# Charge Injection and Transport in a Lossy Capacitor Stressed by a Marx Generator

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## ABSTRACT

A theoretical solution is given for the electric field and charge density of a capacitor with a charge injecting electrode being charged by a Marx capacitor bank.

## INTRODUCTION

Injected charge of density $q$ and mobility $\mu$ increases the effective ohmic conductivity $\sigma$ to $\sigma+|q\mu|$ where $q$ is time and space dependent. Then, open circuit decay curves at moderate voltages can show a dielectric relaxation time that decreases with increasing voltage while at very high voltages the open circuit decay curves can be non-exponential. Such anomalous behavior has been measured in highly purified water [1] and would be present in any material with enough net charge to distort the electric field. A drift dominated conduction model is used to solve for the electric field and charge density distributions and the terminal voltage-current behavior of a lossy capacitor where one electrode injects charge. The capacitor is connected to a Marx capacitor bank so that the injected charge decreases as the Marx voltage decays, the decay rate in turn being in part determined by the injected charge.

## DRIFT DOMINATED CONDUCTION MODEL

We consider parallel plate electrodes located at $x=0$ and $x=l$, where the $x=0$ electrode is assumed to be a source of positive charge with constant mobility $\mu$. The dielectric has constant permittivity $\varepsilon$ and intrinsic ohmic conductivity $\sigma$. Neglecting edge effects, the electric field and conduction current are $x$ directed and all quantities only depend on $x$. The governing equations are then

Irrotational Electric Field: $\nabla\times\overline{E}=0\rightarrow\int_0^l E\,dx=v$ (1)

Gauss's Law: $\nabla\cdot(\varepsilon\overline{E})=q\rightarrow\frac{\partial E}{\partial x}=q/\varepsilon$ (2)

Conservation of Charge: $\nabla\cdot\overline{J}+\frac{\partial q}{\partial t}=0\rightarrow\frac{\partial}{\partial x}\left(J+\varepsilon\frac{\partial E}{\partial t}\right)=0$ (3)

Conduction Constitutive Law: $\overline{J}=\sigma\,\overline{E}+q\mu\overline{E}$ (4)

It is convenient to nondimensionalize these equations by normalizing all variables to the electrode spacing $l$, the initial voltage $V_0$, and nominal injected charge transit time $l^2/(\mu V_0)$:

$$\tilde{x}=x/l,\ \tilde{v}=v/V_0,\ \tilde{E}=El/V_0,\ \tilde{q}=ql^2/(\varepsilon V_0)$$
$$\tilde{J}=Jl^3/(\varepsilon\mu V_0^2),\ \tilde{t}=\mu V_0 t/l^2,\ \tilde{\tau}=\frac{\varepsilon}{\sigma}\mu V_0/l^2 \quad (5)$$

Then (1)-(4) reduce to

$$\frac{\partial\tilde{E}}{\partial\tilde{t}}+\frac{\tilde{E}}{\tilde{\tau}}+\tilde{E}\frac{\partial\tilde{E}}{\partial\tilde{x}}=\tilde{\mathcal{J}}(\tilde{t}) \quad (6)$$

where $\tilde{\mathcal{J}}(\tilde{t})$ is the total terminal current per unit electrode area.

Differentiating (6) gives an equation for the charge density

$$\tilde{q}=\frac{\partial\tilde{E}}{\partial\tilde{x}};\qquad\frac{\partial\tilde{q}}{\partial\tilde{t}}+\frac{\tilde{q}}{\tilde{\tau}}+\tilde{E}\frac{\partial\tilde{q}}{\partial\tilde{x}}+\tilde{q}^2=0 \quad (7)$$

while integrating (6) between electrodes relates the current and voltage

$$\frac{d\tilde{v}}{d\tilde{t}}+\frac{\tilde{v}}{\tilde{\tau}}+\frac{1}{2}[\tilde{E}^2(\tilde{x}=1,\tilde{t})-\tilde{E}^2(\tilde{x}=0,\tilde{t})]=\tilde{\mathcal{J}}(\tilde{t}) \quad (8)$$

