EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

CERN-LEP/RF/85-21

CHARGE PASSING OFF-AXIS THROUGH A CYLINDRICAL RESONATOR WITH BEAM PIPES

by H. Henke

Presented at the 1985 Particle Accelerator Conference,
13-16 May 1985, Vancouver, Canada

Geneva, Switzerland
May 1985

CHARGE PASSING OFF-AXIS THROUGH A CYLINDRICAL RESONATOR WITH BEAM PIPES

H. Henke
CERN, 1211 Geneva 23, Switzerland

Summary

A charge is considered which passes off-axis through a cylindrical resonator with two semi-infinite beam pipes. In the pipe region the fields are represented by the source fields plus a continuous spectrum of waveguide modes and in the resonator region by an infinite, discrete set of resonator modes. Matching the fields at the common interface yields a set of linear equations for the expansion coefficients. The involved integrals are solved with the residuum calculus, where the poles are located so that only waves travelling away from the resonator region are taken into account. By truncating the system, numerical results were derived for the longitudinal and transverse impedance which, contrary to periodic systems, also have real parts due to radiation into the pipes.

Introduction

The present paper treats analytically the passage of a point charge passing off-axis through a cylindrical resonator with semi-infinite beam pipes (Fig. 1). Depending on the dimensions the resonator either represents an RF cavity, one undulation of a bellows, or a vacuum flange gap.

Considerable effort has already been put into the solution of this problem. Most papers deal with closed cylindrical resonators where a complete set of eigenmodes allows for a relatively simple solution. In the case of a resonator with beam pipes the energy loss has been calculated for an axis-symmetric excitation either by estimating the radiation into the pipes[1] or by applying a high-frequency diffraction solution.[2] A class of problems related in some way are the infinite periodic structures. Here the Floquet theorem permits the decomposition of the fields in space-harmonics thus easing either the field matching[3] or the fulfilment of the boundary conditions on a smooth guide[4,5]. All these methods suffer from quite strong restrictions, especially those for the infinite periodic structures which do not give the radiation loss along the pipe.

Certainly the most versatile tool is a purely numerical method[6] if one is looking for a specific result. But on the other hand it does not allow approximate solutions or general parameter dependences to be found. Thus, it was due to analytical methods that the field integration along the beam pipe surface eased considerably the numerical calculation.[7]

Fields in the Different Regions

Referring to Fig. 1 we separate the whole region into two subregions: region I called the tube region and region II called the resonator region. The general solution in region I is given by a particular solution of the inhomogeneous equations (source fields) plus the solution of the homogeneous equations represented by a continuous spectrum of waveguide modes. In the resonator region, limited to $-g < z < g$, the eigenvalues in z direction are fixed and hence a discrete set of modes is obtained.

Cylindrical coordinates ρ, ϕ, z are used and all fields are proportional to $\exp(i\omega t)$ which is omitted throughout the paper.

The charge Q is assumed to have a constant velocity equal to the velocity of light, c_0. It moves along $\rho = r_1$, $\phi = 0$. This yields a charge density

$$q = \frac{Q}{\rho}\delta(\rho-r_1)\delta(\phi)\delta(z-c_0t)$$

$$= \frac{Q}{2\pi\rho}\delta(\rho-r_1)\delta(z-c_0t)\sum_{m=1}^{\infty}\varepsilon_m\cos m\phi \ , \ \varepsilon_m = \begin{cases}1, m=0\\2, m>0\end{cases} \quad (1)$$

