CHARACTERIZING PROPERTIES OF STOCHASTIC OBJECTIVE FUNCTIONS

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ABSTRACT: This paper develops tools for analyzing properties of stochastic objective functions that take the form $V(\mathbf{x},\theta) \equiv \int_{\mathbf{s}} u(\mathbf{x},\mathbf{s})dF(\mathbf{s};\theta)$. The paper analyzes the relationship between properties of the primitive functions, such as the utility functions *u* and probability distributions *F*, and properties of the stochastic objective. The methods are designed to address problems where the utility function is restricted to lie in a set of functions which is a "closed convex cone" (examples of such sets include increasing functions, concave functions, or supermodular functions). It is shown that approaches previously applied to characterize monotonicity of *V* (that is, stochastic dominance theorems) can be used to establish other properties of *V* as well. The first part of the paper establishes necessary and sufficient conditions for *V* to satisfy "closed convex cone properties" such as monotonicity, supermodularity, and concavity, in the parameter θ. Then, we consider necessary and sufficient conditions for monotone comparative statics predictions, building on the results of Milgrom and Shannon (1994). A new property of payoff functions is introduced, called *l*-supermodularity, which is shown to be necessary and sufficient for $V(\mathbf{x},\theta)$ to be quasi-supermodular in **x** (a property which is, in turn, necessary for comparative statics predictions). The results are illustrated with applications.
KEYWORDS: Stochastic dominance, supermodularity, quasi-supermodularity, single crossing properties, concavity, economics of uncertainty, investment, monotone comparative statics.

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1 INTRODUCTION

This paper studies optimization problems where the objective function can be written in the form $V(\mathbf{x},\theta) \equiv \int_{\mathbf{s}} u(\mathbf{x},\mathbf{s})dF(\mathbf{s};\theta)$, where u is a payoff function, F is a probability distribution, and θ and $\mathbf{s}$ are real vectors. For example, the payoff function u might represent an agent's utility or a firm's profits, the vector $\mathbf{s}$ might represent features of the current state of the world, and the elements of $\mathbf{x}$ and θ might represent an agent's investments, effort decisions, other agent's choices, or the nature of the exogenous uncertainty in the agent's environment.

Problems that take this form arise in many contexts in economics. For example, the theory of the firm often considers firm choices about investments that have uncertain returns. And strategic interaction between firms often takes place in an environment of incomplete information, where each firm has private information about its costs or inventory. Examples include signaling games and pricing games. In another example, there is a large literature concerning the comparative statics of portfolio investment decisions and decision-making under uncertainty, where the central questions concern how investment responds to changes in risk preferences, initial wealth, characteristics of the distributions over asset returns, and background risks.[1]

The goal of this paper is to add to the set of methods that can be used to derive comparative statics predictions in such problems. To that end, the paper develops theorems to characterize properties of the objective function, $V(\mathbf{x},\theta)$, based on properties of the payoff function, u. Economic problems often place some structure on the payoff function; for example, a utility function might be assumed to be nondecreasing and concave, while a multivariate profit function might have sign restrictions on cross-partial derivatives. These assumptions then determine a set U of admissible payoff functions. This paper derives theorems that, for a given set U, establish conditions on probability distributions which are equivalent to properties of $V(\mathbf{x},\theta)$. The paper focuses on sets of payoff functions, U, which are restricted to satisfy "closed convex cone" properties, that is, properties which are preserved under affine transformations and limits; examples include the properties nondecreasing or concave.

We proceed to analyze properties of V in two steps. The first step is to characterize conditions under which V satisfies "convex cone" properties. For example, we characterize how restrictions on sets of payoff functions, U, correspond to necessary and sufficient conditions on probability distributions such that V is supermodular in θ, so that investments q_i and q_j are mutually

[1] For examples, see Rothschild and Stiglitz (1970, 1971), Diamond and Stiglitz (1974), Eeckhoudt and Gollier (1995), Gollier (1995), Jewitt (1986, 1987, 1989), Kimball (1990, 1993), Landsberger and Meilijson (1990), Meyer and Ormiston (1983, 1985, 1989), Ormiston (1992), Ormiston and Schlee (1992, 1993), Ross (1981).

complementary for $i \neq j$. Building on these results, the second part of the paper then focuses on characterizing the conditions (single crossing properties and quasi-supermodularity[2]) which are necessary and sufficient for comparative statics predictions (Milgrom and Shannon, 1994). Although single crossing properties and quasi-supermodularity are not closed convex cone properties, we show that the techniques developed for closed convex cones can be generalized to analyze comparative statics properties.

Consider first the analysis of closed convex cone properties of V. Since such properties are preserved by arbitrary sums, closed convex cone properties of $u(\mathbf{x},\mathbf{s})$ in $\mathbf{x}$ are inherited by V. It is somewhat more subtle to analyze properties of V in θ, while exploiting restrictions on the set of admissible payoff functions, U. If the desired property of V is monotonicity, we can build on the existing theory of stochastic dominance. A stochastic dominance theorem gives necessary and sufficient conditions on $F(\mathbf{s};\theta)$ such that $V(\mathbf{x},\theta)$ is nondecreasing in θ, for all payoff functions u in some set U. Thus, each set of payoffs U induces a partial order over probability distributions. This paper begins by introducing some abstract definitions which formalize an approach to stochastic dominance that has been used implicitly, and less often explicitly, in the literature.

An initial result in this paper is that stochastic dominance orders can be completely characterized in the following way. Instead of checking that $V(\mathbf{x},\theta)$ is nondecreasing in θ for all u in U, it is necessary and sufficient to check that $V(\mathbf{x},\theta)$ is nondecreasing in θ for all u in some other set T. Such a theorem is useful if T is smaller and easier to check than U, so that the set T can be thought of as a set of "test functions." We characterize exactly how small T can be: in order to be a valid set of test functions for U, the closed convex cone of T must be equal to the closed convex cone of U. In particular, T might be a set of "extreme points" for U. For example, when U is the set of univariate, nondecreasing functions, we can use a set of test functions T which contains all indicator functions for upper intervals, and the constant functions **1** and **–1**. Applying this approach yields the familiar First Order Stochastic Dominance Order, which requires that $F(s;\theta)$ is nonincreasing in θ for all s.

The approach to stochastic dominance based on convex cones has been applied in previous studies (for example, Brumelle and Vickson, 1975) to characterize monotonicity of $V(\mathbf{x},\theta)$ in θ for a few commonly encountered classes of payoff functions on a case by case basis.[3] This paper formulates and proves general theorems about the approach, and establishes that it can be also be used to check other properties of V in θ. In particular, building directly from our results about

[2] Formal definitions are given in Section 3.1.

[3] Independently, Gollier and Kimball (1995a, 1995b) have advocated a more abstract approach to stochastic dominance theorems analogous to the one in this paper. In contrast to Gollier and Kimball, this paper focuses on characterizing a wide variety of properties of V in θ. Further, we provide necessary conditions for a set of functions T to be a valid set of test functions for U, an exercise which requires us to identify the "right" topology of closure for this purpose.

stochastic dominance, we show that if the property we desire for V (call it property P) in $\boldsymbol{\theta}$ is a closed convex cone property, the "test functions" approach described above is also applicable. Further, for a subset of these closed convex cone properties, we show that the approach cannot be improved upon.

Thus, the paper establishes several potentially interesting new classes of theorems, such as "stochastic supermodularity theorems" and "stochastic concavity theorems." The stochastic supermodularity theorems can be used to derive comparative statics predictions in decision problems and games. For example, they can be used to show when two risky investments are complementary in a firm's profit function, or when investments by two firms are strategic substitutes or complements.

In Section 3, we build on this approach to characterize comparative statics properties of V. We begin with the case where x and q are scalars, and study conditions under which the optimal choice of x is nondecreasing in q. A sufficient condition for comparative statics predictions is that V is supermodular; but if supermodularity is the desired property, we can simply take the first difference of V in x, and apply the theory of stochastic dominance described above, based on the properties of $u(x^H,\mathbf{s})-u(x^L,\mathbf{s})$. However, it will also be useful to characterize the necessary and sufficient condition for comparative statics, the single crossing property (which requires that the incremental returns to x cross zero, at most once and from below, as a function of q). We establish that the conditions required for single crossing are in general weaker than those required for supermodularity. But, a surprising result is that if our set of admissible payoff functions U is convex and further, it is large enough to include at least two functions u and v, where $u(x^H,\mathbf{s})-u(x^L,\mathbf{s})\equiv 1$ and $v(x^H,\mathbf{s})-v(x^L,\mathbf{s})\equiv -1$, then no weakening of the conditions on F can be obtained by considering single crossing as opposed to supermodularity. In other words, if U is large enough in the sense just described, then the optimal choice of x is nondecreasing in q for all $u\in U$, if and only if $V(x,q)$ is supermodular for all $u\in U$. This result is useful because supermodularity is much easier to check than single crossing in this context (in particular, existing stochastic dominance theorems may be applied). To see a simple example, an agent's choice of investment x will be nondecreasing in q for all payoff functions $u(x,s)$ such that the marginal returns to x are non-decreasing in s, if and only if q shifts $F(s;q)$ according to First-Order Stochastic Dominance.[4]

Our final goal is to characterize comparative statics properties of the function V in $\mathbf{x}$ or $\boldsymbol{\theta}$. Milgrom and Shannon (1994) identify a necessary condition for the optimal choices of x_1 and x_2 to jointly increase in response to an exogenous change in some parameter q: V must be quasi-supermodular in (x_1,x_2). However, this property is not preserved by convex combinations, and thus quasi-supermodularity of $u(\mathbf{x},\mathbf{s})$ in $\mathbf{x}$ is not sufficient to establish that V is quasi-supermodular. This

[4] Thus, we have uncovered the general principle underlying the results of Ormiston and Schlee (1992), who establish essentially this result for a few specific classes of payoff functions.

paper identifies a new property, which we call *l*-supermodularity, that is in fact necessary and sufficient to guarantee that V is quasi-supermodular. This property is closely related to quasi-supermodularity, yet it is preserved by convex combinations. Unfortunately, like quasi-supermodularity, it may be difficult to check.

The paper proceeds as follows. Section 2 considers properties of V in θ based on the set of admissible payoff functions U. Section 3 considers comparative statics properties of V. Section 4 concludes.

2 PROPERTIES OF $V(x,\theta)$ IN θ

This section begins by studying monotonicity of V in θ. In this context, we develop the main mathematical constructs and lemmas required for what we call the "closed convex cone" or "test functions" approach to characterizing properties of stochastic objective functions. We then proceed to apply the techniques to characterize other "closed convex cone" properties of V in θ, and we identify a set of properties for which this approach cannot be improved upon.

2.1 Preliminary Definitions

Let M^n be the set of *finite signed measures* on $\Re^n$. Any finite signed measure m has a Jordan decomposition (Royden (1968), pp. 235-236), so that $m = m^+ - m^-$, where each component is a positive, finite measure. We will be especially interested in finite signed measures which have the property that $\int dm = 0$, so that $\int dm^+ = \int dm^-$. Denote the set of all non-zero finite signed measures which have this property by Z^n; we are interested in this set because elements of this set can always be written as $m=a[F^1 - F^2]$, where a is a positive scalar, and F^1 and F^2 are probability distributions; likewise, for any two probability distributions F^1 and F^2, the measure $F^1 - F^2 \in Z^n$.

Let P^n be the set of bounded, measurable payoff functions on $\Re^n$. Define the bilinear functional $b : P^n \times M^n \to \Re$ by $b(u,m) = \int u\,dm$.[5],[6] Unless otherwise noted, for P^n, we will use the coarsest topology such that the set of all continuous linear functionals on P^n is exactly the set

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[5]Then $b(P^n,M^n)$ is a *separated duality*. That is, for any $m_1 \neq m_2$, there is a $u \in P^n$ such that $b(u,m_1) \neq b(u,m_2)$, and for any $u_1 \neq u_2$, there is a $m \in M^n$ such that $b(u_1,m) \neq b(u_2,m)$. Our choice of (P^n,M^n) is somewhat arbitrary: all of our results hold if we let A^n be a subset of measurable payoff functions on $\Re^n$ and we let B^n be any subset of M^n, so long as $b(A^n,B^n)$ is a separated duality.