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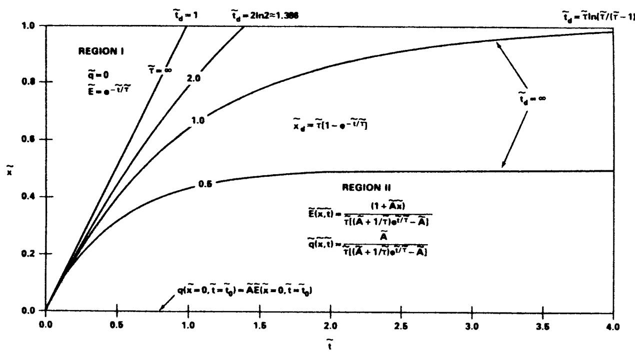


*Fig. 1: Charge transport trajectories for an initially charged but open circuited lossy capacitor where the lower $\tilde{x}=0$ electrode injects charge proportional to the instantaneous local electric field. The demarcation curves emanating from the origin shown for various values of dielectric relaxation time $\tilde{\tau}$, separate the initial value problem with $\tilde{q}(x,\tilde{t}=0)=0$, $\tilde{E}(\tilde{x},\tilde{t}=0)=1$ from the subsequent charge injection problem.*

*Method of Characteristics*

The partial differential equations (6) and (7) can be converted to ordinary differential equations by jumping into the frame of reference of the moving charges.

$$\frac{d\tilde{E}}{d\tilde{t}} = \frac{\partial \tilde{E}}{\partial \tilde{t}} + \frac{d\tilde{x}}{d\tilde{t}}\frac{\partial \tilde{E}}{\partial \tilde{x}} = \tilde{J}(\tilde{t}) - \tilde{E}/\tilde{\tau} \quad ; \quad \frac{d\tilde{x}}{d\tilde{t}} = \tilde{E} \tag{9}$$

$$\frac{d\tilde{q}}{d\tilde{t}} = \frac{\partial \tilde{q}}{\partial \tilde{t}} + \frac{d\tilde{x}}{d\tilde{t}}\frac{\partial \tilde{q}}{\partial \tilde{x}} = -(\tilde{q}^2 + \tilde{q}/\tilde{\tau}) \tag{10}$$

This last equation is directly integrated

$$\tilde{q}(\tilde{t}) = \frac{\tilde{q}_0(\tilde{t}_0)}{(1+\tilde{q}_0(\tilde{t}_0)\tilde{\tau})\exp[(\tilde{t}-\tilde{t}_0)/\tilde{\tau}] - \tilde{q}_0(\tilde{t}_0)\tilde{\tau}} \tag{11}$$

where $\tilde{q}_0(\tilde{t}_0)$ is the injected charge density at time $\tilde{t}$.

The problem is not completely specified until we define the charge injection law. We expect the injected charge density to increase with electrode electric field. Although numerical analysis can treat any injection law, we consider a simple linear law which allows analytic solution

$$q(x=0,t) = AE(x=0,t) \rightarrow \tilde{q}_0 = \tilde{A}\tilde{E}_0 \; ; \; \tilde{A} = Al/\varepsilon \tag{12}$$

## OPEN CIRCUIT DECAY

Consider an initially charged capacitor which is then open circuited for $\tilde{t}>0$ $(\tilde{J}(\tilde{t})=0)$. The solutions to (9) are

$$\tilde{E}(\tilde{t}) = \tilde{E}_0(\tilde{t}_0)\exp[-(\tilde{t}-\tilde{t}_0)/\tilde{\tau}] \; ; \qquad \tilde{x}(\tilde{t}) = \tilde{E}_0(\tilde{t}_0)\tilde{\tau}\left[1-\exp[-(\tilde{t}-\tilde{t}_0)/\tilde{\tau}]\right] \tag{13}$$

where $\tilde{t}_0$ is the time that the charge is injected at $\tilde{x}=0$. For $\tilde{t}_0=0$, there is a charge front which propagates towards the opposite electrode and reaches it at time

