.4) \quad x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times M^2 \rightarrow S^6(1), (t, p) \mapsto \sin(t)N + \cos(t)f(p)$$

is a totally real immersion which satisfies the equality in (1.3). Conversely, every totally real (non-totally geodesic) immersion of M^3 into $S^6(1)$ satisfying

- (1) $\delta_M = 2$,
 - (2) the dimension of \mathcal{D} is constant,
 - (3) \mathcal{D}^\perp is an integrable distribution,
- can be locally obtained in this way.*

2. The nearly Kähler structure on $S^6(1)$. We give a brief explanation of how the standard nearly Kähler structure on $S^6(1)$ arises in a natural manner from Cayley multiplication. For elementary facts about the Cayley numbers and their automorphism group G_2 , we refer the reader to Section 4 of [W] and to [HL].

The multiplication on the Cayley numbers \mathcal{O} may be used to define a vector crossproduct on the purely imaginary Cayley numbers \mathbf{R}^7 using the formula

$$(2.1) \quad u \times v = \frac{1}{2}(uv - vu),$$

while the standard inner product on \mathbf{R}^7 is given by

$$(2.2) \quad \langle u, v \rangle = -\frac{1}{2}(uv + vu).$$

It is now elementary to show that

$$(2.3) \quad u \times (v \times w) + (u \times v) \times w = 2\langle u, w \rangle v - \langle u, v \rangle w - \langle w, v \rangle u,$$

and that the triple scalar product $\langle u \times v, w \rangle$ is skew symmetric in u, v, w .

Conversely, Cayley multiplication of \mathcal{O} is given in terms of the vector crossproduct and the inner product by

$$(2.4) \quad (r + u)(s + v) = rs - \langle u, v \rangle + rv + su + (u \times v), \quad r, s \in \text{Re } \mathcal{O}, \quad u, v \in \text{Im } \mathcal{O}.$$

In view of (2.1), (2.2) and (2.4), it is clear that the group G_2 of automorphisms of \mathcal{O} is precisely the group of isometries of \mathbf{R}^7 which preserve the vector crossproduct.

An ordered orthonormal basis $\{e_1, \dots, e_7\}$ of \mathbf{R}^7 is said to be *canonical* if

$$(2.5) \quad e_3 = e_1 \times e_2, \quad e_5 = e_1 \times e_4, \quad e_6 = e_2 \times e_4, \quad e_7 = e_3 \times e_4.$$

For example, the standard basis of \mathbf{R}^7 is canonical. Moreover, if e_1, e_2, e_4 are mutually orthogonal unit vectors with e_4 orthogonal to $e_1 \times e_2$, then e_1, e_2, e_4 determine a unique canonical basis $\{e_1, \dots, e_7\}$ and (\mathbf{R}^7, \times) is generated by e_1, e_2, e_4 subject to the relations

$$(2.6) \quad e_i \times (e_j \times e_k) + (e_i \times e_j) \times e_k = 2\delta_{ik}e_j - \delta_{ij}e_k - \delta_{jk}e_i.$$

Given any two canonical bases $\{e_1, \dots, e_7\}$ and $\{f_1, \dots, f_7\}$ of \mathbf{R}^7 , there is a unique element $g \in G_2$ such that $ge_i = f_i$; and thus g is uniquely determined by ge_1, ge_2, ge_4 .

Let J be the automorphism of the tangent bundle $TS^6(1)$ of $S^6(1)$ defined by

$$Ju = x \times u, \quad u \in T_x S^6(1), \quad x \in S^6(1).$$

It is clear that J is an almost complex structure on $S^6(1)$ and in fact J is a nearly Kähler structure on $S^6(1)$ in the sense that $(\tilde{\nabla}_u J)u = 0$, for any vector u tangent to $S^6(1)$, where $\tilde{\nabla}$ is the Levi-Civita connection of $S^6(1)$. We define by

$$G(X, Y) = (\tilde{\nabla}_X J)(Y),$$

the corresponding skew-symmetric (2,1)-tensor field. From [S], we know that this tensor field has the following properties:

$$(2.7) \quad G(X, JY) + JG(X, Y) = 0,$$

$$(2.8) \quad (\tilde{\nabla}G)(X, Y, Z) = \langle Y, JZ \rangle X + \langle X, Z \rangle JY - \langle X, Y \rangle JZ,$$

$$(2.9) \quad \langle G(X, Y), Z \rangle + \langle G(X, Z), Y \rangle = 0,$$

$$(2.10) \quad \langle G(X, Y), G(Z, W) \rangle = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Z, Y \rangle \\ + \langle JX, Z \rangle \langle Y, JW \rangle - \langle JX, W \rangle \langle Y, JZ \rangle,$$

$$(2.11) \quad G(X, Y) = X \times Y + \langle X, JY \rangle X.$$

It is clear from the above that G_2 acts transitively on $S^6(1)$ and that the stabilizer of the point $(1, 0, \dots, 0)$ is $SU(3)$. It follows that G_2 , a connected subgroup of $SO(7)$ of dimension 14, is the group of automorphisms of the nearly Kähler structure J .