[6]The boundedness assumption guarantees that the integral of the payoff function exists. It is possible to place other restrictions on the payoff functions and the space of finite signed measures so that the pair is a separated duality, in which case the arguments below would be unchanged; for example, it is possible to restrict the payoff functions and the signed measures using a "bounding function." For more discussions of separated dualities, see Bourbaki (1987, p. II.41).

$\left\{b(\cdot,m)\middle|m\in M^n\right\}$; this is the weak topology $s(P^n,M^n)$ on P^n.[7] Likewise, for M^n, we will use $s(M^n,P^n)$, the coarsest topology such that the set of all continuous linear functionals on M^n is exactly the set $\left\{b(u,\cdot)\middle|u\in P^n\right\}$.

2.2 A Unified Framework for Stochastic Dominance

This section introduces the framework that we will use to discuss stochastic dominance theorems as an abstract class of theorems, and to draw precise parallels between stochastic dominance theorems and other types of theorems.

We allow for multidimensional payoff functions and probability distributions, using the following notation: the set of probability distributions on $\Re^n$ is denoted Δ^n, with typical element $F:\Re^n\to[0,1]$. Further, for a given parameter space Θ, we will use the notation Δ^n_Θ to represent the set of parameterized probability distributions $F:\Re^n\times\Theta\to[0,1]$ such that such that $F(\cdot;\theta)\in\Delta^n$ for all $\theta\in\Theta$. With this notation, we formally define a stochastic dominance pair.

Definition 1 *Consider a pair of sets of payoff functions* (*U*,*T*)*, with typical elements* $u:\Re^n\to\Re$. *The pair* (*U*,*T*) *is a* ***stochastic dominance pair*** *if for all parameter spaces* Θ *with a partial order and all* $F\in\Delta^n_\Theta$,

$\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta)$ *is* ***nondecreasing*** *for all* $u \in U$*, if and only if*
$\int_{\mathbf{s}} t(\mathbf{s})dF(\mathbf{s};\theta)$ *is* ***nondecreasing*** *for all* $t \in T$.

Definition 1 clarifies the structure of stochastic dominance theorems. These theorems identify pairs of sets of payoff functions which have the following property: given a parameterized probability distribution *F*, checking that all of the functions in the set $\left\{\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta)\middle|u\in U\right\}$ are nondecreasing is *equivalent* to checking that all of the functions in the set $\left\{\int_{\mathbf{s}} t(\mathbf{s})dF(\mathbf{s};\theta)\middle|t\in T\right\}$ are nondecreasing. Stochastic dominance theorems are useful because, in general, the set *T* is smaller than the set *U*.

Definition 1 differs from the existing literature (i.e., Brumelle and Vickson, 1975), in that the existing literature generally compares the expected value of two probability distributions, say *F* and *G*, viewing $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s})$ and $\int_{\mathbf{s}} u(\mathbf{s})dG(\mathbf{s})$ as two different linear functionals mapping payoff functions to $\Re$. In contrast, by parameterizing the probability distribution and viewing $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta)$ as a bilinear functional mapping payoff functions and (parameterized) probability distributions to the real

[7] This topology uses as a basis neighborhoods of the form $N(u;e,(m_1,\ldots,m_k))=\left\{\hat{u}\middle|\max_{i=1,\ldots,k}\left|b(u-\hat{u},m_i)\right|<e\right\}$, where there is a neighborhood corresponding to each finite set $(m_1,\ldots,m_k)\subset M^n$ and each $e>0$ (Bourbaki (1987, p. II.43).

line, we are able to create an analogy between stochastic dominance theorems and stochastic supermodularity theorems, an analogy which would not be obvious using the standard constructions, where the formalization emphasizes a relationship between sets of payoff functions and orders over probability distributions. The utility of this definition will become clearer when we formalize the relationship between stochastic dominance and stochastic *P* theorems, for other properties *P* (such as supermodularity). The common theme for the different types of theorems is the use of test sets to characterize the relevant properties.

The first goal of this section is to identify necessary and sufficient conditions for the pair (*U*,*T*) to be a stochastic dominance pair. We will build on the concept of a *closed convex cone*, where a set *G* is a closed convex cone if g^1, $g^2 \in G$ implies $ag^1 + bg^2 \in G$ for α and β positive, and further *G* is closed in the appropriate topology. Thus, an example is the set of nondecreasing functions. The set *ccc*(*G*) is defined to be the smallest closed convex cone which contains *G*. Let $\{\mathbf{1},-\mathbf{1}\}$ denote the set containing the two constant functions, $\{u(s)\equiv 1\} \cup \{u(s)\equiv -1\}$. Then:

Theorem 1 *Consider U,T sets of bounded, measurable functions. Then* (*U*,*T*) *is a stochastic dominance pair if and only if*

$$ccc(U \cup \{\mathbf{1},-\mathbf{1}\}) = ccc(T \cup \{\mathbf{1},-\mathbf{1}\}) \qquad (2.1)$$

Remark: *If we eliminate the requirement that F is a probability distribution in Definition 1, condition (2.1) can be replace by* $ccc(U) = ccc(T)$.

We begin by interpreting Theorem 1; the following subsection establishes the proof in some detail.[8] First, observe that unless *T* is a subset of *U*, there is no guarantee that stochastic dominance theorems provide conditions which are easier to check than $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta)$ nondecreasing in θ for all $u \hat{I} U$. For example, *U* might be a set that is not a closed convex cone (such as the set of single crossing functions, that is, functions which cross zero once from below). Its closed convex cone might be much larger (for the case of single crossing functions, the closed convex cone is the set of all functions). In that case, it might be difficult to find a subset, *T*, of that closed convex cone for which is easier to check that expected payoffs are nondecreasing in θ. Thus, (2.1) indicates that stochastic dominance theorems are most likely to be useful when *U* is itself a closed convex cone.

When *U* contains the constant functions and is a closed convex cone, (2.1) becomes:

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[8] In the context of specific sets of payoff functions *U*, the existing literature identifies similar conditions for the corresponding stochastic dominance theorems, though a different topology is used. For example, Brumelle and Vickson (1975) argue that (2.1) is sufficient for several examples, using the topology of monotone convergence. In this paper, using the abstract definition we have developed for a "stochastic dominance pair," we are able to formally prove that the sufficient conditions hypothesized by Brumelle and Vickson (1975) are in fact sufficient for *any* stochastic dominance theorem, not just particular examples. Further, the result that (2.1) is also necessary for (*U*,*T*) to be a stochastic dominance pair is new here.

$$U = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\}) \tag{2.2}$$

In principle, the most useful T is the smallest set whose closed convex cone is U. However, in general, there will not be a unique smallest set. A set $E(U)$ is a set of *extreme points* of U if, for all u,v in $E(U)$, no convex combination of u and v is in $E(U)$. For simplicity, we will also consider only sets of extreme points that are closed. For example, if U is the set of nondecreasing functions on $\Re$, $E(U)$ is the set of indicator functions $\{\mathbf{1}_{s>a},\ a \in \Re\}$.[9] Finally, because (2.2) is necessary and sufficient for (U,T) to be a stochastic dominance pair when $U=ccc(T \cup \{\mathbf{1}, -\mathbf{1}\}$, we know that we cannot do any better than letting T be a set of extreme points of U.

Of course, even though the approach to establishing stochastic dominance theorems is the same for many different sets of payoff function U, the stochastic dominance orderings themselves (that is, the properties required of the distributions to satisfy the second condition in the Definition) will be quite different for different U.

Table I summarizes the main stochastic dominance theorems which have appeared in the economics or statistics literature to date. There are potentially many other univariate stochastic dominance theorems as well (for example, theorems where the set of payoff functions imposes restrictions on the third derivative of the payoff function); our analysis will indicate how these can be generated.

TABLE I

STOCHASTIC DOMINANCE THEOREMS

	Sets of Payoff Functions, U	Sets of Test Functions, T	Condition on Distribution for Stochastic Dominance
(i)	$U^{FO} \equiv \{u \mid u:\Re \to \Re,\ \text{nondecr.}\}$	$T^{FO} \equiv \{t \mid t(s) = \mathbf{1}_{[a,\infty)}(s),\ a \in \bar{\Re}\}$	$-F(a;\boldsymbol{\theta}) \uparrow$ in $\boldsymbol{\theta}$ $\forall a$.
(ii)	$U^{SO} \equiv \{u \mid u:\Re \to \Re,\ \text{concave}\}$	$T^{SO} \equiv \{t \mid t(s) = -s\} \cup \{t \mid t(s) = \min(a,s),\ a \in \bar{\Re}\}$	$-\int_{-\infty}^{a} F(s;\boldsymbol{\theta})ds$, $\int_{-\infty}^{\infty} F(s,q)ds \uparrow$ in q $\forall a$.
(iii)	$U^{SOM} \equiv \left\{u \,\middle\vert\, \begin{array}{l} u:\Re \to \Re,\ \text{nondecr.,} \\ \text{concave} \end{array}\right\}$	$T^{SOM} \equiv \{t \mid t(s) = \min(a,s),\ a \in \bar{\Re}\}$	$-\int_{-\infty}^{a} F(s,q)dt \uparrow$ in q $\forall a$.

[9] Notice that there are many possible sets of extreme points. We could consider, for example, only indicator functions of sets bounded by rational numbers. And we could multiply the indicator functions by a positive constant.

(iv)	$\left\{u \middle\vert \begin{array}{l} u:\Re^2\to\Re,\ \text{nondecreasing,} \\ \text{supermodular} \end{array}\right\}$	$\left\{t \middle\vert \begin{array}{l} t(s_1,s_2)=\ \mathbf{1}_{[a_1,\infty)}(s_1)\cdot\mathbf{1}_{[a_2,\infty)}(s_2), \\ a_1,a_2\in\bar{\Re} \end{array}\right\}$	$F(a_1,a_2;\boldsymbol{\theta})$ $-F(a_1;\boldsymbol{\theta})-F(a_2;\boldsymbol{\theta})\uparrow$ in $\boldsymbol{\theta}$; $-F(a_1;\boldsymbol{\theta})$, $-F(a_2;\boldsymbol{\theta})$ $\uparrow$ in $\boldsymbol{\theta}$ $\forall(a_1,a_2)$.
(v)	$\left\{u \middle\vert u:\Re^2\to\Re,\ \text{supermodular}\right\}$	$\left\{t \middle\vert \begin{array}{l} t(s_1,s_2)=\ \mathbf{1}_{[a_1,\infty)}(s_1)\cdot\mathbf{1}_{[a_2,\infty)}(s_2), \\ a_1,a_2\in\bar{\Re} \end{array}\right\}$ $\cup\left\{t \middle\vert t(s_1,s_2)=\ -\mathbf{1}_{[a_1,\infty)}(s_1),\ a_1\in\Re\right\}$ $\cup\left\{t \middle\vert t(s_1,s_2)=\ -\mathbf{1}_{[a_2,\infty)}(s_2),\ a_2\in\Re\right\}$	$F(a_1,a_2;\boldsymbol{\theta})\uparrow$ in $\boldsymbol{\theta}$; $F(a_1;\boldsymbol{\theta})$, $F(a_2;\boldsymbol{\theta})$ const in $\boldsymbol{\theta}$ $\forall$ (a_1,a_2).
(vi)	$\left\{u \middle\vert u:\Re^2\to\Re,\ \text{nondecreasing}\right\}$	$\left\{t \middle\vert \begin{array}{l} t(s_1,s_2)=\ \mathbf{1}_A(s_1,s_2),\ \text{where } A\subseteq\Re^2 \\ \text{and } \mathbf{1}_A(s_1,s_2) \text{ nondecreasing} \end{array}\right\}$	$\int_A dF(s;\boldsymbol{\theta})\uparrow$ in $\boldsymbol{\theta}$ $\forall A$ s.t. $\mathbf{1}_A(s_1,s_2)$ is nondecr.

The next section introduces the details of the mathematical results underlying Theorem 1, and further proves two lemmas that will be used throughout the paper.