$$\tilde{t}_d = \tilde{\tau}\ln[\tilde{\tau}/(\tilde{\tau}-1)] \tag{14}$$

The injected charge density of (12) is found by evaluating (6) at $\tilde{x}=0$

$$\frac{d\tilde{E}_0}{d\tilde{t}} + \frac{\tilde{E}_0}{\tilde{\tau}} + \tilde{q}_0\tilde{E}_0 = \frac{d\tilde{E}_0}{d\tilde{t}} + \frac{\tilde{E}_0}{\tilde{\tau}} + \tilde{A}\tilde{E}_0^2 = 0 \tag{15}$$

with solution

$$\tilde{E}_0(\tilde{t}) = \frac{1}{\tilde{\tau}\{(\tilde{A}+1/\tilde{\tau})\exp[\tilde{t}/\tilde{\tau}] - \tilde{A}\}} \tag{16}$$

The solutions for field and charge density are then

$$\tilde{E}(\tilde{x},\tilde{t}) = \begin{cases} \exp[-\tilde{t}/\tilde{\tau}] & \tilde{\tau}(1-\exp[-\tilde{t}/\tilde{\tau}]) \le \tilde{x} \le 1 \\ \dfrac{1+\tilde{A}\tilde{x}}{\tilde{\tau}\{(\tilde{A}+1/\tilde{\tau})\exp[\tilde{t}/\tilde{\tau}] - \tilde{A}\}} & 0 \le \tilde{x} \le \tilde{\tau}(1-\exp[-\tilde{t}/\tilde{\tau}]) \end{cases} \tag{17}$$

$$\tilde{q}(\tilde{x},\tilde{t}) = \frac{\partial \tilde{E}}{\partial \tilde{x}} = \begin{cases} 0 & \tilde{\tau}(1-\exp[-\tilde{t}/\tilde{\tau}]) < \tilde{x} < 1 \\ \dfrac{\tilde{A}}{\tilde{\tau}\{(\tilde{A}+1/\tilde{\tau})\exp[\tilde{t}/\tilde{\tau}] - \tilde{A}} & 0 < \tilde{x} < \tilde{\tau}(1-\exp[-\tilde{t}/\tilde{\tau}]) \end{cases} \tag{18}$$

The open circuit terminal voltage is then

$$\tilde{v}(\tilde{t}) = \int_0^1 \tilde{E}\,d\tilde{x} = \begin{cases} \exp[-\tilde{t}/\tilde{\tau}] - \dfrac{\tilde{A}\tilde{\tau}(1-\exp[-\tilde{t}/\tilde{\tau}])^2}{2[(\tilde{A}+1/\tilde{\tau})\exp[\tilde{t}/\tilde{\tau}] - \tilde{A}} & 0 \le \tilde{t} \le \tilde{t}_d \\ \dfrac{1+\tilde{A}/2}{\tilde{\tau}\{(\tilde{A}+1/\tilde{\tau})\exp[\tilde{t}/\tilde{\tau}] - \tilde{A}\}} & \tilde{t} \ge \tilde{t}_d \end{cases} \tag{19}$$

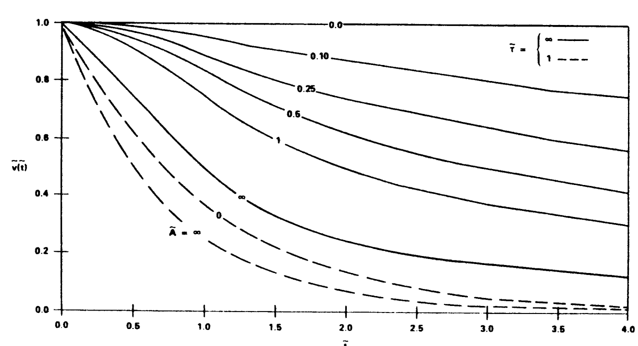


*Fig. 2: Open circuit voltage decay of (19) for various values of $\tilde{A}$ and $\tilde{\tau}$. There is an inflection point if $\tilde{A}\tilde{\tau}^2>1$.*

## MARX CAPACITOR CIRCUIT

Another typical configuration is to have a Marx capacitor bank with total capacitance $C_m$ connected directly across the test dielectric of area $S$. We neglect the Marx shunt resistance and assume the test dielectric is lossless ($\tilde\tau \to \infty$). Then the terminal current is also given by

$$\tilde J = -\tilde C_m \frac{d\tilde v}{d\tilde t} \quad ; \quad \tilde C_m = C_m Sl/\varepsilon \tag{20}$$

To allow closed form solutions we also assume space charge limited injection where $\tilde A = \infty$ so that $\tilde E_0 = 0$.