3. Warped product immersions. Let M_0, \dots, M_k be Riemannian manifolds, M their product $M_0 \times \dots \times M_k$, and let $\pi_i: M \rightarrow M_i$ denote the canonical projection. If $\rho_1, \dots, \rho_k: M_0 \rightarrow \mathbf{R}_+$ are positive-valued functions, then

$$\langle X, Y \rangle := \langle \pi_{0*}X, \pi_{0*}Y \rangle + \sum_{i=1}^k (\rho_i \circ \pi_0)^2 \langle \pi_{i*}X, \pi_{i*}Y \rangle, \quad X, Y \in \Gamma(TM)$$

defines a Riemannian metric on M . We call $(M; \langle \cdot, \cdot \rangle)$ the warped product $M_0 \times_{\rho_1} M_1 \times \dots \times_{\rho_k} M_k$ of M_0, \dots, M_k , and ρ_1, \dots, ρ_k the warping functions.

Let $f_i: N_i \rightarrow M_i, i=0, \dots, k$ be isometric immersions, and define $\sigma_i := \rho_i \circ f_0: N_0 \rightarrow \mathbf{R}_+$ for $i=1, \dots, k$. Then the map $f: N_0 \times_{\sigma_1} N_1 \times \dots \times_{\sigma_k} N_k \rightarrow M_0 \times_{\rho_1} M_1 \times \dots \times_{\rho_k} M_k$ given by $f(p_0, \dots, p_k) := (f_0(p_0), f_1(p_1), \dots, f_k(p_k))$ is an isometric immersion, and is called a warped product immersion.

The decomposition of an immersion into warped product immersions is in particular a very powerful tool when applied to immersions into Euclidean spaces, spheres or hyperbolic spaces. In this respect, the main result from [N] can be stated as follows. Let $f: N_0 \times_{\sigma_1} N_1 \times \dots \times_{\sigma_k} N_k \rightarrow M(c)$ be an isometric immersion into a space of constant curvature c . If h is the second fundamental form of f and $h(X_i, X_j) = 0$, for all vector fields X_i and X_j , tangent to N_i and N_j respectively, with $i \neq j$, then, locally, M is a warped product immersion. The problem of how $M(c)$ can be decomposed into a warped product is solved in [N], see also [DN] for the statement.

Using warped product immersions, we can give a class of examples of minimal submanifolds in a unit sphere which satisfy the equality (1.2). Let $S_+^{n-2}(1) = \{x \in \mathbf{R}^{n-1} \mid \|x\| = 1 \text{ and } x_1 > 0\}$ be an open hemisphere and let $S^{m-n+2}(1)$ be the unit hypersphere of \mathbf{R}^{m-n+3} . Then $\psi: S_+^{n-2}(1) \times_{x_1} S^{m-n+2}(1) \rightarrow S^m(1), (x, y) \mapsto (x_1 y, x_2, \dots, x_{n-1})$ is an isometry onto an open dense subset of $S^m(1)$. This can be considered as a warped product decomposition of $S^m(1)$. Now if N^2 is any minimal surface in $S^{m-n+2}(1)$, immersed by f_1 , then the immersion $f: S_+^{n-2}(1) \times_{x_1} N^2 \rightarrow S^m(1), (x, p) \mapsto \psi(x, f_1(p))$ is an isometric immersion satisfying the equality (1.2). This follows trivially since the dimension of the distribution \mathcal{D} is $n-2$. It is easy to see that the immersion (1.4) is a special case of this family. We now focus on (1.4).

We consider a totally geodesic $S^5(1)$ in $S^6(1)$. Let N denote the unit vector

orthogonal to the hyperplane containing $S^5(1)$. We parametrize the half circle $S^1_+(1)$ by $(-\pi/2, \pi/2) \rightarrow S^1_+(1)$, $t \mapsto (\cos(t), \sin(t))$. Then the isometry ψ of the previous paragraph can also be written as $S^1_+(1) \times_{\cos(t)} S^5(1) \rightarrow S^6(1)$, $(t, p) \mapsto \sin(t)N + \cos(t)p$. Let $f : M^2 \rightarrow S^5(1)$ be an immersion of a surface into $S^5(1)$. Then the associated warped product immersion is given by

$$(3.1) \quad x : (-\pi/2, \pi/2) \times_{\cos(t)} M^2 \rightarrow S^6(1), (t, p) \mapsto \sin(t)N + \cos(t)f(p).$$

We will determine in Section 5 for which immersions f , the warped product immersion x is totally real.

