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# CHARACTERIZING THE MAXIMUM GENUS OF A CONNECTED GRAPH

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In this paper a generalization of Tutte's theorem on perfect matchings and a generalization of Rado's theorem on independent transversals will be used for characterizing the maximum genus of a connected graph.

**0.** By a graph we mean here a graph in the sense of [4], i.e. a pseudograph in the sense of [2]. A graph $G$ is determined by its vertex set $V(G)$, its edge set $E(G)$, and its incidence relation between edges and vertices. A graph in the sense of [2] will be called here a simple graph, similarly as in [4] or [18]. Note that a simple graph $G$ is determined by $V(G)$ and $E(G)$ only.

A trivial graph (i.e. a graph with only one vertex and no edge) will be considered to be 2-edge-connected. Any maximal 2-edge-connected subgraph of a graph $G$ will be referred to as leaf of $G$.

Let $G$ be a graph. We denote by $c(G)$ the number of components of $G$. We define $p(G) = |V(G)|$, $q(G) = |E(G)|$, and $\beta(G) = q(G) - p(G) + c(G)$; thus, if $G$ is connected, then $\beta(G) = q(G) - p(G) + 1$. Moreover, we denote by $b(G)$ or $b^{\lambda}(G)$ the number of components $F_1$ of $G$ such that $\beta(F_1)$ is odd, or the number of leaves $F_2$ of $G$ such that $\beta(F_2)$ is odd, respectively.

Let $G$ be a connected graph. We denote by $\mathscr{A}_G$ the set of all $A \subseteq E(G)$ such that $G - A$ is connected. We denote by $\mathscr{T}(G)$ the set of all spanning trees of $G$. If $T \in \mathscr{T}(G)$, then we denote by $\mathscr{A}_G(T)$ the set of all $A \subseteq E(G) - E(T)$. Clearly,

$$\mathscr{A}_G = \bigcup_{T \in \mathscr{T}(G)} \mathscr{A}_G(T).$$

For every graph $G$ we denote by $\Gamma(G)$ the set of all integers $i$ such that there exists a 2-cell embedding of $G$ into the closed orientable surface of genus $i$ (for the above mentioned concepts of topological graph theory the reader is referred to [17] or to Chapter 5 of [2]). As follows from the properties of 2-cell embeddings, $\Gamma(G)$ is finite for every graph $G$. Moreover, $\Gamma(G) \neq \emptyset$ if and only if $G$ is connected. Duke [5] proved that if $G$ is a connected graph, $i, k \in \Gamma(G)$ and $j$ is an integer such that $i < j < k$, then $j \in \Gamma(G)$. (As was proved in [14], this result does not hold for signed graphs.) For every connected graph $G$, the maximum genus $\gamma_M(G)$ of $G$ is defined as the maximum integer in $\Gamma(G)$. As was shown in [11], $\gamma_M(G) \leqslant [\beta(G)/2]$ for every connected graph $G$. Since the beginning of the seventies many papers concerning the maximum genus have been written. (The maximum nonorientable genus has been also studied. Ringel [13] proved that the maximum nonorientable genus of a connected graph $G$ is equal to $\beta(G)$.)

The maximum genus of a connected graph was determined by Homenko, Ostroverkhy and Kusmenko [8] and independently by Xuong [19]. We will present the result obtained in [19]. The result obtained in [8] looks rather dissimilarly but in substance it is the same.

If $G$ is a connected graph and $T \in \mathscr{T}(G)$, then we denote by $x_G(T)$ the number of components $F$ of $G - E(T)$ such that $|E(F)|$ is odd.

**Theorem A** ([19]). *Let $G$ be a connected graph. Then*

$$\gamma_M(G) = \tfrac{1}{2}\big(\beta(G) - \min_{T \in \mathscr{T}(G)} x_G(T)\big).$$

*For the case when $\gamma_M(G) = [\frac{1}{2}\beta(G)]$, the formula was proved independently by Jungerman* [9].

