

Characterizing the Amount and Speed of Discounting Procedures

Dean T. Jamison and Julian C. Jamison

Abstract:

This paper introduces the concept of categorizing the amount and speed of a discounting procedure in order to generate well-characterized families of procedures for use in social project evaluation. Exponential discounting isolates the concepts of amount and speed into a single parameter that must be disaggregated in order to characterize nonconstant rate procedures. The inverse of the present value of a unit stream of benefits provides a natural measure of the *amount* a procedure discounts the future. We propose geometrical- and time horizon-based measures of how rapidly a procedure acquires its ultimate present value, and we prove these values are the same. This equivalency provides an unambiguous measure of the *speed* of discounting, with values between 0 (slow) and 2 (fast). Exponential discounting has a speed of 1. A commonly proposed approach to aggregating individual discounting procedures into a social one for project evaluation averages the individual functions. We point to a serious shortcoming with this approach and propose an alternative for which the amount and time horizon of the social procedure are the average values of the amounts and time horizons of the individual procedures. We further show that this social procedure will in general be slower than the average of the individual procedures' speeds. We then characterize three families of two-parameter discounting procedures—hyperbolic, gamma, and Weibull—in terms of their discount functions, discount rate functions, amounts, speeds, and time horizons. (The appendix characterizes additional families, including the quasi-hyperbolic procedure.) A one-parameter version of hyperbolic discounting, $d(t)=(1+rt)^{-2}$, has amount r and speed 0. We argue that this zero-speed hyperbolic procedure is a good candidate for use in social project evaluation, although additional empirical work is needed to fully justify a one-parameter simplification of the more general procedure.

Keywords: discounting, social discount rate, project evaluation, hyperbolic discounting

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Discounting at a constant rate has the virtues of familiarity, analytic tractability, time-consistency of preferences, and a well-understood axiomatic foundation. That said, the perceived shortcomings of discounting at a constant rate have increasingly led economists in diverse fields to suggest using procedures in which the discount rate varies over time. Examples occur in contexts involving the full range of time horizons.

With exponential discounting, giving noticeable weight to the far future comes at the cost of having virtually no discounting occur in the short- to medium-term. Hence the literature on economic evaluation of long time-horizon environmental or health projects increasingly contains proposals for using discount rates that decline with time (*slow* discounting), in order to give a “reasonable” weight to far-future outcomes (for example, see Cline 1999; Weitzman 2001; Gollier 2002; Newell and Pizer 2003; and Summers and Zeckhauser 2010). Yet actual practice for the most part continues to use constant discount rates.¹

In economics a recent strand of research imports findings from experimental psychology into explaining economic behavior (Loewenstein 1992; Frederick, Loewenstein, and O’Donoghue 2002). Psychologists and the behavioral economists using their findings and methods focus much of their interest on behavior occurring over relatively short periods—behavior that often seems consistent with discount functions that initially decline slowly (or not at all) and then more rapidly for an interim period (for example, Laibson 1997). This generates interest in accelerating discount rates (*fast* discounting) in a sense that we will make clear.

For financial markets, the time period of interest—typically up to 30 years—lies between the short-term concerns of behavioral economists and the long-term concerns of environmental economists. The rising yield curves that frequently characterize bond markets imply fast discounting, but the empirically observed variation in yield curves includes discounting that is slow as well as fast (Malkiel 1998). Financial economics is thus a third strand of analysis where nonconstant rate discounting is relevant and, indeed, is routine.

Our purpose in this paper is to provide a framework for the increasingly diverse literature using nonconstant rate discounting through exploring the implications of distinguishing the *speed* of discounting from the total *amount* by which the future is discounted. We can characterize the

¹Current guidelines for economic evaluation of health projects (Lipscomb, Weinstein, and Torrance 1996) recommended a constant rate of 3 percent per year. Stern (2006), in an influential assessment of the economics of climate change, used 1.4 percent per year.

heterogeneous collection of proposed discounting procedures along these two dimensions. Among other things, this enables us to better evaluate alternative proposals for aggregating individual procedures into a social procedure. Important work continues on the nature of the factors (for example, time preference, the return to capital, the marginal utility of consumption, and the completeness of financial markets) that influence discounting or the term structure of interest rates. There is a natural division of labor between these topics and the tasks of characterizing the basic properties of discounting procedures and of developing a reasonable framework for aggregating individual procedures into a social procedure. This paper addresses these latter topics and provides families of discount functions of practical utility for social cost-benefit analysis.

Section 1 provides examples of slow and fast discounting and introduces our approach by providing definitions of the amount, the speed, and the time horizon of discounting procedures. It then proves several basic propositions concerning these measures. Section 2 deals with aggregating multiple individual discounting procedures into a socially representative one. It points to fundamental convergence problems—akin to the use of a discount rate at 0—associated with the simple averaging of discount functions as an aggregation procedure and suggests an alternative. This new method has the attractive characteristic that the means of the amounts and of the time horizons of the individual procedures equal the aggregate procedure’s amount and time horizon. Our proposed aggregation procedure results in discount functions which emphasize the preferences of impatient consumers in early years and patient consumers in later years. Section 3 presents three two-parameter discount functions and characterizes their amounts, speeds, and time horizons. It identifies potentially attractive one-parameter fast and slow discount functions as special cases and argues that a specific slow procedure, which we label the zero-speed hyperbolic (ZSH) function, provides a good candidate for an alternative to the exponential procedure for social discounting with long time horizons. An appendix summarizes the relationships between discount functions, yield curves, and other potentially defining functions for discounting procedures. It provides proofs and characterizes six additional discounting procedures.

1 Measuring the Amount and Speed of Discounting

Two methods of defining a discounting procedure are by using a discount function, $d(t)$ —where $d(0) = 1$, $d(t) \geq 0$, and $d(t)$ is nonincreasing for all t —or by using a present value function,

$pv(t)$, that gives the present value of a unit stream of benefits accumulated to time t :

$$(1) \quad slpv(t) = \int_0^t sl d(x) dx.$$

We define the “present value of $d(t)$ ” to be $pv(\infty)$.

In this section we first provide examples that illustrate how, for a given $pv(\infty)$, alternate discounting procedures accumulate present value at different rates. We use this variation to motivate two conceptually different definitions of speed of discounting: a geometrical measure and a measure based on the time horizon. We next prove several basic propositions concerning speed: our alternative definitions of speed are equivalent; our measure of speed lies between 0 and 2; procedures with declining (increasing) discount rates have speeds less than (greater than) 1; and exponential discounting has a speed of 1.

1.1 Examples and Definitions

Consider the following four discount functions:²

$$(2) \quad \text{exponential: } d(t) = e^{-rt}, \text{ where } r = 0.02$$

$$(3) \quad \text{hyperbolic: } d(t) = [1 + (\sigma^2/\mu)t]^{-\mu^2/\sigma^2},$$

where $\mu = .04$ and $\sigma = .029$

$$(4) \quad \text{quasi-hyperbolic: } d(t) = 1 \text{ for } t = 0 \text{ and}$$

$$d(t) = b(1+r)^{-(t-1)} \text{ for } 1 \leq t < \infty \text{ and } t \text{ an integer,}$$

where $b = 0.6$ and $r = .0121$

$$(5) \quad \text{fast Weibull: } d(t) = e^{-rt^{1/s}},$$

where $r = .000314$ and $s = 0.5$.

Equation (2) is the discount function with a constant discount rate of 0.02 and, hence, a present value [= $pv(\infty)$] of $.02^{-1} = 50$. Equation (3) represents a parameterization of the hyperbolic family that Weitzman (2001) used as an aggregation of the exponential discount functions that resulted from a survey of what economists use—with one parameter modified slightly to reduce

²Continuous time formulations simplify most derivations and, with only occasional exception, such as in equation (4), we use continuous time.

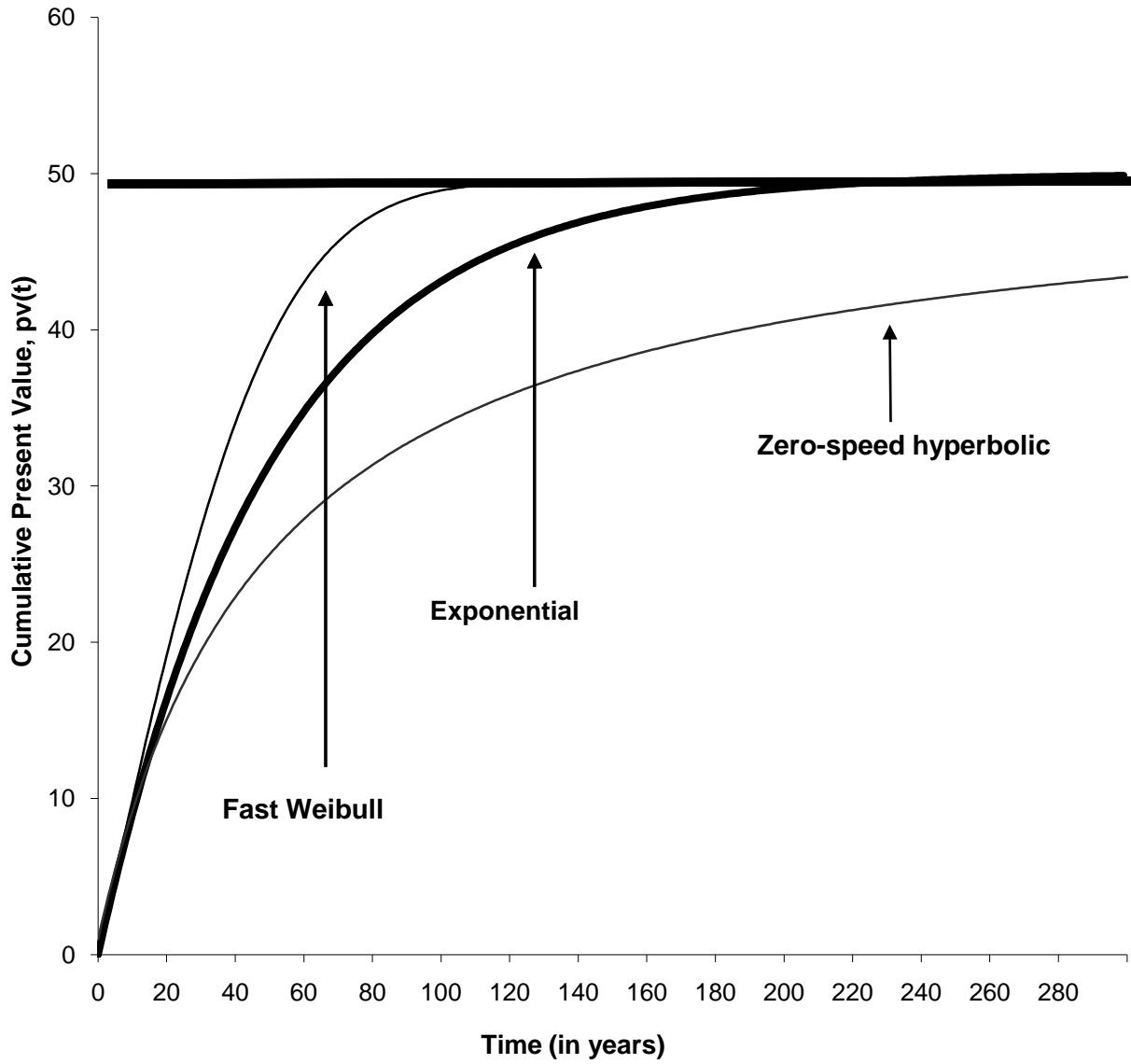
present value from the 54.8 his parameters imply to 50. Equation (4) is the “quasi-hyperbolic” function used by Laibson (1997), again with parameters modified to reduce present value from his implied 60.4 to 50. Read (2001) has suggested the formulation in equation (5), which is an exponential function with the value for time transformed—obtained in this case by squaring it—before being exponentiated. [Any discount function has an associated probability density function (pdf), and for equation (5) the pdf is the Weibull distribution, hence our nomenclature.] We have again chosen parameters so that $pv(\infty) = 50$ for the Weibull discount function.

Although each of these discount functions has a present value of 50, these differ in how rapidly they acquire that present value. Figure 1 illustrates how each present value function rises with time to its asymptote of 50. Note several points: both the hyperbolic and quasi-hyperbolic functions rise more slowly than the exponential function in the sense that, for both of them, $pv(t)$ is strictly less than it is for the exponential for all functions where $t > 0$. They can thus, in this sense, be viewed as slower than the exponential with the same present value. Second, the Weibull is, with the indicated parameters, faster than the exponential. (The Weibull procedure can be either fast or slow.) Third, the differences among the four procedures translate into major differences in the weight given to the far future: while the Weibull procedure has acquired essentially all of its present value within 150 years, and the exponential procedure within 250 years, the hyperbolic has over 6 percent of its total present value still to be acquired after 500 years. Finally, since the present value functions for the hyperbolic and the quasi-hyperbolic cross, neither can be considered strictly slower than the other. The crossing of Lorenz curves provides a close analogy. Just as the Gini coefficient provides one way to complete the inequality ordering on income distributions generated by Lorenz curves, so too will an area-based measure allow completion of the ordering of the speed of discount functions.

The more that one discounts the future, the less a unit stream of benefits will be worth in the present. It is thus natural to define the *amount* of discounting for a procedure, $\alpha(D)$, to be the inverse of its present value:

$$(6) \quad \alpha(D) = \left[\int_0^{\infty} d_D(t) dt \right]^{-1}.$$

Figure 1: Three Discounting Procedures with $pV(\infty)=50$



Source: Authors' calculations

If D is an exponential procedure with constant discount rate r , then of course $\alpha(D)$ is simply r . This makes sense insofar as we think of higher discount rates as discounting the future by a greater amount.

Figure 2 provides a geometrical motivation for the first definition of speed that we introduce. The greater the area between the $pv(\infty)$ of a discounting procedure and its $pv(t)$ function the slower, intuitively, the discounting appears to be. We have denoted this area for the exponential procedure in figure 2 as \mathbf{A} , and the following expression gives its value:

$$(7) \quad \mathbf{A} = \int_0^\infty [r^{-1} - \int_0^t e^{-rx} dx] dt,$$

where the inner integral is, of course, the expression for $pv(t)$ for an exponential procedure, and r^{-1} is $pv(\infty)$ for the exponential procedure (illustrated in figure 2 by the horizontal line at 50). Evaluating the integrals in equation (6) gives:

$$(8) \quad \mathbf{A} = r^{-2}.$$

Likewise \mathbf{B} is the area between the quasi-hyperbolic function and the line r^{-1} . The ratio \mathbf{A}/\mathbf{B} provides a geometrical measure of the speed of this quasi-hyperbolic procedure relative to that of an exponential procedure with the same present value.

