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Charged black holes and phase transitions

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Summary. The thermodynamic behaviour is discussed of a charged black hole in a box in equilibrium with neutral thermal radiation, in the thermodynamic limit (i.e. of a very massive black hole). The heat capacity at constant volume of this system exhibits several types of discontinuities which resemble phase transitions, and in an appropriate phase plane a critical point can be distinguished. The correspondence with normal thermodynamic phase changes is discussed, and various other thought experiments are briefly mentioned.

1 Introduction

In a classical treatment one can assign to a black hole a finite value of the entropy which is equal to a constant times the surface area of the event horizon (Bekenstein 1973). The analogy with thermodynamics can be carried further to the point of formulating four laws of black hole mechanics, corresponding to the four laws of thermodynamics (Bardeen, Carter & Hawking 1973). But the analogy is only a formal one, since a classical black hole cannot emit anything, and the only temperature compatible with the second law of thermodynamics is therefore zero, leading to an infinite entropy (Hawking 1976). The discovery by Hawking that black holes do emit thermal radiation resolved this paradox (Hawking 1974, 1975). In his treatment of quantum fields propagating on a fixed background, he was able to show that the temperature of a black hole is $T = \kappa/2\pi$ where κ is the surface gravity of the hole. As a consequence the entropy is $S = \frac{1}{4}A$, where A is the area of the event horizon. Here and throughout this paper we use units in which $c = \hbar = G = k = 1$.

To explore the thermodynamic properties of black holes, a number of thought experiments can be envisaged. The simplest equilibrium system to be imagined is an uncharged and non-rotating black hole in equilibrium with its thermal radiation. A Schwarzschild black hole formally possesses a negative heat capacity – however as such a statement applies only to closed systems, which a black hole is not, we should rather consider the hole and the radiation to be confined to a box with perfectly reflecting walls to obtain a system in a stable equilibrium state (Gibbons & Perry 1976a, b). There are certain difficulties in defining the general relativistic energy and entropy density of the radiation, but these can be over-

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come (Perry 1976). For a box much larger than the Schwarzschild radius of the hole the relevant expressions tend to the well-known special relativistic ones, which we shall use in this paper. There appear to be possible two equilibrium states; either the entire energy is present in the form of radiation, or a black hole has condensed out, having the same temperature as the remaining radiation. This black hole condensation resembles a first-order phase change.

It would be interesting to extend these investigations to the full Kerr–Newman family of black holes. For, in contrast with the Schwarzschild case, extremely highly charged or rapidly rotating black holes have a positive heat capacity at constant charge and angular momentum. At intermediate states there appears an infinite discontinuity in the heat capacity, which again reminds one of a phase transition (Davies 1977). There are, however, a number of complications in the treatment of charged and rotating thermal radiation fields, and in the prescription of appropriate boundary conditions to them. Therefore, in this paper we discuss what seems to be the simplest generalization of the Schwarzschild case, namely a charged black hole in equilibrium with neutral radiation. This can be realized by taking a very massive black hole, since the emission rate of electrons and positrons, the lightest charged particles, falls off exponentially with increasing mass of the black hole (Carter 1974; Gibbons 1975). In this thermodynamic limit the system already exhibits some interesting properties different from the Schwarzschild case. The treatment of a rotating, uncharged system is presently being undertaken by Perry & Stewart (private communication).

2 Equilibrium states

One of the fundamental aspects of black hole thermodynamics is the existence of the explicit fundamental relation for the Kerr–Newman family between its characteristic parameters

$$M = \left[\frac{S}{4\pi} + \frac{\pi}{S}(J^2 + \tfrac{1}{4}Q^4) + \frac{Q^2}{2}\right]^{1/2} \tag{1}$$

where M is the mass or energy of the hole, S the entropy, and J and Q are the angular momentum and charge respectively. An infinitesimal change in the energy relates to changes in the other parameters as follows

$$dM = TdS + \Omega dJ + \Phi dQ \tag{2}$$

where T is the temperature of the hole, Ω the angular velocity and Φ the electromagnetic potential at the event horizon. We refer to Hawking's paper for more information and the original references (Hawking 1976). Equation (2) has the usual thermodynamic form, and we can therefore call S, J and Q together with M, extensive parameters, and T, Ω and Φ the corresponding intensive parameters. However, there is an important difference from the ordinary thermodynamic situation, in that a black hole cannot be divided into smaller parts. Therefore, it is not surprising that M is a homogeneous function of degree ½ in S, J and Q^2 in this case, while normally the energy is a homogeneous linear function in the extensive parameters.