Then with $\tilde E(\tilde x=1, \tilde t) = \tilde E_1$, (8)-(11) reduce to

$$\tilde J = -\tilde C_m \frac{d\tilde v}{d\tilde t} = \frac{\tilde C_m \tilde E_1^{\,2}}{2(1+\tilde C_m)} \tag{21}$$

$$(1+\tilde C_m)\frac{d\tilde v}{d\tilde t} + \frac{1}{2}\tilde E_1^{\,2} = 0 \tag{22}$$

$$\frac{d\tilde E}{d\tilde t} = \tilde J = -\tilde C_m \frac{d\tilde v}{d\tilde t} \rightarrow \tilde E + \tilde C_m \tilde v = \text{constant} \tag{23}$$

$$\frac{d\tilde x}{d\tilde t} = \tilde E \tag{24}$$

$$\tilde q = \begin{cases} 0 & \tilde t_0 = 0, \tilde x(\tilde t_0) \neq 0 \\ 1/\tilde t & \tilde t_0 = 0, \tilde x(\tilde t_0) = 0 \\ 1/(\tilde t - \tilde t_0) & \tilde t_0 > 0, \tilde x(\tilde t_0) = 0 \end{cases} \tag{25}$$

The charge transport trajectories obtained by solving (24) break the solution into three spatial regions.

*Initial Value Problem*

Ahead of the charge front, $\tilde q = 0$, so that the electric field is uniform. The field $\tilde E_1$ at $\tilde x = 1$ is found using (6) with (21) for $\tilde q = 0$ and $\tilde\tau = \infty$

$$\frac{d\tilde E_1}{d\tilde t} - \frac{\tilde C_m}{2(1+\tilde C_m)}\tilde E_1 = 0 \rightarrow \tilde E_1 = \frac{1}{1 - \frac{\tilde C_m \tilde t}{2(1+\tilde C_m)}} \tag{26}$$

This solution is valid until the time $\tilde t_1$ when the initially injected charge first reaches the opposite electrode

$$\tilde t_1 = \frac{2(1+\tilde C_m)}{\tilde C_m}\{1 - \exp[-\tilde C_m/(2(1+\tilde C_m))]\} \tag{27}$$

The voltage and current for $0 \le \tilde t \le \tilde t_1$ are

$$\tilde J = \frac{d\tilde E_1}{d\tilde t} = \frac{\tilde C_m}{2(1+\tilde C_m)\left[1 - \frac{\tilde C_m \tilde t}{2(1+\tilde C_m)}\right]^2} \tag{28}$$

$$\tilde v = 1 - \frac{(\tilde E_1 - 1)}{\tilde C_m} = \frac{2 - \tilde t}{2 - \frac{\tilde C_m \tilde t}{(1+\tilde C_m)}} \tag{29}$$

*Initial Injection Problem*

Approaching the $\tilde x = 0$, $\tilde t = 0$ origin from along the $\tilde x$ axis has $\tilde E(\tilde x=0, \tilde t=0) = 1$ while approaching the origin from the $\tilde t$ axis has $\tilde E(\tilde x=0, \tilde t=0) = 0$ for space charge limited conditions. This discontinuity results in a family of injected charge at $\tilde t = 0$ all with charge density $\tilde q = 1/\tilde t$. As the charge propagates, Coulombic repulsion causes the charge to separate maintaining a uniform charge density and thus a linear electric field. Using (6) at $\tilde x = 1$ with $\tilde\tau = \infty$ and (21) yields for $\tilde t > \tilde t_1$ a single differential equation in the variable $\tilde t \tilde E_1$