More precisely, our geometric measure of the *speed* of a discounting procedure³, D , is $\rho_1(D)$ and the preceding discussion suggests the following:

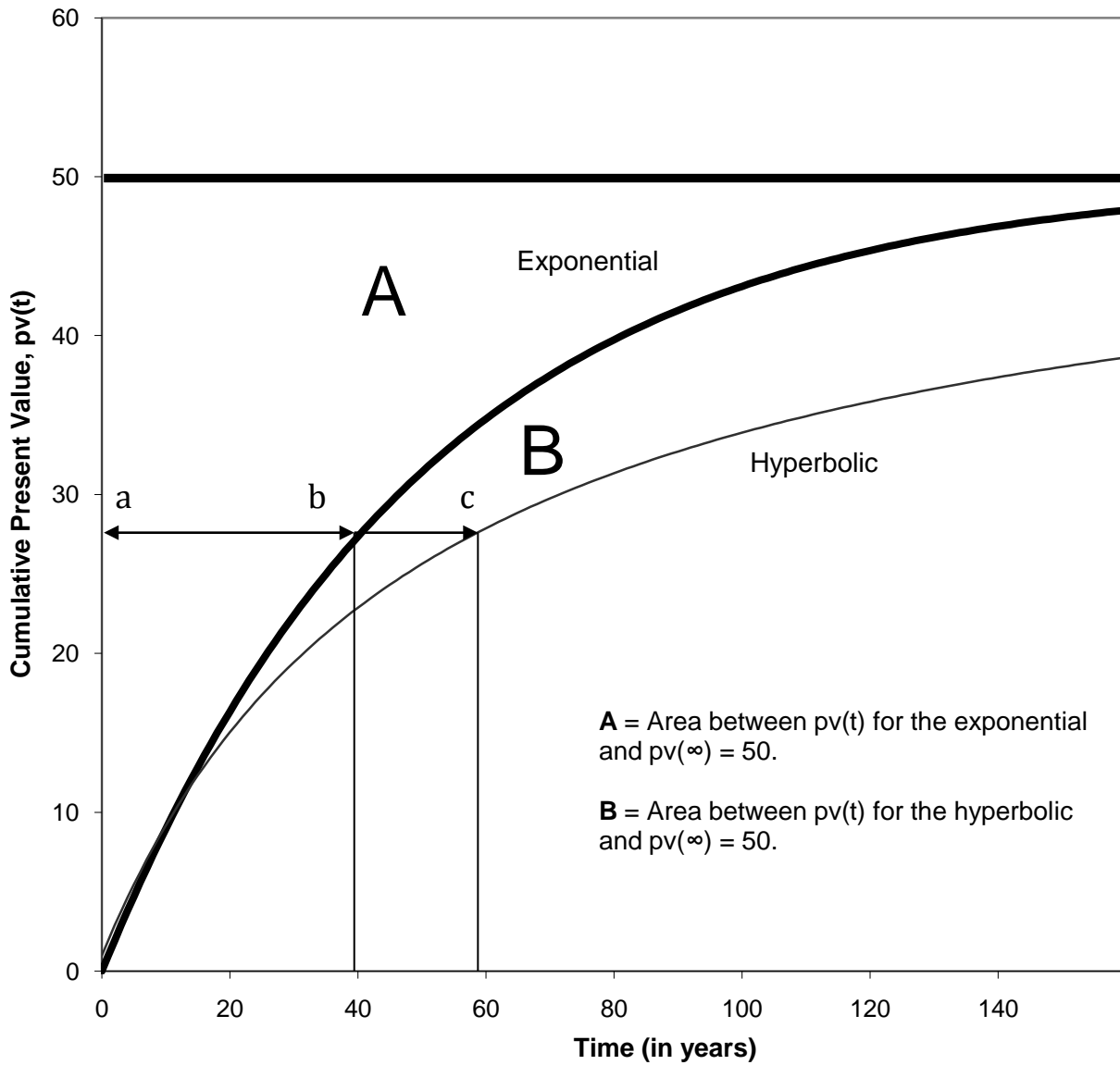
$$(9) \quad \mathbf{A}(D) = \text{area between } pv_D(\infty) \text{ and the exponential with the same present value as } D, \text{ that is, the exponential with } r = pv_D(\infty)^{-1};$$

$$(10) \quad \mathbf{B}(D) = \text{area between } pv_D(\infty) \text{ and } pv_D(t); \\ = \int_0^\infty [pv_D(\infty) - pv_D(t)] dt, \text{ if this integral converges; and}$$

$$(11) \quad \rho_1(D) = \mathbf{A}(D)/\mathbf{B}(D).$$

³A discounting *procedure* can be defined in terms of its associated discount function, $d_D(t)$, its present value function, $pv_D(t)$, its discount rate function, $r_D(t)$, its yield curve or an associated probability density function. The appendix states the relations among alternative definitions.

Figure 2: The Time Horizon and Relative Speed of Discounting Procedures



Note: This figure displays $pv(t)$ for the same exponential and hyperbolic discounting procedures shown in figure 1.

Source: Authors' calculations

Another characteristic of potential interest for evaluating a discounting procedure concerns the length of time the procedure takes to build up present value. We refer to the time required for accumulating present value as a procedure's *time horizon* and introduce two distinct definitions. The line segments (a,b) and (a,c) in figure 2 show how many years it takes the exponential and quasi-hyperbolic procedures to accumulate half of their present values of 50. The times are, respectively, 35 and 58 years. We define a procedure's "median time to accumulation of present value" to be the time required for it to accumulate 50 percent of its present value. We label this $\tau(D)$, which is given by the following expression:

$$(12) \quad \tau(D) = t^* \text{ such that } \frac{\int_0^{t^*} d_D(t) dt}{pv(D)} = 0.5.$$

A plausible second definition of a procedure's speed would relate how long an exponential procedure (with the same α as the procedure of interest) takes to reach half its present value to how long the procedure of interest does. (In this example the speed would be 35/58.) A variant of this approach proves more tractable.

Just as there is a median time to accumulation of present value so, too, can one define a mean time, which can be thought of as how far from time zero, on average, the present value is being accumulated.⁴ We label the mean time to accumulation as $\theta(D)$, given by:

$$(13) \quad \theta(D) = \frac{\int_0^\infty t d_D(t) dt}{pv(D)}.$$

Our second definition of the speed of a discounting procedure relates the procedure's time horizon, $\theta(D)$, to the time horizon of an exponential procedure that discounts the future by the same amount as D , meaning by $\alpha(D)$. The time horizon for an exponential procedure with rate r is simply r^{-1} , so the time horizon for the exponential procedure that has the same amount of discounting as D will be $\alpha(D)^{-1}$. Our time horizon-based definition of speed, $\rho_2(D)$, is the ratio of the horizon for the equivalent exponential procedures to the time horizon for the procedure being considered:

$$(14) \quad \rho_2(D) = \alpha(D)^{-1} / \theta(D).$$

⁴A physical analogy may be helpful. Think of $d(t)$ as distributing mass along one arm of an imaginary balance. A given total mass can be spread far along the arm or concentrated toward the beginning. θ is the point such that if all the mass could be concentrated at that distance on the other arm, it would exactly balance $d(t)$.

This equation reflects the intuition that if D has a longer time horizon than the exponential procedure with the same present value, it can be viewed as a slower procedure.

Before proceeding further it is important to deal briefly with questions of convergence. Initially, proposed variants of hyperbolic discounting took the form:⁵

$$(15) \quad d_h(t) = 1/(1 + at).$$

Like exponential discounting with a zero discount rate, $d_h(t)$ has infinite present value. To put this slightly differently, an improved outcome in each time period by a finite amount x , however large, up to time t^* , also arbitrarily large, would be more than counterbalanced in present value terms using the discount function $d_h(t)$, by a decrement y , however small, to all outcomes after t^* . It is precisely this property of making outcome changes over any finite time horizon irrelevant compared to tiny but sustained changes in the extremely distant future—even without infinite time horizons—that leads to objections to using exponential discounting with a rate of 0.

The point of this example is simply to illustrate the importance of paying attention to the issue of convergence when selecting a discounting procedure for evaluating investments that have a long time horizon.⁶ Even with a finite but long time horizon, present values can be quite sensitive to the discount rate that applies to later years—hence the importance of explicit consideration of the total amount, $\alpha(D)$, by which a procedure discounts the future.

The implications for $r(t)$ —the discount rate function—are that it must go to zero very slowly,

⁵See, for examples, Harvey’s (1986, 1994) early important work on nonconstant rate discounting and in the psychology literature, the work of Herrnstein (1981). Loewenstein and Prelec (1992) discuss and provide an axiomatic foundation for a “generalized hyperbolic” that places an exponent, k , on the denominator of equation (15). Many parameterizations are possible [for example equation (3)] and, in this paper, we refer to the generalized hyperbolic family simply as hyperbolic. The hyperbolic procedure will have a finite present value if and only if $k > 1$.

⁶The issue of convergence assumes less importance in modeling phenomena over shorter periods as in the psychological literature or some areas of science and engineering. Sokolnikoff and Redheffer (1958, 176) observe, for example that “divergent Fourier series often arise in practice, for example in the theory of Brownian motion, in problems of filtering and noise, or in analyzing the ground return to a radar system. Even when divergent the Fourier series represents the main features of $f(x)$ ”.

or not at all, in order to ensure convergence.⁷ If

$$(16) \quad \lim_{t \rightarrow \infty} r(t) = r^*, \quad r^* > 0,$$

then the limit $d(t+1) \leq \frac{d(t)}{1+r^*}$ guarantees convergence. Analogous results hold for continuous time. Therefore all exponential discount functions, or ones that are ultimately exponential—in the sense that equation (16) holds—will yield finite present values. On the other hand, if for all t greater than some t^* , $r(t) = 0$ [but $d(t) > 0$ still] then the present value will be infinite (as is the case for the discounting procedure proposed in Weitzman (2001), table 2).

Having a convergent discounting procedure, however, does not necessarily entail convergence of the integrals defining our concepts of the speed of discounting [equation (10) and equation (13)]. Equation (13) illustrates the question of convergence more clearly since its integrand, $td(t)$, is the product of a function going to zero and a function going to infinity. Stronger conditions on $d(t)$ must, obviously, obtain for this integral to converge than for the integral of $d(t)$ to converge.

We define a discounting procedure to be *strongly convergent* if the integrals both of its related discount function, $d(t)$, and of $td(t)$ converge. We define a procedure to be *weakly convergent* if $d(t)$ converges but $td(t)$ fails to converge. Weakly convergent procedures have a speed of zero and the potentially desirable property of giving infinite weight to the future—in the sense that the average time to accumulation of present value is infinite—while still having a finite present value. Weakly convergent $d(t)$ s can thus provide an alternative to zero-rate discounting for those who—like Ramsey (1928)—hold a preference for giving genuine weight to outcomes in the extremely distant future. Among the discounting procedures discussed in this paper only one, the hyperbolic, includes weakly convergent (or zero speed) procedures within the family. In section 3.2 we pay particular attention to the one-parameter procedure that we label the zero-speed hyperbolic (ZSH).

1.2 Main results concerning the speed of discounting

The concepts introduced in the preceding subsection are not independent of one another, and our initial proposition identifies and proves the central relationship between our two definitions of speed.

⁷To take an example: if $r(t) = r_0 e^{-kt}$ then r will decline from an initial level of r_0 to 0 too rapidly for the present value of the procedure to be finite.

Proposition 1: For all D , $\rho_1(D) = \rho_2(D)$.

Proof: We have already seen that the area $\mathbf{A}(D)$ is α^{-2} , so starting from the definition:

$$\rho_1(D) = [\alpha^2 \int_0^\infty (\alpha^{-1} - pv_D(t)) dt]^{-1} = [\alpha^2 \int_0^\infty \left(\int_0^\infty d(x) dx - \int_0^t d(x) dx \right) dt]^{-1} = [\alpha^2 \int_0^\infty \int_t^\infty d(x) dx dt]^{-1}.$$

Changing the order of integration, this is $[\alpha^2 \int_0^\infty \int_0^x d(x) dt dx]^{-1} = \alpha^{-1} [\alpha \int_0^\infty d(x) \left(\int_0^x dt \right) dx]^{-1} = \alpha^{-1} [\alpha \int_0^\infty x d(x) dx]^{-1} = \alpha^{-1} [\theta(D)]^{-1} = \rho_2(D)$.

Because of the equivalence of ρ_1 and ρ_2 , from this point on we unambiguously use ρ to denote speed, and the following corollary follows immediately from Proposition 1 and equation (14):

Corollary: $\alpha(D)\rho(D)\theta(D) = 1$.

We now turn to the range of values possible for the speed of discounting:

Proposition 2: For all D , $0 \leq \rho(D) \leq 2$.

Proof: Fixing α , the mean time θ is minimized by concentrating as much weight as possible as early as possible (that is, onto small values of t in the integrand). Given that $d(t)$ must start at 1 and be weakly decreasing, this is accomplished by setting $d(t) = 1$ for $t \leq 1/\alpha$ and $d(t) = 0$ for $t > 1/\alpha$. Such a discounting procedure has a $\theta = 1/(2\alpha)$, while the equivalent exponential has $\theta = 1/\alpha$. Hence the maximum possible ρ (corresponding to the minimum θ) is 2. Meanwhile, the maximum θ is infinite (even with a finite present value and thus $\alpha > 0$), as occurs for any weakly convergent procedure. This proves that the minimum ρ is 0.

Finally, we wish to relate the speed of a discounting procedure to its discount rate function, $r(t)$. In particular, intuition suggests that a decreasing discount rate yields a procedure that is slow. It is obvious that such a procedure is “slower” in the sense of having a longer time horizon, θ , than the constant-rate procedure that starts at the same discount rate and stays there, but on the other hand these two will not have the same total present value. Our third proposition states that, even relative to an exponential *with the same amount of discounting*, any decreasing-rate procedure is slow.

Proposition 3: If the discount rate $r(t)$ is weakly decreasing (increasing), then the corresponding discounting procedure is slow (fast), that is, $\rho \leq 1$ (resp. $\rho \geq 1$). Furthermore, this result is tight in the sense that if $r(t)$ is weakly decreasing everywhere and strictly decreasing somewhere, then the inequality is strict.

Proof: Let $d(t)$ be the discount function with decreasing rate $r(t)$, and let $\mu = pv(\infty) =$

$\int_0^\infty d(t)dt$. We can think of d as the survivor function for a failure density⁸, in which case $r(t)$ decreasing is exactly the definition of a decreasing failure rate (and μ is the mean time to failure). Then for any strictly increasing function f on $[0, \infty)$, $\int_0^\infty f(t)d(t)dt \geq \int_0^\infty f(t)e^{-t/\mu}dt$ by Theorem 4.8 (p. 32) of Barlow and Proschan (1967), with equality only if $d(t) = e^{-t/\mu}$ identically. So let $f(t) = t$: $\int_0^\infty td(t)dt \geq \int_0^\infty te^{-t/\mu}dt = \mu^2 = [\int_0^\infty d(t)dt]^2$, implying that

$$\rho = \frac{[\int_0^\infty d(t)dt]^2}{\int_0^\infty td(t)dt} \leq 1,$$

as desired. Equality implies that d must be exponential with amount $\alpha = 1/\mu$. Likewise, $r(t)$ increasing corresponds to an increasing failure rate, and all inequalities are reversed.

We can alternately interpret this conclusion as saying that the exponential is the fastest discounting procedure within the family of those with weakly decreasing discount rates. As we have seen, the exponential discounting procedure $d(t) = e^{-rt}$ also has a total present value equal to its mean time horizon: $pv_d(\infty) = \theta = 1/r$. Proposition 3 allows us to answer the question of whether or not it is the unique discounting procedure with this property.

Corollary: If $d(t)$ is a discount function with monotone discount rate $r(t)$ and satisfying $pv_d(\infty) = \theta(D)$, then d is exponential: $d(t) = e^{-t/\theta}$.

Proof: From the definitions, $pv_d(\infty) = \theta$ means $\int_0^\infty d(t)dt = \frac{\int_0^\infty td(t)dt}{\int_0^\infty d(t)dt}$, that is, $[\int_0^\infty d(t)dt]^2 = \int_0^\infty td(t)dt$. But this again corresponds to equality in the proof of Proposition 3, so d must identically equal the equivalent exponential.

2 Social Discounting

Suppose that we start with a population of individuals, each of whom uses some discounting procedure. The individual procedures may differ both in their parameter values and in their actual functional forms. We wish to aggregate these procedures to achieve a social discounting procedure that appropriately reflects the preferences of all members of society. In this section

⁸Failure analysis arises in the study of systems reliability. The primitive in these analyses is the probability of no failures before time t ; this is the role played by $d(t)$ in our setting. In reliability studies this is referred to as the survivor or reliability function, and it is the inverse *cdf* of the failure density (that is, the probability of failure at any given time). The failure rate as a function of time is $-d'/d$, which is thus exactly our discount rate $r(t)$.

we first discuss existing approaches to generating a social discounting procedure from individual ones and propose an alternative procedure, the average normalized discount function, that avoids the shortcomings of those in the literature. We next prove results that relate the amount, speed and time horizon of the social discount function to the amounts, speeds and time horizons of the individual ones. We conclude by providing specific examples for when all the individual discounting procedures are exponential.

2.1 Alternatives for generating a social discounting procedure from individual ones

One obvious aggregation option is to average discount *rate* (ADR) functions across individuals. For example if person A uses a standard constant-rate procedure with $d(t) = e^{-rt}$ (that is, a rate of r), and person B uses $d(t) = e^{-st}$, then this method would yield a social discounting procedure characterized by a constant rate equal to $\frac{1}{2}(r + s)$. One advantage of using the ADR is that even if only one of the individuals uses nonconvergent discounting (for example, if $s = 0$), the aggregate social procedure will be convergent (a notion of robustness). As Weitzman (2001) has emphasized, averaging rates has a downside: when society decides how to trade between two given time points in the future, this method counts everyone’s opinion on that question equally, even those who do not care much about the future. Thus while a procedure that averages rates is able to successfully aggregate *amounts* of discounting (in the sense that the amount of the aggregate procedure is the average of the amounts of the individual procedures), it does less well on the shape over time that discounting should take.