4. Totally real submanifolds in $S^6(1)$. A submanifold M in $S^6(1)$ is called totally real if for any vector field X , tangent to M , JX is a normal vector field.

The dimension of M can be 2 or 3. Totally real surfaces in $S^6(1)$ were first studied in [DOVV]. Totally real 3-dimensional submanifolds were first studied in Ejiri [E3], who proved that a 3-dimensional totally real submanifold of $S^6(1)$ is orientable and minimal. In both cases it can be proved that $G(X, Y)$ is orthogonal to M , for tangent vectors X and Y .

We denote the Levi-Civita connection of M by ∇ . The formulas of Gauss and Weingarten are respectively given by

$$(4.1) \quad D_X Y = \nabla_X Y + h(X, Y),$$

$$(4.2) \quad D_X \xi = -A_\xi X + \nabla_X^\perp \xi,$$

for tangent vector fields X and Y and normal vector fields ξ . The second fundamental form h is related to A_ξ by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

From (4.1) and (4.2), we find that

$$(4.3) \quad \nabla_X^\perp JY = J\nabla_X Y + G(X, Y) + (Jh(X, Y))^n,$$

$$(4.4) \quad A_{JY} X = -(Jh(X, Y))^t,$$

where $(Jh(X, Y))^n$ and $(Jh(X, Y))^t$ denote the normal and tangential parts of $Jh(X, Y)$. Obviously, if $\dim M = 3$, then $Jh(X, Y)$ is tangent.

The above formulas immediately imply that $\langle h(X, Y), JZ \rangle$ is totally symmetric. If we denote the curvature tensors of ∇ and ∇^\perp by R and R^\perp , respectively, then the equations of Gauss, Codazzi and Ricci are given by

$$(4.5) \quad R(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y + A_{h(Y, Z)} X - A_{h(X, Z)} Y,$$

$$(4.6) \quad (\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z),$$

$$(4.7) \quad \langle R^\perp(X, Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta]X, Y \rangle,$$

where X, Y, Z (respectively, η and ξ) are tangent (respectively, normal) vector fields to M and ∇h is defined by

$$(\nabla h)(X, Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

5. Totally real surfaces in $S^6(1)$. We now continue to investigate the immersion (3.1).

Let X denote a vector field tangent to M^2 . Then

$$x_*(\partial/\partial t) = \cos(t)N - \sin(t)f(p), \quad x_*(X) = \cos(t)f_*(X),$$

and

$$Jx_*(\partial/\partial t) = N \times f(p), \quad Jx_*(X) = \cos(t)\sin(t)N \times f_*(X) + \sin^2(t)Jf_*(X).$$

From this it is easy to check that x is totally real if and only if

$$(5.1) \quad \langle N \times f(p), f_*(X) \rangle = 0,$$

$$(5.2) \quad \langle N \times f_*(X), f_*(Y) \rangle = 0,$$

$$(5.3) \quad \langle Jf_*(X), f_*(Y) \rangle = 0,$$

for all tangent vector fields X and Y to M . Now (5.3) simply says f has to be totally real; from (2.11), we obtain that (5.2) is equivalent to $\langle G(f_*(X), f_*(Y)), N \rangle = 0$; (5.1) is equivalent to $\langle Jf_*(X), N \rangle = 0$. We have reduced the condition that x is totally real to conditions depending only on f . From now on, for simplicity, we identify M with $f(M)$, so we do not write f_* if there is no confusion.

Differentiating (5.1) gives us

$$(5.4) \quad \langle N \times Y, X \rangle + \langle N \times p, h(X, Y) \rangle = 0.$$

Since the first term in (5.4) is skew symmetric and the second is symmetric, both terms have to be zero. Therefore (5.1) implies (5.2). Now take any orthonormal basis $\{e_1, e_2\}$ of T_pM . From the properties of G , we obtain that $G(e_1, e_2)$ is orthogonal to e_1, e_2, Je_1, Je_2 and p ; (5.2) implies that $G(e_1, e_2)$ is also orthogonal to N . On the other hand, (5.1) implies that $N \times p$ is orthogonal to e_1 and e_2 ; clearly $N \times p$ is orthogonal to p and N , and a straightforward calculation shows that $N \times p$ is orthogonal to Je_1 and Je_2 . Therefore $G(e_1, e_2) = \pm p \times N$. After changing the sign of e_1 , if necessary, we can make sure that $e_1 \times e_2 = JN$. This implies that $e_1 \times N = -Je_2, e_2 \times N = Je_1$. Note that the normal space of M in $S^5(1)$ at p is spanned by Je_1, Je_2 and JN .