Since the structure is axis-symmetric one can solve for the m^{th} Fourier component independently. Due to the velocity of light the source fields are purely transverse and can be gained from Maxwell's equations with $\partial/\partial z = \partial/\partial t = 0$ and $\vec{j} = qc_0\vec{e}_z$. In the ω domain they are given by

$$E^S_\phi = \frac{\varepsilon_m QZ_0}{2\pi\rho}\left\{\left(\frac{r_1}{\rho}\right)^m - \left(\frac{r_1}{a}\right)^m\left(\frac{\rho}{a}\right)\right\}\sin m\phi\, e^{-ik_0z}$$

$$H^S_\phi = \frac{\varepsilon_m Q}{2\pi\rho}\left\{\left(\frac{r_1}{\rho}\right)^m + \left(\frac{r_1}{a}\right)^m\left(\frac{\rho}{a}\right)\right\}\cos m\phi\, e^{-ik_0z} \quad (2)$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ and $k_0 = \omega/c_0$ are the free-space impedance and wave number respectively. The expressions in equation (2) are valid for $\rho > r_1$ and are determined so that $E_\phi(\rho = a) = 0$ to ease later use.

The source-free fields in regions I and II have to be a superposition of H- and E-modes, due to the azimuthal dependence, which can be derived from vector potentials in z direction

$$\begin{Bmatrix}A^H\\A^E\end{Bmatrix} = \begin{Bmatrix}F^H(k_0,k_z)\cos m\phi\\F^E(k_0,k_z)\sin m\phi\end{Bmatrix}[J_m(K\rho)+CN_m(K\rho)][e^{-ik_zz}+De^{ik_zz}]$$

$$i\vec{E} = \vec{\nabla}\times\vec{A}^E + \frac{1}{k_0}\vec{\nabla}\times\vec{\nabla}\times\vec{A}^H \quad (3)$$

$$Z_0\vec{H} = \frac{1}{k_0}\vec{\nabla}\times\vec{\nabla}\times\vec{A}^E + \vec{\nabla}\times\vec{A}^H \ , \quad K = \sqrt{k_0^2-k_z^2}$$

In region II the boundary conditions

$$E^{II}_\phi(z = \pm g) = 0 \qquad E^{II}_z(\rho = b) = 0 \quad (4)$$

determine the constants in such a way that the vector potentials become

$$A^{IIH} = \cos m\phi\left[\sum_{0,2}A^H_nR(K_n\rho)\cos k_{zn}z + \sum_{1,3}B^H_nR(K_n\rho)\sin k_{zn}z\right]$$

$$A^{IIE} = \sin m\phi\left[\sum_{2,4}A^E_nP(K_n\rho)\sin k_{zn}z + \sum_{1,3}B^E_nP(K_n\rho)\cos k_{zn}z\right] \quad (5)$$

with

$$R(K_n\rho) = N_m(k_n b)J_m(K_n\rho) - J_m(K_n b)N_m(K_n\rho)$$

$$P(K_n\rho) = N_m'(k_n b)J_m(K_n\rho) - J_m'(K_n b)N_m(K_n\rho)$$

$$K_n = \sqrt{k_0^2 - k_{zn}^2} \quad , \quad k_{zn} = n\pi/2g$$

N_m, J_m are the Neumann and Bessel functions, respectively.

In region I no boundary conditions exist. The region is unbound in z direction and in ρ direction the fields are unknown at ρ = a. The only requirement is regularity at ρ = 0, i.e. C = 0. So one has to sum over all radial (K) or longitudinal (k_z) wave numbers. We chose k_z, yielding

$$\begin{Bmatrix} A^{IH} \\ A^{IE} \end{Bmatrix} = \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\phi \int_{-\infty}^{\infty} \begin{Bmatrix} F^H(k_z) \\ F^E(k_z) \end{Bmatrix} J_m(K\rho) e^{-jk_z z} \, dk_z \qquad (6)$$

with $K = \sqrt{k_0^2 - k_z^2}$

Field Matching

In region I the complete solution is given by the source fields [equation (2)] and source-free fields [equation (6)], whereas in II we only have source-free fields [equation (5)]. The expansion coefficients A_n, B_n and the amplitude functions $F(k_z)$, all unknown, can now be determined by matching the tangential fields at the interface ρ = a. Since