Another natural option is simply to average the discount functions (ADF) among individuals. In the example with two exponentially discounting individuals, this would lead to a social discount function of $\frac{1}{2}(e^{-rt} + e^{-st})$. This aggregate procedure has a discount rate that starts at $\frac{1}{2}(r + s)$ for $t = 0$, and then declines over time to the minimum of r and s . To redress the inadequacy of aggregation by averaging rates, Weitzman (2001) advocates using ADF and interprets this process as valuing a dollar at time t according to any particular discounting procedure weighted by the “probability of correctness” of that procedure. In our example, this probability is 0.5 for each of the two individual procedures in the domain and is 0 for all others.

One immediate concern about the ADF process is that even if one individual’s discounting

procedure is nonconvergent, then the social procedure will also be, no matter how large the society (assuming it is finite). Indeed, even if no individual in the population has an infinite present value (nonconvergent) procedure, it turns out to be quite possible for the aggregate procedure generated by ADF to have infinite present value. This shortcoming alone makes ADF unviable as a general aggregation procedure.

The convergence problem could be avoided by arbitrarily ruling out nonconvergent procedures in the first place, but it speaks to a second important concern. Basically, under ADF, individuals are weighted equally (in terms of either discount functions or discount rates) at the beginning of time, after which those who discount less are increasingly favored. ADF yields social discounting procedures whose discount rate functions are both declining over time (as patient individuals are increasingly favored) *and* that are low overall (since they start at the social average and go down). At no point in time does ADF aggregation differentially reflect the preferences of *impatient* individuals even though, as time goes on, the preferences of patient individuals become increasingly consequential. The shape of the ADF social discount function faithfully reflects a weighted average of the individual shapes (weighted by their own sense of relative time preference), but it skews the amount of discounting toward those who discount less.

We seek an aggregating process that avoids the problems associated with the simple averaging of either discount rates or functions. That is, we feel an aggregation process should satisfy two criteria:

(i) the aggregate procedure should discount the future by an amount that is the average of the individual amounts—as averaging the discount rates does⁹; and

(ii) the aggregate procedure’s discount rates in the future should place greater weight on individuals who value the future more highly—as averaging the discount functions does. (This implies, combined with criterion 1, that near-term discount rates will be weighted toward those of individuals who place greater weight on the present.)¹⁰

Both criteria can be met by averaging the *normalized discount functions* for each individual

⁹Note that this criterion refers to aggregation of individual procedures into a social one. It has less relevance for generating an expected value when there is underlying uncertainty in interest rates, for example, the situation considered in Newell and Pizer (2003).

¹⁰Gollier and Zeckhauser (2005) develop a market-oriented aggregation procedure for exponentials that results in high discount rates (relative to the average) in early periods and low discount rates in later periods. ANDF has this feature and we conjecture that any market-derived aggregation procedure will also.

(the normalized function is $\alpha d(t)$ and has a total present value of 1). We then divide by the value at 0 of this average function in order to unnormalize and recover a valid discount function (with $d(0) = 1$, as is necessary). We label this the average normalized discount function (ANDF) aggregation process. The ANDF process results in a shape equal to the average shape, and we will prove that it has an amount exactly equal to the average amount, $\bar{\alpha}$ (and a mean time horizon equal to the average mean time, $\bar{\theta}$). In our running example, the normalized individual discount functions are re^{-rt} and se^{-st} , respectively, so the ANDF social procedure is

$$(17) \quad d_{ANDF}(t) = \frac{re^{-rt} + se^{-st}}{r + s}.$$

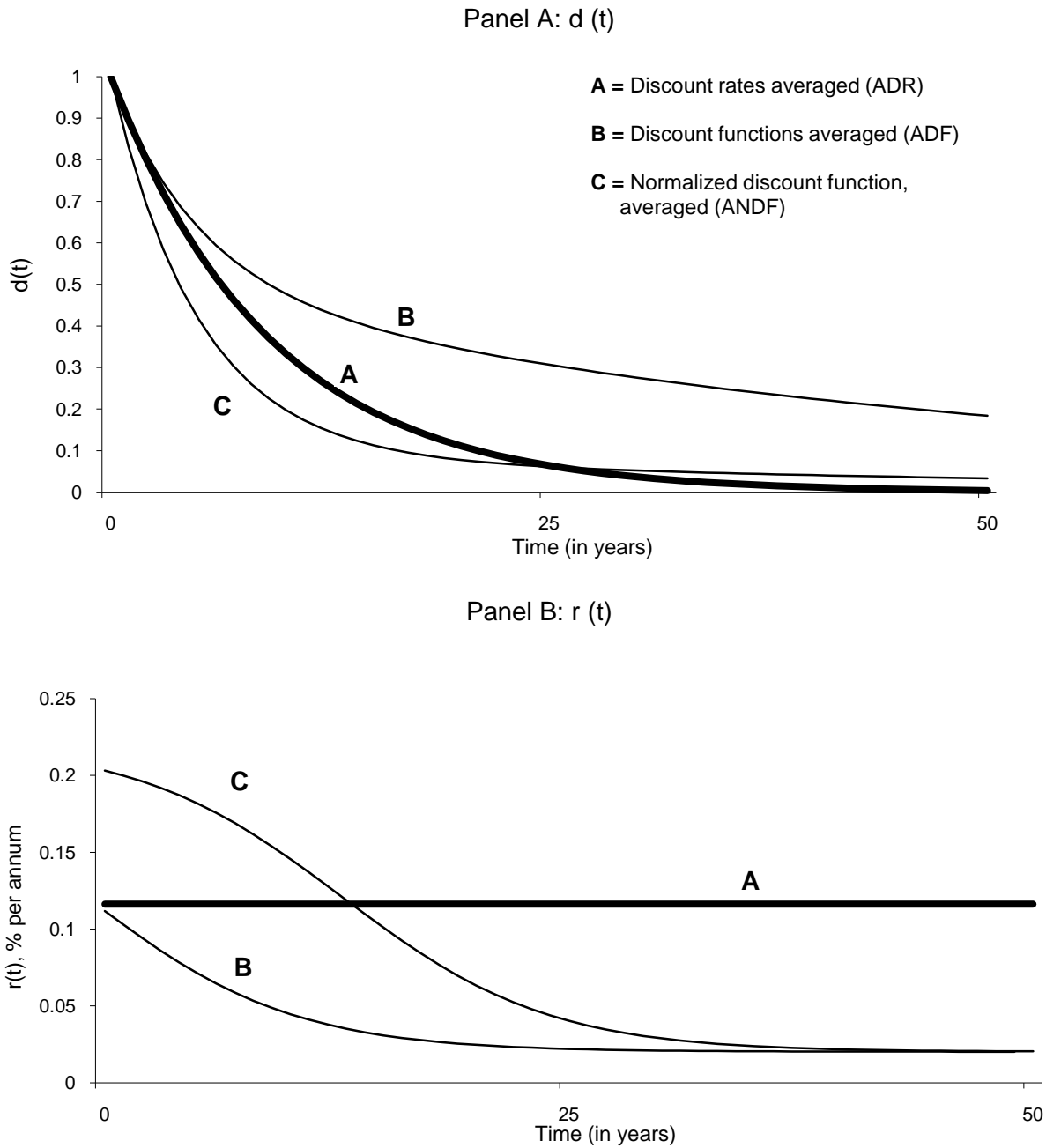
Here $\alpha = (r + s)/2$ and $\theta = (r^{-1} + s^{-1})/2$. Figure 3 illustrates the $d(t)$ and $r(t)$ resulting from each of the three ways of aggregating two exponential procedures, in this case with $r = 0.02$ and $s = 0.20$. For the ANDF process applied to these two exponentials, $\alpha = 0.11$, $\theta = 27.5$, and $\rho = 0.33$ (so it is slow). Note in particular in panel B of figure 3 that the discount rate function for averaging the discount functions (ADF) is below the average discount rate for all $t > 0$, whereas ANDF falls from above the average to 0.02.

One interpretation of the ANDF is that it gives each individual a total weight of 1 to spread across the future in a way that reflects his or her preferred time path of consumption. ANDF averages those shapes and then unnormalizes, which retrieves the average amount of discounting. This separation of amount and shape ensures that there is no problem with convergence and that more patient members of society are favored at later times relative to earlier times, but not overall.

2.2 The amount and time horizon of social discounting

Formally, assume that we have a collection of individuals with discounting procedures parameterized by $x \in X$ (possibly multivariate), with frequency distribution across parameters of $f(x)$, so that in particular $\int_X f(x)dx = 1$. Then if individual x uses a discount function $d(t; x)$ with associated amount $\alpha(x)$, we define the ANDF aggregate procedure \bar{D} by its discount function as

Figure 3: Three Procedures for Aggregating Individual Discount Functions



Note: This figure shows the functions $d(t)$ and $r(t)$ that result from different ways of aggregating two discount functions: $d_1(t) = e^{-.2t}$ and $d_2(t) = e^{-.02t}$.

Source: Authors' calculations

w A1

follows:¹¹

$$(18) \quad d_{\overline{D}}(t) = \frac{\int_X \alpha(x)d(t;x)f(x)dx}{\int_X \alpha(x)f(x)dx}.$$

As required, $\int_X \alpha(x)d(0;x)f(x)dx = \int_X \alpha(x)(1)f(x)dx = \int_X \alpha(x)f(x)dx$, $d_{\overline{D}}(0) = 1$. The definition in equation (18) leads to a convergent aggregate procedure if there is a nonzero proportion of the population using convergent procedures.

To each individual discount function $d(t;x)$ there corresponds a discount rate function $r(t;x)$ satisfying $\dot{d}(t;x) = -r(t;x)d(t;x)$, where the superscript dot denotes a time derivative. For the aggregate procedure \overline{D} , since the $d(t;x)$ in the numerator is the only term involving time,

$$(19) \quad \dot{d}_{\overline{D}}(t) = \frac{-\int_X \alpha(x)r(t;x)d(t;x)f(x)dx}{\int_X \alpha(x)f(x)dx},$$

and therefore

$$(20) \quad r_{\overline{D}}(t) = \frac{-\dot{d}_{\overline{D}}(t)}{d_{\overline{D}}(t)} = \frac{\int_X \alpha(x)r(t;x)d(t;x)f(x)dx}{\int_X \alpha(x)d(t;x)f(x)dx}.$$

We now prove three core results that relate the characteristics of the ANDF aggregation procedure to the corresponding characteristics of the individual procedures that were aggregated. Let $Y = \{x \in X | \alpha(x) = 0\}$, that is, Y is the set of parameters that correspond to nonconvergent discounting procedures.

Proposition 4: If $\int_Y f(x)dx = 0$, then, for the ANDF process defined above, the amount $\alpha(\overline{D})$ and mean time horizon $\theta(\overline{D})$ of the aggregate procedure \overline{D} are the average amount $\bar{\alpha}$ and the average mean time $\bar{\theta}$ respectively.¹²

Proof: Since $\alpha(x) = 0$ for all $x \in Y$, anytime the integrand involves $\alpha(x)$ we can switch the domain of integration between X and $X \setminus Y = X - Y$ as we wish. We first verify that $\int_0^\infty \int_X \alpha(x)d(t;x)f(x)dxdt = \int_0^\infty \int_{X \setminus Y} \alpha(x)d(t;x)f(x)dxdt = \int_{X \setminus Y} \alpha(x)f(x) \left[\int_0^\infty d(t;x)dt \right] dx =$

¹¹We require only that $\int_X \alpha(x)f(x)dx > 0$, that is, that at least some nonzero fraction of the population uses convergent discounting procedures. If $\int_X \alpha(x)f(x)dx = 0$, then there is no need to normalize (essentially all individuals already have the same α , namely $\alpha = 0$), so we define $d_{\overline{D}}(t)$ as simply $\int_X d(t;x)f(x)dx$.

¹²Thus the bar above a particular characteristic denotes expectation with respect to the density f . Exactly because of the proposition, this will not prove to be confusing terminology.

$\int_{X \setminus Y} \alpha(x) f(x) \frac{1}{\alpha(x)} dx$ (note that $\alpha(x) > 0$ on $X \setminus Y$ so $\frac{1}{\alpha(x)}$ is well-behaved) $= \int_{X \setminus Y} f(x) dx = \int_X f(x) dx - \int_Y f(x) dx = 1 - 0 = 1$, which makes sense since this was an average normalized function. Now note that the denominator of the aggregate function $d_{\bar{D}}$ is constant in t , and thus, using the calculation we just made, the aggregate amount is

$$\alpha(\bar{D}) = \left[\int_0^\infty d_{\bar{D}}(t) dt \right]^{-1} = \left[\frac{1}{\int_X \alpha(x) f(x) dx} \right]^{-1} = \int_X \alpha(x) f(x) dx,$$

which is precisely the formula for the average amount $\bar{\alpha}$ of discounting across the population, as desired.

For the mean time horizon, we compute θ for the aggregate procedure as

$$\begin{aligned} \theta(\bar{D}) &= \alpha(\bar{D}) \int_0^\infty t d_{\bar{D}}(t) dt \\ &= \alpha(\bar{D}) \int_0^\infty t \left[\frac{\int_X \alpha(x) d(t; x) f(x) dx}{\alpha(\bar{D})} \right] dt \\ &= \int_0^\infty \int_X t \alpha(x) d(t; x) f(x) dx dt \\ &= \int_X \left[\alpha(x) \int_0^\infty t d(t; x) dt \right] f(x) dx \\ &= \int_X \theta(x) f(x) dx \\ &= \bar{\theta}. \end{aligned}$$

In general, if $\int_Y f(x) dx > 0$, $\alpha(\bar{D})$ will simply be the average over all strictly positive α in the population: formally, $\alpha(\bar{D}) = \int_{X \setminus Y} \alpha(x) f(x) dx$ (with an analogous outcome for θ). The intuitive interpretation is that any individual who chooses $\alpha = 0$ (that is, a nonconvergent procedure), effectively suggesting an infinite present value, ends up spreading his/her normalized weight so thinly over time that it has no effect at all on the aggregate.¹³ We turn next to the speed of the aggregate procedure.

Proposition 5: If the amount $\alpha(x)$ and the mean time $\theta(x)$ are negatively covariant within a population X , then the speed resulting from the ANDF process is lower than the average speed,

¹³It is possible to construct an aggregate discounting procedure whose amount is *always* equal to the average amount, inclusive of x such that $\alpha(x) = 0$. In this case the shape of the aggregate function is identical to the aggregate as defined, so the larger present value means that $pv(t)$ does not converge to $pv(\infty)$, and thus of course $\theta(\bar{D}) = \infty$ and $\rho(\bar{D}) = 0$. Details are available upon request.

that is, $\rho(\bar{D}) \leq \bar{\rho}$, with equality only if all individuals have the same amount of discounting α and all individuals have the same speed ρ .

Proof: By Propositions 1 and 4, $\rho(\bar{D}) = \theta(\bar{D}E)/\theta(\bar{D}) = (\bar{\alpha}\bar{\theta})^{-1}$, while $\bar{\rho} = \int_X \rho(x)f(x)dx = \int_X (\alpha(x)\theta(x))^{-1} f(x)dx = \overline{(\alpha\theta)^{-1}}$, where the bar continues to denote expectation with respect to f , and $\bar{D}E$ refers to the exponential that discounts by the same amount as \bar{D} . But since $(\cdot)^{-1}$ (that is, taking inverses) is a convex function on \mathbb{R} , Jensen's inequality implies that the inverse of the average is weakly less than the average of the inverses, that is, $(\bar{\alpha}\bar{\theta})^{-1} \leq \overline{(\alpha\theta)^{-1}}$. Now $cov(\alpha, \theta) = \overline{\alpha\theta} - \bar{\alpha}\bar{\theta}$ by definition, and this is negative by assumption. Hence $\overline{\alpha\theta} \leq \bar{\alpha}\bar{\theta}$, so $\rho(\bar{D}) = (\bar{\alpha}\bar{\theta})^{-1} \leq (\overline{\alpha\theta})^{-1} \leq \overline{(\alpha\theta)^{-1}} = \bar{\rho}$. Furthermore, since $(\cdot)^{-1}$ is in fact strictly convex, the inequality is strict unless both $cov(\alpha, \theta) = 0$ and $(\alpha\theta)^{-1}$ is constant across the population. But these cannot hold simultaneously unless α and θ are themselves constant, which is equivalent to α and ρ being constant.