Differentiating (5.2), we obtain

$$(5.5) \quad \langle N \times h(Y, Z), X \rangle + \langle N \times Y, h(X, Z) \rangle = 0,$$

for all tangent vector fields X, Y and Z to M . Putting $X=e_1$ and $Y=e_2$, we obtain that

$$0 = \langle Je_2, h(e_2, Z) \rangle + \langle Je_1, h(e_1, Z) \rangle = \langle h(e_2, e_2), JZ \rangle + \langle h(e_1, e_1), JZ \rangle,$$

such that the mean curvature vector H of M in $S^5(1)$ is orthogonal to Je_1 and Je_2 ; from (5.4) we obtain that H is orthogonal to JN . Therefore H can only be zero. Then (5.4) immediately implies that h is of the form

$$h(e_1, e_1) = \alpha Je_1 + \beta Je_2, \quad h(e_1, e_2) = \beta Je_1 - \alpha Je_2, \quad h(e_2, e_2) = -\alpha Je_1 - \beta Je_2,$$

such that the ellipse of curvature of M at p is a circle (possibly a point).

Conversely, let M^2 be a minimal, totally real surface of $S^6(1)$ whose ellipse of curvature is a circle. Then we can use exactly the same computations as in the proof of [DOVV, Lemma] to obtain that $h(X, Y)$ is contained in $J(TM)$. Let $\{E_1, E_2\}$ be any local orthonormal basis of TM . Then $G(E_1, E_2)$ does not depend on the choice of $\{E_1, E_2\}$ up to sign. Hence we can define a subbundle B of the normal bundle by $B(p) = J(T_p M) \oplus \langle G(E_1, E_2) \rangle$. From (4.3), (2.8) and the minimality of M , we obtain that B is ∇^\perp -parallel. Hence by the Erbacher theorem, M lies in a 5-dimensional totally geodesic hypersphere of $S^6(1)$. Let N be a unit vector orthogonal to this $S^5(1)$. By construction, JX is tangent to $S^5(1)$ and hence orthogonal to N for all X tangent to M . Therefore (5.1) is satisfied; (5.2) follows from (5.1) and (5.3) is true since M is totally real. Even if M is totally geodesic, this 5-dimensional unit sphere is uniquely determined as follows: take any point p in M . Then $S^5(1)$ is the unique great hypersphere of $S^6(1)$ through p , tangent to the $T_p M \oplus B(p)$. If M is not totally geodesic, it follows again as in [DOVV] that M is not contained in a totally geodesic 4-sphere. Hence we have proved the following theorem.

THEOREM 5.1. (1) *Let $f: M^2 \rightarrow S^6(1)$ be a minimal non-totally geodesic totally real immersion in $S^6(1)$ whose ellipse of curvature is a circle. Then M^2 is contained in a unique totally geodesic S^5 and the warped product immersion (3.1) is totally real.*

(2) *Let f and x as in Section 3. Then x is totally real if and only if f is totally real and $J(f_*X)$ is tangent to $S^5(1)$ for all X tangent to M .*

(3) *Let f and x as in Section 3. If x is totally real, then f is totally real, minimal and has ellipse of curvature a circle.*

Other examples of totally real 3-dimensional submanifolds in S^6 were constructed by Ejiri in [E1] in the following way. Let $f: M^2 \rightarrow S^6$ be a linearly full superminimal (in the sense of [BVW]) almost complex immersion. Let U and V be local orthonormal vector fields, defined on a neighborhood W , which span the second normal bundle. Then for any real number γ ($0 < \gamma < \pi$) we can define the tube of radius γ in the direction of the second normal bundle by

$$F_\gamma: W \times S^1 \rightarrow S^6, (x, \theta) \mapsto \cos \gamma f(x) + \sin \gamma (\cos \theta U + \sin \theta V).$$

Then F_γ defines a totally real immersion if and only if either $\cos \gamma = 0$ or $\tan^2 \gamma = 4/5$. A similar construction of totally real submanifolds of CP^3 can be found in [E2].

A straightforward computation shows that all tubes of radius $\pi/2$ satisfy the equality in (3.1). In particular, starting from the Veronese immersion $f: S^2(1/6) \rightarrow S^6$ one obtains a 3-dimensional totally real submanifold with constant scalar curvature, which corresponds to Example 3.1 of [CDVV1]. It is also possible to show that for this class of tubes $F_{\pi/2}$ the distribution \mathcal{D}^\perp is never integrable. As for the second possibility ($\tan^2 \gamma = 4/5$), one can show that the equality is never realized.