We shall further restrict our treatment to the Reissner–Nordström family of black holes which have $J = \Omega = 0$ in equation (2) and (3) and consider a system formed by a hole in a box with volume V in equilibrium with thermal radiation. The walls are taken to be rigid and perfectly reflecting. As the charge of the hole is constant in the thermodynamic limit where $M \to \infty$ we can restrict ourselves to considering only neutral radiation. For a given Q

and V, we can find the equilibrium state of the system by maximizing the entropy

$$S = 2\pi M^2[1 + \sqrt{(1 - Q^2/M^2)}] - \pi Q^2 + \tfrac{4}{3}aVT^3 \tag{3}$$

while keeping the total energy constant which is given by

$$E = M + aVT^4. \tag{4}$$

The 'constant' a, determining the radiation contributions, depends in general on temperature, but if we consider massless fields only it is a real constant, given by

$$a = \frac{\pi^2}{15}(n_b + \tfrac{7}{8}n_f + \tfrac{1}{2}n_s) \tag{5}$$

where n_b, n_f and n_s are the number of boson fields with non-zero spin, the number of fermion fields and the number of scalar fields respectively. Since the energy density of the blackground radiation in the universe is finite, among all elementary particles there must be one or more having the smallest mass, say m_0. If $T \ll m_0$, the radiation will indeed contain massless particles only, and such will be the case in the thermodynamic limit of very large black hole mass we are considering here.

Introducing the parameters

$$x = \frac{M}{E};\ y = \frac{1}{3\pi}\left(\frac{aV}{E^5}\right)^{1/4};\ z = 2^{-1/2}(3\pi)^{4/5}(aV)^{-1/5}Q$$

the (meta)stable equilibrium states can be found by maximizing the function

$$f_{y,z}(x) = x^2[1 + (1 + 2y^{8/5}z^2x^{-2})^{1/2}] + 2y(1 - x)^{3/4} \tag{6}$$

with respect to x on the interval $(2^{1/2}y^{4/5}z, 1)$. The parameter x has to be greater than this lower limit to avoid a (nearly) naked singularity where $Q \geq M$. The global maximum of f indicates a stable equilibrium, a local maximum indicates a metastable equilibrium, whereas a minimum gives an unstable equilibrium state. A qualitative picture of the behaviour of the system is given in Fig. 1. When Q and V are kept constant and E is varied slowly there are four different possibilities:

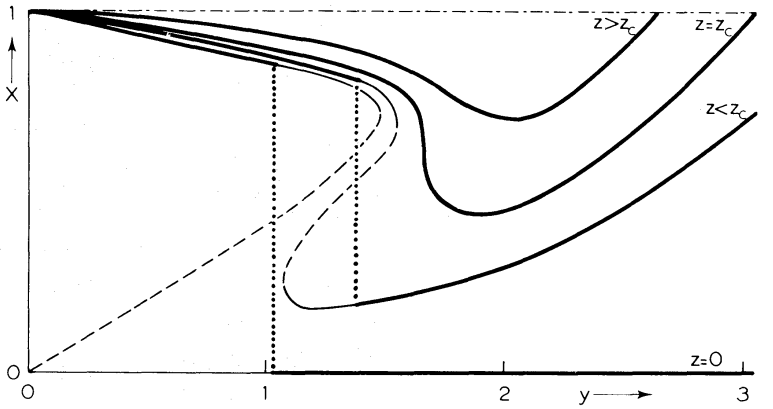

Figure 1. The values of x are shown for which $f_{yz}(x)$ is extremized, for four different values of z. Heavy lines indicate a global maximum, thin drawn lines a local maximum, and dashed lines a minimum. In physical terms: $f_{yz}(x)$ is a measure of the entropy, and the heavy, thin and dashed lines indicate a stable, a metastable and an unstable equilibrium, respectively.