In general, we do expect α and θ to have negative covariance, since placing less total present value on the future often implies that that present value is reached more quickly. This tendency is confirmed by looking at the expressions in table 1, but it need not hold for every possible choice of distributions f over the underlying parameters. Thus we expect an aggregate procedure to be slower overall than its components, but it is possible for it to be faster under certain circumstances.¹⁴ In the special case of constant relative speed, however, we can rule out this possibility.

Corollary: If $\rho(x)$ is constant across x , then $\rho(\bar{D}) \leq \bar{\rho}$, with equality only if $\alpha(x)$ is also constant.

Proof: From the corollary to Proposition 1, $\rho = (\alpha\theta)^{-1}$, so $\rho(x)$ constant implies $\alpha(x)\theta(x) = C$ for all x . Then we can compute $cov(\alpha, \theta) = cov(\alpha, C/\alpha) = \overline{\alpha(C/\alpha)} - \bar{\alpha}\overline{(C/\alpha)} = C - \bar{\alpha}C\bar{\alpha}^{-1} = C(1 - \bar{\alpha}\bar{\alpha}^{-1})$. But Jensen's inequality once again implies that $(\bar{\alpha})^{-1} \leq \overline{\alpha^{-1}}$, or $1 \leq \bar{\alpha}\bar{\alpha}^{-1}$. Hence $cov(\alpha, \theta) \leq 0$ and Proposition 5 applies. The inequality is clearly strict unless α is constant across the population.

We turn now to the relationship between the aggregate rate $r_{\bar{D}}(t)$ and the individual discount

¹⁴Hara, Huang, and Kuzmics (2007) prove a conceptually similar result in the context of risk aversion. Specifically, they show that the representative consumer exhibits strictly decreasing relative risk aversion, ranging from that of the most risk averse individual to that of the least risk averse as the aggregate consumption level increases.

rates $r(t; x)$. In particular, one variable of interest is the limiting discount rate $r^*(x) = \lim_{t \rightarrow \infty} r(t; x)$, if it exists, which gives the asymptotic discount rate for individual x . For any potential social limiting rate $r \geq 0$, let $A(r) = \{x \in X \text{ s.t. } \alpha(x) > 0, r^*(x) \text{ exists, and } r^*(x) \leq r\}$; this is the set of individuals who use convergent procedures and whose limit is no larger than r . Finally, we define $r_{\min}^* = \inf \left\langle r \mid \int_{A(r)} f(x) dx > 0 \right\rangle$. This is the lowest rate r such that at least some nonzero fraction of the population has a limiting rate no higher than r . Dybvig, Ingersoll, and Ross (1996) and Weitzman (1998) showed that if the social discount function is constructed by simple averaging of a finite number of individual discount functions, then r_{\min}^* is exactly the asymptotic discount rate for the aggregate.¹⁵ Our next proposition states that, without assuming a finite number of individuals, the same is true for the ANDF process—that is, that in the limit the social discount rate is in some sense the smallest of any across the population:

Proposition 6: The asymptotic social discount rate for the ANDF aggregation process is given by: $\lim_{t \rightarrow \infty} r_{\overline{D}}(t) = r_{\min}^*$.

Proof: See appendix, section 4.

2.3 Aggregation of exponential procedures

The aggregation processes defined above can be carried out no matter what the underlying discounting procedures are. If, however, we hope for simple closed-form outcomes, we will need to make some specific assumptions. For the remainder of this section, we assume that each individual i uses a constant-rate procedure given by $d(t) = e^{-r_i t}$. This not only simplifies the analysis, but also allows us to compute empirical results from the data collected by Weitzman (2001), who surveyed 2,160 economists and asked each to supply the exponential discount rate they used.

We first observe that when aggregating exponentials, all of which have $\rho(x) = 1$, the corollary to Proposition 5 applies, so the speed of the ANDF aggregate is strictly less than 1 unless all individuals use the same constant discount rate, meaning these are identical.

We begin by considering discrete aggregation of exponentials, that is, the case when there is a finite population. Let the number of individuals be n , with respective constant rates r_1, \dots, r_n

¹⁵The analogous result concerning mortality has been standard in the demography literature for some time. Just as the lowest discount rate becomes dominant in later years so, too, in a population that is an aggregate of distinct subpopulations, the cohort mortality rate in later years will approach that of the subpopulation with the lowest mortality rate. See Vaupel, Manton, and Stallard (1979) for an early treatment of this subject.

(possibly with multiplicity, of course). We assume $r_i > 0$ for all i (or equivalently that there are n individuals with $r_i > 0$ and the rest can be ignored; see comment after Proposition 4). In this case the ANDF aggregate procedure results in

$$(21) \quad \bar{\alpha} = \frac{1}{n} \sum_{i=1}^n r_i,$$

$$(22) \quad \bar{\theta} = \frac{1}{n} \sum_{i=1}^n r_i^{-1}, \text{ and}$$

$$(23) \quad \rho(\bar{D}) = \frac{n^2}{\left(\sum_{i=1}^n r_i\right) \left(\sum_{i=1}^n r_i^{-1}\right)}.$$

These results follow directly from Proposition 4, given that individual i 's amount α_i is r_i and i 's mean time horizon θ_i is r_i^{-1} . Then, using Proposition 1, $\rho(\bar{D})$ can be computed as $(\bar{\alpha}\bar{\theta})^{-1} = \left[\frac{1}{n^2} \left(\sum_{i=1}^n r_i\right) \left(\sum_{i=1}^n r_i^{-1}\right)\right]^{-1}$, as stated.

We next study continuous populations by considering a specific but very general density on the underlying distribution of discount rates, the gamma distribution, for $x > 0$:

$$(24) \quad f(x) = \frac{a^b}{\Gamma(b)} e^{-ax} x^{b-1},$$

with $a, b > 0$. This has mean $\mu = b/a$ and variance $\sigma^2 = b/a^2$. Such a distribution fits the Weitzman data well (although, unless $b = 1$, it has the disadvantage of putting no weight at 0, whereas almost 2.5 percent of the respondents chose discount rates at or below 0). Weitzman, aggregating by averaging of the $d(t)$ s, showed that the gamma distribution¹⁶ on parameters leads to:

$$(25) \quad d_\gamma(t) = \left[\frac{1}{1 + t/a} \right]^b.$$

¹⁶Axtell and McRae (2007) obtain the same result as a special case of their much more general findings on ADF aggregation of exponential procedures. They obtain their results very simply by observing that the aggregation of exponentials involves taking the Laplace transform of the aggregating distribution. They also underscore the importance of “potential problems” with convergence in showing that averaging of exponential discount functions whose parameters are exponentially distributed always leads to a nonconvergent aggregate procedure.

Although Weitzman labels this function “gamma discounting” the procedure is generally known as hyperbolic discounting, and we use the standard terminology.

Proposition 7: The ANDF aggregation process for individuals with exponential discount functions, whose parameters are distributed according to the gamma function above, leads to a social discount function given by:

$$(26) \quad d_{\bar{D}}(t) = \left[\frac{1}{1 + t/a} \right]^{1+b}.$$

Proof: See appendix, section 4.

So it is also hyperbolic, but with an exponent that is greater by 1 than for $d_{\gamma}(t)$, which resolves the convergence issue. We can compute $\bar{\alpha} = b/a$ and $\rho(\bar{D}) = 1 - 1/b$. In terms of the mean μ and variance σ^2 of the density f , we get

$$(27) \quad \bar{\alpha} = \mu \text{ and } \rho(\bar{D}) = 1 - \frac{\sigma^2}{\mu^2}.$$

The ADF aggregate—given in the parameterization of equation (3)—has

$$(28) \quad \alpha_{\gamma} = \mu - \frac{\sigma^2}{\mu} \text{ and } \rho_{\gamma} = 1 - \frac{\sigma^2}{\mu^2 - \sigma^2}.$$

These values imply, among other things, that even after ruling out individual discount rates equal to zero (of which there were many in Weitzman’s sample), the aggregate function $d_{\gamma}(t)$ fails to converge when $\sigma \geq \mu$, as it almost is in Weitzman’s sample (where σ is roughly 3 percent and μ is roughly 4 percent). If the ANDF aggregation process (which is always convergent) is used when $\sigma \geq \mu$, then the aggregate discounting procedure \bar{D} remains convergent, but only weakly so.

3 Specific Discounting Procedures

This section discusses a number of discounting procedures, some generalized from the literature and some new. The appendix states and proves basic properties of the discounting procedures discussed in sections 1 and 2, including the quasi-hyperbolic discounting procedure. More procedures could be derived from the demography literature on parametric survival curves (see Carrière 1992). Subsection 3.1 summarizes the properties of the specific procedures we focus on. Subsection 3.2 then examines in more detail a specific procedure, the zero-speed hyperbolic (ZSH), which we argue to be particularly well-suited for discounting in social decisionmaking.

3.1 The amount and speed of selected discounting procedures

This section states the properties of a number of familiar discounting procedures as well as characterizing and stating the properties of several that are novel procedures. Three relatively general procedures require two parameters and are characterized in table 1. The first column of table 1 provides a parameterization of the hyperbolic discount function (HYP) that allows amount and speed to be expressed directly in terms of its parameters r and s : $\alpha(HYP) = r$ and $\rho(HYP) = s$. The table's first row gives the discount function, and the second row gives the discount rate function. Mean time to acquisition of present value for the hyperbolic is $(rs)^{-1}$. Note that the hyperbolic procedure is always slow and that its speed can take on the value 0. Each probability density function, $f(t)$, defined on $[0, \infty)$ forms the basis for a discounting function through the formula:

$$(29) \quad d(t) = \int_t^{\infty} f(x)dx,$$

as discussed in the appendix. One such probability density function (pdf), the gamma, forms the basis for procedures that result in a strictly positive asymptotic value of r . Starting with a gamma, the result is for $d(t)$ is

$$(30) \quad d(t) = \frac{\Gamma(s+1, (s+1)rt)}{\Gamma(s+1)} \text{ with } s > -1 \text{ and } r > 0,$$

a somewhat complex formulation involving the incomplete gamma function, $\Gamma(\cdot, \cdot)$. See derivations in the appendix. The virtues of this procedure are that it allows both fast and slow speeds and that the amount and speed are simply given: $\alpha = r$ and $\rho = 2 - \frac{2}{s+2}$. For the slow or declining discount rate gamma the limiting value of $r(t)$ is $r(s+1)$.

Finally, we wish to include procedures that can be either fast or slow but that can have a limiting discount rate of zero as time goes to infinity. A procedure that falls into this class was introduced by Read (2001):

$$(31) \quad d(t) = e^{-rt^{1/s}}.$$

Table 1: Properties of Hyperbolic, Gamma and Weibull Discounting Procedures

Property	Hyperbolic	Gamma	Weibull
1. discount function, $d(t)$	$[1 + r(1 - s)t]^{-(1 + \frac{1}{1-s})}$ ($r > 0; s < 1$)	$\frac{\Gamma(s+1, (s+1)rt)}{\Gamma(s+1)}$ ($r > 0; s > -1$)	$\exp(-rt^{1/s})$ ($r, s > 0$)
2. discount rate, $r(t)$	$r \frac{2-s}{1+r(1-s)t}$	$\frac{[(s+1)r]^{s+1} t^s e^{-(s+1)rt}}{\Gamma(s+1, (s+1)rt)}$	$\frac{r}{s} t^{(1-s)/s}$
$r(0)$	$r(2 - s)$	$\begin{cases} \infty & \text{if } s < 0 \\ r & \text{if } s = 0 \\ 0 & \text{if } s > 0 \end{cases}$	$\begin{cases} 0 & \text{if } s < 1 \\ r & \text{if } s = 1 \\ \infty & \text{if } s > 1 \end{cases}$
$r(\infty)$	$\begin{cases} 0 & \text{if } s < 1 \\ r & \text{if } s = 1 \\ \infty & \text{if } s > 1 \end{cases}$	$r(s + 1)$	$\begin{cases} \infty & \text{if } s < 1 \\ r & \text{if } s = 1 \\ 0 & \text{if } s > 1 \end{cases}$
3. amount, α	r	r	$r^s / \Gamma(s + 1)$
4. speed, ρ	s ($\rho < 1$)	$2 - \frac{2}{s+2}$ ($0 < \rho < 2$)	$\Gamma(s)\Gamma(s + 1) / \Gamma(2s)$ ($0 < \rho < 2$)
5. median time, τ	$r^{-1}(2^{1-s} - 1) / (1 - s)$	<i>no closed form</i>	<i>no closed form</i>
6. mean time, θ	$(rs)^{-1}$	$\frac{s+2}{2(s+1)r}$	$r^{-s}\Gamma(2s) / \Gamma(s)$

This discount function's associated pdf is Weibull and we denote this as Weibull discounting. Here the parameter s either expands or contracts time relative to constant rate exponential discounting.¹⁷ Characteristics of the Weibull discounting procedure appear in Table 1; for derivations see the supplement. If $s = 2$ (which corresponds to contracted time), then $\alpha = r^2/2$, $\rho = 1/3 < 1$, and $\theta = 6/r^2$; this is an example of slow discounting (and more generally $\rho < 1$

¹⁷One can, in principle, transform time with a broad range of functions $g(t)$ to get $d(t) = e^{-rg(t)}$. For example Roelofsma (1996) discusses Weber's Law from psychology, for which $g(t) = \ln t$. In this case, interestingly, the resulting discounting procedure can be shown to be a member of the hyperbolic family.

exactly when $s > 1$). We label this the “slow Weibull” discounting procedure. Having $s = 1/2$ entails squaring t in the formula for $d(t)$ effectively expanding time. We label this the “fast Weibull”. For the fast Weibull, which appeared as an example in equation 5, $\alpha = \sqrt{4r/\pi}$, $\rho = \pi/2$, and $\theta = 1/\sqrt{r\pi}$.

Not surprisingly the procedures listed in table 1 reduce to exponentials under certain parameter values, namely the Weibull for $s = 1$ and the gamma for $s = 0$. Although we do not allow $s = 1$ explicitly in the hyperbolic procedure, if we take the limit as s approaches 1 (from below) we get an exponential with discount rate r , as expected. This can be shown directly, but it also follows from the discount rate function for the hyperbolic.

We reiterate that $s > 0$ is not required for the hyperbolic procedure. In fact, $s \leq 0$ is perfectly legitimate; this corresponds to weak convergence in our terminology. Technically, the speed as a ratio of areas will be identically zero for any weakly convergent procedure, but it is useful to allow $s < 0$ in the definition of a hyperbolic without referring to this as a speed. That said, the case where $s = 0$ is of particular interest in that it anchors one end of a spectrum of hyperbolic procedures that is anchored at the other by the exponential ($\rho = 1$). And, as with the exponential, the zero-speed hyperbolic is a single parameter procedure: $d(t) = (1 + rt)^{-2}$.

The preceding paragraphs have pointed to a number of special cases of the general discounting procedures characterized in Table 1. Each case results from fixing the speed of a general procedure so that the remaining single parameter in the procedure is the amount of discounting or a simple function of that amount. Table 2 presents and characterizes five single parameter procedures with speeds (row 4) ranging from $\rho = 0$ (zero-speed hyperbolic or ZSH) to $\rho = \pi/2$ (fast Weibull). In between are the slow Weibull procedure ($\rho = 1/3$), the exponential procedure ($\rho = 1$), and the gamma procedure with $s = 1$ ($\rho = 4/3$). Taken together these five procedures present a menu of simple and analytically tractable discounting procedures that span almost the full range of speeds. The fast procedures represent single parameter alternatives to the the-parameter quasi-hyperbolic procedures (appendix, section 2) for use in psychology and behavioral and neuroeconomics.¹⁸ The

¹⁸The basic observation in this literature is that individuals appear to discount very little over a “present” period that extends a short but usually unspecified time into the future. After that the discount function drops sharply and then resumes a slow decline. The fast Weibull with speed $\pi/2$ provides a natural single parameter alternative to the 3-parameter hyperbolic. (Note that even the discrete time quasi-hyperbolic has 3 parameters—the parameters b and r in equation 4 plus the specification of the units in which time is measured.)

slow procedures provide approaches to discounting for long time horizon social investments, and the next subsection further discusses the ZSH in that context.

