

Charged magnetic brane solutions in AdS_5 and the fate of the third law of thermodynamics¹

Eric D'Hoker and Per Kraus

*Department of Physics and Astronomy, University of California,
Los Angeles, CA 90095, U.S.A.*

E-mail: dhoker@physics.ucla.edu, pkraus@physics.ucla.edu

ABSTRACT: We construct asymptotically AdS_5 solutions to 5-dimensional Einstein-Maxwell theory with Chern-Simons term which are dual to 4-dimensional gauge theories, including $\mathcal{N} = 4$ SYM theory, in the presence of a constant background magnetic field B and a uniform electric charge density ρ . For the solutions corresponding to supersymmetric gauge theories, we find numerically that a small magnetic field causes a drastic decrease in the entropy at low temperatures. The near-horizon $\text{AdS}_2 \times R^3$ geometry of the purely electrically charged brane thus appears to be unstable under the addition of a small magnetic field. Based on this observation, we propose a formulation of the third law of thermodynamics (or Nernst theorem) that can be applied to black holes in the AdS/CFT context.

We also find interesting behavior for smaller, non-supersymmetric, values of the Chern-Simons coupling k . For $k = 1$ we exhibit exact solutions corresponding to warped AdS_3 black holes, and show that these can be connected to asymptotically AdS_5 spacetime. For $k \leq 1$ the entropy appears to go to a finite value at extremality, but the solutions still exhibit a mild singularity at strictly zero temperature.

In addition to our numerics, we carry out a complete perturbative analysis valid to order B^2 , and find that this corroborates our numerical results insofar as they overlap.

KEYWORDS: Gauge-gravity correspondence, Black Holes in String Theory, AdS-CFT Correspondence

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1 Introduction and summary of results

The dynamics of gauge theories at finite temperature, charge density and background electromagnetic fields may be studied at strong coupling via AdS/CFT dual supergravity solutions. The supergravity approximation is valid for large N and large 't Hooft coupling, but may be consistently truncated further to Einstein/Maxwell theory in the bulk when studying these electromagnetic effects. Temperature arises as the solutions to this Einstein/Maxwell theory exhibit a horizon, while charge density and background electromagnetic fields are introduced by imposing boundary conditions on the bulk Maxwell field. Thermodynamics and transport properties in gauge theories at strong coupling may then be obtained from suitable black hole or black brane solutions in this relatively simple Einstein/Maxwell bulk theory.

This program has been applied extensively to 2 + 1-dimensional gauge theory, which is realized holographically through 3 + 1-dimensional Einstein/Maxwell theory. Its key solution is the AdS_4 black brane with electric charge ρ , and magnetic field B . This brane solution is known analytically for all ρ and B ; its spectrum of small fluctuations may be obtained systematically, and used to compute physical quantities such as electric and thermal conductivities [1–9, 11, 12].

While several important condensed matter problems in an external magnetic field, such as the Quantum Hall Effect and high T_c superconductivity, are driven by 2+1-dimensional physics, it is clearly urgent to obtain results for 3+1-dimensional gauge theories as well. For instance, strong magnetic fields are created in collisions at RHIC, giving rise to observable effects which have been the subject of much recent discussion, e.g., [13–18].

In this paper, we shall present a systematic study of the thermodynamic properties of 3 + 1-dimensional gauge theories with finite electric charge density ρ in the presence of a constant magnetic field B . Their holographic duals should be electrically and magnetically charged black brane solutions to 4 + 1-dimensional Einstein/Maxwell theory with a Chern-Simons term. The Chern-Simons coupling k captures the strength of the anomaly of the

boundary current, and is required to take a specific value $k = 2/\sqrt{3}$ (in our conventions) if the Einstein/Maxwell theory is to be the bosonic truncation of minimal $D = 5$ supergravity; see e.g. [19].¹ It has been proven that any $\mathcal{N} = 1$ superconformal theory with an AdS_5 supergravity dual obtained by compactification from IIB or M-theory admits a consistent truncation to $D = 5$ minimal gauged supergravity [20–22]. Thus the results we find pertain to a large class of theories, of which $\mathcal{N} = 4$ super Yang-Mills is but one example.