If $G$ is a connected graph and $A \subseteq E(G)$, then we denote

$$y_G(A) = c(G - A) + b(G - A) - 1 - |A|.$$

**Proposition A.** *If $G$ is a connected graph, then*

$$\begin{aligned}\max_{A_0 \subseteq E(G)} \big(b^\lambda(G - A_0) - |A_0|\big) &= \max_{A \subseteq E(G)} y_G(A) \\ &= \max_{A_1 \in \mathscr{A}_G} \big(b^\lambda(G - A_1) - |A_1|\big).\end{aligned}$$

Proof (outlined). Let $A \subseteq E(G)$; there exists $A' \subseteq A$ such that $G - A'$ is connected and $|A - A'| = c(G - A) - 1$; we can see that $b^\lambda(G - A') \geqslant b(G - A)$. Let $A_1 \in \mathscr{A}_G$; there exists $A'' \subseteq E(G)$ such that $A_1 \subseteq A''$ and the set of components of $G - A''$ is the same as the set of leaves of $G - A_1$; hence $|A'' - A_1| = c(G - A'') - 1$. Finally, let $A_0 \subseteq E(G)$; there exists $A^* \subseteq A_0$ such that $A^* \in \mathscr{A}_G$ and $b^\lambda(G - A^*) = b^\lambda(G - A_0)$. The result of the proposition easily follows. □

Homenko and Glukhov [7] and independently Nebeský [10] have found that for any connected graph $G$,

$$\min_{T\in\mathscr{T}(G)} x_G(T)$$

can be expressed as the maximum of a function. Homenko and Glukhov [7] proved that if $G$ is a connected graph, then

$$\min_{T\in\mathscr{T}(G)} x_G(T) = \max_{A\subseteq E(G)} \left(b^{\lambda}(G-A) - |A|\right).$$

The present author proved the following theorem:

**Theorem B** ([10]). *If $G$ is a connected graph, then*

$$\min_{T\in\mathscr{T}(G)} x_G(T) = \max_{A\subseteq E(G)} y_G(A).$$

Note that Širáň and Škoviera [15] generalized Theorems A and B to signed graphs. In Section 2 of the present paper an extension of Theorem B will be given.

**1.** Let $G$ be a connected graph different from a tree, and let $T \in \mathscr{T}(G)$. It is clear that if $e_1$ and $e_2$ are distinct edges in $E(G) - E(T)$, then the subgraph $T + e_1 + e_2$ of $G$ has at least one and at most two nontrivial (i.e. cyclic) leaves. We denote by $G\#T$ the simple graph with

$$V(G\#T) = E(G) - E(T)$$

and with the property that

$ef \in E(G\#T)$ if and only if the subgraph $T + e + f$ of $G$ has only one nontrivial leaf

for any distinct $e, f \in E(G) - E(T)$.

**Lemma 1.** *Let $G$ be a nontrivial 2-edge-connected graph, and let $T \in \mathscr{T}(G)$. Then $G\#T$ is connected.*

P r o o f. We assume, to the contrary, that $G\#T$ is not connected. Then there exist $E_1$, $E_2 \subseteq \mathscr{A}_G(T)$ such that $E_1 \neq \emptyset \neq E_2$, $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E(G) - E(T)$, and that $T + e_1 + e_2$ has two nontrivial leaves for any $e_1 \in E_1$ and $e_2 \in E_2$. We denote by $\mathscr{E}$ the set of all $E \in \mathscr{A}_G(T)$ with the properties that $E \cap E_1 \neq \emptyset \neq E \cap E_2$ and the subgraph $T + E$ of $G$ has only one nontrivial leaf. Clearly, $\mathscr{E} \neq \emptyset$.