2.3 Mathematical Underpinnings and a Proof of Theorem 1

The results in this section are variations on theorems from the theories of linear functional analysis, topological vector spaces, and linear algebra; the main contribution of this section is to define the appropriate function spaces and topology and restate the problem in such a way that we can adapt these theorems to solve the stochastic dominance problem. These results will be applied throughout the paper.

This section analyzes inequalities of the form $\int u\ dm \geq 0$, where $m\in Z^n$. Recalling our discussion of Section 2.1, we observe that since positive scalar multiples will not affect this inequality, and because the integral operator is linear, we can without loss of generality interpret this inequality as $\int_{\mathbf{s}} u(\mathbf{s})dF^1(\mathbf{s}) \geq \int_{\mathbf{s}} u(\mathbf{s})dF^2(\mathbf{s})$ for the pair of probability distributions F^1 and F^2 such that $m=\alpha[F^1-F^2]$ for some $\alpha>0$. However, the former notation will be easier to work with in terms of proving our main results, and further it will be useful in proving results in Section 4 about stochastic P theorems for properties P other than "nondecreasing."

Given a set $U\in P^n$, the *dual cone* of the set U, denoted U^*, is defined by $U^*\equiv\{m\text{: } \int u(\mathbf{s})dm(\mathbf{s})\geq 0$ for all $u\in U\}$. Likewise, for a set M of measures, the dual cone is $M^*\equiv\{u\text{: } \int u(\mathbf{s})dm(\mathbf{s})\geq 0$ for all $m\in M\}$. If $M=U^*$, we refer to $M^*=U^{**}$ as the *second dual* of U. Dual cones can be characterized in the following way (for example, see Bourbaki (1987)):

Lemma 1 *Consider (P^n,M^n) with the weak topology. Suppose U Ì P^n and M Ì M^n. Then U^* is a closed convex cone. Further, U^{**}=ccc(U) and M^{**}=ccc(M).*

Lemma 1 can be related to our condition (2.1) in the following way.

Lemma 2 *ccc(T)=ccc(U), if and only if U^*=T^*.*

Proof: Suppose *ccc*(*T*)= *ccc(U)*. By Lemma 1, T^{**}=*ccc*(*T*)= *ccc(U)*. Also by Lemma 1, U^* and T^* are closed convex cones, and thus U^{***}=U^* and T^{***}=T^*. Taking the dual of T^{**}=*ccc(U)*=U^{**} yields T^{***}=U^{***}. Now suppose T^*=U^*. Then it follows that T^{**}=U^{**}, which by Lemma 1 implies *ccc*(*T*)= *ccc(U)*.

Lemma 2 refers to a condition that is close, but not exactly the same as, (2.1): the constant functions are not included. The following Lemma specializes Lemma 2 to the case where we restrict attention to measures $m \in Z^n$, and states the result in a way which will map most closely into our stochastic dominance formulation. The proof of this Lemma is based directly on the Hahn-Banach theorem, rather than relying on Lemma 2, in order to clarify the structure of the result and the role of the constant functions in the analysis.

Lemma 3 *Consider a pair of sets of payoff functions* (*U*,*T*), *where U and T are subsets of P^n. Then the following two conditions are equivalent:*

(i) $\left\{m \in Z^n \,\middle|\, \int u \; dm \geq 0 \;\; \forall u \in U\right\} = \left\{m \in Z^n \,\middle|\, \int u \; dm \geq 0 \;\; \forall u \in T\right\}$.

(ii) $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$.

Proof: First consider (ii) implies (i). We begin by establishing that:

$$\left\{m \in Z^n \,\middle|\, \int u \; dm \geq 0 \;\; \forall u \in U\right\} = \left\{m \in Z^n \,\middle|\, \int u \; dm \geq 0 \;\; \forall u \in ccc(U \cup \{\mathbf{1}, -\mathbf{1}\})\right\}. \quad (2.3)$$

Since $U \subset ccc(U \cup \{\mathbf{1}, -\mathbf{1}\})$, it suffices to show that *m* is in the RHS set of (2.3) whenever it is in the LHS set of (2.3). It follows for the constant functions because $\int dm = -\int dm = 0$. This follows for positive combinations of functions in $U \cup \{\mathbf{1}, -\mathbf{1}\}$ because $\int u_1 \; dm \geq 0$ and $\int u_2 \; dm \geq 0$ implies $\int [a_1 u_1 + a_2 u_2] \; dm = a_1 \int u_1 \; dm + a_2 \int u_2 \; dm \geq 0$ when $a_1, a_2 \geq 0$. It follows for the closure of the set because we have chosen the weakest topology such that $\int u dm$ is continuous in *u*, and for a continuous function *f*, $f(\bar{A}) \subseteq \overline{f(A)}$.

Under the hypothesis that $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$, we can apply (2.3) replacing *U* with *T* to get the result.

Now we prove that (i) implies (ii). Define $\tilde{U} \equiv ccc(U \cup \{\mathbf{1}, -\mathbf{1}\})$ and $\tilde{T} \equiv ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$. Suppose (without loss of generality) that there exists a $\hat{u} \in \tilde{U}$ such that $\hat{u} \notin \tilde{T}$. We know that the $s(P^n, M^n)$ topology is generated from a family of open, convex neighborhoods. Recall from above that the set of continuous linear functionals on P^n is exactly the set $\left\{ b(\cdot,m) \middle| m \in M^n \right\}$. Using these facts, a corollary to the Hahn-Banach theorem implies that since $\tilde{T}$ is closed and convex, there exists a constant c and an $m_* \in M^n$ (a separating hyperplane)·so that $b(u,m_*) \geq c$ for all $u \in \tilde{T}$, and $b(\hat{u},m_*) < c$.

Since $\{\mathbf{1},-\mathbf{1}\} \in \tilde{T}$ and is $\tilde{T}$ convex, $0 \in \tilde{T}$ as well. Thus, $b(0,m_*) = 0 \geq c$. Now we will argue we can take $c = 0$ without loss of generality. Suppose not. Then there exists a $\hat{u} \in \tilde{T}$ such that $c \leq b(\hat{u},m_*) = \hat{c} < 0$. Choose any positive scalar r such that $r > \frac{c}{\hat{c}} \geq 1$ (which implies that $r\hat{c} < c$). Since $\tilde{T}$ is a cone, $r\hat{u} \in \tilde{T}$. But, $b(r\hat{u},m_*) = r\hat{c} < c$, contradicting the hypothesis that $b(u,m_*) \geq c$ for all $u \in \tilde{U}$. So, we let $c = 0$.

Because $\{\mathbf{1},-\mathbf{1}\} \in \tilde{T}$, and $b(u,m_*) \geq 0$ for all $u \in \tilde{T}$, we conclude that $b(1,m_*) = -b(1,m_*) = 0$, and thus $\int dm_* = 0$.[10] So, $m_* \in Z^n$, and we have shown that $\int u \; dm_* \geq 0$ for all $u \in \tilde{T}$, but $\int \hat{u} dm_* < 0$, which violates condition (i).

The proof uses a separating hyperplane argument, and thus the topology, which determines the meaning of closure, is critical for determining the form of the hyperplane. By appropriately choosing the topology and the two spaces of functions, Lemma 3 has produced the following result: if (ii) fails, there exists a measure $m_* \in Z^n$ such that $\int u \; dm_* \geq 0$ for all $u \in ccc(T \cup \{\mathbf{1},-\mathbf{1}\})$, but $\int \hat{u} dm_* < 0$ for some $\hat{u} \in ccc(U \cup \{\mathbf{1},-\mathbf{1}\})$. This m_* is the separating hyperplane.[11]

The key consequence of our construction is that $m_* \in Z^n$, which implies that there exists F^1 and F^2 such that $\alpha m_* = F^1 - F^2$ for some $\alpha > 0$. Thus, this separating hyperplane can be used to generate a counterexample to a stochastic dominance theorem, by letting $\Theta = \{\theta_H, \theta_L\}$, and letting $F(\cdot;\theta_H) = F^1$ and $F(\cdot;\theta_L) = F^2$. Then, we have $\int_{\mathbf{s}} u(\mathbf{s}) dF(\mathbf{s};\theta)$ nondecreasing in θ $\forall u \in T$, but $\int_{\mathbf{s}} u(\mathbf{s}) dF(\mathbf{s};\theta)$ is decreasing for $\hat{u} \in U$. This counter-example establishes the proof of Theorem 1.

Theorem 1 makes use of the weak topology, which might be less familiar than some others; in particular, some existing stochastic dominance theorems have been proved by showing that the closure under monotone convergence of the convex cones of the two sets are equal (Brumelle and Vickson, 1975; Topkis, 1968). However, recall that $s(P^n, M^n)$ is the coarsest topology that makes the bilinear functional continuous. Since it is locally convex, and since any two locally convex topologies

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10 See also McAfee and Reny (1992) for a related application of the Hahn-Banach theorem, where the separating hyperplane also takes the form of an element of Z^n.

11 Notice that if F is not restricted to be a probability distribution, it is not important that $m_* \in Z^n$, and thus Lemma 2 establishes the remark following Theorem 1.

that are compatible with $s(P^n, M^n)$ have the same closed convex sets, results obtained elsewhere using other topologies are no different that those obtained here.

2.4 *Other Closed Convex Cone Properties of* $V(\mathbf{x},\theta)$ *in* θ.

This subsection derives necessary and sufficient conditions for the objective function, $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta)$, to satisfy properties P other than "nondecreasing in θ," for example, the properties supermodular or concave in θ. We ask two questions: (i) For what properties P is the closed convex cone method of proving stochastic P theorems valid? (ii) For what properties P can we establish that the closed convex cone approach is exactly the right one, as in the case of stochastic dominance?

To begin, we introduce a construct that we will call *stochastic P theorems*, which are precisely analogous to stochastic dominance theorems. We are interested in properties P together with parameter spaces Θ_P that are defined so that, given a function $h:\Theta_P \to \Re$, the statement "$h(\theta)$ satisfies property P on Θ_P" is well-defined and takes on the following values: "true" or "false." Let $\bar{\Theta}_P$ denote the set of all such parameter spaces Θ_P. When P represents concavity, Θ_P can be any convex set, while for supermodularity, it must be a lattice. Further, for a given property P, we will define the set of admissible parameter spaces together with probability distributions parameterized on those spaces:

$$D_P^n \equiv \left\{(F,\Theta_P)\middle|\Theta_P \in \bar{\Theta}_P \text{ and } F \in \Delta^n_{\Theta_P}\right\}.$$

Definition 2 *Consider a pair of sets of payoff functions (U,T), with typical elements* $u:\Re^n \to \Re$.
The pair (U,T) is a ***stochastic P pair*** *if for all* $(F,\Theta_P) \in D_P^n$:

$\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta)$ ***satisfies property P*** *on* Θ_P *for all u Î U, if and only if*

$\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta)$ ***satisfies property P*** *on* $_P$ *for all u Î T.*

With this definition in place, it will be straightforward to show that the closed convex cone method of proving stochastic dominance theorems can be extended to all stochastic P theorems, if P is a closed convex cone property, or a "CCC property," defined as follows:

Definition 3 *A property P is a CCC property if the set of functions* $g \quad _P$

Note that we are using closure under the topology of pointwise convergence for properties P, while we are using closure under the weak topology (as defined in Section 2.1) for sets of payoff functions, U and T.

The properties nondecreasing, concave, and supermodular are all CCC properties, as is the property "constant." Further, any property that places a sign restriction on a mixed partial derivative is CCC. Finally, since the intersection of two closed convex cones is itself a closed convex cone, any of these properties can be combined to yield another CCC property. For example, the property "nondecreasing and convex" is a CCC property.

The following result shows that the closed convex cone approach applies to CCC properties.[12]

Theorem 2 *Suppose property P is a CCC property. If* (*U*,*T*) *is a stochastic dominance pair, then* (*U*,*T*) *is a stochastic P pair. Equivalently, if* $ccc(U \cup \{\mathbf{1},-\mathbf{1}\}) = ccc(T \cup \{\mathbf{1},-\mathbf{1}\})$, *then* (*U*,*T*) *is a stochastic P pair.*

Proof: We establish a series of implications, referring to the following condition:

$$\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta) \text{ satisfies } P \qquad (2.4)$$

First, (i) (2.4) holds $\forall u \in T$ implies (ii) (2.4) holds $\forall u \in T \cup \{\mathbf{1},-\mathbf{1}\}$, because $\int_s dF(\mathbf{s};q) = 1$, and constant functions satisfy *P* by definition.