$$E_\phi^S + E_\phi^I = 0 \, , \quad E_z^I = 0 \quad \text{for } \rho = a, \ |z| > g$$

$$E_\phi^I = E_\phi^{II}, \ E_z^I = E_z^{II} \quad \text{for } \rho = a, \ -g < z < g \qquad (7)$$

we expand the E^I fields in terms of the coefficients A_n and B_n

$$2\pi F^H(k_z)K^2 J_m(Ka) = \sum_{0,2} A_n^H K_n^2 R(K_n a) C_n(k_z) + \sum_{1,3} B_n^H K_n^2 R(K_n a) S_n(k_z)$$

$$2\pi\left(\frac{im}{k_0 a} F^H(k_z) k_z J_m(Ka) - F^E(k_z) K J_m'(Ka)\right) =$$

$$= \sum_{2,4} \left(m \frac{k_{zn}}{k_0 a} R(K_n a) A_n^H - K_n P'(K_n a) A_n^E\right) S_n(k_z) - \qquad (8)$$

$$- \sum_{1,3} \left(m \frac{k_{zn}}{k_0 a} R(K_n a) B_n^H - K_n P'(K_n a) B_n^E\right) C_n(k_z)$$

$$\text{with } \begin{Bmatrix} C \\ S \end{Bmatrix}_n (k_z) = \int_{-g}^{g} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} k_{zn} z \cdot e^{ik_z z} dz$$

Substituting the functions $F^{E,H}(k_z)$, equations (8) into (6), one obtains for the tangential magnet fields in region I

$$\frac{\pi k_0 Z_0 H_z^I}{\sin m\phi} = i \frac{m}{k_0} \sum_{0,2} (-) K_n^2 R(K_n a) Is_n^{(4)} A_n^H + \frac{m}{k_0} \sum_{1,3} (-) K_n^2 R(K_n a) Ic_n^{(4)} B^H$$

$$+ i \sum_{2,4} (-) k_{zn} \left(K_n a P'(K_n a) A_n^E - m \frac{k_{zn}}{k_0} R(K_n a) A_n^H\right) Is_n^{(2)} -$$

$$- \sum_{1,3} (-) k_{zn} \left(K_n a P'(K_n a) B_n^E + m \frac{k_{zn}}{k_0} R(K_n a) B_n^H\right) Ic_n^{(2)} \qquad (9)$$

$$-\frac{\pi Z_0 H_\phi}{\cos m\phi} = a \sum_{0,2} (-) K_n^2 R(K_n a) Is_n^{(1)} A_n^H -$$

$$- ia \sum_{1,3} (-) K_n^2 R(K_n a) Ic_n^{(1)} B_n^H -$$

$$- \frac{m}{k_0 a} \sum_{2,4} (-) k_{zn} \left(K_n a P'(K_n a) A_n^E - m \frac{k_{zn}}{k_0} R(K_n a) A_n^H\right) Is_n^{(3)} -$$

$$-i \frac{m}{k_0 a} \sum_{1,3} (-) k_{zn} \left(K_n a P'(K_n a) B_n^E + m \frac{k_{zn}}{k_0} R(K_n a) B_n^H\right) Ic_n^{(3)}$$

where (-) means $(-1)^{n/2}$ for n even and $(-1)^{n-1/2}$ for n odd. The integrals Ic_n, Is_n are complicated functions of n, k_0 and z. At first one expands the fractions $J_m'(x)/xJ_m(x)$ and $J_m(x)/xJ_m'(x)$ into series of algebraic terms which, then, can be integrated by means of the residuum calculus. Attention must be paid to the location of the poles. Those lying on the real axis have to be shifted in such a way that for a slightly lossy structure the waves excited at the tube ports of the resonator are damped and travel away from the resonator.