(1) $z = 0$. An uncharged black hole can be in metastable equilibrium with radiation if $y < 2^5 3^{-1} 5^{-5/4} = 1.4266$, and in stable equilibrium if $y < 1.0144$. For pure radiation it is always possible to exist in (meta)stable equilibrium.

(2) $z < z_c$. For small charges the line of constant z turns over between two non-zero values of y. Unlike the previous case a metastable equilibrium is no longer possible for an arbitrarily small volume.

(3) $z = z_c = 0.2914$ (see Appendix 2). At this critical value of z no metastable states are possible any more.

(4) $z > z_c$. For large Q all states are stable. ($z > z_c$ implies $Q > 0.5626E$).

Fig. 1 seems to indicate that as E decreases keeping Q and V constant the system reaches a point where $x = 1$, i.e. there is no radiation left but only a maximally charged black hole ($Q = M = E$). This limiting situation however cannot be reached since by that time the hole will start to emit charged particles. The point is that for supermassive black holes the rate of emission of charged particles will drop much more rapidly with increasing mass of the hole than the rate of emission of neutral massless particles. This can be seen by the following heuristic argument: As the hole grows, the difference in electrostatic potential energy of the members of a virtual pair of particles drops far lower than the rest mass of the lightest charged particles, electrons and positrons. Therefore the superradiant discharge modes cease effectively (Carter 1974). Thus in the thermodynamic limit where $E \to \infty$ and $V \to \infty$ keeping y constant, i.e. $V \sim E^5$, Fig. 1 is valid over the whole energy range where $x < 1$.

To obtain a clearer physical picture of the behaviour of the system, the temperature is plotted against the energy in Fig. 2 for the four different cases. For example, if the black hole has a small charge, Fig. 2(b), we can perform the following thought experiment. Starting with a large amount of energy the hole will be very massive, having a low temperature. When the energy is decreased, the temperature will increase, the hole becomes lighter and the radiation field more intense, until the system will jump to the other branch, the hole losing suddenly an amount of energy to the radiation. If the total energy is varied very

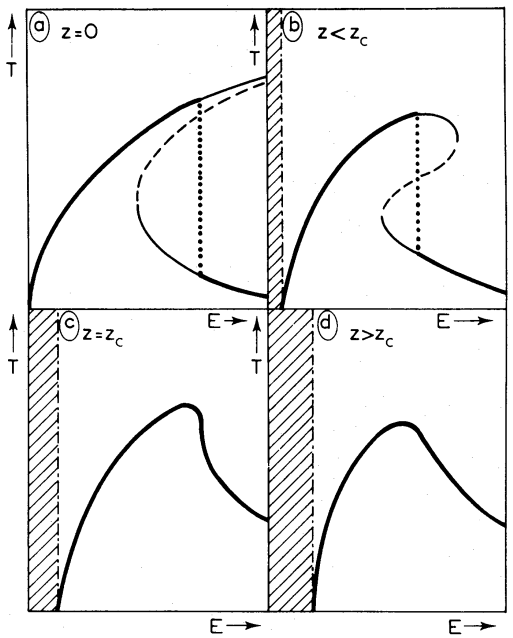

Figure 2. The temperature is plotted against the total energy in the box for a constant volume. The lines have the same meaning as in Fig. 1. The dotted lines indicate the jump which the system will make if the total energy is varied very slowly.

slowly, the jump will occur at the end of the stable part of the equilibrium curve, initiated by a huge fluctuation. If the total energy is decreased faster, but still reversibly, the jump will occur near the turning point only, at the end of the metastable region, on the timescale of evaporation of the hole, which varies as the third power of the mass (Hawking 1974).