Consider $E_0 \in \mathscr{E}$ such that no proper subset of $E_0$ belongs to $\mathscr{E}$. We can see that $|E_0| \geqslant 3$. Without loss of generality we will assume that $|E_0 \cap E_2| \geqslant 2$. Consider an arbitrary $e_0 \in E_0 \cap E_2$. Obviously, $E_0 - \{e_0\} \notin \mathscr{E}$. According to the definition, $T + (E_0 - \{e_0\})$ has at least two nontrivial leaves. Clearly, there exists a leaf $F_1$ of $T + (E_0 - \{e_0\})$ such that $E(F_1) \cap E_1 \neq \emptyset$. Denote $E^* = E(F_1) - E(T)$. Since $T + (E_0 - \{e_0\})$ has at least two nontrivial leaves, we conclude that $E^*$ is a proper subset of $E_0 - \{e_0\}$, and therefore $E^* \cup \{e_0\}$ is a proper subset of $E_0$. Hence $E^* \cup \{e_0\} \notin \mathscr{E}$.

On the other hand, $F_1$ is a nontrivial leaf of $T+(E_0-\{e_0\})$ and $T+(E_0-\{e_0\})+e_0$ has only one nontrivial leaf. It is easy to see that $T + \big(E(F_1) - E(T)\big) + e_0$ has only one nontrivial leaf. Thus we get $E^* \cup \{e_0\} \in \mathscr{E}$, which is a contradiction. The lemma is proved. □

**Corollary.** *Let $G$ be a connected graph different from a tree, and let $T \in \mathscr{T}(G)$. Then there exists a bijection $\varphi$ of the set of all nontrivial leaves of $G$ onto the set of all components of $G\#T$ such that*

$$V\big(\varphi(F)\big) = E(F) - E(T)$$

*for each nontrivial leaf $F$ of $G$.*

Proof is obvious.

Let $G$ be a graph. If $M$ is a matching in $G$ and $u \in V(G)$ is such that $u$ is incident with no edge in $M$, then we say that $u$ is an unsaturated vertex of $M$. A matching $M$ in $G$ is referred to as a maximum matching in $G$ if $|M_0| \leqslant |M|$ for every matching $M_0$ in $G$.

If $H$ is a graph, then we denote by $c_0(H)$ the number of components $F$ of $H$ such that $p(F)$ is odd. We shall need the following theorem:

**Theorem C** (Berge [3]). *Let $G$ be a graph. Then the number of unsaturated vertices of a maximum matching in $G$ is equal to*

$$\max_{U \subset V(G)} \big(c_0(G - U) - |U|\big).$$

Note that Theorem C is a generalization of Tutte's theorem on perfect matchings [16].

If $G$ is a connected graph different from a tree and $T \in \mathscr{T}(G)$, then we shall denote by $z_G(T)$ the number of unsaturated vertices of a maximum matching in $G\#T$.

**Lemma 2.** *Let $G$ be a connected graph different from a tree, and let $T \in \mathscr{T}(G)$. Then*

$$z_G(T) = \max_{A \in \mathscr{A}_G(T)} \big(b^\lambda(G - A) - |A|\big).$$

P r o o f. According to Theorem C,

$$z_G(T) = \max_{A \subset E(G) - E(T)} \Big(c_0\big((G\#T) - A\big) - |A|\Big).$$

Consider an arbitrary $A \subset E(G) - E(T)$. The corollary implies that

$$c_0\big((G - A)\#T\big) = b^\lambda(G - A).$$

It is easy to see that

$$(G\#T) - A = (G - A)\#T.$$

Obviously, $b^\lambda\big(G - \big(E(G) - E(T)\big)\big) = 0$. Hence, the statement of the lemma follows. □

In the next section we will prove that if $G$ is a connected graph different from a tree, then there exists $T \in \mathscr{T}(G)$ such that

$$\min_{T_0 \in \mathscr{T}(G)} x_G(T_0) = x_G(T) = z_G(T) = \max_{T_1 \in \mathscr{T}(G)} z_G(T_1).$$

**2.** The following proposition can be easily proved:

**Proposition B.** *If $G$ is a connected graph, then*

$$y_G(A) \equiv \beta(G) \pmod 2$$

*for every $A \subseteq E(G)$.*

For the proof see [10].