Second, (ii) implies (iii) (2.4) holds $\forall u \in cc(T \cup \{\mathbf{1},-\mathbf{1}\})$, because the integral is a linear functional and *P* is a CCC property.

Third, (iii) implies (iv) (2.4) holds $\forall u \in ccc(T \cup \{\mathbf{1},-\mathbf{1}\})$. To see this, recall that a subset of a topological space is closed if and only if it contains the limits of all the convergent nets of elements in that set. Since the linear functional $b(\cdot;m)$ is continuous for all *m*, for any net u_a in $T \cup \{\mathbf{1},-\mathbf{1}\}$ such that $u_a \to u$, then given q_0, $\int_{\mathbf{s}} u_a(\mathbf{s})\, dF(\mathbf{s};q_0) \to \int_{\mathbf{s}} u(\mathbf{s})\, dF(\mathbf{s};q_0)$. Since *P* is a CCC property, and $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};q)$ is the pointwise limit of a net of functions which satisfy *P*, then $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};q)$ satisfies *P* as well.

Finally, (iv) implies (2.4) holds $\forall u \in U$, by the hypothesis of the theorem and since $U \subset ccc(U \cup \{\mathbf{1},-\mathbf{1}\})$. Precisely analogous arguments establish the symmetric implication.

This theorem generates many new classes of stochastic *P* theorems, where the (*U*,*T*) pairs which have been identified in the large literature on stochastic dominance are also stochastic *P* pairs. As a special case, this theorem generalizes a result by Topkis (1968), who proves that the (*U*,*T*) pair corresponding to nondecreasing functions and indicator functions of nondecreasing sets, respectively, is a stochastic *P* pair when *P* is a *CCC* property.

[12] It should be noted that the following is critical for the result: constant functions satisfy property *P*, whenever *P* is a CCC property. Without that assumption, (*U*,*T*) is a stochastic *P* pair if and only if *ccc*(*U*)=*ccc*(*T*). If *U* does not contain constant functions (an example, single crossing at a point, will be discussed below in Section 3), this distinction is substantive.

2.4.1 Some Simple Examples

Using Theorem 2, we can apply the results of Table I to problems of stochastic supermodularity. Consider the following optimization problem:

$$\max_z \int_s u(s)dF(s;z,t) - c(z)$$

In our first example, the optimizer is a firm making investments in research and development (*z*) to improve its production process, and in particular it searches for ways to reduce its unit production costs (*s*). The returns to research and development are uncertain, and the probability distribution over the firm's future production costs is parameterized by *t*. Suppose that the firm's payoffs are nonincreasing in its production costs, and that the firm's investment has a cost, *c*(*z*). Then, Theorem 2 (together with a comparative statics theorem of Milgrom and Shannon (1994)) implies that investment (*z*) is nondecreasing in *t* for all *c* and all *u* nonincreasing, if and only if $F(s;z,t)$ is supermodular in (*z*,*t*). Intuitively, the goal of the firm's investment in research is to shift probability weight towards lower realizations of its unit cost. When *F* is supermodular, the parameter *t* indexes the "sensitivity" of the probability distribution to investments in research: higher values of *t* correspond to probability distributions where research is more effective at lowering unit costs.

In the second example, we consider a risk-averse worker's choice of effort (*z*), as in the classic formulation of the principal-agent problem (Holmstrom, 1979). Let *s* represent the agent's wealth (which might be determined by a sharing rule, though we will not model that here), let *z* be the choice of effort, let *c*(*z*) be the cost of effort, and let *t* be a parameter which describes the job assignment or production technology. Then expected utility (ignoring *c*(*z*) for the moment) is nondecreasing in *z* for all *u* nondecreasing and concave, if and only if *z* shifts *F* according to second order monotonic stochastic dominance, that is, $\int_{s=-\infty}^{a} F(s;z,t)ds$ nonincreasing in *z* for all *a*.

Next, observe that the optimal choice of effort is nondecreasing for all *c*(*z*), if and only if $-\int_{s=-\infty}^{a} F(s;z,t)ds$ is supermodular in (*z*,*t*) for all *a*. Finally, consider the analogous "stochastic concavity" result: expected utility is concave in *z* if and only if $-\int_{s=-\infty}^{a} F(s;z,t)ds$ is concave for all *a*. The latter result is particularly useful, since it can be used to establish that the First-Order Approach is valid in the analysis of principal-agent problems. In fact, Jewitt (1988) establishes just this result, applying the more standard approach of integration by parts.

2.4.2 Necessary and Sufficient Conditions for "Stochastic P Theorems"

This section considers the question of whether the "test functions" approach can be improved upon for "stochastic *P* theorems," when *P* is a property other than nondecreasing. To be more precise, it will be helpful to introduce some additional notation. We let Σ_{SDT} be the set of (*U*,*T*) pairs that are stochastic dominance pairs, and we let Σ_{SPT} be the set of all stochastic *P* pairs for some property *P*. Then, observe that we have *not* established that $\Sigma_{SPT} = \Sigma_{SDT}$ for arbitrary closed convex cone properties *P*. This opens the door for the possibility that one could check a stochastic *P* theorem for a set *U* *without* necessarily checking all of the extreme points of *U*, *E*(*U*). Our final endeavor in this section is to identify which properties *P* are such that (*U*,*T*) satisfies a stochastic *P* theorem *if and only if* (2.1) holds.

To see an example where (2.1) is too strong, consider a simpler example of a CCC property, the property "constant in **θ**." Take the case of U^{FO}, the set of all univariate, nondecreasing payoff functions, and the set $\hat{T} = -U^{FO}$. The pair $(U^{FO}, \hat{T})$ satisfies a "stochastic constant theorem," since $\int u(s)dF(s;q)$ is constant in **θ** if and only if $-\int u(s)dF(s;q)$ is constant in **θ**. However, $(U^{FO}, \hat{T})$ clearly is not a stochastic dominance pair.

In general, the result that $\Sigma_{SPT} = \Sigma_{SDT}$ is useful because it may be easier to verify whether or not (*U*,*T*) is a stochastic P pair by drawing from the existing literature on stochastic dominance. For example, if the characteristics of a set *U* are determined by an economic problem, and this set *U* has been analyzed in the stochastic dominance literature, then the corresponding stochastic supermodularity theorem is immediate. This allows us to bypass the step of checking whether *U* and *T* have the same closed convex cone directly (further, most of the stochastic dominance literature does not make such a statement explicitly).

Thus motivated, we begin by giving an example of a second property, supermodular, for which (*U*,*T*) are a stochastic *P* pair *if and only if* $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$. Before proceeding, we formally define supermodularity and some related concepts from lattice theory. Given a set *X* and a partial order ≥, the operations "meet" (∨) and "join" (∧) are defined as follows: $\mathbf{x} \vee \mathbf{y} = \inf\{\mathbf{z} | \mathbf{z} \geq \mathbf{x}, \mathbf{z} \geq \mathbf{y}\}$ and $\mathbf{x} \wedge \mathbf{y} = \sup\{\mathbf{z} | \mathbf{z} \leq \mathbf{x}, \mathbf{z} \leq \mathbf{y}\}$. A lattice is a set *X* together with a partial order, such that the set is closed under meet and join. A function $h: X \to \Re$ is **supermodular** if, for all $\mathbf{x}, \mathbf{y} \in X$, $h(\mathbf{x} \vee \mathbf{y}) + h(\mathbf{x} \wedge \mathbf{y}) \geq h(\mathbf{x}) + h(\mathbf{y})$. If *h* is smooth and $X = \Re^n$, supermodularity corresponds to the restriction that $\frac{\partial^2}{\partial x_i \partial x_j} h(\mathbf{x}) \geq 0$ for all **x** and all $i \neq j$ (Topkis, 1978).

Recall that, in Lemma 3, we showed that if $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) \neq ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$, there exists a $m_* \in Z^n$ (normalized so that $\int dm_*^+ = 1$) such that $\int u \, dm_* \geq 0$ for all $u \in ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$, but $\int \hat{u} dm_* < 0$ for some $\hat{u} \in ccc(U \cup \{\mathbf{1}, -\mathbf{1}\})$. This m_*, our separating hyperplane, can be expressed as the difference between two probability distributions. But, observe that it can also be expressed as follows:

$$m_* = \tfrac{1}{2}[m_*^+ - m_*^- - [m_*^- - m_*^+]].$$

Now, let $\Theta = \{\theta^1 \vee \theta^2, \theta^1, \theta^1 \wedge \theta^2, \theta^2\}$. Define a parameterized distribution, $G(\cdot;\theta)$, as follows: $G(\cdot;\theta^1 \vee \theta^2) = G(\cdot;\theta^1 \wedge \theta^2) = \frac{m_*^+}{\int dm_*^+}$, and $G(\cdot;\theta^1) = G(\cdot;\theta^2) = \frac{m_*^-}{\int dm_*^+}$. Then, $\int u\ dm_* \geq 0$ if and only if $\int u\ dG(\cdot;\theta)$ is supermodular, since the latter entails

$$\int u(\mathbf{s})dG(\mathbf{s};\theta^1 \vee \theta^2) - \int u(\mathbf{s})dG(\mathbf{s};\theta^1) - \left[\int u(\mathbf{s})dG(\mathbf{s};\theta^2) - \int u(\mathbf{s})dG(\mathbf{s};\theta^1 \wedge \theta^2)\right] \geq 0.$$

Thus, we can conclude that for this G, $\int u\ dG(\cdot;\theta)$ is supermodular for all $u \in ccc(T \cup \{\mathbf{1}, \mathbf{-1}\})$, yet $\int \hat{u}\ dG(\cdot;\theta)$ fails to be supermodular, and we contradict the definition of a "stochastic P theorem." The critical feature of supermodularity in this example is that the separating hyperplane, an element of Z^n, can be used to generate the needed counterexample to the stochastic supermodularity theorem.

Another way to understand this result is to recognize that supermodularity is checked by looking at differences of differences of functions (evaluated at different parameter vectors). Since differences of differences of probability distributions live in the same space as differences of probability distributions, namely Z^n, we can use the same approach to characterize the property supermodularity that we used to characterize monotonicity.

We can formalize this requirement with the following definition:

Definition 4 *A property P satisfies the* ***single inequality condition*** *if, for any m Î* Z^n *and any u Î* P^n, *there exists* $(F,\Theta) \in D_P^n$ *such that* $\int udm \geq 0$ *if and only if* $\int u(\mathbf{s})dF(\mathbf{s};\theta)$ *satisfies P on* Θ.

The single inequality is a strong condition. We have already established that nondecreasing and supermodular satisfy it. Consider some additional examples. First, observe that any discrete generalization of a (single) sign restriction on a mixed partial derivative is a difference of differences: if $\mathrm{T}_e^i f(\mathbf{x}) = f(\mathbf{x}) - f(x_i - e, \mathbf{x}_{-i})$, then $\mathrm{T}_e^i \circ \cdots \circ T_e^k \int u(\mathbf{s})dF(\mathbf{s};\theta) \geq 0$ if and only if $\int udm \geq 0$ for $m(\mathbf{s}) = \mathrm{T}_e^i \circ \cdots \circ T_e^k F(\mathbf{s};\theta) \in Z^n$. Finally, we consider concavity.