There are of course other ways of changing the system reversibly. We could have kept Q and E constant, while varying V, or we could have carried out adiabatic changes, keeping Q and S constant. In both cases the qualitative behaviour would be the same as in the previous case with a critical point at the border of a region where metastable equilibria are possible.

3 Phase changes

The heat capacity at constant volume C_V of the system is plotted as a function of the total energy in Fig. 3. Before discussing the different types of behaviour of $C_V(E)$ we shall review briefly the simpler situation of one isolated Reissner–Nordström black hole only (Davies 1977). Expressions for T and Φ can be obtained from equations (1) and (2), and with the definition $C_x = T(\partial S/\partial T)_x$ we get

$$C_Q = (MS^3T)/(2\pi^2Q^4 - 8\pi T^2S^3) \tag{7}$$

$$C_\Phi = -(S/4\pi) \tag{8}$$

where

$$T = \frac{1}{8M}\left(1 - \frac{\pi^2Q^4}{S^2}\right). \tag{9}$$

Further details are given in Appendix 1.

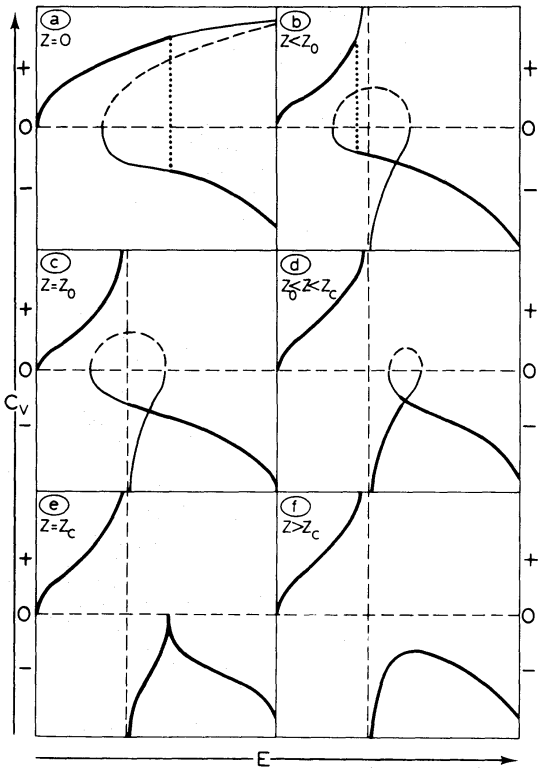

Figure 3. The heat capacity at constant volume versus the total energy, for a constant volume. The lines have the same meaning as in Fig. 2, but the vertical dashed lines indicate the asymptotes where $C_V \to \pm\infty$, and the horizontal dashed lines connect the points where C_V vanishes.

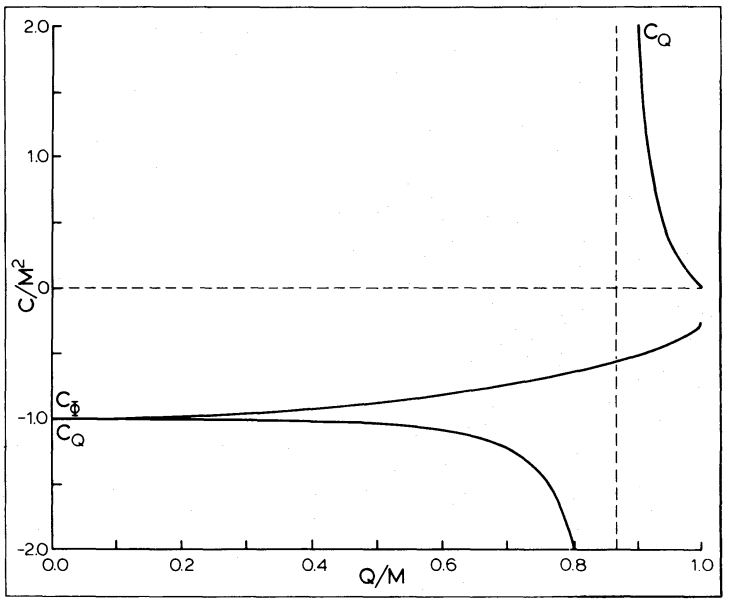

Figure 4. The heat capacities at constant charge C_Q and at constant potential C_Φ for a Reissner–Nordström black hole.