We now elaborate some more on totally real minimal surfaces whose ellipse of curvature is a circle. For simplicity, assume that $N = e_4$. We denote by π the Hopf map from S^5 to CP^2 given by

$$\pi(x_1, x_2, x_3, 0, x_5, x_6, x_7) = [x_1 + ix_5, x_2 + ix_6, x_3 + ix_7].$$

Then the following theorem from [BVW] gives a relation between minimal totally real surfaces in $S^6(1)$ whose ellipse of curvature is a circle and minimal totally real surfaces in CP^2 .

THEOREM 5.2 (cf. [BVW]). *If $f: M^2 \rightarrow S^5 \subset S^6(1)$ is a minimal totally real isometric immersion, not totally geodesic, whose ellipse of curvature is a circle, then $\pi(f): M^2 \rightarrow CP^2$ is a totally real, not totally geodesic, minimal isometric immersion of M^2 into CP^2 . Conversely, if M^2 is simply connected and if $\psi: M^2 \rightarrow CP^2$ is a totally real, not totally geodesic, weakly conformal harmonic map, then there is a minimal totally real immersion $f: M^2 \rightarrow S^5$ whose ellipse of curvature is a circle such that $\psi = \pi(f)$.*

In this respect Theorem 5.1 should be compared with [BVW, Theorem 7.1]. In its turn, minimal totally real immersions of a surface in CP^2 can be characterized as follows:

THEOREM 5.3. *Let $(M^2, \langle \cdot, \cdot \rangle)$ be a simply connected surface with Gaussian curvature K satisfying $K < 1$. Then the following two conditions are equivalent:*

- (1) $\Delta \log(1 - K) = 6K$;
- (2) *there exists a totally real minimal immersion $f: M^2 \rightarrow CP^2(4)$.*

PROOF. The fact that (2) implies (1) follows from a straightforward computation, in view of the basic formulas for a totally real submanifold in $CP^2(4)$ from [CO].

Let us now prove the converse. We take isothermal coordinates on M^2 . So, we have a local non-zero function E such that $\langle \partial/\partial u, \partial/\partial u \rangle = E^2 = \langle \partial/\partial v, \partial/\partial v \rangle$ and $\langle \partial/\partial u, \partial/\partial v \rangle = 0$. Then $K = -\Delta \log E$. We now define a function ϕ by

$$\phi^2 = \frac{E^6}{2} (1 - K).$$

Then, by the assumption of the theorem, we get that

$$\begin{aligned} \Delta \log \phi &= \frac{1}{2} \Delta \log \phi^2 = \frac{1}{2} \Delta \log \frac{E^6}{2} (1-K) \\ &= 3 \Delta \log E + \frac{1}{2} \Delta \log (1-K) = -3K + 3K = 0. \end{aligned}$$

Hence there exist a function

$$\psi(u, v) = F(u, v) - iG(u, v),$$

holomorphic in $z = u + iv$ such that $F^2 + G^2 = \phi^2$. Put

$$f(u, v) = \frac{F(u, v)}{E^2(u, v)}, \quad g(u, v) = \frac{G(u, v)}{E^2(u, v)},$$

and define $\alpha: TM^2 \times TM^2 \rightarrow TM^2$ by

$$\begin{aligned} \alpha(\partial/\partial u, \partial/\partial u) &= f(u, v)\partial/\partial u + g(u, v)\partial/\partial v, \\ \alpha(\partial/\partial u, \partial/\partial v) &= \alpha(\partial/\partial v, \partial/\partial u) = g(u, v)\partial/\partial u - f(u, v)\partial/\partial v, \\ \alpha(\partial/\partial v, \partial/\partial v) &= -f(u, v)\partial/\partial u - g(u, v)\partial/\partial v. \end{aligned}$$