If $G$ is a connected graph, then we denote by $\mathscr{M}(G)$ the set of all $A \subseteq E(G)$ such that

$$y_G(A) = \max_{A' \subseteq E(G)} y_G(A')$$

and $y_G(A'') < y_G(A)$ for every $A'' \subseteq E(G)$ such that $A$ is a proper subset of $A''$.

A complete proof of the next Lemma can be found in [10].

**Lemma A.** *Let $G$ be a connected graph, let $A \in \mathscr{M}(G)$, and let $F$ be a component of $G - A$. If $\beta(F)$ is even, then $q(F) = 0$. If $\beta(F)$ is odd, then $F - e$ is connected and*

$$\max_{A_F \subseteq E(F-e)} y_{F-e}(A_F) = 0$$

*for each $e \in E(F)$.*

Proof (outlined). The case when $\beta(F)$ is even is clear. Let $\beta(F)$ be odd. Consider an arbitrary $e \in E(F)$. Since $A \in \mathscr{M}(G)$, we get that $F - e$ is connected. Let $A_F \subseteq E(F - e)$. Then

$$y_G(A) > y_G(A \cup \{e\} \cup A_F) = y_G(A) + y_{F-e}(A_F) - 2,$$

and thus $y_{F-e}(A_F) < 2$. Proposition B implies that $y_{F-e}(A_F) \leqslant 0$. Since $y_{F-e}(\emptyset) = 0$, the proof is complete. □

We shall need a theorem from the intersection of matroid theory and transversal theory; see Wilson [18], for example. Corollary 33B in [18] can be reformulated as follows:

**Theorem D.** *Consider a matroid on a finite nonempty set $A$ with rank function $r$. Let $D_1, \ldots, D_k$ ($k \geqslant 1$) be nonempty subsets of $A$. Denote $\mathscr{D} = (D_1, \ldots, D_k)$. Then the maximum size of an independent transversal of $\mathscr{D}$ is equal to*

$$k - \max_{I \subseteq \{1,\ldots,k\}} \Big(|I| - r\Big(\bigcup_{i \in I} D_i\Big)\Big).$$

Clearly, Theorem D is a generalization of Rado's theorem on independent transversals [12].

We are now prepared to prove the main result of the present paper.

**Theorem 1.** *Let $G$ be a connected graph different from a tree. Then there exists $T \in \mathscr{T}(G)$ such that*

$$\min_{T_0 \in \mathscr{T}(G)} x_G(T_0) = x_G(T) = \max_{A \in E(G)} y_G(A) = z_G(T) = \max_{T_1 \in \mathscr{T}(G)} z_G(T_1).$$

Proof. For every connected graph $H$ we denote

$$x_H = \min_{T \in \mathscr{T}(G)} x_H(T) \qquad \text{and} \qquad y_H = \max_{A \subseteq E(G)} y_H(A).$$

We shall prove that

(I) there exists $T \in \mathscr{T}(G)$ such that $x_G(T) \leqslant y_G \leqslant z_G(T)$ and $y_G \leqslant x_G$ and that

(II) $\max_{T_1 \in \mathscr{T}(G)} z_G(T_1) = y_G$.