Table 2: Properties of Five Single-Parameter Discounting Procedures

Property	Zero-speed	Slow			Fast
	hyperbolic ($s = 0$)	Weibull ($s = 2$)	Exponential	Gamma ($s = 1$)	Weibull ($s = \frac{1}{2}$)
1. discount function, $d(t)$	$(1 + rt)^{-2}$	$\exp(-rt^{1/2})$	e^{-rt}	$e^{-2rt}(1 + 2rt)$	$\exp(-rt^2)$
2. discount rate, $r(t)$	$\frac{2r}{1+rt}$	$\frac{r}{2\sqrt{t}}$	r	$\frac{4r^2t}{1+2rt}$	$2rt$
$r(0)$	$2r$	∞	r	0	0
$r(\infty)$	0	0	r	$2r$	∞
3. amount, α	r	$r^2/2$	r	r	$\sqrt{4r/\pi}$
4. speed, ρ	0	$1/3$	1	$4/3$	$\pi/2$
5. median time, τ	r^{-1}	—	$(\ln 2)/r$	$2r\tau = \ln(2 + 2r\tau)$	—
6. mean time, θ	∞	$6/r^2$	r^{-1}	$3/4r$	$1/\sqrt{r\pi}$

3.2 Zero-speed hyperbolic (ZSH) discounting for social choice

The preceding subsection discussed the three very general discounting procedures presented in table 1 and provided one-parameter special cases spanning a range of speeds in table 2. This offers a broad menu for choice of a social discounting procedure. We would look for several things when selecting a procedure from this menu:

(i) Because the ANDF aggregation procedure yields a social procedure that is slower than the average of the speeds of the individual procedures (Proposition 6) it would in general be desirable for a social procedure to be slow ($\rho < 1$). While it is possible that the individual procedures being aggregated would be fast it is more likely that they would be exponential and in any case aggregation will result in a slower discounting procedure.¹⁹

¹⁹For those who might prefer the ADF procedure, it too will generate a slower aggregate procedure that the

(ii) It would be desirable for the social discounting procedure to be a single parameter function and for that parameter to be simply related to the amount of discounting.

Table 2 presents two candidates that meet these criteria—the zero-speed hyperbolic (ZSH) with $\rho = 0$ and the slow Weibull with $\rho = 1/3$. The ZSH is a slightly simpler expression than the slow Weibull and its single parameter r is equal to the amount of discounting whereas with the slow Weibull the amount of discounting is $r^2/2$. For these reasons we propose the ZSH as a simple yet flexible procedure for social discounting.

We designate the discount and discount rate functions for the ZSH by $d_z(t)$ and $r_z(t)$, and if the amount of discounting is r and the speed is zero we have:

$$(32) \quad d_z(t) = (1 + rt)^{-2} \text{ and}$$

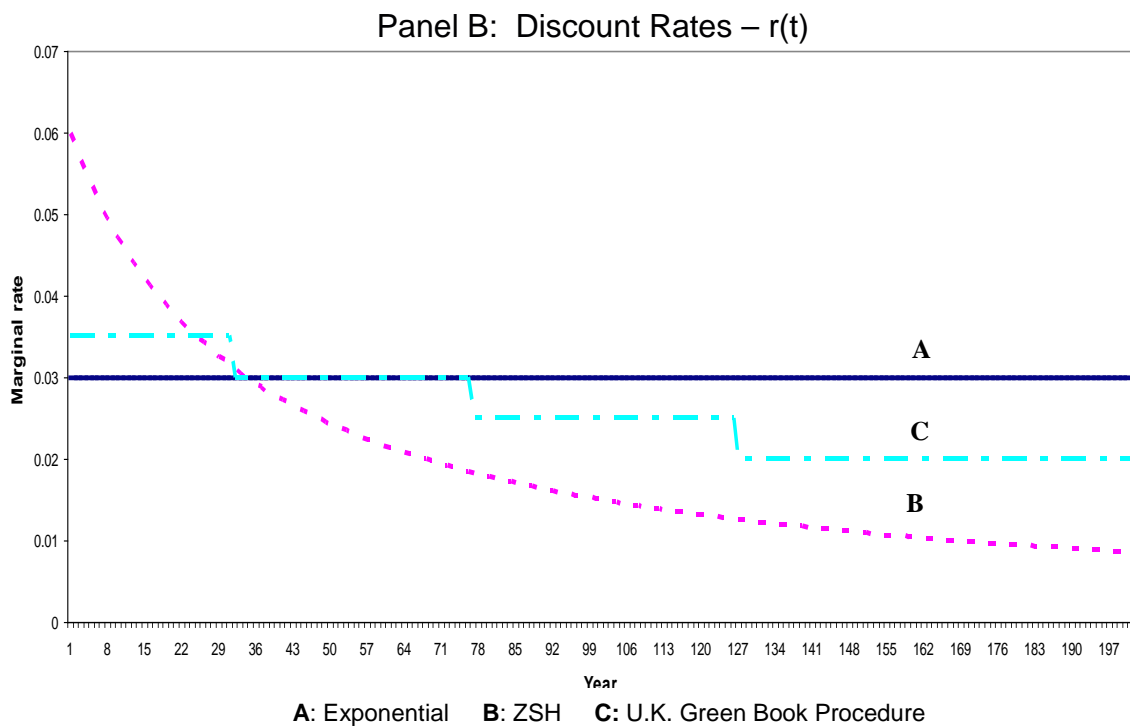
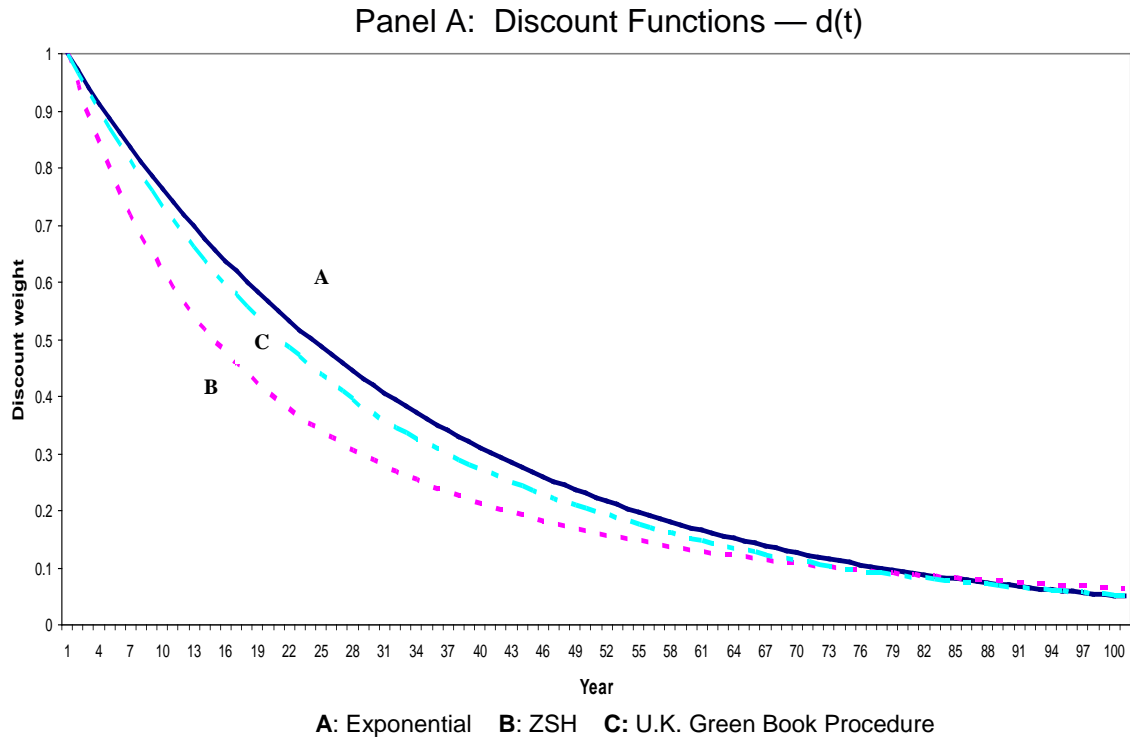
$$(33) \quad r_z(t) = 2r/(1 + rt).$$

Thus $r_z(t)$ declines from a value of $2r$ at 0 to a value of 0 at ∞ , and equals r when $t = r^{-1}$, that is, at the time by which half of the present value of the ZSH has been accumulated. (Half the present value of an equivalent exponential will have been accumulated earlier by a factor of 0.69.) Although the median time for accumulation of present value for the ZSH is r^{-1} , the mean time is infinite so the procedure puts substantial weight on the far future.

Newell and Pizer (2003) used historic volatility of discount rates in bond markets to generate an expected value for a potential social discounting function. In a study for the Office of the Deputy Prime Minister of the United Kingdom, the consulting firm Oxera (2002) simplified their results into a proposed procedure for long horizon project analysis in the United Kingdom which was used in the U.K. Green Book (HM Treasury 2003). The proposed procedure starts with a discount rate of 3.5 percent per year that then declines in increments to 1 percent per year after 300 years. Figure 4 compares the discount and discount rate functions of the proposed U.K. procedure, which has an amount $\alpha = 0.0318$, to $d_z(t)$ and $r_z(t)$ and to the exponential with the same amount. While there are broad similarities, $r_z(t)$ starts higher (at slightly above 6 percent) and declines to zero rather than to 1 percent, which has the effect of placing substantially greater weight on the far future.

average of the individual procedures.

Figure 4: Comparison of Discounting Procedures with $\alpha = 0.0318$: Exponential, U.K. Green Book Procedure and Zero-Speed Hyperbolic (ZSH)



Source: Authors' calculations

Table 3 shows the amount of present value remaining to be accumulated at various times for the U.K. Green Book procedure and for ZSH and exponential procedures with amount 0.0318. After 100 years the exponential has 4.2 percent of its present value still to be accumulated and the Green Book procedure has 7.5 percent. The ZSH, however, has 23.9 percent remaining, and even after 400 years still has as much present value to accumulate as the Green Book procedure had at 100 years. The ZSH, with its ultimately 0 value of $r_z(t)$, places more weight on the far future than does a procedure with stepwise declining exponential rates that discounts by the same amount. The recently published Stern report on climate change (Stern 2007) provides another example of a potential social discounting function. This procedure gives weight to the far future by (effectively) choosing a low discount rate of 1.4 percent per year. But even that low rate leaves only 6.1 percent of present value to be accumulated after 200 years whereas the ZSH with $r = 0.014$ has 26.3 percent still to be accumulated after two hundred years. The ZSH has 8 percent to accumulate after 800 years whereas, for the exponential, there is virtually nothing left by that time. In addition, short-term interest rates of 2.8 percent with the ZSH are, while low, still more reasonable than 1.4 percent.

A switch to ZSH discounting, with an r of about 0.03, gives far more weight to the far future than does exponential discounting at any reasonable rate. It does this while preserving realistic discount rates over short horizons and with a tractable functional form. The ZSH results from ANDF aggregation of individual exponential discount functions whose parameters follow a gamma distribution with its standard deviation equal to its mean (equation 27). [In Weitzman's (2001) data the standard deviation was 3 percent around a mean of 4 percent, and the gamma fit fairly well except at 0 percent.] It would be worthwhile to conduct sensitivity analyses of long horizon policy assessments (such as those of the Stern report) to the use of ZSH discounting.

Important to selecting the appropriate parameter for the amount of a social discounting procedure is being clear about whether one is discounting utility or money. In a careful experimental study, Anderson et al. (2008) find that joint elicitation of risk and time preferences in individuals leads to substantially lower discount rates than a separate elicitation of time preference rates: expected utility is discounted less than money. This finding needs to be incorporated into the establishment of social discounting procedures.

When selecting the parameter for the amount of a social discounting procedure, it is

Table 3: Present Value Remaining as a Function of Time for Selected Exponential and Zero-Speed Hyperbolic (ZSH) Procedures

After year	Percent of present value remaining				
	$\alpha = 0.014$ (Stern Report)		$\alpha = 0.0318$		
	Exp	ZSH	Exp	ZSH	Green Book procedure
0	100	100	100	100	100
25	70.4	74.1	45.2	55.7	47.1
50	49.7	58.8	20.4	38.6	24.3
100	24.7	41.7	4.2	23.9	7.5
200	6.1	26.3	0.2	13.6	1.4
400	0.4	15.2	0	7.3	0.2
800	0	8.2	0	3.8	0

Notes:

1. The amount of discounting is designated α . For exponential discounting $d(t) = e^{-\alpha t}$; for ZSH discounting $d(t) = (1 + \alpha t)^{-2}$.
2. The Stern Report on climate change Stern (2006) effectively used a discount rate of 1.4 percent per year.
3. The procedure developed for the U.K. Green Book (OXERA, 2002) prescribes discounting with rates fixed over time intervals, beginning at 3.5 percent per year and declining to 1 percent per year after 300 years. For this procedure it turns out that $\alpha = 0.0318$.

important to be clear about whether one is discounting social utility or money. In a careful experimental study, Andersen et al. (2008) find that joint elicitation of risk and time preferences leads to substantially lower discount rates than when these preferences are elicited separately: expected utility is discounted less than money. This finding needs to be incorporated into the establishment of social discounting procedures.

3.3 Conclusions

This paper proceeds from the observation that economists in a range of specialized fields use nonexponential discounting functions. Within the exponential framework, alternative discounting procedures align along the single dimension of the discount rate. Until recently, its simplicity, along with the intuitive appeal of the axiomatic formulation of constant rate discounting and its analytic tractability, have ensured the exponential framework's dominance.²⁰ This paper develops a framework for evaluating nonconstant rate discounting procedures that arrays these options along the two dimensions of amount and speed, thereby facilitating a systematic comparison of these procedures and enhancing their tractability.

A question addressed by Arrow (1999), one which is antecedent to discounting in the context of assessing long-term investments, concerns the extent to which decisions made today will have their influence attenuated (or eliminated) by compensatory decisions made by subsequent generations. This issue is very real, and arises also in the design of other sequential decision procedures, such as foreign aid programs and transfers across levels of government. It has not been our purpose in this paper to address this intergenerational issue but, more simply, to improve the tools available in

²⁰The axiomatic foundation for constant rate discounting goes back to Koopmans (1960) who introduced a “stationarity” axiom concerning preferences over time streams of outcomes and established plausible conditions that require discounting to be exponential. A number of authors have explored weaker axiom systems that allow representation of intertemporal preferences by a discounting procedure that is not necessarily constant rate—see Jamison (1969), (Fishburn and Rubinstein (1982), Bleichrodt and Gafni (1996), and Bleichrodt and Johannesson (2001). The simple existence of an axiomatic foundation, therefore, is no argument in favor of constant- over variable-rate discounting; the question is one of assessing the descriptive, normative, and tractability consequences of adding the strong stationarity axiom. In particular the normative appeal of stationarity must be weighed against its implication that at, say, a 3 percent annual discount rate we should be prepared to pay about 25 times as much to realize a given benefit in 200 rather than 300 years.

circumstances where discounting procedures are being used. Similarly, although we acknowledge the existence and practical relevance of time-inconsistency issues with regard to variable rate discounting—initially addressed by Strotz (1956)—our focus has been on a different facet of the problem.

Key results of the paper include:

- (i) precise formulations for the concepts of “amount”, “speed”, and “time horizon” used in discounting procedures;
- (ii) proofs of key relationships among the concepts of amount, speed, and time horizon;
- (iii) identification of inadequacies in existing approaches to aggregating individual discounting procedures into a socially representative one and formulation of an alternative process—averaging of normalized discount functions or ANDF—that overcomes these inadequacies;
- (iv) proof that under reasonable assumptions the ANDF aggregate procedure will be slower than the average of the speeds of the individual procedures;
- (v) descriptions of a range of existing and new discounting procedures and provision of closed form characterizations relating amount, speed, and time horizon to their underlying parameters; and
- (vi) use of a particular variant of hyperbolic discounting, which we label the zero-speed hyperbolic (ZSH), would provide an analytically tractable way of giving substantial weight to the far future in policy analyses while preserving reasonable discount rates in the short term.