The two specific heats are plotted in Fig. 4 as a function of the charge. While C_Φ is finite and negative everywhere, C_Q has an infinite discontinuity at $Q/M = \frac{1}{2}\sqrt{3} \simeq 0.87$. At this point its sign changes too: C_Q is positive for more highly charged black holes. This remarkable behaviour of C_Q occurs because the Q, T coordinates are double valued, which is illustrated in Fig. 5. The Q, Φ coordinates, however, form a non-degenerate coordinate system everywhere. The physical reason for the double valued $M(Q, T)$ is that the temperature of a black hole tends to zero not only in the limit $M \to \infty$, but also in the lower limit $M \downarrow Q$ if Q is kept constant, since in both cases the surface gravity vanishes. This two-valuedness is also responsible for the occurrence of the metastable regions in Figs 1 and 2.

In order to interpret the discontinuity in C_Q in terms of a phase transition, the appropriate thermodynamic potential to consider would be the Helmholtz free energy $F(T, Q) = M - TS$ instead of the usual Gibbs free energy $G(T, \Phi) = F - Q\Phi$, since the heat capacities can be written as $C_Q = -T(\partial^2 F/\partial T^2)$ and $C_\Phi = -T(\partial^2 G/\partial T^2)$, respectively.

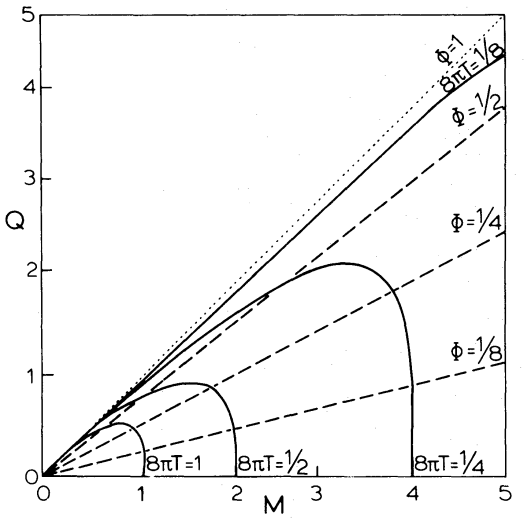

Figure 5. Isotherms and equipotentials for a Reissner–Nordström black hole.

In analogy to the Ehrenfest classification of phase transitions, we might call the discontinuity in C_Q a first order phase transition, since $(\partial F/\partial T)_Q$ becomes infinite at $Q/M = \frac{1}{2}\sqrt{3}$. But this terminology does not seem to be very appropriate here, as there is no such thing as a latent heat in this case, and most other thermodynamic functions are well behaved (see Appendix 1).

Returning to our charged black hole in a box, the finite discontinuity of the Schwarzschild case, Fig. 3(a), together with the infinite discontinuity we have discussed above are seen to be present in Fig. 3(b–f). For small charges the infinite jump lies in the metastable part, Fig. 3(b), and therefore the behaviour of the stable state is completely analogous to that in the uncharged case. There is a definite value of z, $z_0 = 0.2784$, above which both discontinuities occur in the stable state. But $z_c > z_0$, which can be seen as follows. At the critical point, a small change in M leaves E, V and Q all constant in first order (Fig. 1). If $dM > 0$, the energy in the form of radiation decreases, and so does the temperature of the radiation. But in equilibrium the temperature of the hole is equal to that of the radiation, and therefore the former drops at the same rate. Thus we have at the critical point $C_Q < 0$ and since metastable regions occur only for smaller values of z, we have $z_0 < z_c$.