Example 1 *"Concave" satisfies the single inequality condition.*

Proof: Let $\Theta = [0,1]$. Consider $m_* \in Z^n$ and $u \in P^n$. Define $G(\cdot;0) = G(\cdot;1) = \frac{m_*^-}{\int dm_*^+}$,

$G(\cdot;\theta) = (1-2\theta)\frac{m_*^-}{\int dm_*^+} + 2\theta\frac{m_*^+}{\int dm_*^+}$ for 0<θ≤1/2, and $G(\cdot;\theta) = (2\theta-1)\frac{m_*^-}{\int dm_*^+} + (2-2\theta)\frac{m_*^+}{\int dm_*^+}$ for 1/2≤θ<1. Since G is linear in θ on [0,1/2] and (1/2,1], $\int_s u(\mathbf{s})dG(\mathbf{s};\theta)$ is concave if and only if $\int_s u(\mathbf{s})dG(\mathbf{s};\tfrac{1}{2}) \geq \tfrac{1}{2}\int_s u(\mathbf{s})dG(\mathbf{s};0) + \tfrac{1}{2}\int_s u(\mathbf{s})dG(\mathbf{s};1)$. But, with the above definition, this is equivalent to $\int_s u(\mathbf{s})dm_*^+ \geq \tfrac{1}{2}\int_s u(\mathbf{s})dm_*^- + \tfrac{1}{2}\int_s u(\mathbf{s})dm_*^-$, or $\int_s u(\mathbf{s})dm_* \geq 0$.

It is interesting to note that in general, the intersection of two properties that satisfy the single inequality condition does not necessarily satisfy the single inequality condition. For example, consider the property nondecreasing and concave. Let Θ =[0,2], and let $F(\mathbf{s};\boldsymbol{\theta})$ be linear in $\boldsymbol{\theta}$ on [0,1/2) and (1/2,1]. Then, $\int u(\mathbf{s})dF(\mathbf{s};\boldsymbol{\theta})$ is nondecreasing and concave if and only if $\int udm_1 \geq 0$, $\int udm_2 \geq 0$, and $\int udm_3 \geq 0$, where $m_1 = F(\cdot;1) - F(\cdot;1/2)$, $m_2 = F(\cdot;1/2) - F(\cdot;0)$, and $m_3 = F(\cdot;1/2) - F(\cdot;0) - [F(\cdot;1) - F(\cdot;1/2)]$.

However, since the intersection of two CCC properties is a CCC property, we can always apply the closed convex cone approach for the intersection of two properties that satisfy the single inequality condition. Given this, it is interesting to note that supermodularity, which for a suitably differentiable function $f(\mathbf{x})$ can be defined as requiring that $\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) \geq 0$ for all $i \neq j$, satisfies the single inequality condition. This is true because supermodularity is a property which (i) can be defined on an arbitrary lattice and (ii) when the lattice has four or fewer points, supermodularity of a function on that lattice can be expressed in terms of a single inequality, so that the single inequality condition can be satisfied.

We now state the final result of this section, which is that if P is an LDP and satisfies the single inequality condition, then the same mathematical structure underlies the stochastic P theorem as a stochastic dominance theorem.

Theorem 3 *If P is a CCC and the single inequality condition holds, then conditions (i)-(iii) are equivalent:*

(i) (U,T) *is a stochastic P pair.*

(ii) (U,T) *is a stochastic dominance pair.*

(iii) $ccc(U \cup \{\mathbf{1},-\mathbf{1}\}) = ccc(T \cup \{\mathbf{1},-\mathbf{1}\})$.

Proof: Theorem 1 establishes the equivalence of (ii) and (iii), and Theorem 2 gives (iii) implies (i). So it remains to show that (i) implies (iii). Suppose (iii) fails, and there exists a $\hat{u} \in \tilde{U}$ such that $\hat{u} \notin \tilde{T}$ (the sets are short-hand for the closed convex cones, as in Lemma 3). Then by the proof of Lemma 3, there exists $m_* \in Z^n$ such that $\int u \, dm_* \geq 0$ for all $u \in \tilde{T}$, but $\int \hat{u} dm_* < 0$. By the single inequality condition, this implies that there exists $(F,\Theta) \in D_P^n$ such that $\int \hat{u}(\mathbf{s})dF(\mathbf{s};\boldsymbol{\theta})$ fails P on Θ, but $\int u(\mathbf{s})dF(\mathbf{s};\boldsymbol{\theta})$ satisfies P for all $u \in \tilde{T}$.

The consequence of Theorem 3 can be stated as follows: if P satisfies the single inequality condition, then $\Sigma_{SPT} = \Sigma_{SDT}$. This result establishes that the closed convex approach to stochastic P theorems exactly the right one for this class of properties. Note that Theorem 3 holds without any assumptions about continuity, differentiability, or other such properties. Linearity of the functional $b(u,m)$ in m, however, is critical.

2.5 Applications: Stochastic Supermodularity Theorems with Supermodular Payoffs

This section explores applications of the new stochastic supermodularity theorems that follow from Theorem 3. By Theorem 3, each of the (existing) stochastic dominance results corresponds to a (new) stochastic supermodularity result, which may then be applied to solve comparative statics problems. The following example is derived by applying Theorem 3 to the stochastic dominance theorem described in Table I (v):

Example 2 $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta)$ *is supermodular for all* $u:\Re^2\rightarrow\Re$ *supermodular, if and only if (i) for all* $\mathbf{s}$, $F(\mathbf{s};\theta)$ *is supermodular in* θ; *and (ii) for i=1,2, and for all* s_i, *both* $F_i(s_i;\theta)$ *and* $-F_i(s_i;\theta)$ *are supermodular in* θ.

Part (i) requires the parameters are complements in increasing the joint distribution. When Θ is a product set, part (ii) requires that the marginal distributions are additively separable in the components of θ. To understand this result, it is helpful to consider the intuition for the stochastic dominance theorem for bivariate, supermodular payoff functions. The conditions in Table I (v) require that the marginal distribution of each random variable is unchanged by the parameter θ. To see why, observe that a supermodular function might be additively separable, and either monotone decreasing *or* monotone increasing. Then, a FOSD shift in a marginal distribution will raise expected profits for some supermodular payoff functions, and lower expected payoffs for others.

Now, consider a partition of the space $(s_1,s_2)\in\Re^2$ into four quadrants, delineated by the axes $s_1=a_1$ and $s_2=a_2$. Then the requirement that the function $\iint_{s_1,s_2}\mathbf{1}_{[a_1,\infty)}(s_1)\cdot\mathbf{1}_{[a_2,\infty)}(s_2)\cdot dF(s_1,s_2;\theta)$ must be nondecreasing in θ specifies that θ shifts probability mass into the northeast quadrant. When the marginal distributions are constant in θ, that is equivalent to requiring that $F(a_1,a_2;\theta)$ is nondecreasing in θ. We will refer to this type of shift as an increase in the "interdependence" of the random variables; such a shift is beneficial when the random variables are complementary in increasing the payoff function.

The intuition for the stochastic supermodularity theorem for the set of supermodular payoffs builds directly from the stochastic dominance intuition. For two parameters to be complementary in increasing the expected value of a supermodular payoff function, they must not interact in the marginal distribution functions, and further they must be complementary in increasing the interdependence of the random variables.

To see a special case of how Table I (v) might be used, consider the following example. A firm engages in product design for a product with two components, and the random variables, s_1 and s_2, represent the qualities of the two components. The components fit together in such a way that increasing the quality of one component increases the returns to quality in the other component. The two product design teams work to develop the two different components. Each team's output is stochastic, but the outputs of the two teams are correlated (perhaps due to realizations of random events that affect the whole firm, or due to communication between the two teams). The firm wishes to set target qualities (or incentive contracts) for each group, where an increase in the target quality increases the expected quality the group will produce.

The above example gives necessary and sufficient conditions for supermodularity of the firm's objective. This result would be have the following implications: increasing the target quality for one group increases the returns to increasing the target quality of the second group. Further, in response to an exogenous decrease in the cost of producing quality for team one, the firm would find it optimal to raise the targets for *both* teams.

The conditions for "stochastic supermodularity" can be verified in a specific functional form example. Suppose that the random variables have a bivariate normal distribution with a positive covariance ($(s_1, s_2) \sim BVN(m_1, m_2, s_1^2, s_2^2, s_{12})$; $s_{12} \geq 0$). Then checking the conditions given in Table I (v) establishes that if u is supermodular, the expected profits are supermodular in the means of each marginal distribution. That is, $\int_{s_1, s_2} u(s_1, s_2) dF(s_1, s_2; m_1, m_2, s_{12})$ is supermodular in (m_1, m_2), for all u supermodular, if $s_{12} \geq 0$. Intuitively, when the payoff function is such that under certainty, one random variable increases the returns to the other, then in a stochastic environment, raising the mean of one random variable increases the returns to raising the mean of the other. Further, since the random variables have a positive covariance, increasing the mean of one does not decrease the effectiveness of the other in terms of shifting probability weight into regions where the random variables realize high or low values together. Of course, the requirement that the parameters do not interact in the marginal distribution functions is satisfied in this example. Thus, in the product design example, we conclude that if the outputs of the product design teams are joint normal with positive correlation, the target qualities of the two teams will be complements in the firm's profit function.

Now we turn briefly to multivariate, supermodular payoff functions. Multivariate payoff functions pose a particularly complicated problem because changes in the joint probability distribution can potentially affect the co-movements of many random variables simultaneously; thus, the high-order mixed partial derivatives between all of the arguments of the payoff function are relevant for stochastic dominance results. However, it is often difficult to place economic interpretations on such derivatives. One approach, followed by Meyer (1990) in the context of an application to income inequality, is to consider objective functions where higher-order derivatives are assumed to be zero. Meyer (1990) considered supermodular objectives where interactions between three or more variables are ruled out. Supermodular payoffs of this form can be used to represent a social welfare function that is averse to inequality. Our results could be used in that context as well; a stochastic supermodularity theorem could be used to check when two policies are complementary in increasing expected social welfare.

The following result takes an alternative approach, studying the case where the random variables are independent.

Theorem 4 *Let* **s** *be a vector of independent random variables. Further, suppose that* θ *is a vector of parameters such that for i=1,...,n, the marginal distribution of* s_i *is given by* $F_i(s_i;q_i)$. *Then the following two conditions are equivalent:*

(i) For all supermodular payoff functions $u:\Re^n \to \Re$, $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};q)$ *is supermodular in* θ.

(ii) Either (a) or (b) is true:

(a) For $i=1,..,n$, $F_i(s_i;q_i^H) \geq F_i(s_i;q_i^L)\ \ \forall q_i^H \geq q_i^L,\ \ \forall s_i \in \Re$.

(b) For $i=1,..,n$, $F_i(s_i;q_i^H) \leq F_i(s_i;q_i^L)\ \ \forall q_i^H \geq q_i^L,\ \ \forall s_i \in \Re$.

Proof: For all (*i*,*j*) pairs, rewrite the expectation as

$$\int_{s_i,s_j} \int_{\mathbf{s}_{n\backslash ij}} u(\mathbf{s}_{n\backslash ij};s_i,s_j)dF_{n\backslash ij}(\mathbf{s}_{n\backslash ij};\boldsymbol{\theta}_{n\backslash ij}) \cdot dF(s_i;q_i) \cdot dF(s_j;q_j).$$

Note that the inner integral is supermodular in (s_i,s_j) since supermodularity is preserved by sums. Apply Table I (v) and check that the supermodularity condition on the distributions reduces to (ii) (a) or (b).

This result says that parameters that induce FOSD shifts in independent random variables are complementary in increasing the expected value of supermodular payoff functions.[13] Thus, if a payoff function is supermodular in a group of random variables, increasing one variable in the stochastic sense (of FOSD) is complementary with increasing the others stochastically as well. Returning to the joint normal distribution, if the random variables are independent, the conditions are satisfied when q_i represents the mean of random variable s_i.

Theorem 4 result has potential applications in the study of coordination problems in firms (recall the product design example from above) as well as in general investment problems. Athey and Schmutzler (1995) apply this result to analyze a firm's choices over investments in product and process innovation.

Theorem 4 can also be applied to investment games. For example, consider a game between *n* firms, where each firm *i* makes an investment q_i that shifts the distribution over the firm's own state

[13]Since a supermodular function is also supermodular in the negative of all of its arguments, we allow for either case (ii)(a) or (ii)(b) in Theorem 4.6.

variable, s_i. If a higher level of an opponent's state variable (s_j, $j \neq i$) increases the returns to a firm's own state variable (that is, a firm's payoff is supermodular in **s**), and if firm i's investment q_i results in an FOSD improvement in the firm's own state variable, the investments will be strategic complements, and the theory of supermodular games (i.e. Topkis (1979), Milgrom and Roberts (1990), and Vives (1990)) can be applied.