Frederick, Loewenstein, and O’Donoghue (2002), Chapman (2003), and Groom et al. (2005) provide valuable compilations of the recent literature on time preference and discounting. Poulos and Whittington (2000) extend the literature on the discounting of lives saved to several developing country contexts. Yet transforming the empirical literature into useful discounting procedures for policy applications will require two additional steps. First, to the extent it is practical, the data underlying the reported literature needs to be characterized in terms of estimates of the amount and speed of individual discounting procedures. Second, the ANDF aggregation algorithm can be used to generate candidate social discounting procedures. We feel that our proposed approach to discounting both undermines many of the practical objections to expanding the use of nonconstant rate procedures and provides a needed framework for integrating and comparing results in the existing literature.

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4 Appendix

This appendix has four sections. Section 1 describes the relationships between the discount function and other defining functions. Section 2 derives properties of discounting procedures found in the text. Section 3 presents derivations for the characteristics of quasi-hyperbolic discounting, generalized to continuous time in two separate ways, and then for four nonconstant-rate extensions of exponential discounting. Section 4 provides proofs for Propositions 6 and 7 in the main text.

4.1 Relations between the discount function and other defining functions

A discounting procedure can be defined in multiple ways, each of which leads to a discount function, $d(t)$, that assigns to each future time t a coefficient by which the net outcome at that time can be converted to its present value or the time 0 “equivalent” of the time t outcome. Appendix table 1 lists five functions that can, in addition to $d(t)$ itself, be used to define a discounting procedure and relates each of these to $d(t)$. Four of the potential defining functions—the discount rate, discount factor, yield curve, and present value—are familiar, although they often appear in different contexts. We add a fifth defining function in addition to these four, which we label the “associated probability density function” (or *pdf*).

Table 4 brings together in one place the relation between each of the defining functions and $d(t)$. For reference the relation to $d(t)$ is given for both discrete and continuous time formulations. Row 1, for example shows the relation of $d(t)$ to the discount rate function $r(t)$ by giving $d(t)$ as a function of $r(t)$ and vice versa. Row 2 shows the relation of discount factors—the standard representation of discounting in game theory, for instance—to $d(t)$.

Row 3 relates the yield curve from the finance literature to $d(t)$. The relation here is a useful one for interpreting results in the health-related literature on assessment of discount rates since they are frequently reported (implicitly) as yield curves.²¹ Row 4 relates $d(t)$ to the cumulative

²¹See Malkiel (1998) or Cochrane (2001, Chap. 19), for discussions of yield curves and the term structure of interest rates. Cropper, Aydede, and Portney (1994), in an important early paper on discounting of lives saved, report a 16.8 percent discount rate over 5 years declining to 3.4 percent over 100 years, a “yield curve” formulation.

present value function. Finally, Row 5 results from the observation that the inverse cumulative²² of any *pdf* defined on $[0, \infty)$ will, in fact, be a discount function with a present value equal to the expectation of a random variable with that *pdf*. Thus if f is a *pdf* on $[0, \infty)$, then

$$(34) \quad d_f(t) = 1 - \int_0^t f(x)dx = \int_t^\infty f(x)dx$$

will be a discount function.

Finally it is worth noting that the “survival curve” of demography is formally equivalent to $d(t)$ and a survival curve’s “hazard function” corresponds to $r(t)$. Life expectancy is then equal to $pv(\infty)$.

Cairns and van Der Pol (1997) provide further strong evidence for slow discounting in the context of saving lives.

²²The inverse cumulative of a *pdf* is 1 minus the cumulative.

Appendix Table 1: Relations Between the Discount Function and Other Defining Functions

<u>Definitional function</u>	<u>Discrete time</u>	<u>Continuous time</u>
1. discount rate, $r(t)$	$d(t) = \prod_{i=1}^t [1 + r(i)]^{-1}$ $r(t) = \frac{d(t-1)}{d(t)} - 1$	$d(t) = e^{-\int_0^t r(x)dx}$ $r(t) = \frac{-d'(t)}{d(t)}$
2. discount factor, $\delta(t)$	$d(t) = \prod_{i=1}^t \delta(i)$ $\delta(t) = \frac{d(t)}{d(t-1)}$	$d(t) = e^{\int_0^t \ln \delta(x)dx}$ $\delta(t) = e^{d'(t)/d(t)}$
3. yield curve, $y(t)$	$d(t) = [1 + y(t)]^{-t}$ $y(t) = [d(t)]^{-1/t} - 1$	$d(t) = e^{-y(t)t}$ $y(t) = -\frac{\ln d(t)}{t}$
4. present value, $pv(t)$	$d(t) = pv(t) - pv(t-1)$ $pv(t) = \prod_{i=0}^t d(i)$	$d(t) = pv'(t)$ $pv(t) = \int_0^t d(x)dx$
5. associated <i>pdf</i> , $f(t)$	$d(t) = \prod_{i=t}^{\infty} f(i)$ $f(t) = d(t) - d(t+1)$	$d(t) = \int_t^{\infty} f(x)dx$ $f(t) = -d'(t)$

Note: Our conventions here are the usual ones, namely that the empty product is equal to 1 and the unit stream begins to accrue immediately at time 0. In discrete time, $r(t)$ refers to the discount rate between time $t-1$ and time t , and similarly for $\delta(t)$. Also in discrete time, $pv(t)$ is the present value accumulated up through and inclusive of time t .

4.2 Derivation of properties of discounting procedures characterized in the text

Table 1 in the main paper provided formulas for discount rates, amounts, relative speeds, median time horizons, and mean time horizons for the hyperbolic, Weibull, and Gamma discounting procedures. The results not derived in the text are derived below.

4.2.1 Hyperbolic Discounting

For the hyperbolic discount function, $d(t) = [1 + r(1 - s)t]^{-\frac{1}{1-s}-1}$, the discount rate is given by

$$r(t) = -\left(-\frac{1}{1-s} - 1\right)r(1-s) \frac{[1 + r(1-s)t]^{-\frac{1}{1-s}-2}}{[1 + r(1-s)t]^{-\frac{1}{1-s}-1}} = \frac{r(2-s)}{1 + r(1-s)t}.$$

To obtain the amount, α , we first calculate

$$\int_0^\infty d(t)dt = \frac{1}{r(1-s)} \frac{1}{-1/(1-s)} [1 + r(1-s)t]^{-\frac{1}{1-s}} \Big|_0^\infty = \frac{1}{r},$$

and so by definition $\alpha = r$ as claimed. For the speed, we begin by observing that

$$\int_0^\infty td(t)dt = \int_0^\infty \frac{t}{[1 + r(1-s)t]^{1+\frac{1}{1-s}}} dt = \frac{1}{r^2(1-s)^2} \int_0^\infty \frac{x}{[1+x]^{1+\frac{1}{1-s}}} dx,$$

with the change of variable $x = r(1-s)t$. The beta function $B(y, z)$ can be written

$$B(y, z) = \int_0^\infty \frac{x^{y-1}}{[1+x]^{y+z}} dx,$$

so inserting $y = 2$ and $z = \frac{1}{1-s} - 1 = \frac{s}{1-s}$ yields

$$\int_0^\infty td(t)dt = \frac{1}{r^2(1-s)^2} B\left(2, \frac{s}{1-s}\right).$$

But the beta function can be expressed in terms of the gamma function (Gradshteyn and Ryzhik

1980, section 8.38) as

$$B(y, z) = \frac{\Gamma(y)\Gamma(z)}{\Gamma(y+z)},$$

which itself has the property that $\Gamma(y+1) = y\Gamma(y)$. Plugging in (and recalling that $\Gamma(n) = (n-1)!$ for integer n), we find that

$$B(2, \frac{s}{1-s}) = \frac{\Gamma(2)\Gamma(\frac{s}{1-s})}{\Gamma(\frac{s}{1-s}+2)} = \frac{1 \cdot \Gamma(\frac{s}{1-s})}{(\frac{s}{1-s}+1)\Gamma(\frac{s}{1-s}+1)} = \frac{\Gamma(\frac{s}{1-s})}{(\frac{1}{1-s})(\frac{s}{1-s})\Gamma(\frac{s}{r-s})} = \frac{(1-s)^2}{s}.$$

Finally,

$$\rho = \left[\alpha^2 \int_0^\infty t d(t) dt \right]^{-1} = \left[r^2 \frac{1}{r^2(1-s)^2} \frac{(1-s)^2}{s} \right]^{-1} = s$$

as desired. And of course θ is then $(rs)^{-1}$. For the median time, we set $pv(\tau) = pv(\infty)/2$:

$$\begin{aligned} \frac{1 - [1 + r(1-s)\tau]^{-\frac{1}{1-s}}}{r} &= \frac{1}{2r} \implies \\ 1 + r(1-s)\tau &= 2^{1-s} \implies \\ \tau &= \frac{2^{1-s} - 1}{r(1-s)}. \end{aligned}$$

Note that, as expected, $\lim_{s \rightarrow 1} \tau = r^{-1} \ln 2$ (use L'Hôpital's rule), which is the value for the standard exponential.

4.2.2 Weibull Discounting

For $d(t) = \exp(-rt^{1/s})$, we can immediately observe that $r(t) = (r/s)t^{1/s-1}$. To calculate amount and speed, we will use the following formula (Gradshteyn, I.S. and Ryzhik, I.M., section 3.4781):

$$\int_0^\infty x^{y-1} \exp(-rx^z) dx = \frac{1}{z} r^{-y/z} \Gamma(y/z)$$

with $z = 1/s$. First substituting $y = 1$, we find

$$\int_0^\infty d(t)dt = \frac{1}{1/s} r^{-s} \Gamma(s) = \frac{s\Gamma(s)}{r^s} = \frac{\Gamma(s+1)}{r^s}$$

using the same identity $\Gamma(y+1) = y\Gamma(y)$ as before. The inverse of this is then α , as claimed in the main text. To calculate the speed, we instead substitute $y = 2$, implying that

$$\int_0^\infty td(t)dt = \frac{1}{1/s} r^{-2s} \Gamma(2s) = \frac{s\Gamma(2s)}{r^{2s}}$$

so that

$$\rho = \left[\alpha^2 \int_0^\infty td(t)dt \right]^{-1} = \frac{s^2 [\Gamma(s)]^2 r^{2s}}{r^{2s} s\Gamma(2s)} = s \frac{[\Gamma(s)]^2}{\Gamma(2s)} = \frac{\Gamma(s)\Gamma(s+1)}{\Gamma(2s)},$$

also as claimed. The mean time follows immediately:

$$\theta = \alpha \int_0^\infty td(t)dt = \frac{\Gamma(2s)}{r^s \Gamma(s)}.$$

Unfortunately, there is no closed form integral for $d(t)$ and thus no closed form expression for the median time horizon.

We can further investigate several properties of the Weibull discounting procedure. The binomial coefficient $\binom{n}{m}$ is by definition

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)},$$

and the formulation in terms of gamma functions can be used to define a continuous binomial coefficient for non-integer n, m . In that case, the relative speed is the inverse of

$$\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1)} = \frac{\Gamma((2s-1)+1)}{\Gamma((2s-1)-s+1)\Gamma(s+1)} = \binom{2s-1}{s},$$

which is strictly increasing in s for $s > 0$ (since it is an extension of the binomial coefficient). Thus the speed is strictly decreasing in s , and of course it equals $\binom{2-1}{1} = 1$ when $s = 1$. This validates our claim that the transformed time discounting procedure is slow (that is, $\rho < 1$) exactly when $s > 1$, and it is fast ($\rho > 1$) when $s < 1$. It also provides a simple way to calculate ρ when s is an integer. The claim is also a direct consequence of Proposition 3, once we note that the derivative of the discount rate has the same sign as $1 - s$:

$$r'(t) = (1 - s) \frac{r}{s^2} t^{(1/s)-2}.$$

So when $s > 1$, the discount rate is declining over time (slow discounting), and vice-versa.

Finally, we examine the limit cases for extreme values of s . As $s \rightarrow \infty$, $d(t)$ starts to look almost constant at a value of e^{-r} , so the procedure is less and less convergent (though of course for any finite s there is no actual problem). This means that $\alpha \rightarrow 0$. Since $\Gamma(y)$ grows “factorially” in y , we find that ρ goes to 0 as well (and θ grows without bound), which does not automatically follow from the result for α . That is, the limit discounting procedure is slow even relative to the equivalent exponential (which has, of course, a zero discount rate in the limit). In the other extreme, as $s \rightarrow 0$, $d(t)$ starts to appear constant at 1 until $t = 1$, where it precipitously drops to 0. This suggests that α should be near 1, and indeed,

$$\lim_{s \rightarrow 0} \alpha = \lim_{s \rightarrow 0} \frac{r^s}{\Gamma(s+1)} = \frac{r^0}{\Gamma(1)} = 1,$$

since both the numerator and denominator are continuous at $s = 0$. For the speed, we make use of the doubling formula for the gamma function:

$$\Gamma(2y) = \frac{2^{2y-1}}{\sqrt{\pi}} \Gamma(y) \Gamma(y + \frac{1}{2}),$$

from which we obtain

$$\rho = \frac{\Gamma(s)\Gamma(s+1)}{\Gamma(2s)} = \frac{\sqrt{\pi}}{2^{2s-1}} \frac{\Gamma(s+1)}{\Gamma(s+1/2)}$$

and thus (using the known value $\Gamma(1/2) = \sqrt{\pi}$)

$$\lim_{s \rightarrow 0} \rho = \lim_{s \rightarrow 0} \frac{\sqrt{\pi}}{2^{2s-1}} \frac{\Gamma(s+1)}{\Gamma(s+1/2)} = \frac{\sqrt{\pi}}{2^{0-1}} \frac{1}{\sqrt{\pi}} = 2,$$

again by continuity at $s = 0$ of all functions involved. This is the “fastest” that we can get with this type of discounting procedure. Therefore, θ approaches $1/2$, and it is clear in this case that τ approaches $1/2$ as well.

4.2.3 Gamma

We start with a gamma formulation $x^s e^{-(s+1)rx}$, so $s = 0$ is the basic exponential case. Parameter restrictions are $s > -1$ and $r > 0$. We need to first turn this into a legitimate *pdf* function: $\int_0^\infty x^s e^{-(s+1)rx} dx = \frac{\Gamma(s+1)}{[(s+1)r]^{s+1}}$, so we normalize to get

$$f(x) = \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} x^s e^{-(s+1)rx}$$

as our starting point. We turn this *pdf* into a discount function as in the paper:

$$d(t) = \int_t^\infty f(x) dx = \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_t^\infty x^s e^{-(s+1)rx} dx = \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \frac{\Gamma(s+1, (s+1)rt)}{[(s+1)r]^{s+1}}$$

(where $\Gamma(x, z) = \int_z^\infty e^{-y} y^{x-1} dy$ is by definition the incomplete gamma function), ending up with

$$d(t) = \frac{\Gamma(s+1, (s+1)rt)}{\Gamma(s+1)}.$$

The discount rate function is thus

$$r(t) = \frac{-d'(t)}{d(t)} = \frac{[(s+1)r]^{s+1} t^s e^{-(s+1)rt}}{\Gamma(s+1, (s+1)rt)}.$$

So $r(0) = 0$ for $s > 0$, $r(0) = r$ for $s = 0$, and $r(0) = \infty$ for $s < 0$. To find $r(\infty)$, use L'Hôpital's Rule:

$$\lim_{t \rightarrow \infty} r(t) = \lim_{t \rightarrow \infty} \frac{[(s+1)r]^{s+1} t^{s-1} e^{-(s+1)rt} [s - (s+1)rt]}{-[(s+1)r]^{s+1} t^s e^{-(s+1)rt}} = \lim_{t \rightarrow \infty} \left[(s+1)r - \frac{s}{t} \right] = (s+1)r$$

for any s .