We shall now discuss the physical significance of the behaviour of C_V. In the usual thermodynamic treatment of phase transitions the appropriate thermodynamic potential is $G(p, T)$, since normally the systems are kept at constant pressure and temperature. But in the present case our system cannot be in stable equilibrium with a heat bath, due to the occurrence of a negative C_V. Instead of the Gibbs free energy we must rather consider the entropy $S(E, V)$, since the system is kept at constant E and V. Indeed, generalizing the Ehrenfest classification, we can call the finite jump in C_V a first order phase charge, since $(\partial/\partial E)\,S(E, V)$ is discontinuous there.

This phaseline separating both stable phases is drawn in Fig. 6. Note that in our case an E, V phase plane takes the place of the usual p, T plane. Furthermore, a generalized

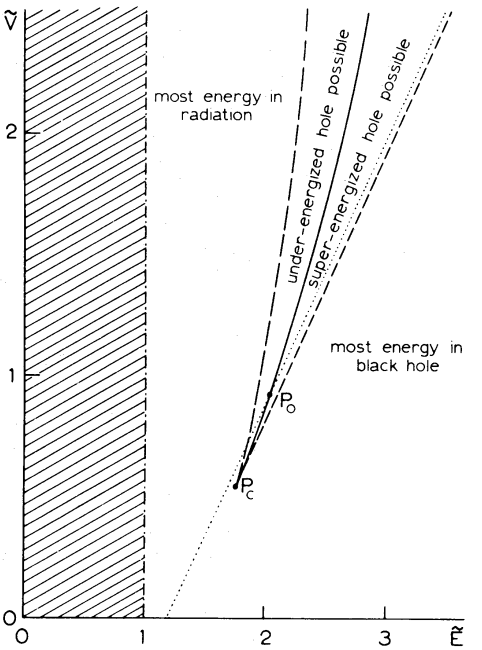

Figure 6. The phase plane for an isolated box containing a charged black hole in equilibrium with neutral radiation. The dashed region is unphysical, since the total energy must exceed the charge of the black hole. The full line indicates the first order phase transition, the dashed lines the boundaries of the metastable regions, and the dotted line indicates the infinite discontinuity in C_V. The coordinates $\tilde{E}$ and $\tilde{V}$ are a measure for E and V and are defined in the text.

Clausius–Clapeyron equation can be derived. Along the phase line $S_0 = S_1$, where a subscript 0 denotes the 'hot' side of the phase line, a subscript 1 the 'cool' side (a smaller or bigger hole respectively). Using

$$dS = \frac{1}{T}\,dE + \frac{P}{T}\,dV$$

at the two sides of the phase line we obtain

$$(\partial S_0/\partial E)_V\,dE + (\partial S_0/\partial V)_E\,dV = (\partial S_1/\partial E)_V\,dE + (\partial S_1/\partial V)_E\,dV.$$

A combination of these two relations gives

$$dE/dV = (p_1/T_1 - p_0/T_0)/(1/T_0 - 1/T_1)$$

and using $p = \frac{1}{3}aT^4$ we finally get

$$dE/dV = \tfrac{1}{3}aT_0T_1(T_0^2 + T_0T_1 + T_1^2) \tag{10}$$

for the slope of the phase line. Note furthermore that the critical point is really a critical point in the thermodynamic sense, it being the terminal point of a phaseline. The coordinates $\tilde{E}$ and $\tilde{V}$ in Fig. 6 are defined in order to have constant values at the critical point, independent of Q. The definitions are

$$\tilde{E} = E/Q \tag{11}$$

$$\tilde{V} = 2^{-4}3^{-6}\pi^{-4}aVQ^{-5}. \tag{12}$$

Instead of calling the metastable states superheated or undercooled, it seems more apt here to call them superenergized or underenergized.

Although the finite discontinuity in C_V can be classified as a first order phase transition, the infinite discontinuity cannot be considered as a phase change at all. Here the entropy is continuous together with all its derivatives with respect to E and V. But this infinite discontinuity has a physical significance which in a way transcends that of a phase transition. Although the internal characteristics of the system do not change here, the way the system can be brought into stable equilibrium with an external environment changes. On the side with a higher Q/E, $C_V > 0$, and the system can be in equilibrium with a heat bath. But at the lower Q/E side, $C_V < 0$, and the system can only be in equilibrium if it is isolated from the outside world. Thus instead of a phase transition we encounter here a transition from a region where only a microcanonical ensemble can be used to a region where both a canonical and a microcanonical ensemble are appropriate.