Then

$$\begin{aligned} (\nabla \alpha)(\partial/\partial u, \partial/\partial u, \partial/\partial v) &= \left(g_u - 3f \frac{E_v}{E} - g \frac{E_u}{E} \right) \partial/\partial u - \left(f_u + 3g \frac{E_v}{E} - f \frac{E_u}{E} \right) \partial/\partial v, \\ (\nabla \alpha)(\partial/\partial v, \partial/\partial u, \partial/\partial u) &= \left(f_v - f \frac{E_v}{E} - 3g \frac{E_u}{E} \right) \partial/\partial u + \left(g_v - g \frac{E_v}{E} + 3f \frac{E_u}{E} \right) \partial/\partial v, \\ (\nabla \alpha)(\partial/\partial v, \partial/\partial u, \partial/\partial v) &= \left(g_v - 3f \frac{E_u}{E} - g \frac{E_v}{E} \right) \partial/\partial u + \left(-f_v + 3g \frac{E_u}{E} + f \frac{E_v}{E} \right) \partial/\partial v, \\ (\nabla \alpha)(\partial/\partial u, \partial/\partial v, \partial/\partial v) &= \left(-f_u + f \frac{E_u}{E} - 3g \frac{E_v}{E} \right) \partial/\partial u + \left(-g_u + 3f \frac{E_v}{E} + g \frac{E_u}{E} \right) \partial/\partial v, \end{aligned}$$

showing that $\nabla \alpha$ is totally symmetric if and only if

$$f_u + g_v = -2 \left(f \frac{E_u}{E} + g \frac{E_v}{E} \right), \quad f_v - g_u = 2 \left(g \frac{E_u}{E} - f \frac{E_v}{E} \right),$$

which by the definition of f and g is satisfied indeed. Since

$$\frac{E^2}{2} (1-K) = \frac{\phi}{E^4}, \quad \phi^2 = F^2 + G^2 = E^4 (f^2 + g^2),$$

we get that

$$R(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y + \alpha(\alpha(Y, Z), X) - \alpha(\alpha(X, Z), Y).$$

Applying the basic existence theorem (cf. [CDVV2]) then completes the proof of the theorem.

6. Proof of the main theorem. We first recall some results of [CDVV1] for 3-dimensional totally real (and therefore minimal) submanifolds of $S^6(1)$.

LEMMA 6.1. *Let M be a 3-dimensional totally real submanifold of $S^6(1)$. Then*

$$\delta_M \leq 2.$$

Equality holds at a point p of M if there exists a tangent basis $\{e_1, e_2, e_3\}$ of T_pM such that

$$\begin{aligned} h(e_1, e_1) &= \lambda J e_1, & h(e_1, e_3) &= 0, \\ h(e_1, e_2) &= -\lambda J e_2, & h(e_2, e_3) &= 0, \\ h(e_2, e_2) &= -\lambda J e_1, & h(e_3, e_3) &= 0, \end{aligned}$$

where λ is a positive number determined by the scalar curvature τ according to

$$2\lambda^2 = 3 - \tau(p).$$

So if we define a distribution \mathcal{D} by

$$\mathcal{D}(p) = \{X \in T_pM \mid h(X, Y) = 0, \forall Y \in T_pM\},$$

we see that $\mathcal{D}(p)$ is either 3-dimensional, in which case p is a totally geodesic point, or 1-dimensional. From now on, we assume that the dimension of $\mathcal{D}(p)$ is constant on M . Then, exactly as in Lemma 5.3 of [CDVV1], we obtain:

LEMMA 6.2. *Let M^3 be a totally real submanifold of $S^6(1)$ satisfying the equality in (1.3). Assume also that the dimension of the distribution \mathcal{D} is constantly equal to 1 and let $p \in M$. Then, there exists local orthonormal vector fields E_1, E_2, E_3 on a neighborhood of p such that*

$$\begin{aligned} h(E_1, E_1) &= \lambda J E_1, & h(E_1, E_3) &= 0, \\ h(E_1, E_2) &= -\lambda J E_2, & h(E_2, E_3) &= 0, \\ h(E_2, E_2) &= -\lambda J E_1, & h(E_3, E_3) &= 0, \end{aligned}$$

where λ is a non-zero local function determined by the scalar curvature by $2\lambda^2 = 3 - \tau$.

Let us take the basis from the previous lemma. By (2.7) and (2.9), we see that $G(E_1, E_2)$ is in the direction of $J E_3$. From (2.10), we obtain that $G(E_1, E_2)$ is a unit vector. Replacing E_3 by $-E_3$ if necessary, we may assume

$$G(E_1, E_2) = J E_3, \quad G(E_2, E_3) = J E_1, \quad G(E_3, E_1) = J E_2.$$

From now on, assume that we take this choice of orthonormal basis.

Throughout this section, M^3 is assumed to be a (non-totally geodesic) totally real submanifold in $S^6(1)$ which at every point p of M satisfies the equality in (1.3). We also assume that

- (1) the dimension of \mathcal{D} is constant on M ,
- (2) the distribution \mathcal{D}^\perp is integrable.