We may now calculate the amount α and relative speed ρ .

$$\begin{aligned}
\int_0^\infty d(t)dt &= \int_0^\infty \left(\frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_t^\infty x^s e^{-(s+1)rx} dx \right) dt \\
&= \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_0^\infty \int_t^\infty x^s e^{-(s+1)rx} dx dt \\
&= \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_0^\infty \int_0^x x^s e^{-(s+1)rx} dt dx \\
&= \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_0^\infty x^s e^{-(s+1)rx} \int_0^x dt dx \\
&= \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_0^\infty x^{s+1} e^{-(s+1)rx} dx \\
&= \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \frac{\Gamma(s+2)}{[(s+1)r]^{s+2}} = \frac{1}{r}
\end{aligned}$$

so $\alpha = r$ quite simply. For the mean time and relative speed:

$$\begin{aligned}
\int_0^\infty td_s(t)dt &= \int_0^\infty t \left(\frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_t^\infty x^s e^{-(s+1)rx} dx \right) dt \\
&= \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_0^\infty \int_t^\infty tx^s e^{-(s+1)rx} dx dt \\
&= \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_0^\infty \int_0^x tx^s e^{-(s+1)rx} dt dx \\
&= \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_0^\infty x^s e^{-(s+1)rx} \int_0^x t dt dx \\
&= \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \int_0^\infty \left(\frac{1}{2}\right)x^{s+2} e^{-(s+1)rx} dx \\
&= \frac{[(s+1)r]^{s+1}}{\Gamma(s+1)} \frac{\Gamma(s+3)}{2[(s+1)r]^{s+3}} = \frac{(s+2)}{2(s+1)r^2}.
\end{aligned}$$

But θ is α times this integral, and therefore $\theta = \frac{s+2}{2(s+1)r}$. Finally, $\rho = (\alpha\theta)^{-1}$ so

$$\rho = \left(\frac{s+2}{2(s+1)} \right)^{-1} = 2 - \frac{2}{s+2}.$$

Thus the relative speed depends only on s (not r) and is increasing in s toward a limit of 2 (the

maximum possible relative speed of any discounting procedure) as s increases to infinity, with a lower limit of 0 as s goes to -1 . Unfortunately, there is no closed form for the median time τ , since one cannot analytically integrate the incomplete gamma function in $d(t)$.

4.3 Properties of other discounting procedures

4.3.1 Split function quasi-hyperbolic discounting

Another class of functions derives from continuous time versions of the quasi-hyperbolic discount function. The discrete time quasi-hyperbolic has a value of 1 at $t = 0$ and of $\beta(1+r)^{-t}$ for $t \geq 1$. This function can be generalized in a variety of ways. First, and most obviously, the point of abrupt transition can be made to be at any time, not just $t = 1$ (which was in any case already a parameter due to the need to choose a time scale). Second, an option that we do not pursue is to introduce transitions at multiple time points. Third, the generalization to continuous time can be undertaken either by introducing a discontinuity in $r(t)$ —and, hence, in the derivative of $d(t)$ —or by introducing a discontinuity in $d(t)$ itself. We label these the “split rate quasi-hyperbolic” and the “split function quasi-hyperbolic.” The latter allows $r(t)$ to remain constant (except for a Dirac delta function at the time of the discontinuity in $d(t)$) and has been utilized by Harris and Laibson (2004) and, in quite a different way, by Cline (1999). The most natural generalization of the discrete-time quasi-hyperbolic is the split function version, and this subsection includes a definition of this version and shows its properties. In the next subsection, we explore properties of the split rate quasi-hyperbolic procedures.

We first consider the procedure that splits the discount function rather than the discount rate:

$$d(t) = \begin{cases} e^{-rt} & \text{if } t \leq t^* \\ \lambda e^{-rt} & \text{if } t > t^* \end{cases}$$

where $r > 0$ and $\lambda < 1$. The discount rate for this procedure is constant at r except at $t = t^*$, where it is infinite. If we again denote e^{-rt^*} by β , we see that

$$\int_0^\infty d(t)dt = \int_0^{t^*} e^{-rt} dt + \lambda \int_{t^*}^\infty e^{-rt} dt = (1 - \beta)\frac{1}{r} + \lambda\frac{1}{r}\beta = \frac{1 - (1 - \lambda)\beta}{r},$$

and the amount of discounting is the inverse of this:

$$\alpha = \frac{r}{1 - (1 - \lambda)\beta}.$$

For the speed,

$$\begin{aligned} \int_0^\infty td(t)dt &= \int_0^{t^*} te^{-rt}dt + \lambda \int_{t^*}^\infty te^{-rt}dt \\ &= \frac{1}{r}((1 - \beta)\frac{1}{r} - t^*\beta) + \lambda\frac{1}{r}(t^*\beta + \frac{1}{r}\beta) \\ &= \frac{1}{r^2}[1 - (1 - \lambda)\beta(1 + rt^*)] \end{aligned}$$

and so

$$\rho = \frac{[1 - (1 - \lambda)\beta]^2}{1 - (1 - \lambda)\beta(1 + rt^*)}.$$

From here, θ follows as always. To find bounds on ρ for this procedure, we first consider the extreme case $\lambda = 0$: here $\rho = f(\beta) = (1 - \beta)^2/(1 - \beta + \beta \ln \beta)$. So the maximum value for ρ is 2, achieved as β goes to 1 (i.e. as either t^* or r goes to 0). Note that if $\lambda > 0$, then ρ is maximized at an interior choice of β , but the arg max goes to 1 (and the maximum value goes to 2) as λ goes to 0. On the other hand, fixing $\lambda > 0$ and letting β go to 1, ρ is approximately λ —so we can make it as small as we want (greater than 0). Thus the order of limits makes a big difference. Finally, we point out that for any fixed $\lambda \geq 0$, ρ goes to 1 as β goes to 0.²³

For the median time, we again distinguish two cases: either τ is smaller than t^* (and τ occurs before the λ jump) or it is larger than t^* (where the expression for $pv(t)$ is different due to the λ factor). The boundary case will be if $\tau = t^*$, which occurs exactly when $pv(t^*) = pv(\infty)/2$, meaning if $1/r - \beta/r = [1 - (1 - \lambda)\beta]/2r$. This is true when $\beta = 1/(1 + \lambda)$, that is, for $t^* = r^{-1} \ln(1 + \lambda)$. If t^* is larger than this (so that τ is in the initial range), $pv(\tau) = 1/r - e^{-r\tau}/r$ and

²³It may seem odd that the quasi-hyperbolic procedure can be either fast or slow, but in fact it has elements of both. In the “short run” (where it is primarily used), it crosses the equivalent exponential from above at t^* and thus appears fast, which is how most practitioners think of it. In most applications this happens within hours or days, so it has little effect on the overall amount. Thus the eventual rate r for a quasi-hyperbolic procedure is going to be lower than for the equivalent exponential, since it has to overcome the downward scaling factor, meaning it will eventually cross again from below and is overall slow for most reasonable parameter choices.

we need to set this equal to $pv(\infty)/2 = [1 - (1 - \lambda)\beta]/2r$. Solving,

$$\begin{aligned} 1 - e^{-r\tau} &= \frac{1 - (1 - \lambda)\beta}{2} \implies \\ e^{-r\tau} &= \frac{1 + (1 - \lambda)\beta}{2} \implies \\ \tau &= r^{-1} \ln \frac{2}{1 + (1 - \lambda)\beta}. \end{aligned}$$

If t^* is smaller than the cutoff value (so that $\tau > t^*$), then $pv(\tau) = \int_0^{t^*} e^{-rt} dt + \beta \int_{t^*}^{\tau} \lambda e^{-r(t-t^*)} dt = (1 - \beta)/r + \lambda(\beta - e^{-r\tau})/r$ and we solve

$$\begin{aligned} \frac{1 - \beta}{r} + \frac{\lambda(\beta - e^{-r\tau})}{r} &= \frac{1 - (1 - \lambda)\beta}{2r} \implies \\ \lambda(\beta - e^{-r\tau}) &= \frac{1}{2} [\lambda\beta - (1 - \beta)] \implies \\ e^{-r\tau} &= \frac{\lambda\beta + (1 - \beta)}{2\lambda} \implies \\ \tau &= r^{-1} \ln \frac{2\lambda}{1 - (1 - \lambda)\beta}. \end{aligned}$$

4.3.2 Split rate quasi-hyperbolic discounting

If $d(t) = e^{-rt}$ for $t \leq t^*$ and $d(t) = \beta e^{-s(t-t^*)}$ for $t > t^*$ (where $\beta = e^{-rt^*}$), then the discount rate is immediate (it is in fact the defining characteristic for this procedure), though it is undefined at $t = t^*$ where d is not differentiable²⁴. For the amount,

$$\int_0^{\infty} d(t) dt = \int_0^{t^*} e^{-rt} dt + \beta \int_{t^*}^{\infty} e^{-s(t-t^*)} dt = (1 - \beta) \frac{1}{r} + \beta \frac{1}{s},$$

from which it is clear that the inverse is indeed a [weighted] harmonic mean:

$$\alpha = \frac{rs}{\beta r + (1 - \beta)s}.$$

For the discounting speed, we calculate

²⁴If $s > r$, say, we can think of $r(t)$ as having a Dirac delta function spike at t^* .

$$\begin{aligned}
\int_0^\infty td(t)dt &= \int_0^{t^*} te^{-rt}dt + \beta \int_{t^*}^\infty te^{-s(t-t^*)}dt \\
&= \left[-\frac{1}{r}te^{-rt} - \frac{1}{r^2}e^{-rt} \right]_0^{t^*} + \beta \int_0^\infty (t+t^*)e^{-st}dt \\
&= \frac{1}{r}(-t^*\beta - \frac{\beta}{r} + 0 + \frac{1}{r}) + \beta \int_0^\infty te^{-st}dt + \beta t^* \int_0^\infty e^{-st}dt \\
&= \frac{1}{r}((1-\beta)\frac{1}{r} - t^*\beta) + \beta \frac{1}{s^2} + \beta t^* \frac{1}{s} \\
&= \frac{1}{rs} \left[(1-\beta)\frac{s}{r} - t^*\beta s + \beta \frac{r}{s} + \beta t^*r \right]
\end{aligned}$$

so that the relative speed is the inverse of

$$\alpha^2 \int_0^\infty td(t)dt = \frac{\beta r^2 + (1-\beta)s^2 + \beta t^*rs(r-s)}{[\beta r + (1-\beta)s]^2},$$

as claimed. The formula for θ also follows from this computation. It is easy to calculate that $\rho \geq 1$ as $s \geq r$. To find bounds for ρ , we first consider $s \gg r$: in this case, ρ is approximately $f(\beta) \equiv (1-\beta)^2/(1-\beta+\beta \ln \beta)$. The function f is monotonically increasing in β , with $\lim_{\beta \rightarrow 0} f(\beta) = 1$ and $\lim_{\beta \rightarrow 1} f(\beta) = 2$ (use L'Hôpital's rule twice), so ρ can get arbitrarily close to 2. On the other side, for $r \gg s$, ρ is approximately $\beta/(1+st^*)$, which is obviously minimal for β near 0 (meaning rt^* is very large), and hence ρ near 0.

For the median time, we need to distinguish two cases: either τ is smaller than t^* (in which case τ occurs while the discount rate is still r and we can use the appropriate expression for $pv(t)$) or it is larger than t^* (where the rate is s and the expression for $pv(t)$ is different). The boundary case will be if $\tau = t^*$, which occurs exactly when $pv(t^*) = pv(\infty)/2$, i.e. if $1/r - \beta/r = [\beta r + (1-\beta)s]/2rs$. This is true if $\beta r = (1-\beta)s$ (which is intuitively reasonable), i.e. for $\beta = e^{-rt^*} = \frac{s}{r+s}$. Thus the cutoff value for t^* is $r^{-1} \ln \frac{r+s}{s}$. If t^* is larger than this (so that τ is in the initial range), $pv(\tau) = 1/r - e^{-r\tau}/r$ and we need to set this equal to $pv(\infty)/2 = [\beta r + (1-\beta)s]/2rs$. Solving,

$$\begin{aligned}
1 - e^{-r\tau} &= \frac{1}{2} \frac{\beta r + (1 - \beta)s}{s} \implies \\
e^{-r\tau} &= \frac{s - \beta(r - s)}{2s} \implies \\
\tau &= r^{-1} \ln \frac{2s}{s - \beta(r - s)}.
\end{aligned}$$

If t^* is smaller than the cutoff value (so that $\tau > t^*$), then $pv(\tau) = \int_0^{t^*} e^{-rt} dt + \beta \int_{t^*}^{\tau} e^{-s(t-t^*)} dt = (1 - \beta)/r + \beta (1 - e^{-s(\tau-t^*)})/s$ and we solve

$$\begin{aligned}
\frac{1 - \beta}{r} + \frac{\beta (1 - e^{-s(\tau-t^*)})}{s} &= \frac{1}{2} \frac{\beta r + (1 - \beta)s}{rs} \implies \\
\beta r e^{-s(\tau-t^*)} &= \frac{1}{2} [\beta r + (1 - \beta)s] \implies \\
e^{-s(\tau-t^*)} &= \frac{\beta r + (1 - \beta)s}{2\beta r} \implies \\
\tau &= t^* + s^{-1} \ln \frac{2\beta r}{\beta r + (1 - \beta)s}.
\end{aligned}$$

4.3.3 Split-rate exponential discounting

If $d(t) = e^{-rt}$ for $t \leq t^*$ and $d(t) = \beta e^{-s(t-t^*)}$ for $t > t^*$ (where $\beta = e^{-rt^*}$), then the discount rate is immediate (it is in fact the defining characteristic for this procedure), though it is undefined at $t = t^*$ where d is not differentiable. For the amount,

$$\int_0^{\infty} d(t) dt = \int_0^{t^*} e^{-rt} dt + \beta \int_{t^*}^{\infty} e^{-s(t-t^*)} dt = (1 - \beta) \frac{1}{r} + \beta \frac{1}{s},$$

from which it is clear that the inverse is indeed a [weighted] harmonic mean:

$$\alpha = \frac{rs}{\beta r + (1 - \beta)s}.$$

For the speed, we calculate

$$\begin{aligned}
\int_0^\infty td(t)dt &= \int_0^{t^*} te^{-rt}dt + \beta \int_{t^*}^\infty te^{-s(t-t^*)}dt \\
&= \left[-\frac{1}{r}te^{-rt} - \frac{1}{r^2}e^{-rt} \right]_0^{t^*} + \beta \int_0^\infty (t+t^*)e^{-st}dt \\
&= \frac{1}{r}(-t^*\beta - \frac{\beta}{r} + 0 + \frac{1}{r}) + \beta \int_0^\infty te^{-st}dt + \beta t^* \int_0^\infty e^{-st}dt \\
&= \frac{1}{r}((1-\beta)\frac{1}{r} - t^*\beta) + \beta \frac{1}{s^2} + \beta t^* \frac{1}{s} \\
&= \frac{1}{rs} \left[(1-\beta)\frac{s}{r} - t^*\beta s + \beta \frac{r}{s} + \beta t^*r \right]
\end{aligned}$$

so that the relative speed is the inverse of

$$\alpha^2 \int_0^\infty td(t)dt = \frac{\beta r^2 + (1-\beta)s^2 + \beta t^*rs(r-s)}{[\beta r + (1-\beta)s]^2},$$

as claimed. The formula for θ also follows from this computation. For the median time, we need to distinguish two cases: either τ is smaller than t^* (in which case it occurs while the discount rate is still r and we can use the appropriate expression for $pv(t)$) or it is larger than t^* (where the rate is s and the expression for $pv(t)$ is different). The boundary case will be if $\tau = t^*$, which occurs exactly when $pv(t^*) = pv(\infty)/2$, meaning if $1/r - \beta/r = [\beta r + (1-\beta)s]/2rs$. This is true if $\beta r = (1-\beta)s$ (which is intuitively reasonable), meaning for $\beta = e^{-rt^*} = \frac{s}{r+s}$. Thus the cutoff value for t^* is $r^{-1} \ln \frac{r+s}{s}$. If t^* is larger than this (so that τ is in the initial range), $pv(\tau) = 1/r - e^{-r\tau}/r$ and we need to set this equal to $pv(\infty)/2 = [\beta r + (1-\beta)s]/2rs$. Solving,

$$\begin{aligned}
1 - e^{-r\tau} &= \frac{1}{2} \frac{\beta r + (1-\beta)s}{s} \implies \\
e^{-r\tau} &= \frac{s - \beta(r-s)}{2s} \implies \\
\tau &= r^{-1} \ln \frac{2s}{s - \beta(r-s)}.
\end{aligned}$$