The tangential magnetic fields [equations (9)] are undetermined on the tube walls and have to be continuous at the interface:

$$\begin{aligned} H_\phi^S + H_\phi^I &= H_\phi^{II} \\ H_z^I &= H_z^{II} \end{aligned} \quad \text{for } \rho = a, \ g < z < g \qquad (10)$$

which requires the expansion of H^{II} with respect to H^I and H^S, i.e. multiplying (10) with $\cos k_{zp} z / \sin k_{zp} z$ and integrating over $-g < z < g$. This yields two separate inhomogeneous systems of linear equations for the expansion coefficients A_n, B_n with even and odd indices.

Having determined A_n, B_n the fields in region II are known through equation (5) and the fields in region I can be calculated by substituting A_n, B_n into (8) and then into (6).

Derivation of the Impedances

Wake potentials are defined as the integrated force acting on a probing particle at position $\rho = r_2$, $\phi = \phi_2$, $z = c_0(t-\tau)$

$$W_z(\tau) = -\frac{1}{Q} \int_{-\infty}^{\infty} E_z(r_2, \phi_2, t = z/c_0 + \tau) dz$$

$$W_\rho(\tau) = \frac{1}{Q} \int_{-\infty}^{\infty} [E_\rho(r_2, \phi_2, t = Z/c_0 + \tau) - Z_0 H_\phi(r_2, \phi_2, t = z/c_0 + \tau)] dz \qquad (11)$$

For $\tau < 0$ the potentials are zero. The exciting charge is assumed to be of unit quantity.

Taking the Fourier components of the fields and the Fourier transformation of (11) one obtains

$$\tilde{W}_z(\omega) = -\frac{2}{Q} \left(\frac{r_2}{a}\right)^m \cos m\phi_2 \left[\cos k_0 g \sum_{1,3} (-)^{\frac{n-1}{2}} R(K_n a) B_n^H + \right.$$

$$\left. + \ i \sin k_0 g \sum_{0,2} (-)^{\frac{n}{2}} R(K_n a) A_n^H \right] \qquad (12)$$

$$\tilde{W}_\rho(\omega) = \frac{im}{k_0 r_2} \tilde{W}_z(\omega)$$

The wake potentials [equations (12)] equal the ratio of voltage over current, both taken as time and spatial Fourier components. They represent the impedances seen by a unit current in z direction varying like $\exp i(\omega t-k_0z)$ and by a unit current in y direction varying like $\exp i(\omega t-k_0z+\pi/2)$. Since $W_z \sim r_1^m r_2^m$ and $W_r \sim r_1^m r_2^{m-1}$, one normally defines the longitudinal and transverse impedance for $r_1 = r_2 = r$

$$Z_\parallel(\omega) = \tilde{W}_z(\omega)/r^{2m}$$

$$Z_\perp(\omega) = \tilde{W}_\rho(\omega)/ir^{2m-1} = \frac{m}{k_0} Z_\parallel(\omega) \qquad (13)$$

in order to be independent of the offset of the charge.

Numerical Results

For some special cases such as short gap length, 2g, or low frequencies approximate formulae for the coefficients A_n, B_n can easily be derived. In general, however, the system of linear equations has to be solved. For this purpose a computer code ICYRP (Impedance of Cylindrical Resonator with Pipes) has been written which calculates the coefficients and then the impedance [equation (12)], for arbitrary azimuthal mode number m. In order to do so it truncates the matrices to a finite size.

Some results obtained by ICYRP are presented in Figs. 2 and 3. Figure 2 shows the longitudinal, m = 0, impedance of an RF cavity. For a = 5 cm the fundamental, E_{010}, resonance is at 350 MHz. The other resonances at k_0a = 0.845, 0.875, 1.165, 1.335 correspond to E_{020}, E_{011}, E_{021}, E_{030} modes. The impedance is purely imaginary and will only have a real part above the pipe cut-off k_ca = 2.405. In Fig. 3 the transverse, m = 1, impedance of a very small cavity is depicted. This cavity corresponds to one undulation of an RF bellows with dimensions used in [5]. Now, the pipe cut-off is k_ca = 1.841 where the impedance becomes complex onwards. At k_0a = 1.97 and 3.8 resonances occur. In the same graph results from [5] are shown. Taking into account the different geometry (smooth sinusoidal) both results agree well for low frequencies. But for higher frequencies the infinite periodic approach has neither the same cut-off nor the resonance behaviour.