Since M is assumed to be non-totally geodesic, we have that $\dim \mathcal{D} = 1$. Let $p \in M$. We introduce local functions γ_{ij}^k by

$$\gamma_{ij}^k = \langle \nabla_{E_i} E_j, E_k \rangle .$$

Since $\{E_1, E_2, E_3\}$ is an orthonormal basis, $\gamma_{ij}^k + \gamma_{ik}^j = 0$. Then, we have the following lemma.

LEMMA 6.3. *We have*

- (1) $\gamma_{33}^1 = \gamma_{33}^2 = 0$,
- (2) $\gamma_{11}^3 = \gamma_{22}^3$,
- (3) $\gamma_{12}^3 = -\gamma_{21}^3$,
- (4) $\gamma_{31}^2 = -\frac{1}{3}(\gamma_{12}^3 + 1)$.

Moreover, the function λ satisfies the following system of differential equations.

- (5) $E_1(\lambda) = -3\lambda\gamma_{21}^2$,
- (6) $E_2(\lambda) = 3\lambda\gamma_{11}^2$,
- (7) $E_3(\lambda) = -\lambda\gamma_{13}^1$.

PROOF. Since

$$(\nabla h)(E_1, E_3, E_3) = \nabla_{E_1}^\perp h(E_3, E_3) - 2h(\nabla_{E_1} E_3, E_3) = 0$$

and

$$(\nabla h)(E_3, E_1, E_3) = \nabla_{E_3}^\perp h(E_1, E_3) - h(\nabla_{E_3} E_1, E_3) - h(E_1, \nabla_{E_3} E_3) = -h(E_1, \nabla_{E_3} E_3),$$

Codazzi's equation yields $\nabla_{E_3} E_3 = 0$. Next we compute

$$\begin{aligned} (\nabla h)(E_3, E_1, E_1) &= \nabla_{E_3}^\perp h(E_1, E_1) - 2h(\nabla_{E_3} E_1, E_1) \\ &= E_3(\lambda)JE_1 + \lambda J\nabla_{E_3} E_1 + \lambda JE_2 + 2\langle \nabla_{E_3} E_1, E_2 \rangle \lambda JE_2 \end{aligned}$$

and

$$\begin{aligned} (\nabla h)(E_1, E_3, E_1) &= \nabla_{E_1}^\perp h(E_3, E_1) - h(\nabla_{E_1} E_3, E_1) - h(E_3, \nabla_{E_1} E_1) \\ &= -\langle \nabla_{E_1} E_3, E_1 \rangle \lambda JE_1 + \langle \nabla_{E_1} E_3, E_2 \rangle \lambda JE_2 . \end{aligned}$$

Since $\langle \nabla_{E_3} E_3, E_1 \rangle = 0$, by comparing components, we get that

$$(6.1) \quad E_3(\lambda) = \lambda \langle \nabla_{E_1} E_1, E_3 \rangle = \lambda \gamma_{11}^3, \quad 3 \langle \nabla_{E_3} E_1, E_2 \rangle = - \langle \nabla_{E_1} E_2, E_3 \rangle - 1.$$

This proves (4) and (7). Similarly, from $(\nabla h)(E_3, E_2, E_2) = (\nabla h)(E_2, E_3, E_2)$ we obtain

$$(6.2) \quad \gamma_{22}^3 = \gamma_{11}^3.$$

Finally, we have

$$\begin{aligned} (\nabla h)(E_2, E_1, E_1) &= \nabla_{E_2}^\perp h(E_1, E_1) - 2h(\nabla_{E_2} E_1, E_1) \\ &= E_2(\lambda) J E_1 + \lambda J \nabla_{E_2} E_1 - \lambda J E_3 + 2 \langle \nabla_{E_2} E_1, E_2 \rangle \lambda J E_2 \end{aligned}$$

and

$$\begin{aligned} (\nabla h)(E_1, E_2, E_1) &= \nabla_{E_1}^\perp h(E_2, E_1) - h(\nabla_{E_1} E_2, E_1) - h(E_2, \nabla_{E_1} E_1) \\ &= -E_1(\lambda) J E_2 - \lambda J \nabla_{E_1} E_2 - \lambda J E_3 - \langle \nabla_{E_1} E_2, E_1 \rangle \lambda J E_1 + \langle \nabla_{E_1} E_1, E_2 \rangle \lambda J E_1 \\ &= -\lambda J \nabla_{E_1} E_2 - \lambda J E_3 + 2\lambda \langle \nabla_{E_1} E_1, E_2 \rangle J E_1. \end{aligned}$$

So, by comparing components, we get

$$\langle \nabla_{E_2} E_1 + \nabla_{E_1} E_2, E_3 \rangle = 0, \quad E_2(\lambda) = 3\lambda \langle \nabla_{E_1} E_1, E_2 \rangle, \quad E_1(\lambda) = -3\lambda \langle \nabla_{E_2} E_2, E_1 \rangle.$$

This completes the proof of the lemma.