If t^* is smaller than the cutoff value (so that $\tau > t^*$), then $pv(\tau) = \int_0^{t^*} e^{-rt}dt + \beta \int_{t^*}^\tau e^{-s(t-t^*)}dt = (1-\beta)/r + \beta(1 - e^{-s(\tau-t^*)})/s$ and we solve

$$\begin{aligned}
\frac{1-\beta}{r} + \frac{\beta(1 - e^{-s(\tau-t^*)})}{s} &= \frac{1}{2} \frac{\beta r + (1-\beta)s}{rs} \implies \\
\beta r e^{-s(\tau-t^*)} &= \frac{1}{2} [\beta r + (1-\beta)s] \implies \\
e^{-s(\tau-t^*)} &= \frac{\beta r + (1-\beta)s}{2\beta r} \implies \\
\tau &= t^* + s^{-1} \ln \frac{2\beta r}{\beta r + (1-\beta)s}.
\end{aligned}$$

4.3.4 Blended-rate exponential discounting

For this procedure, we begin with $r(t) = \left(\frac{a}{a+t}\right)r + \left(\frac{t}{a+t}\right)s$ as our defining characteristic, and must first derive an expression for the discount function itself:

$$\begin{aligned} \int_0^t r(x)dx &= \int_0^t \left(\frac{a}{a+x}\right)rdx + \int_0^t \left(1 - \frac{a}{a+x}\right)sdx \\ &= [ar \ln(a+x) + sx + as \ln(a+x)]_0^t \\ &= st + a(r-s)(\ln(a+t) - \ln a) \\ &= st - \ln\left(\frac{a}{a+t}\right)^{a(r-s)}. \end{aligned}$$

Thus

$$d(t) = \exp\left(-\int_0^t r(x)dx\right) = e^{-st} \left(\frac{a}{a+t}\right)^{a(r-s)}.$$

In order to calculate the amount we need to integrate this, so we make the change of variable $y = s(a+t)$, implying that $dt = \frac{1}{s}dy$ and

$$\int_0^\infty d(t)dt = \int_{as}^\infty e^{-(y-as)} \left(\frac{as}{y}\right)^{a(r-s)} \frac{1}{s}dy = \frac{(as)^{a(r-s)}}{se^{-as}} \int_{as}^\infty e^{-y}y^{a(s-r)}dy.$$

But the definition of the incomplete gamma function is $\Gamma(x, z) = \int_z^\infty e^{-y}y^{x-1}dy$, so

$$\int_0^\infty d(t)dt = \frac{(as)^{a(r-s)}}{se^{-as}} \Gamma(a(s-r) + 1, as)$$

and α is simply the inverse of this, as claimed. Calculating the speed is similar but slightly more involved:

$$\begin{aligned} \int_0^\infty td(t)dt &= \int_{as}^\infty \left(\frac{y}{s} - a\right)e^{-(y-as)} \left(\frac{as}{y}\right)^{a(r-s)} \frac{1}{s}dy \\ &= \frac{1}{s} \frac{(as)^{a(r-s)}}{se^{-as}} \int_{as}^\infty e^{-y}y^{a(s-r)+1}dy - a \frac{(as)^{a(r-s)}}{se^{-as}} \int_{as}^\infty e^{-y}y^{a(s-r)}dy \\ &= \frac{(as)^{a(r-s)}}{se^{-as}} \left[\frac{1}{s} \Gamma(a(s-r) + 2, as) - a \Gamma(a(s-r) + 1, as) \right] \end{aligned}$$

but $\Gamma(x+1, z) = x\Gamma(x, z) + z^x e^{-z}$ (see Gradshteyn and Ryzhik 1980, section 8.3, 562), so when we apply this simplification and multiply through by the α above, we get

$$\begin{aligned}
& \alpha \int_0^\infty t d(t) dt \\
&= \frac{1}{\Gamma(a(s-r)+1, as)} \left[\left(\frac{a(s-r)+1}{s} - a \right) \Gamma(a(s-r)+1, as) + \frac{(as)^{a(s-r)+1} e^{-as}}{s} \right] \\
&= \frac{1}{s} \left[a(s-r)+1 - as + \frac{ase^{-as}}{(as)^{a(r-s)}} \frac{1}{\Gamma(a(s-r)+1, as)} \right] \\
&= \frac{1}{s} [1 - ar + a\alpha],
\end{aligned}$$

which is therefore θ . The relative speed ρ is then $(\alpha\theta)^{-1}$ as always. Unfortunately, since there is no closed form indefinite integral available for $d(t)$, we cannot solve explicitly for τ , though of course it always exists and can be computed numerically for any given parameters.

4.3.5 Split function exponential discounting

We can also consider the procedure that splits the discount function rather than the discount rate:

$$d(t) = \begin{cases} e^{-rt} & \text{if } t \leq t^* \\ \lambda e^{-rt} & \text{if } t > t^* \end{cases}$$

where $r > 0$ and $\lambda < 1$. The discount rate for this procedure is constant at r except at $t = t^*$, where it is infinite. If we again denote e^{-rt^*} by β , we see that

$$\int_0^\infty d(t) dt = \int_0^{t^*} e^{-rt} dt + \lambda \int_{t^*}^\infty e^{-rt} dt = (1-\beta) \frac{1}{r} + \lambda \frac{1}{r} \beta = \frac{1 - (1-\lambda)\beta}{r},$$

and the amount of discounting is the inverse of this:

$$\alpha = \frac{r}{1 - (1-\lambda)\beta}.$$

For the speed,

$$\begin{aligned}
\int_0^\infty t d(t) dt &= \int_0^{t^*} t e^{-rt} dt + \lambda \int_{t^*}^\infty t e^{-rt} dt \\
&= \frac{1}{r} \left((1-\beta) \frac{1}{r} - t^* \beta \right) + \lambda \frac{1}{r} \left(t^* \beta + \frac{1}{r} \beta \right) \\
&= \frac{1}{r^2} [1 - (1-\lambda)\beta(1 + rt^*)]
\end{aligned}$$

and so

$$\rho = \frac{[1 - (1 - \lambda)\beta]^2}{1 - (1 - \lambda)\beta(1 + rt^*)}.$$

From here, θ follows as always, and τ can be computed as it was for the split rate exponential (with two different expressions depending on how large t^* is relative to some cutoff).

4.3.6 Blended function exponential discounting

This is a discounting procedure with initial rate $r \geq 0$, asymptotic rate $s > 0$, and parameter $a \geq 0$ ($a = 0$ is again the standard exponential with rate s):

$$d(t) = \left(\frac{a}{a+t}\right) e^{-rt} + \left(\frac{t}{a+t}\right) e^{-st}.$$

Here we find

$$\begin{aligned} \int_0^\infty d(t)dt &= \int_0^\infty \left(\frac{a}{a+t}\right) e^{-rt} dt + \int_0^\infty \left(1 - \frac{a}{a+t}\right) e^{-st} dt \\ &= a \int_{ar}^\infty \frac{1}{x} e^{-(x-ar)} dx + \int_0^\infty e^{-st} dt - a \int_{as}^\infty \frac{1}{x} e^{-(x-as)} dx \\ &= ae^{ar} \int_{ar}^\infty \frac{e^{-x}}{x} dx + \frac{1}{s} - ae^{as} \int_{as}^\infty \frac{e^{-x}}{x} dx \\ &= \frac{1}{s} - ae^{ar} Ei(-ar) + ae^{as} Ei(-as), \end{aligned}$$

using a change of variable in the second line, where $Ei(x)$ is the exponential-integral function $Ei(x) = -\int_{-x}^\infty (e^{-y}/y)dy$ for $x < 0$. The amount of discounting is then the inverse of this:

$$\alpha = \left[\frac{1}{s} + a(e^{as} Ei(-as) - e^{ar} Ei(-ar)) \right]^{-1}.$$

For the speed,

$$\begin{aligned} \int_0^\infty td(t)dt &= \int_0^\infty \left(\frac{at}{a+t}\right) e^{-rt} dt + \int_0^\infty \left(t - \frac{at}{a+t}\right) e^{-st} dt \\ &= a \int_0^\infty \left(1 - \frac{a}{a+t}\right) e^{-rt} dt + \int_0^\infty te^{-st} dt - a \int_0^\infty \left(1 - \frac{a}{a+t}\right) e^{-st} dt \\ &= a \left[\frac{1}{r} + ae^{ar} Ei(-ar) \right] + \frac{1}{s^2} - a \left[\frac{1}{s} + ae^{as} Ei(-as) \right] \\ &= \frac{1}{s^2} + \frac{a}{r} - a \left[\frac{1}{s} + a(e^{as} Ei(-as) - e^{ar} Ei(-ar)) \right] \end{aligned}$$

so that

$$\rho = \left[\alpha^2 \int_0^\infty td(t)dt \right]^{-1} = [\alpha^2(s^{-2} + a/r) - a\alpha]^{-1} = \frac{\alpha^{-1}}{\alpha s^{-2} + a(\alpha - r)/r}$$

and the mean time is

$$\theta = \frac{\alpha}{s^2} + \frac{a(\alpha - r)}{r}.$$

Once again, the median time cannot be computed in closed form.

4.4 Proofs of Propositions in the Main Text

4.4.1 Proof of Proposition 6

Proposition 6 relates the limiting value of the ANDF social discount rate function to the minimum of the limiting values of the individual discount rate functions. Recall that

$$r_{\overline{D}}(t) = \frac{\int_X \alpha(x)r(t;x)d(t;x)f(x)dx}{\int_X \alpha(x)d(t;x)f(x)dx},$$

which we can rewrite as a weighted average

$$r_{\overline{D}}(t) = \int_X \beta(t;x)r(t;x)dx$$

with time-dependent weights $\beta(t;x)$ given by

$$\beta(t;x) = \frac{\alpha(x)d(t;x)f(x)}{\int_X \alpha(x)d(t;x)f(x)dx}.$$

Obviously, $\int_X \beta(t;x)dx = 1$ for any t .

For any $\varepsilon > 0$, let us partition X into A_1 , $A_2(\varepsilon)$, and $A_3(\varepsilon)$ as follows: $A_1 = \{x \in X \text{ s.t. } r^*(x) < r_{\min}^*\}$; $A_2(\varepsilon) = \{x \in X \text{ s.t. } r_{\min}^* \leq r^*(x) \leq r_{\min}^* + \varepsilon\}$; and $A_3(\varepsilon) = \{x \in X \text{ s.t. } r_{\min}^* + \varepsilon < r^*(x)\}$. Then we can write $\lim_{t \rightarrow \infty} r_{\overline{D}}(t) = \lim_{t \rightarrow \infty} \int_X \beta r dx = \lim_{t \rightarrow \infty} \int_{A_1} \beta r dx + \lim_{t \rightarrow \infty} \int_{A_2(\varepsilon)} \beta r dx + \lim_{t \rightarrow \infty} \int_{A_3(\varepsilon)} \beta r dx$.

For any x with $\alpha(x) = 0$, $\beta(t;x) = 0$ identically. So ignoring any such x and noting that $\int_{A(r_{\min}^*)} f(x)dx = 0$ by definition of r_{\min}^* , we see that $\int_{A_1} \beta(t;x)dx = 0$ as well (for all t), since $\alpha(x)$ and $d(t;x)$ are finite (and α, d, f are all weakly positive). But then $\lim_{t \rightarrow \infty} \int_{A_1} \beta(t;x)r(t;x)dx = 0$, so these values of x have no effect on $\lim_{t \rightarrow \infty} r_{\overline{D}}(t)$. This establishes that $\lim_{t \rightarrow \infty} r_{\overline{D}}(t) \geq r_{\min}^*$; it remains to be shown that $\lim_{t \rightarrow \infty} r_{\overline{D}}(t) \leq r_{\min}^*$.

Also by definition of r_{\min}^* , $\int_{A(r_{\min}^* + \varepsilon)} f(x) dx > 0$ for all $\varepsilon > 0$. But, as we just saw, $\int_{A_1} f(x) dx = 0$. Hence, for any ε , we can pick $x_2 \in A_2(\varepsilon)$ with $\alpha(x_2) > 0$ and $f(x_2) > 0$; now take any $x_3 \in A_3(\varepsilon)$. Then

$$\frac{\beta(t; x_3)}{\beta(t; x_2)} = \frac{\alpha(x_3)d(t; x_3)f(x_3)}{\alpha(x_2)d(t; x_2)f(x_2)}$$

Rewriting $d(t; x)$ as $\exp\left(-\int_0^t r(\tau; x) d\tau\right)$, we get

$$\frac{\beta(t; x_3)}{\beta(t; x_2)} = \frac{\alpha(x_3)f(x_3) \exp\left(-\int_0^t r(\tau; x_3) d\tau\right)}{\alpha(x_2)f(x_2) \exp\left(-\int_0^t r(\tau; x_2) d\tau\right)} = M \exp\left(-\int_0^t [r(\tau; x_3) - r(\tau; x_2)] d\tau\right),$$

where $M = \frac{\alpha(x_3)f(x_3)}{\alpha(x_2)f(x_2)}$ is constant in t (and is finite by the choice of x_2). Therefore

$$\lim_{t \rightarrow \infty} \frac{\beta(t; x_3)}{\beta(t; x_2)} = M \lim_{t \rightarrow \infty} \exp\left(-\int_0^t [r(\tau; x_3) - r(\tau; x_2)] d\tau\right).$$

But of course $x_2 \in A_2(\varepsilon)$ and $x_3 \in A_3(\varepsilon)$ imply that $\lim_{t \rightarrow \infty} r(t; x_3) > r_{\min}^* + \varepsilon \geq \lim_{t \rightarrow \infty} r(t; x_2)$, from which it is clear that $\lim_{t \rightarrow \infty} \int_0^t [r(\tau; x_3) - r(\tau; x_2)] d\tau = \infty$ and thus

$$\lim_{t \rightarrow \infty} \frac{\beta(t; x_3)}{\beta(t; x_2)} = 0.$$

This implies that, for any $x_3 \in A_3(\varepsilon)$, we must have $\lim_{t \rightarrow \infty} \beta(t; x_3) = 0$. This in turn yields $\lim_{t \rightarrow \infty} \int_{A_3(\varepsilon)} \beta(t; x) dx = 0$ (so that $\lim_{t \rightarrow \infty} \int_{A_2(\varepsilon)} \beta(t; x) dx = 1$), but it also implies the stronger conclusion that $\lim_{t \rightarrow \infty} \int_{A_3(\varepsilon)} \beta(t; x) r(t; x) dx = 0$. Therefore only $x \in A_2(\varepsilon)$ influence $\lim_{t \rightarrow \infty} r_{\overline{D}}(t)$, and in particular $\lim_{t \rightarrow \infty} r_{\overline{D}}(t) \leq r_{\min}^* + \varepsilon$ for any $\varepsilon > 0$. But that means exactly that $\lim_{t \rightarrow \infty} r_{\overline{D}}(t) \leq r_{\min}^*$, as needed.

4.4.2 Proof of Proposition 7

Proposition 7 states that the social discounting procedure arising from ANDF aggregation of exponentials whose parameters are gamma distributed is hyperbolic. We first start with an underlying gamma distribution for the parameter of the individual exponential discount functions:

$$f(x) = \frac{a^b}{\Gamma(b)} e^{-ax} x^{b-1},$$

where $x \in (0, \infty)$ denotes the discount rate parameter in a standard exponential, and $a, b > 0$.

Thus $\alpha(x) = x$, $d(t; x) = e^{-xt}$, and $f(x)$ is as above. Then

$$\int_0^\infty \alpha(x) d(t; x) f(x) dx = \frac{a^b}{\Gamma(b)} \int_0^\infty x^b e^{-(a+t)x} dx = \frac{a^b}{\Gamma(b)} \frac{\Gamma(b+1)}{(a+t)^{b+1}} = \frac{b}{a} \left[\frac{a}{a+t} \right]^{b+1}.$$

But

$$\bar{\alpha} = \int_0^\infty \alpha(x) f(x) dx = \frac{a^b}{\Gamma(b)} \int_0^\infty x^b e^{-ax} dx = \frac{a^b}{\Gamma(b)} \frac{\Gamma(b+1)}{a^{b+1}} = \frac{b}{a},$$

so $d_{\bar{D}}(t) = [1 + t/a]^{-(1+b)}$ as claimed. The speed follows from Theorem 3 with $r = b/a$ and $s = 1 - 1/b$.