3. COMPARATIVE STATICS PROPERTIES OF $V(\mathbf{x},\theta)$ IN $(\mathbf{x},\theta)$

This section builds on the analysis of Section 2 to characterize interactions between components of **x** and components of **θ** in the function V. Particular emphasis will be placed on properties that are necessary and sufficient for monotone comparative statics predictions.

3.1 Definitions

Since this section will focus on properties relevant to comparative statics predictions, we begin by introducing several relevant definitions. Recall our lattice-theoretic definitions from Section 2.3.2. Then, consider the following definitions (where we restate the definition of supermodularity for ease of comparison with the other properties):

Definition 5 *Let* $(X,\geq)$ *be a lattice, and let* $h: X \rightarrow \Re$. *(i) h is* ***supermodular*** *if, for all* $\mathbf{x},\mathbf{y} \in X$, $h(\mathbf{x} \vee \mathbf{y}) + h(\mathbf{x} \wedge \mathbf{y}) \geq h(\mathbf{x}) + h(\mathbf{y})$. *(ii) h is* ***quasi-supermodular*** *if, for all* $\mathbf{x},\mathbf{y} \in X$, $h(\mathbf{x} \wedge \mathbf{y}) - h(\mathbf{y}) \geq (>) 0$ *implies* $h(\mathbf{x} \vee \mathbf{y}) - h(\mathbf{x}) \geq (>) 0$. *(iii) If h is positive, it is* ***log-supermodular (log-spm)***[14] *if, for all* $\mathbf{x},\mathbf{y} \in X$, $h(\mathbf{x} \vee \mathbf{y}) \cdot h(\mathbf{x} \wedge \mathbf{y}) \geq h(\mathbf{x}) \cdot h(\mathbf{y})$. *(iv) h has* ***increasing differences*** *in* $(\mathbf{x_k};\mathbf{x_{-k}})$ *if* $h(\mathbf{x_k^H},\mathbf{x_{-k}}) - h(\mathbf{x_k^L},\mathbf{x_{-k}})$ *is nondecreasing in* $\mathbf{x_{-k}}$ *for all* $\mathbf{x_k^H} > \mathbf{x_k^L}$.

We will be particularly interested in the special case where our function of interest is bivariate. In that case, if h is smooth, supermodularity corresponds to the restriction that $\frac{\partial^2}{\partial x_1 \partial x_2} h(\mathbf{x}) \geq 0$ (Topkis, 1978); h is log-supermodular if $\log(h)$ is supermodular. Equivalently, supermodularity requires that $h(x_1^H, x_2) - h(x_1^L, x_2)$ is nondecreasing in x_2 for all $x_1^H > x_1^L$, and log-supermodularity requires the same of $h(x_1^H, x_2) / h(x_1^L, x_2)$. Quasi-supermodularity (Milgrom and Shannon, 1994) requires something weaker: the incremental returns, $h(x_1^H, x_2) - h(x_1^L, x_2)$, must satisfy a single crossing property. The following definition lays out several variants of single crossing properties that will be useful in our analysis.

[14] Karlin and Rinott (1980) called log-supermodularity multivariate total positivity of order 2.

Definition 6 *Let X and Θ be partially ordered sets. (i) g(q) satisfies single crossing (**SC1**) in q if, for all* $q_H>q_L$, $g(q_L)\geq(>)0$ *implies* $g(q_H)\geq(>)0$. *(ii)* $h(\mathbf{x},q)$ *satisfies single crossing of incremental returns (**SC2**) in* $(\mathbf{x};q)$ *if, for all* $\mathbf{x}_H\geq\mathbf{x}_L$, $g(q)=h(\mathbf{x}_H;q)-h(\mathbf{x}_L;q)$ *satisfies SC1.*

Topkis (1978, 1979) introduced some basic comparative statics theorems based on supermodularity. Building on his work, Milgrom and Shannon (1994) show that $\mathbf{x}^*(q,B)=\arg\max_{\mathbf{x}\in B} h(\mathbf{x},q)$ is nondecreasing in (q,B) in the "strong set order"[15] if and only if h is quasi-supermodular in $\mathbf{x}$ and satisfies SC2 in $(\mathbf{x};q)$. If $\mathbf{x}\in\Re$, the result simply requires that h satisfies SC2 in $(\mathbf{x};q)$. Thus, we will focus special attention on necessary and sufficient conditions for $V(\mathbf{x},q)$ to satisfy SC2 in $(\mathbf{x};q)$.

3.2 Stochastic Single Crossing when Primitives Form a Closed Convex Cone

The last subsection motivates the study of necessary and sufficient conditions for $V(\mathbf{x},q)$ to satisfy SC2 in $(\mathbf{x};q)$. Since this property does not concern interactions between components of $\mathbf{x}$ or components of $\boldsymbol{\theta}$, we change notation to consider properties of $V(x,q)$, where the italic variables are real numbers. In this context, it is clearly equivalent to study when $V(x,q)$ satisfies SC2, and to study when $V(x^H,q)-V(x^L,q)=\int[u(x^H,\mathbf{s})-u(x^L,\mathbf{s})]dF(\mathbf{s};q)$ satisfies SC1 in q.

However, in other contexts, we might wish to study SC2 of $V(x,q)$ in $(q;x)$, or quasi-supermodularity of $V(x,q)$. Thus, we present a theorem that allows the roles of the utility function and the probability distribution to be interchanged: we analyze $\int u(\mathbf{s})dK(\mathbf{s};q)$, but do not restrict K to be a probability distribution.

Theorem 5 *Suppose that ccc(T)=G. Let* $K(\cdot;q)$ *be a finite signed measure. Then the following two conditions are equivalent:*

(i) $V(q)\equiv\int g(\mathbf{s})dK(\mathbf{s};q)$ *satisfies SC1 in q for all* $g\in G$.

(ii) For all $q_H\geq q_L$, *there exists a* $l>0$ *such that* $\int t(\mathbf{s})dK(\mathbf{s};q_H)\geq l\int t(\mathbf{s})dK(\mathbf{s};q_L)$ *for all* $t\in T$ *(that is,* $K(\mathbf{s},q_H)-l\cdot K(\mathbf{s},q_L)\in T^*$*).*

Proof: Observe that (a) $V(q_L)\geq0$ implies $V(q_H)\geq0$ for all $g\in G$, if and only if (b) $g\in G$ and $g\in K(\mathbf{s},q_L)^*$ implies $V(q_H)\geq0$, if and only if (c) $K(\mathbf{s},q_H)\in(K(\mathbf{s},q_L)^*\cap G)^*=ccc(K(\mathbf{s},q_L)\cup G^*)$. (The latter equality follows because $(K(\mathbf{s},q_L)\cup G^*)^*=K(\mathbf{s},q_L)^*\cap G^{**}$, simply applying definitions, and G is assumed to be a closed convex cone, so that $G=G^{**}$). In turn, (c) holds if and only if (d) $K(\mathbf{s},q_H)=\lambda K(\mathbf{s},q_L)+\alpha\, m(\mathbf{s})$ for some $m(\mathbf{s})\in G^*$, and some $\alpha,\lambda\geq0$. But then,

[15] $A\geq B$ in the strong set order if, for any $a\in A$ and $b\in B$, $a\vee b\in A$ and $a\wedge b\in B$.

$\int g(\mathbf{s})dK(\mathbf{s};q_H) = \lambda \int g(\mathbf{s})dK(\mathbf{s};q_L) + \alpha \int g(\mathbf{s})dm(\mathbf{s})$. Since $T^*=G^*$ by Lemma 2, and $\int g(\mathbf{s})dm(\mathbf{s}) \geq 0$ by definition of G^*, it follows that $\int g(\mathbf{s})dK(\mathbf{s};q_H) \geq \lambda \int g(\mathbf{s})dK(\mathbf{s};q_L)$.

Now suppose that $l=0$, and suppose that $V(q_L)>0$ for some $g\in G$. Then, we can find a $K(\mathbf{s},q_H) \in T^*$ such that $\int g(\mathbf{s})dK(\mathbf{s};q_H)=0$. But $l=0$ implies that $K(\mathbf{s},q_H)-\lambda K(\mathbf{s},q_H)\in T^*$ as well.

Then, we have $V(q_L)>0$ and $V(q_H)=0$ for some $K(\mathbf{s},q_H)$ where $K(\mathbf{s},q_H)-\lambda K(\mathbf{s},q_H)\in T^*$, contradicting the definition of SC1.

Closely related results have been applied in the economics literature by a few authors; we first interpret the theorem, and then return to discuss the literature in the next subsection.

Sufficiency in Theorem 5 (that is, (ii) implies (i)) is straightforward. If (ii) holds, then whenever $\int t(\mathbf{s})dK(\mathbf{s};q_L) \geq(>)0$, it will follow that $\int t(\mathbf{s})dK(\mathbf{s};q_H) \geq l \int t(\mathbf{s})dK(\mathbf{s};q_L) \geq(>)0$. Taking convex combinations and limits of sequences or nets of these t's yields the implication required.

Now consider necessity. To recast the result in a more familiar form, consider taking the infimum of $\int g(\mathbf{s})dK(\mathbf{s};q_H)$ with respect to g, where g is restricted to lie in G (a closed convex cone), subject to the constraint that $\int g(\mathbf{s})dK(\mathbf{s};q_L) \geq 0$. Then, if the single crossing property holds, the infimum must be nonnegative. This implies that there must exist a Lagrange multiplier λ such that

$$\inf_{g\in G} \int g(\mathbf{s})dK(\mathbf{s};q_H) - l \int g(\mathbf{s})dK(\mathbf{s};q_L) \geq 0.$$

Then it will follow that there exists a l such that

$$\int g(\mathbf{s})dK(\mathbf{s};q_H) - l \int g(\mathbf{s})dK(\mathbf{s};q_L) \geq 0 \text{ for all } g\in G.$$

Of course, this is equivalent to checking that $K(\mathbf{s};q_H) - l\, K(\mathbf{s};q_L) \in G^*$. Finally, we can apply Lemma 2, which establishes that $G^* = T^*$ when $ccc(T)=G$.

Clearly, Theorem 5 and Theorem 1 are very closely related. If we relax the restriction in Theorem 1 that F is a probability distribution and apply it to the problem above, stochastic monotonicity requires that $K(\mathbf{s},q_H) - K(\mathbf{s},q_L) \in T^*$. In contrast, Theorem 5 requires that there exists a positive l such that $K(\mathbf{s},q_H) - l\cdot K(\mathbf{s},q_L) \in T^*$. The latter condition is less stringent, and yields a weaker conclusion (single crossing as opposed to monotonicity).

However, if G contains the constant functions and K is a probability distribution, it turns out that Theorem 1 and Theorem 5 coincide. In particular, we have the following corollary:

Corollary 5.1 *Suppose that* $\{\mathbf{1},-\mathbf{1}\}\in G$ *and* $G=ccc(T\cup\{\mathbf{1},-\mathbf{1}\})$. *Suppose further that* $\int dK(\mathbf{s};q)$ *is constant in q. Then the following three conditions are equivalent:*

(i) $\int g(\mathbf{s})dK(\mathbf{s};q)$ *satisfies SC1 in q for all g*$\in G$*.*

(ii) $\int g(\mathbf{s})dK(\mathbf{s};q)$ *is nondecreasing in q for all g*$\in G$*.*

(iii) $\int t(\mathbf{s})dK(\mathbf{s};q)$ *is nondecreasing in q for all t*$\in T$*.*

Proof: (ii) and (iii) are equivalent by Theorem 1. Clearly, (iii) implies (i), since (iii) implies that condition (B) of Theorem 5 holds when λ=1. If $\{\mathbf{1},-\mathbf{1}\}\in G$, then condition (B) of Theorem 5 requires that $\int dK(\mathbf{s};q_H)\geq\lambda\int dK(\mathbf{s};q_L)$ and $\int dK(\mathbf{s};q_H)\leq\lambda\int dK(\mathbf{s};q_L)$. This holds only if $\int dK(\mathbf{s};q_H)=\lambda\int dK(\mathbf{s};q_L)$. If $\int dK(\mathbf{s};q)$ is constant in *q*, then it must follow that λ=1.