4 Conclusion

We have studied the thermodynamic behaviour of a charged black hole in a box in equilibrium with neutral radiation. As often occurs when treating systems where gravitation is important, there are regions where the system has a negative heat capacity, and this implies that we cannot keep the system in stable equilibrium with a heat bath (Lynden-Bell & Wood 1968; Lynden-Bell, D. & Lynden-Bell, R. M., private communication). Instead of keeping T and p fixed, as usual when discussing phase changes, we have now to keep E and V fixed. Now, instead of minimizing the Gibbs free energy $G(p, T)$ we have to maximize the entropy $S(E, V)$ to obtain the stable equilibrium state. But in other respects the situation is entirely analogous to that of ordinary thermodynamics, in that we encounter

first-order phase changes, a critical point in the phase plane, and Clausius–Clapeyron equations can be derived.

But C_V is not negative everywhere. For relatively highly charged black holes, having $Q > \frac{1}{2}\sqrt{3}M$, the system has a positive heat capacity. In this region a stable equilibrium can be attained with the system being in thermal contact with a heat bath. The equilibrium points can then be found by minimizing the Helmholtz free energy $F(V, T)$. But since $p = (1/3)aT^4$, the pressure is not an independent quantity, and the system cannot be kept in stable equilibrium at constant p and T. At $Q/M = (1/2)\sqrt{3}$ the heat capacity has an infinite discontinuity. Although this does not affect the internal state of the system as in the case of a phase transition, it is physically important: It indicates a transition from a region where only a microcanonical ensemble is appropriate to a region where a canonical ensemble too can be used to describe the system.

Although we concentrated our attention on the behaviour of $C_V(E)$ we could have studied other thermodynamic quantities as well, like $\kappa_S = -1/V(\partial V/\partial p)_S$, the adiabatic compressibility. It vanishes at the critical point, where a slight change in temperature leaves V constant in first order in adiabatic changes. Note that $C_p = T(\partial S/\partial T)_p$ and $\kappa_T = -(1/V)(\partial V/\partial p)_T$ are not well-defined for the system under consideration, since the pressure depends on temperature only.

Finally we can consider a large number of replicas of our system in thermal contact with each other. If they are all in a region of positive heat capacity they can remain in metastable equilibrium with each other. But as soon as one of them enters the region of negative specific heat, e.g. by a fluctuation, it will grow until it has used up most of the energy available from the other systems. The black hole of this system will then be nearly a Schwarzschild hole, while the other holes tend to the extreme limit where their masses equal their charges. The heat capacity of the combination of all these systems, looked at as one large system, will be negative, and the ensemble would in every respect behave as just one big Schwarzschild black hole in a box.

Acknowledgments

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Appendix 1: infinite discontinuities in $C_{Q,J}$

In the Reissner–Nordström family of black holes $C_Q \to \pm\infty$ if $Q \to \frac{1}{2}\sqrt{3}M$. But only those

functions are affected which depend on the locally degenerated coordinates T and Q. We can show this by explicitly calculating them, for example

$$\alpha \equiv -1/Q[\partial Q/\partial T]_{\Phi} = 1/8\pi T$$

$$\kappa_S \equiv 1/Q[\partial Q/\partial \Phi]_S = 1/\Phi$$

$$\kappa_T \equiv 1/Q[\partial Q/\partial \Phi]_T = -(3\Phi^2 - 1)/(4\pi Q)$$

which are analogous to the coefficient of thermal expansion, the adiabatic and the isothermal compressibility, respectively. At $Q = \frac{1}{2}\sqrt{3}M$ we have

$$\alpha = 9M/8; \kappa_S = \sqrt{3}M; \kappa_T = 0$$

As expected, only κ_T is affected.