The purely electric solution ($B = 0$) is the Reissner-Nordstrom black hole in AdS_5 ; its analytic form and thermodynamic properties are well-known. The existence of purely magnetic solutions ($\rho = 0$) was demonstrated numerically in a previous paper [23], but no analytical solutions are available at present. Purely magnetic solutions interpolate between AdS_5 and $AdS_3 \times R^2$ near the horizon. As a function of temperature T , their entropy density behaves as T^3 for large T , and vanishes as BT for small T . These limits agree with $\mathcal{N} = 4$ SYM calculations at zero gauge coupling, up to factors of $3/4$ and $\sqrt{4/3}$ respectively. On the $\mathcal{N} = 4$ SYM side the low temperature thermodynamics is governed by fermions in the lowest Landau level; an appealing feature of these supergravity solutions is that they reproduce this low temperature behavior.

Here we wish to extend these results to nonzero ρ and B . Besides their clear usefulness for applications of AdS/CFT, this investigation has conceptual implications for the status of extremal black holes, as we now pause to discuss.

1.1 Extremal black holes, Nernst’s theorem, the third law of thermodynamics, and all that

A striking feature of the Reissner-Nordstrom black brane solution, in any dimension, is that it possesses a smooth zero temperature limit with nonzero entropy density. This extremal solution exhibits a near-horizon $AdS_2 \times R^{D-2}$ region, the existence of which has played a central role in recent holographic descriptions of non-Fermi liquids [24–26]. However, a much discussed cause for concern is that while the extremal entropy apparently plays a crucial role in this analysis, it is not expected from the point of view of interacting fermions, nor from the point of view of the dual field theory where the existence of massless charged bosons suggests that Bose condensation should rule. One possibility is that the large ground state degeneracy should be understood as an artifact of the large N limit, as discussed in [9]. Another is that one should focus on alternative bulk theories where the extremal entropy vanishes, as discussed recently in the case of gauge fields coupled to massless scalars in [10]. While this may be the case, we would like to propose another resolution, based on the results we find for the response to magnetic fields.

The tension between the existence of extremal black hole entropy and the thermodynamic behavior of typical systems has been discussed periodically over the years (see e.g. [27]), and can be phrased in terms of a clash with Nernst’s “theorem” and the third “law” of thermodynamics. These statements can be expressed in various ways, but essentially they stipulate that the entropy density s should go to zero at zero temperature (see

¹Whenever $B = 0$, the Chern-Simons coupling is immaterial, and does not enter into the physical quantities considered here.

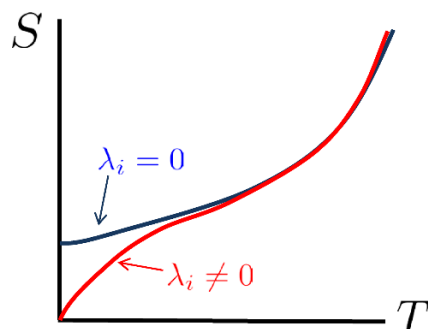


Figure 1. Illustration of fine tuning required to maintain nonzero extremal entropy. Small nonzero couplings λ_i lead to no appreciable effect at high temperature, but cause the low temperature entropy to flow to zero.

e.g. [28]). Despite the name, this “law”/“theorem” is actually meant to be a phenomenological observation characterizing the behavior of observed physical systems. Indeed, only a moment’s thought is required to realize that it is trivial to concoct theoretical counterexamples based on free systems. However, such counterexamples are to be thought of as being fine-tuned to an unphysical degree, as any realistic system will exhibit some degree of interactions, and these will typically lift the ground state degeneracy. The relevant question is whether the ground state degeneracy is stable under adding generic weak interactions or perturbations of the system. If λ_i represent some set of coupling constants, we should consider

$$\lim_{\lambda_i \rightarrow 0} \lim_{T \rightarrow 0} s(\lambda_i, T) . \tag{1.1}$$

If this limit gives zero for “typical” couplings λ_i , then we may conclude that a fine tuning is required to sustain the entropy.