This result is potentially quite powerful, since it implies that in many problems, single crossing holds for all payoff functions in the desired class if and only if the probability distribution is ordered according to stochastic dominance. Thus, Corollary 5.1 reduces a potentially more difficult problem to a well-studied problem. For example, it implies that if *F* is a probability distribution, $\int g(\mathbf{s})dF(\mathbf{s};q)$ satisfies SC1 for all *g* nondecreasing, if and only if *F* is ordered according to FOSD. Likewise, $\int g(\mathbf{s})dF(\mathbf{s};q)$ satisfies SC1 for all *g* concave, if and only if *F* is ordered according to SOSD. Below, we will apply this result to comparative statics problems.

3.2.1 Relation to the Literature

The approach used in our formal proof of Theorem 5 builds on the approach taken in a note by Jewitt (1986) to characterize the notion of "more risk averse" (Ross (1981)). Gollier and Kimball (1995a, b) also prove this result for the case where $\mathbf{s}\in\Re$ or $\mathbf{s}\in\Re^2$ and **s** independent. They call the "diffidence theorem." The Gollier and Kimball's diffidence theorem can be stated in our language as follows (taking *h*, a density, as given, and assuming that $s\in\Re$): $\int k(s,q)h(s)ds - \int k(s,q)f(s)ds$ satisfies SC1 in *q* for all densities *f*, if and only if there exists a $l>0$ such that $\int h(s)k(s,q_H)ds - k(s,q_H) \geq l\, [\int h(s)k(s,q_L)ds - k(s,q_L)]$ a.e. We can place their result in the above framework as follows. If we fix some probability density $h(s)$ and consider $G_h=\{g{:}g=h-f$ for some probability density $f\}$, then we can let $T(G_h) = \{g{:}g(s)=h(s)-\mathbf{1}_{(a-e,a+e)}(s)$ for $e>0$, $a\in\Re\}$. When we restrict attention to a single random variable, their result is equivalent to Theorem 6. Their method of proof is different from the one used here.

Gollier and Kimball show that a wide variety of problems in the theory of investment under uncertainty can be usefully studied with the diffidence theorem, and their papers provide many examples and applications. To see the simplest example where the diffidence theorem applies, let *q* index preferences. Then SC1 of $\int h(s)k(s,q)ds - \int f(s)k(s,q_H)ds$ is interpreted as follows: if an agent with preferences q_L likes the gamble corresponding to density h better than gamble corresponding to f, all agents with preferences $q_H > q_L$ will also prefer gamble h to gamble f.

The contribution of Theorem 5 thus is to provide a statement and formal proof of the theorem at the appropriate level of generality for our purposes, using the basic theorems of topological vector spaces, and then derive theorems which establish its consequences for comparative statics analysis. Thus, we extend the domain of applicability of the tools, and show that these tools can be usefully connected to Milgrom and Shannon (1994)'s methods for comparative statics analysis. Of particular interest is Corollary 5.1, which does not appear in the previous literature; this corollary, which gives conditions under which stochastic dominance and stochastic single crossing are equivalent, will imply that in many economic situations, comparative statics theorems based on supermodularity of a stochastic objective cannot be improved upon. This is taken up in the next section.

3.3 Single Choice Problems and Comparative Statics

This section examines conditions under which $V(x,q) \equiv \int u(x,\mathbf{s})dK(\mathbf{s};q)$ satisfies SC2 in $(x;q)$, which is necessary and sufficient for comparative statics in univariate choice problems. Since we allow for K to be a signed measure (not necessarily a probability distribution), our analysis applies to problems where the choice variable affects either a probability distribution, or a utility function.

Corollary 5.2 *Let K be a finite signed measure, and let U_1, T be sets of payoff functions mapping $\Re^k \to \Re$ such that $ccc(U_1)=ccc(T)$, and let $U_2 = \{u: X \times \Re^k \to \Re$: X a partially ordered set, and $u(x_H,\mathbf{s}) - u(x_L,\mathbf{s}) \in U_1$ for all $x_H > x_L\}$.*

(A) $x^*(q) = \text{argmax}_x \int u(x,\mathbf{s})dK(\mathbf{s};q)$ is nondecreasing in q for all $u \in U_2$, if and only if there exists a $l>0$ such that $\int t(\mathbf{s})dK(\mathbf{s};q_H) \geq l \int t(\mathbf{s})dK(\mathbf{s};q_L)$ for all $t \in T$.

(B) Suppose that F is a probability distribution, and that $\{\mathbf{1},-\mathbf{1}\} \in U_1$. Then $x^*(q) = \text{argmax}_x \int u(x,\mathbf{s})dF(\mathbf{s};q)$ is nondecreasing in q for all $u \in U_2$, if and only if $\int t(\mathbf{s})dF(\mathbf{s};q)$ is nondecreasing in q for all $t \in T$.

> **Proof:** Follows from Theorem 5 and Milgrom and Shannon (1994), except for the following caveat: Milgrom and Shannon (1994) show that SC2 is necessary for the conclusion that $x^*(q,B)$ is nondecreasing in (x,B). The quantification over B can be dropped in our case because we are asking that the comparative statics result hold for all $u \in U_2$, and thus, effectively, for all constraint sets B.

To see how this result could be applied, consider first part (B). Three examples illustrate the result:

Example 3 *$x^*(q)$=argmax$_x \int u(x,s)dF(s;q)$ is nondecreasing in q for all u:$\Re^2 \to \Re$ supermodular, if and only if q shifts F according to FOSD.*

Example 4 *$x^*(q)$=argmax$_x \int u(x,s)dF(s;q)$ is nondecreasing in q for all u such that $u(x_H,s)-u(x_L,s)$ is concave in s for all $x_H>x_L$, if and only if q shifts F according to SOSD.*

Example 5 *$x^*(q)$=argmax$_x \int u(x,s_1,s_2)dF(s_1,s_2;q)$ is nondecreasing in q for all $u(x,s_1,s_2)$ that satisfy increasing differences in $(x;s_1)$ and $(x;s_2)$, if and only if $\int \mathbf{1}_A(s_1,s_2)dF(s_1,s_2;q)$ is nondecreasing in q for all A such that $\mathbf{1}_A(s_1,s_2)$ is nondecreasing.*

These results show that not only are stochastic dominance shifts sufficient to generate comparative statics predictions, they are also necessary, so long as we allow the payoff function to have constant incremental returns to *x* (or, more generally, we ask for the result to hold across a variety of payoff functions that differ only in the level of the incremental returns). Ormiston and Schlee (1992) show, using different techniques, that the stochastic dominance orderings are necessary for comparative statics in a few specific classes of payoff functions; thus, Corollary 5.2 (B) identifies the general principle behind the results.

To see how this result might be applied, consider a noisy signaling game, as studied by Maggi (1996). Suppose that there are two firms, a first mover and a second mover, and each firm chooses the quantity to produce. The first mover has private information about his own marginal cost, γ. After observing γ, the first mover chooses a quantity (q_1), but the second mover observes only a noisy signal (z) of the incumbent's choice. The second mover forms a posterior $F(q_1|z)$. Suppose that $\pi_2(q_1,q_2)$ is the expected profit to the entrant when the quantities chosen are q_1 and q_2, and suppose (as is usual in such models) that π_2 is submodular (that is, $-\pi_2$ is supermodular). Then the second mover's best policy is given by

$$q_2(z) = \arg\max_q \int p_2(q_1, q)\,dF(q_1 \mid z).$$

Then, Corollary 5.2 (B) establishes that $q_2(z)$ is nonincreasing in z for all π_2 submodular, if and only if $F(q_1|z)$ is ordered according to FOSD. Now consider the first mover's problem. Let the first mover's profits be given by $\pi_1(q_1,q_2,g)$, and assume that this function is submodular. Then, the first mover's policy is given by

$$q_1(g) = \arg\max_q \int p_1(q, q_2(z), g)\,dG(z \mid q).$$

Then, by Corollary 5.2(B), $q_1(g)$ is nonincreasing for all π_1 submodular and for all q_2 nonincreasing, if and only if and $G(z|q)$ is ordered by FOSD. By Athey (1997), whenever each player's best response function is nondecreasing, a pure strategy Nash equilibrium exists in this game (also see Vives (1990) for an existence result for supermodular games with incomplete information).

While it might seem that most economic problems fall under the domain of Corollary 5.2 (B), Section 3.3.1 below discusses a class of payoffs which does not satisfy the relevant assumptions.

Now, return to part (A). Part (A) can be illustrated with the following examples:

Example 6 $x^*(q)$=argmax$_x \int g(s;q)dH(s;x)$ *is nondecreasing in q for all H such that x shifts H according to SOSD, if and only if there exists a $l>0$ and a concave function k(s), such that* $g(s,q_H)=l \cdot g(s,q_L)+k(s)$.

Example 7 $x^*(q)$=argmax$_x \int g(s_1,s_2;q)dH(s_1,s_2;x)$ *is nondecreasing in q for all H such that* $H(s_1,s_2;x)$ *is nondecreasing in x for all* (s_1,s_2) *and each marginal distribution is constant in x, if and only if there exists a $l>0$ and a supermodular function* $k(s_1,s_2)$*, such that* $g(s_1,s_2;q_H)=l \cdot g(s_1,s_2;q_L)+k(s_1,s_2)$.

These examples illustrate problems where agents' utility functions are parameterized, and the agent must choose between two distributions. We wish to characterize how the choice of distributions changes with a parameter of the agent's preferences. In Example 6, consider two agents, corresponding to q_H and q_L, whose payoffs are not globally concave. The agent q_H will always choose a distribution that is better in the sense of SOSD than that chosen by agent q_L, if and only if agent q_H has payoffs that are "more concave" than agent q_L, in the sense that agent q_H's payoffs are a convex combination of a concave function and agent q_L's payoffs. The latter result is due to Ross (1981), and Jewitt (1986) showed how the method used in Theorem 5 could be used to establish Ross' result. The intuitions for the other examples are similar.

3.3.1 Single Crossing at a Point and the Portfolio Problem

It might at first seem that most sets of payoff functions that arise in economics contain the constant functions. However, portfolio theory motivates a class of payoff functions, which does not: the class of functions which cross zero at a fixed point. In particular, consider the following optimization problem:

$$\max_{x\in[0,1]} \int u(xs+(1-x)r)dF(s;q).$$

This is the standard portfolio problem. In this problem, the marginal returns to investment for a given realization of s are given by $(s-r)\cdot u'(xs+(1-x)r)$. When utility is nondecreasing, this function crosses zero from below, and it always crosses at the same point: $s=r$. Let us say that a function g satisfies *weak single crossing at s_0* if $s>(<)s_0$ implies $g(s)\geq(\leq)0$. Notice that, while the set of single crossing (SC1) functions is not a closed convex cone, the set of functions that satisfy weak single crossing at s0 *is* a closed convex cone. Further, observe that a set of "test functions" for this set of functions can be described as follows (for some $\beta>0$):

$$T(s_0;\beta)\equiv\{t: \exists a\in\Re,\ \varepsilon,\delta<\beta \text{ such that } t(s)=(1-2\cdot\mathbf{1}\{s<s_0\})\cdot(\mathbf{1}\{\min(a-e,s_0)\leq s\leq\max(a+e,s_0)\}+\delta)\}$$

Notice that the constant functions are not in this set of test functions, and thus Corollary 5.1 and Corollary 5.2(B) do not apply.

Athey (1998) shows that problems where payoffs (or incremental returns) are weak single crossing about s_0 can be analyzed using a different, and seemingly unrelated, set of tools. The approach is based on the fact that single crossing properties are preserved when integrating with respect to log-supermodular densities. While the statistics literature (for example, see the introduction to Karlin and Rinott (1980)) has typically drawn sharp distinctions between methods based on convex cones, and methods based on "majorization" (Marshall and Olkin, 1979), we will now briefly establish a connection between the two approaches.