(I) We proceed by induction on $q(G)$. Since $G$ is different from tree, we get that $q(G) \geqslant 1$. The case when $q(G) = 1$ is obvious. Let $q(G) \geqslant 2$. Consider an arbitrary $A \in \mathscr{M}(G)$. Let $\mathscr{B}$ denote the set of all components $F$ of $G - A$ such that $\beta(F)$ is odd. We put $k = b(G - A)$. Since $G$ is not a tree and $A \in \mathscr{M}(G)$, we can see that $k \geqslant 1$. There exist mutually distinct components $B_1, \ldots, B_k$ of $G - A$ such that $\mathscr{B} = \{B_1, \ldots, B_k\}$. For every $i \in \{1, \ldots, k\}$ we denote by $N_i$ the set of all $e \in A$ such that $e$ is incident with a vertex of $B_i$. For $I \subseteq \{1, \ldots, k\}$ we denote

$$N_I = \bigcup_{i \in I} N_i.$$

Let $r$ denote the mapping of exp $A$ into the set of integers defined as follows:

$$r(A_0) = |A_0| - c(G - A_0) + 1 \qquad \text{for every } A_0 \subseteq A.$$

If $I \subseteq \{1, \ldots, k\}$, then

$$|I| - r(N_I) = |I| - |N_I| + c(G - N_I) - 1 \leqslant y_G(N_I) \leqslant y_G.$$

It is easy to see that

$$k - r(N_{\{1,\ldots,k\}}) = y_G(A).$$

Thus,

$$\max_{I \subseteq \{1,\ldots,k\}} (|I| - r(N_I)) = y_G.$$

It is not difficult to see that $r$ is the rank function of a matroid on $A$. According to Theorem D, the maximum size of an independent partial transversal of $(N_1, \ldots, N_k)$ is equal to $k - y_G$. Thus, without loss of generality we will assume that there exists an independent transversal of $(N_1, \ldots, N_{k-y_G})$. This means that there exist mutually distinct $a_1, \ldots, a_{k-y_G} \in A$ such that

$$a_i \in N_i, \qquad \text{for each } i \in \{1, \ldots, k - y_G\},$$

and $G - a_1 - \ldots - a_{k-y_G}$ is connected. Denote

$$A^* = A - \{a_1, \ldots, a_{k-y_G}\}.$$

We can see that $|A^*| = c(G - A) - 1$.

Let $i \in \{1, \ldots, k\}$. We choose an edge $e_i$ of $B_i$ such that if $i \leqslant k - y_G$, then there exists a vertex incident with both $a_i$ and $e_i$. According to Lemma $A$, $B_i - e_i$ is connected and $y_{B_i - e_i} = 0$. If $B_i - e_i$ is not a tree, then it follows from the induction hypothesis that there exists $T_i \in \mathscr{T}(B_i - e_i)$ such that $x_{B_i - e_i}(T_i) = 0$. If $B_i - e_i$ is a tree, we put $T_i = B_i - e_i$.

We denote by $T$ the subgraph of $G$ induced by the set of edges

$$A^* \cup E(T_1) \cup \ldots \cup E(T_k).$$

Clearly, $T$ is a spanning tree of $G$. It is easy to see that $x_G(T) \leqslant y_G$.

According to Lemma 2,

$$z_G(T) = \max_{A_0 \in \mathscr{A}_G(T)} \left(b^\lambda(G - A_0) - |A_0|\right).$$

Since $|A^*| = c(G - A) - 1$, we can see that

$$b^\lambda(G - \{a_1, \ldots, a_{k-y_G}\}) - |\{a_1, \ldots, a_{k-y_G}\}| = y_G.$$

Hence $y_G \leqslant z_G(T)$.

Consider $T' \in \mathscr{T}(G)$ such that $x_G(T') = x_G$. Let

$\mathscr{B}_{\text{con}} = \{B \in \mathscr{B};$ the subgraph of $T'$ induced by $V(B)$ is connected$\}$.

It is not difficult to see that $q(F)$ is odd for at least $|B_{\text{con}}| - |A - E(T)|$ components $F$ of $G - E(T')$. Thus

$$x_G(T') \geqslant |B_{\text{con}}| - |A - E(T')| = |B_{\text{con}}| - |A| + |A \cap E(T')|.$$

Moreover, we can see that

$$|A \cap E(T')| \geqslant c(T' - A) - 1 \qquad \text{and} \qquad c(T' - A) \geqslant c(G - A) + |B - B_{\text{con}}|.$$

We get that $x_G(T') \geqslant y_G(A)$, and thus $x_G \geqslant y_G$.