References

[1] O.A. Kolpakov and V.I. Kotov, "Radiation of a charge passing a cylindrical resonator," Sov. Phys.-Tech. Phys., vol. 9, no. 8, pp. 1072-1074, Feb. 1965.

[2] J.D. Lawson, "Radiation from a ring charge passing through a resonator," Rutherford Lab. report RHEL/M 144, April 1968.

[3] E. Keil and B. Zotter, "Longitudinal stability of a coasting beam in a corrugated resistive vacuum chamber," Particle Acc., vol. 3, pp. 11-20, 1972.

[4] R.K. Cooper, S. Krinsky and P.L. Morton, "Transverse wake force in periodically varying waveguide", Particle Acc., vol. 12, pp. 1-12, 1982.

[5] S. Kheifets and B. Zotter, "Longitudinal and transverse impedances of bellows in the low-frequency range", Paper to be presented at this conference.

[6] T. Weiland, "Transverse beam cavity interaction", Nucl. Instr. Meth., vol. 212, pp. 13-21, 1983.

[7] T. Weiland, "Comment on wake field computation in time domain", Nucl. Instr. Meth., vol. 216, pp. 31-34, 1983.

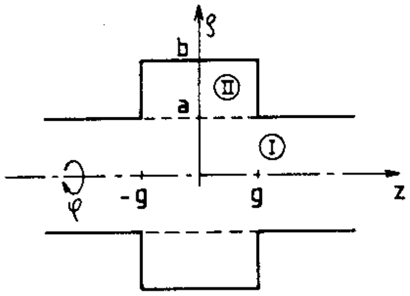

Fig. 1. Geometry of cylindrical resonator with pipes

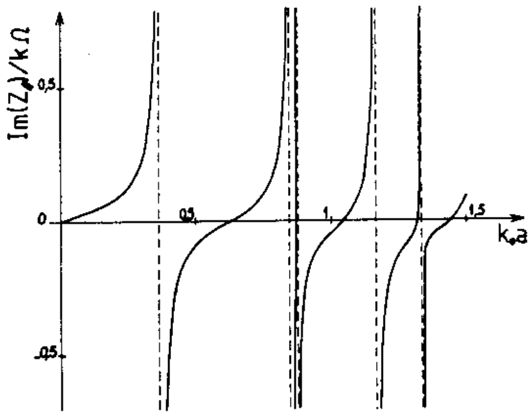

Fig. 2. Longitudinal impedance of an RF cavity; azimuthal mode number m = 0; b/a = 6.56, g/a = 1.98.

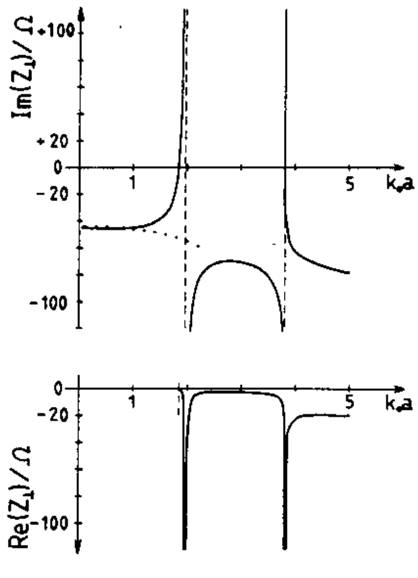

Fig. 3. Transverse impedance of one undulation of RF bellows; azimuthal mode number m = 1; b/a = 1.18, g/a = 0.05; dots are values from [5].