Phrased in this way, Nernst’s “theorem” admits a natural formulation in terms of black holes in the AdS/CFT correspondence. We should ask whether the extremal black branes exhibiting finite entropy at extremality are fine-tuned in the same sense when we change the interactions. In AdS/CFT we change the Lagrangian of the CFT by changing boundary conditions for fields in the bulk, and thus we can ask whether the extremal entropy persists even in the presence of nontrivial boundary conditions for a suitable set of bulk fields. In a scenario in which the entropy is to be regarded as requiring fine-tuning, we would expect to see behavior like that in figure 1.

In this work we study the case in which λ_i corresponds to turning on a constant external magnetic field coupling to the R-current of $\mathcal{N} = 4$ SYM (or other superconformal field theories described holographically), and we will present evidence that the extremal entropy indeed is unstable in the above sense under the inclusion of a magnetic field. Since we proceed numerically, and our numerics break down at very low temperature, we are not able to follow the entropy all the way down to zero, but the simplest extrapolation suggests a picture in concordance with figure 1. In further support of this interpretation, we will see that an attempt to construct a finite entropy solution perturbatively in B breaks down at very low temperature. These results suggest that conclusions drawn from the extremal black

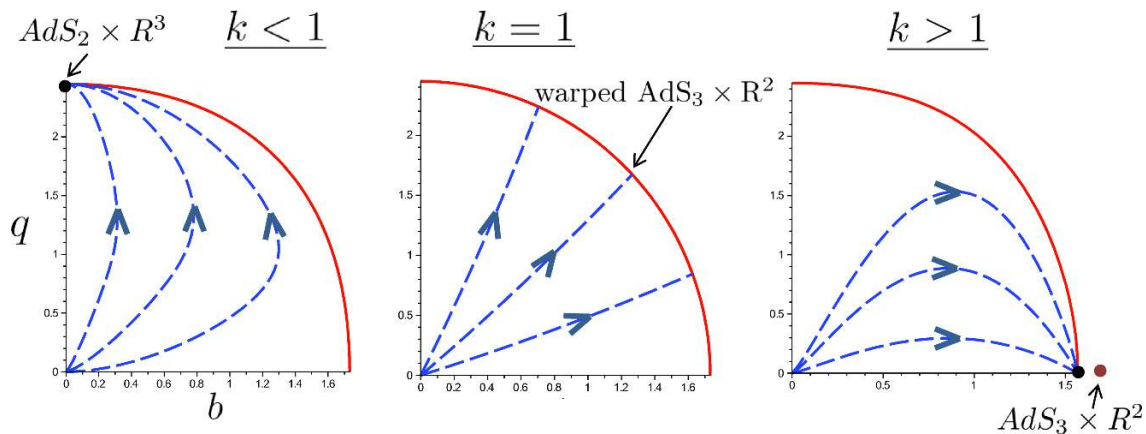


Figure 2. Schematic illustration of flows in parameter space for three ranges of k . Blue lines are flows at various fixed values of B^3/ρ^2 . Arrows indicate direction of decreasing temperature. Red lines indicate the boundary of allowed (b, q) values where nonsingular solutions are possible. The near-horizon geometries at the end points of the flows are indicated in the $k < 1$ and $k = 1$ cases. For $k > 1$ there exists an $AdS_3 \times R^2$ solution at $b = \sqrt{3}$, indicated by the dot, but the flows do not reach this point. In the $k < 1$ and $k > 1$ cases, near the endpoint of the flow B^3/ρ^2 becomes a very sensitive function of (b, q) , depending on the precise direction of approach.

brane solution should be viewed with caution, as they can be drastically affected by even a small (perhaps even arbitrarily small) magnetic field. It is an interesting question to explore the effect of introducing boundary conditions for other fields and to study their effect on extremal branes in various dimensions [29]. In this regard, it is worth noting that although the asymptotically AdS_4 extremal brane solution maintains its entropy in the presence of a magnetic field, the question remains regarding more general perturbations [29].