Consider the following result:

Corollary 5.3 *Fix s_0 and suppose $k(s;q)$ is nonnegative, and suppose further that k is continuous and strictly positive at s_0 for all q. Then the following are equivalent:*

(i) $\int g(s)k(s;q)ds$ *satisfies SC1 for all g which satisfy weak single crossing about s_0.*

(ii) *For all $q_H>q_L$, $k(s;q_H)/k(s;q_L)-k(s_0;q_H)/k(s_0;q_L)$ satisfies weak single crossing about s_0 almost everywhere.*

Proof: By Theorem 5 and using the set of test functions $T(s_0;\beta)$ for β arbitrarily small, $\int g(s)k(s;q)ds$ satisfies SC1 for all g which satisfy weak single crossing about s_0, if and only if there exists $\lambda>0$ such that $k(s;q_H)\geq\lambda k(s;q_L)$ for almost all $s\geq s_0$, and $k(s;q_H)\leq\lambda k(s;q_L)$ for almost all $s\leq s_0$. This implies that $k(s_0;q_H)=\lambda k(s_0;q_L)$. Substituting and applying the definition of single crossing gives the desired conclusion.

To interpret condition (ii) of Corollary 5.3, observe that $k(s;q_H)/k(s;q_L)$ is a "likelihood ratio" for the realization *s* of the risky asset. Condition (ii) requires that for realizations of *s* above the crossing point, the likelihood ratio is greater than it is at the crossing point. Likewise, for realizations of *s* below the crossing point, the likelihood ratio is lower than at the crossing point.

To connect this result to results about log-supermodular densities, notice that if we do not specify the crossing point of *g* in advance, then we will be forced to check that $k(s;q_H)/k(s;q_L)-k(s_0;q_H)/k(s_0;q_L)$ satisfies weak single crossing about s_0 for every possible crossing point, s_0. But that is equivalent to checking that $k(s;q_H)/k(s;q_L)$ is nondecreasing in *s*, or that *k* is log-supermodular. Equivalently, *k* satisfies a monotone likelihood ratio property. Thus, despite the fact that neither single crossing nor log-supermodularity is a closed convex cone property, Theorem 5 can be used to establish a result relating the two properties.

To see how this result can be applied, return to the portfolio problem. We conclude that an agent will invest more in a risky asset in response to an increase in q, if and only if q shifts the density of asset returns according to condition (ii) in Corollary 5.3, when $s_0=r$.

3.4 Multivariate Choice Problems and l-Supermodularity

This section derives comparative statics properties of stochastic objective functions when the decision-maker's choice set is a lattice (for example, a vector of real numbers). Consider first the question of when $U(\mathbf{x},q)$ is quasi-supermodular in $\mathbf{x}$. In order to apply Theorem 5 to this problem, we need to introduce two new properties.

Definition 7 *For a given* $l \in \Re_+$*, consider a function* $h:X\rightarrow\Re$ *(X a lattice). Then h is* ***l-supermodular at* (x,y) *with parameter*** *l if* $h(\mathbf{x}\vee\mathbf{y})-h(\mathbf{x}) \geq \lambda[h(\mathbf{y})-h(\mathbf{x}\wedge\mathbf{y})]$. *If h:X´Θ®Â, where Θ is a partially ordered set, then* $h(\mathbf{x},q)$ *satisfies* ***l-increasing differences (l-ID) on Y´Γ with parameter*** *l if, for all* $\mathbf{x}_H>\mathbf{x}_L\in Y$ and $q_H>q_L\in\Gamma$, $h(\mathbf{x}_H,q_H)-h(\mathbf{x}_L,q_H) \geq \lambda[h(\mathbf{x}_H,q_L)-h(\mathbf{x}_L,q_L)]$.

It is clear that *l*-supermodularity implies quasi-supermodularity. Further, if h is bounded and quasi-supermodular, then for each $(\mathbf{x},\mathbf{y})$, there exists some $\lambda>0$ such that h is *l*-supermodular at $(\mathbf{x},\mathbf{y})$ with parameter λ. But, a critical property of *l*-supermodularity, as opposed to quasi-supermodularity, is that if, for a fixed λ, two functions are both *l*-supermodular with parameter λ at $(\mathbf{x},\mathbf{y})$, the convex combination of the two functions will inherit the *l*-supermodularity property. In other words, the definition of *l*-supermodularity gives us a way to make two quasi-supermodular functions comparable, so that we may take convex combinations without upsetting quasi-supermodularity.

We will say that the function is *l*-supermodular on a sublattice Y if, for each $\mathbf{x},\mathbf{y}\in Y$, there exists some $\lambda\geq 0$ such that the function is *l*-supermodular at $(\mathbf{x},\mathbf{y})$ with parameter λ, and likewise for *l*-ID. Clearly, supermodularity is stronger than *l*-supermodularity, since the definition is satisfied for $\lambda=1$. However, *l*-supermodularity allows for some flexibility in the scaling of a function at different points. For example, consider the sublattice $Y=\{(0,0),(0,1),(1,0),(1,1)\}$ and a function $h(\mathbf{x})$ which satisfies increasing differences in $(x_1;x_2)$ on $\{0,1\}\times\{0,1\}$. If the function is transformed by taking (the same) affine transformation of both $h(0,0)$ and $h(1,0)$, increasing differences is not necessarily preserved, while *l*–I.D. in $(x_1;x_2)$ will be. Intuitively, taking an affine transformation of $h(\cdot,0)$ does not affect the sign of the incremental returns to x_1.

When attention is restricted to a four-point sublattice (i.e. $\{\mathbf{x},\mathbf{y},\mathbf{x}\vee\mathbf{y},\mathbf{x}\wedge\mathbf{y}\}$), log-supermodularity is also stronger than *l*-supermodularity for positive functions h. To see this, let $\lambda=h(\mathbf{x})/h(\mathbf{x}\wedge\mathbf{y})$. However, since this constant depends on the choice of $\mathbf{x}$ and $\mathbf{y}$, we cannot necessarily find a single λ for all $(\mathbf{x},\mathbf{y})$ pairs in a given sublattice.

The following result characterizes quasi-supermodularity and the single crossing property in stochastic problems (where, for a distribution K, μ is a measure such that K is absolutely continuous with respect to μ and k is the density).

Theorem 6 *Consider a lattice X. Let $k:X\times\Re^n\rightarrow\Re$. Suppose ccc(T)=G.*

(A) The following conditions are equivalent:

(i) For every $\mathbf{y},\mathbf{z}\in X$, there exists a $\lambda>0$ such that, for all $t\in T$, $\int t(\mathbf{s})k(\mathbf{x},\mathbf{s})d\mu(\mathbf{s})$ is λ-supermodular at $(\mathbf{y},\mathbf{z})$ with parameter λ.

(ii) $V(\mathbf{x})\equiv\int g(\mathbf{s})k(\mathbf{x},\mathbf{s})d\mu(\mathbf{s})$ is quasi-supermodular in $\mathbf{x}$ for all $g\in G$.

(B) Let Θ be a partially ordered set, and let $k:X\times\Theta\times\Re^n\rightarrow\Re$. The following two conditions are equivalent:

(iii) For every subset $Y=\{\mathbf{x}_H,\mathbf{x}_L\}\subset X$ such that $\mathbf{x}_H>\mathbf{x}_L$, and every subset $\Gamma=\{q_H,q_L\}\subset\Theta$ such that $q_H>q_L$, there exists a $\lambda>0$ such that, for all $t\in T$, $\int t(\mathbf{s})k(\mathbf{x},q,\mathbf{s})d\mu(\mathbf{s})$ satisfies λ-increasing differences on $Y\times\Gamma$ with parameter λ.

(iv) $V(\mathbf{x},q)\equiv\int g(\mathbf{s})k(\mathbf{x},q,\mathbf{s})d\mu(\mathbf{s})$ satisfies SC2 in $(\mathbf{x};q)$ for all $g\in G$.

Proof: Consider part (A). $V(\mathbf{x})$ is quasi-supermodular in $\mathbf{x}$ if and only if, for every sublattice $Y=\{\mathbf{y},\mathbf{z},\mathbf{y}\wedge\mathbf{z},\mathbf{y}\vee\mathbf{z}\}\subset X$, $\int g(\mathbf{s})[k(\mathbf{y}\vee\mathbf{z},\mathbf{s})-k(\mathbf{y},\mathbf{s})]d\mu(\mathbf{s})\geq(>)0$ implies $\int g(\mathbf{s})[k(\mathbf{z},\mathbf{s})-k(\mathbf{y}\wedge\mathbf{z},\mathbf{s})]d\mu(\mathbf{s})\geq(>)0$. Applying Theorem 5, this in turn holds if and only if there exists a $\lambda>0$ such that $\int t(\mathbf{s})[k(\mathbf{y}\vee\mathbf{z},\mathbf{s})-k(\mathbf{y},\mathbf{s})]d\mu(\mathbf{s})\geq\lambda\int t(\mathbf{s})[k(\mathbf{z},\mathbf{s})-k(\mathbf{y}\wedge\mathbf{z},\mathbf{s})]d\mu(\mathbf{s})$ for all $t\in T$. But this in turn is equivalent to the requirement that $\int t(\mathbf{s})k(\mathbf{x},\mathbf{s})d\mu(\mathbf{s})$ is λ-supermodular with parameter λ at $(\mathbf{y},\mathbf{z})$ for all $t\in T$. Part (B) is analogous.

The comparative statics consequence of Theorem 6 is that $\mathbf{x}^*(q)\equiv\text{argmax}_{\mathbf{x}\in B}\int g(\mathbf{s})k(\mathbf{x},q,\mathbf{s})d\mu(\mathbf{s})$ is nondecreasing in (q,B) for all $g\in G$, if and only if (i) and (iii) are satisfied.

To interpret this result, consider first the case where G is the set of all probability densities, and suppose μ is Lebesgue. Then (i) requires that for $\mathbf{y},\mathbf{z}\in X$, there exists a $\lambda\geq 0$ such that $k(\mathbf{x},\mathbf{s})$ is λ-supermodular at $(\mathbf{y},\mathbf{z})$ with parameter λ *for almost all* $\mathbf{s}$. The order of the quantifiers is critical: there must be a single λ for almost all $\mathbf{s}$. As we discussed above, one sufficient condition is that k is supermodular in $\mathbf{x}$. Since sums of supermodular functions are supermodular, $V(\mathbf{x})$ will clearly be supermodular in $\mathbf{x}$ as well, a stronger result than is required.

Consider an example:

Example 8 *$x^*(q)\equiv\text{argmax}_{x\in B}\int u(s)dF(s;x,q)$ is nondecreasing for all u nondecreasing, if and only if for all $x_H>x_L$ and $q_H>q_L$ there exists a $\lambda>0$ such that $F(s;x_L,q_H)-F(s;x_H,q_H)\geq\lambda\cdot[F(s;x_L,q_L)-F(s;x_H,q_L)]$ for all s.*

We can interpret the condition of the example as follows: the effect on the cumulative distribution of increasing x at q_H, $F(s;x_L,q_H)–F(s;x_H,q_H)$, is equal to a convex combination of $F(s;x_L,q_L)–F(s;x_H,q_L)$ and a difference $H_H(s)–H_L(s)$, where H is ordered by FOSD ($H_H(s)\leq H_L(s)$). Thus, we can interpret the result as requiring that the shift in the distribution due to a change in x at q_H is "more FOSD" than the shift due to a change in x at q_L. It is interesting to return to the result of Theorem 3 as applied to the property supermodularity and the set of nondecreasing functions: a necessary condition for $V(x,q)$ to be supermodular for all g nondecreasing is that $1–F(s;x,q)$ is supermodular in (x,q) for all s. Thus, the necessary condition for comparative statics is a bit weaker. This result could be applied to study how a worker's choice of effort (x) varies with different task assignments (q).

4. CONCLUSIONS

This paper develops systematic tools for analyzing properties of stochastic objective functions, in particular properties that are relevant for deriving comparative statics predictions. The paper introduces abstract definitions which allow parallels to be drawn between classes of theorems, such as stochastic dominance theorems and theorems which characterize other properties of stochastic objective functions, in particular properties which arise in the study of comparative statics analysis such as supermodularity, single crossing properties, and quasi-supermodularity. The main results are then stated as mappings between two sets of functions (such as restricted classes payoff functions and probability distributions), where the mappings provide characterizations of when a stochastic objective function satisfies desired properties. The results can be applied to many economic problems, including portfolio theory, investment games, and games of incomplete information.

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