$$\frac{d(\tilde t \tilde E_1)}{d\tilde t} - \frac{\tilde C_m}{2(1+\tilde C_m)}\frac{(\tilde t \tilde E_1)^2}{\tilde t} = 0 \rightarrow \tilde E_1 = \frac{\tilde E_s \tilde t_1}{\tilde t\left[1 - \frac{\tilde C_m \tilde t_1 \tilde E_s}{2(1+\tilde C_m)}\ln \tilde t/\tilde t_1\right]} \tag{30}$$

where $\tilde E_s = \tilde E_1(\tilde t = \tilde t_1)$ is found from (26). This solution is valid until time $\tilde t_2$ when the last of the charge injected at $\tilde t = 0$ reaches $\tilde x = 1$. For later times it is not possible to obtain analytic solutions, and numerical integration is necessary.

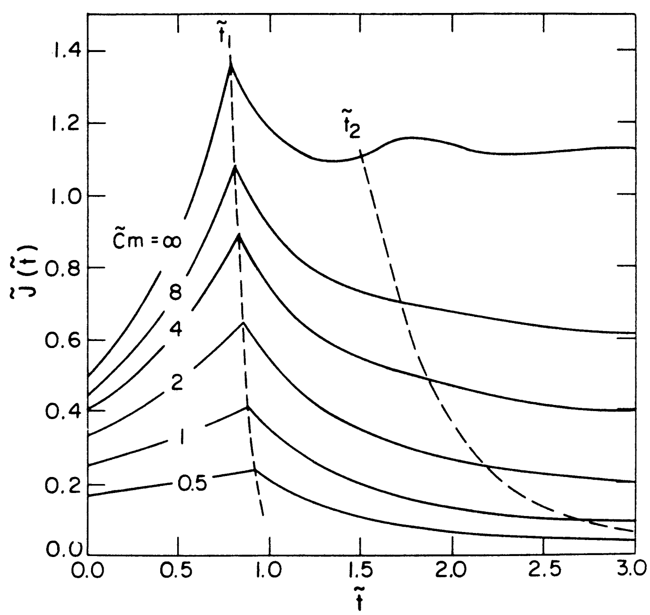


*Fig. 3: The terminal current $\tilde J(\tilde t)$ of (21) for various values of $\tilde C_m$. Analytic solutions are given in the time intervals $0 \le \tilde t \le \tilde t_1$ and $\tilde t_1 \le \tilde t \le \tilde t_2$. Numerical integration is required to find the solution for $\tilde t \ge \tilde t_2$. The dashed curves show $\tilde t_1$ and $\tilde t_2$ for each case. When $\tilde C_m = \infty$, the solution is that of a step voltage from rest.*

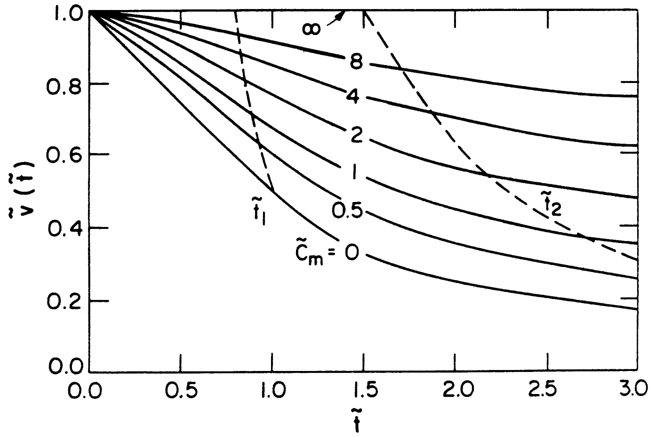


*Fig. 4: Terminal voltage $\tilde{v}(\tilde{t})$ for various values of $\tilde{C}_m$*

## REFERENCES

[1] M. Zahn, D. B. Fenneman, S. Voldman, and T. Takada, "Charge Injection and Transport in High Voltage Water/Glycol Capacitors", J. Appl. Phys. 54, pp. 315-325 (1983).

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