Although only charged black holes were considered, with respect to the infinities in C_Q this is not much of a restriction. Similar phenomena occur in qualitatively the same fashion in the Kerr family with respect to C_J. In the general Kerr–Newman family the same picture applies with respect to $C_{J,Q}$ (Davies 1977), the heat capacity at constant angular momentum and charge. It is given by

$$C_{J,Q} = (MS^3T)/(8\pi J^2 + 2\pi^2 Q^4 - 8\pi T^2 S^3).$$

Now there is a whole line along which $C_{J,Q} = \pm\infty$, which can be parametrized by

$$\alpha = J^2/M^4 \qquad 0 \leqslant \alpha \leqslant 2\sqrt{3} - 3 \simeq 0.46.$$

To show explicitly that here too all principle thermodynamic functions are well behaved, we give them as a function of α along the line where the heat capacity is infinite

$$S = \pi/4\ (\alpha + 1)(\alpha + 9)M^2$$

$$T = [\pi(\alpha + 9)]^{-1}M^{-1}$$

$$J = \sqrt{\alpha}M$$

$$\Omega = \frac{4\sqrt{\alpha}}{(\alpha + 1)(\alpha + 9)}M^{-1}$$

$$Q = \tfrac{1}{2}\sqrt{3 - \alpha^2 - 6\alpha} - M$$

$$\Phi = \sqrt{3 - \alpha^2 - 6\alpha}\ \frac{\alpha + 3}{(\alpha + 1)(\alpha + 9)}$$

Appendix 2: some analytical relations

To compute $\tilde{E}_c$ and $\tilde{V}_c$ (Fig. 6), we have to put $(d/dx)f_{y,z}(x)$ equal to zero, as given by equation (6), together with the second and third derivatives. We then have three equations relating x_c, y_c and z_c. By a happy coincidence an equation for x_c can be separated

$$831875x_c^6 - 4719000x_c^5 + 10979100x_c^4 - 13407840x_c^3 + 9066816x_c^2 - 3220992x_c + 470016 = 0$$

and solved numerically giving as the physical root $x_c = 0.7319$. Expressions for y_c and z_c can be found giving

$$y_c = \tfrac{2}{3}(1-x_c)^{1/4}\left\{2x_c + \frac{65x_c^2 - 108x_c + 48}{2[15(1-x_c)(4-5x_c)]^{1/2}}\right\} \simeq 1.4757$$

$$z_c = 2^{-1/2} y_c^{-4/5}[\tfrac{1}{5}(-55x_c^2 + 108x_c - 48)]^{1/2} \simeq 0.2914.$$

The resulting values for $\tilde{E}_c$ and $\tilde{V}_c$ are $\tilde{E}_c \simeq 1.7774$ and $\tilde{V}_c \simeq 0.5842$.

In the simpler case of a Schwarzschild black hole in a box, the phase change occurs at the point where $aV = (3\pi y_1)^4 E^5$, where y_1 can be calculated explicitly to be

$$y_1 = \tfrac{8}{3} x_1 (1-x_1)^{1/4}$$

where

$$x_1 = 1 + \tfrac{1}{15}[17 - 2^{10/3}5^{-1/3}[(29 + 9\sqrt{11})^{1/3} + (29 - 9\sqrt{11})^{1/3}]].$$

Finally it can be seen that the curves of constant Q/M are straight lines in the $E-V$ plane, as drawn in Fig. 6 for the case $Q = \frac{1}{2}\sqrt{3}M$. For if Q/M is kept constant, and if Q is kept constant by varying only E and V, then M must remain invariant. An increase in E can thus only feed the energy in the form of radiation. But since the hole does not change, its temperature too is constant, as must be the temperature of the radiation. So the increase of energy in the form of radiation must be compensated for by an expansion, giving $dE = aT^4 dV$.

Therefore the curves of constant Q/M are straight lines. In terms of $\tilde{E}$ and $\tilde{V}$ they turn out to be represented by

$$\tilde{V} = 3^{-6}\beta^{-4}[2 + 2(1-\beta^2)^{1/2} + \beta^2(1-\beta^2)^{-1/2}]^4 \cdot (\tilde{E} - \beta^{-1})$$

where $\beta = Q/M$.