In order to simplify the notation, we introduce local functions a, b, c and d by

$$a = \gamma_{11}^3, \quad b = \gamma_{12}^3, \quad c = \gamma_{11}^2, \quad d = \gamma_{21}^2.$$

Then Lemma 6.3 implies that

$$\begin{aligned} \nabla_{E_1} E_1 &= c E_2 + a E_3, & \nabla_{E_1} E_2 &= -c E_1 + b E_3, & \nabla_{E_1} E_3 &= -a E_1 - b E_2, \\ \nabla_{E_2} E_1 &= d E_2 - b E_3, & \nabla_{E_2} E_2 &= -d E_1 + a E_3, & \nabla_{E_2} E_3 &= b E_1 - a E_2, \\ \nabla_{E_3} E_1 &= -\frac{1}{3}(b+1)E_2, & \nabla_{E_3} E_2 &= \frac{1}{3}(b+1)E_1, & \nabla_{E_3} E_3 &= 0, \end{aligned}$$

and

$$E_1(\lambda) = -3\lambda d, \quad E_2(\lambda) = 3\lambda c, \quad E_3(\lambda) = \lambda a.$$

Let us now use the assumption that the distribution \mathcal{D}^\perp , which is locally spanned by the vector fields E_1 and E_2 is an integrable distribution. Then, the above formulas imply that $b=0$. Then, we have:

LEMMA 6.4. *The local function a , under the assumptions made above, satisfies the following system of differential equations:*

$$E_1(a) = 0, \quad E_2(a) = 0, \quad E_3(a) = 1 + a^2.$$

PROOF. Using the Gauss equation, we find that

$$\begin{aligned} 0 &= \langle R(E_1, E_2)E_1, E_3 \rangle = \langle \nabla_{E_1} \nabla_{E_2} E_1 - \nabla_{E_2} \nabla_{E_1} E_1 - \nabla_{\nabla_{E_1} E_2 - \nabla_{E_2} E_1} E_1, E_3 \rangle \\ &= -ca - E_2(a) + ca = -E_2(a). \end{aligned}$$

Similarly, it follows from the Gauss equation $0 = \langle R(E_2, E_1)E_2, E_3 \rangle$ that $E_1(a) = 0$. Finally, in order to prove that $E_3(a) = 1 + a^2$, we use again the Gauss equation. We have

$$1 = \langle R(E_1, E_3)E_3, E_1 \rangle = \langle \nabla_{E_1} \nabla_{E_3} E_3 - \nabla_{E_3} \nabla_{E_1} E_3 - \nabla_{\nabla_{E_1} E_3 - \nabla_{E_3} E_1} E_3, E_1 \rangle = E_3(a) - a^2. \quad \square$$

LEMMA 6.5. *Let M be as above and let $p \in M$. Then, in a neighborhood of the point p , M is warped product of an interval $(-\varepsilon, \varepsilon)$ and N^2 , the leaf of the distribution \mathcal{D}^\perp through p .*

PROOF. We check Hiepko's condition [H], using the formalism of [N, §3]. In particular, we have to check that \mathcal{D} is totally geodesic and that \mathcal{D}^\perp is spherical. Since $\nabla_{E_3} E_3 = 0$, the first assumption is trivially satisfied. For the second assertion, we first have for $i, j \in \{1, 2\}$ that

$$\langle \nabla_{E_i} E_j, E_3 \rangle = \delta_{ij} a E_3,$$

which shows that \mathcal{D}^\perp is totally umbilical in M with mean curvature vector $\eta = aE_3$. Since, by the previous lemma, $E_1(a) = E_2(a) = 0$, the mean curvature vector is parallel. So, we get that \mathcal{D}^\perp is spherical. \square

The warping function can be determined from Lemma 6.4, but we do not need an explicit expression. Now we can finish the proof. Indeed, we know that M is locally a warped product and that the distributions on M , determined by the product structure, coincide with \mathcal{D} and \mathcal{D}^\perp . Moreover, since $h(\mathcal{D}, \mathcal{D}^\perp) = 0$, we obtain that M^3 (locally) is immersed as a warped product; further, the first factor is totally geodesic, and therefore we can assume that the first factor of the corresponding warped product decomposition of $S^6(1)$ is 1-dimensional. Since the decomposition of $S^6(1)$ into a warped product whose first factor is 1-dimensional is unique up to isometries, we obtain that M^3 is immersed as described by (3.1).

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