(II) If we combine Lemma 2 with Proposition A, we obtain

$$\max_{T_1 \in \mathscr{T}(G)} z_G(T_1) = \max_{T_1 \in \mathscr{T}(G)} \max_{A_{T_1} \in \mathscr{A}_G(T_1)} (b^\lambda(G - A_{T_1}) - |A_{T_1}|)$$
$$= \max_{A \in \mathscr{A}_G} (b^\lambda(G - A') - |A'|) = y_G.$$

The proof of the theorem is complete. □

Theorem 1 is an extension of Theorem B. In the proof of Theorem 1 some ideas from [10] were utilized. On the other hand, the proof of Theorem 1 shows that Theorem B can be proved by using Theorem D; then the role of Theorem D is similar to the role of Hall's theorem on distinct representatives [6] in Anderson's proof [1] of Tutte's theorem on perfect matchings.

*References*


[1] *I. Anderson*: Perfect matchings of a graph, J. Combinatorial Theory *B 10* (1971), 183–186.
[2] *M. Behzad, G. Chartrand and L. Lesniak-Foster*: Graphs & Digraphs, Prindle, Weber & Schmidt, Boston, 1979.
[3] *C. Berge*: Théorie des graphes et ses applications, Dunod, Paris, 1958.
[4] *J. A. Bondy and U.S.R. Murty*: Graph Theory with Applications, MacMillan, London, 1976.
[5] *R. A. Duke*: The genus, regional number, and Betti number of a graph, Canad. J. Math *18* (1966), 817–822.
[6] *P. Hall*: On representatives of subsets, J. London Math. Soc. *10* (1935), 26–30.
[7] *N. P. Homenko and A. D. Glukhov*: Single-component 2-cell embeddings and the maximum genus of a graph, In: Some Topological and Combinatorial Properties of Graphs, preprint 80.8 (N. P. Homenko, ed.), IM AN USSR, Kiev, 1980, pp. 5–23. (In Russian.)
[8] *N. P. Homenko, N. A. Ostroverkhy and V. A. Kusmenko*: The maximum genus of a graph, In: $\varphi$-Transformations of Graphs (N. P. Homenko, ed.), IM AN URSR, Kiev, 1973, pp. 180–207.
[9] *M. Jungerman*: A characterization of upper-embeddable graphs, Trans. Amer. Math. Soc. *241* (1978), 401-406.
[10] *L. Nebeský*: A new characterization of the maximum genus of a graph, Czechoslovak Math. J. *31 (106)* (1981), 604–613.
[11] *E. A. Nordhaus, B. M. Stewart and A. T. White*: On the maximum genus of a graph, J. Combinatorial Theory *B 11* (1971), 258–267.
[12] *R. Rado*: A theorem on independence relations, Quart. J. Math. (Oxford) *13* (1942), 83–89.
[13] *G. Ringel*: The combinatorial map color theorem, J. Graph Theory *1* (1977), 141–155.
[14] *J. Širáň*: Duke's theorem does not extend to signed graph embeddings, Discrete Math. *94* (1991), 233–238.
[15] *J. Širáň and M. Škoviera*: Characterization of the maximum genus of a signed graph, J. Combinatorial Theory *B 52* (1991), 124–146.
[16] *W. T. Tutte*: The factorization of linear graphs, J. London Math. Soc. *22* (1947), 107–111.
[17] *A. T. White*: Graphs, Groups, and Surfaces, North Holland, Amsterdam, 1973.
[18] *R. J. Wilson*: Introduction to Graph Theory, Longman Group, London, 1975.
[19] *N. H. Xuong*: How to determine the maximum genus of a graph, J. Combinatorial Theory *B 26* (1979), 217–225.



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