1.2 Summary of results

One of the main results of the present paper is that the low temperature thermodynamics of solutions carrying nonzero charge density ρ and magnetic field B depends crucially on the value of the Chern-Simons coupling k . There are three qualitatively distinct cases: $k < 1$, $k = 1$, and $k > 1$. In our conventions, the supersymmetric value is $k = 2/\sqrt{3}$, and so falls into the $k > 1$ category. At high temperatures there is no significant distinction between the three cases. However, as we take the temperature to zero, holding fixed the dimensionless ratio B^3/ρ^2 , we find markedly different behavior, as shown in figure 2.

We parameterize our solutions by b and q , which represent the values of the magnetic field and charge density at the horizon in a particular coordinate system. They differ from the physical magnetic field and charge density, which we are calling B and ρ ; the latter are measured at spatial infinity, and the relation between the two sets of parameters is determined numerically. For nonsingular solutions, b and q take values in a bounded region, which we can scan over numerically. As we lower the temperature holding B^3/ρ^2 fixed² we flow along lines in the (b, q) plane, in what are essentially renormalization group

²Holding B and ρ fixed independently is not meaningful, as they are dimensionful parameters and are

flows. The flows for the various values of k are shown schematically in figure 2. The figures representing our actual numerical data will be presented in section 7, specifically in figure 6.

The flows are driven towards three distinct endpoints, depending on the value of k .

- For $k < 1$ we are driven towards $(b = 0, q = \sqrt{6})$. This is the Reissner-Nordstrom black brane with near horizon geometry $\text{AdS}_2 \times R^3$.
- For $k = 1$ we flow towards the curve $q^2 + 2b^2 = 6$; the solutions along this curve have a near-horizon geometry corresponding to warped $\text{AdS}_3 \times R^2$. These warped AdS_3 geometries have attracted attention recently in the context of topologically massive gravity [30] and the Kerr/CFT correspondence [31]; here we find that they emerge as solutions of 4 + 1 dimensional Einstein-Maxwell theory, and can be connected to asymptotically AdS_5 spacetimes. As we move along the curve the near-horizon geometries interpolate continuously between $\text{AdS}_2 \times R^3$ and $\text{AdS}_3 \times R^2$.
- For $k > 1$, including the supersymmetric value, we flow towards $(b_c, q = 0)$, where b_c is a k dependent number that starts out at $\sqrt{3}$ for $k = 1$ and then decreases with k . To the right of this end point, at $(\sqrt{3}, 0)$, is the $\text{AdS}_3 \times R^2$ solution that was studied in [23]. At the supersymmetric value of the Chern-Simons coupling, $k = 2/\sqrt{3}$, the values of q and b are bounded by a critical curve, which to about 0.5% accuracy, is given by the relation $q^2 + \alpha b^2 = 6$ with $\alpha \approx 2.44149$ (this value of α is chosen to give high precision at the endpoint of the curve.)

As would be expected from the flow diagrams, the behavior of the entropy at low temperature depends on k . In figure 3 we show our numerical results for the supersymmetric value $k = 2/\sqrt{3}$ with B^3/ρ^2 fixed at 0 and approximately .15. We plot dimensionless versions of the entropy density and temperature, since the dimensionful versions have no intrinsic meaning.

This plot illustrates that a small value of B causes a large decrease in the entropy. This behavior is representative of the $k > 1$ case in general, and the effect seems to get more pronounced with increasing k . Our numerics break down at low temperatures due to our choice of gauge fixing, and we have stopped our numerics in a regime where the results are still reliable. Extrapolating further, it is possible that a singularity or instability arises at some finite temperature. However, what cannot happen is that we end up at a smooth finite entropy extremal black hole, as we will show that no such solution exists at nonzero B (within our Ansatz, which assumes such properties as translation invariance).

In the $k < 1$ and $k = 1$ cases the situation is dramatically different, as the entropy appears to go to a finite value (which depends on the value of B^3/ρ^2), as shown for $k = 0$ in figure 4. In the $k = 1$ case the thermodynamics is governed by spacelike warped AdS_3 black hole solutions, which represent exact near-horizon geometries for our theory. At strictly zero temperature, for both $k < 1$ and $k = 1$, the full interpolating solutions acquire a relatively mild singularity at the horizon, unless $B = 0$ in which case we recover the Reissner-Nordstrom solution. These singularities can be understood from a perturbative

thus changed by a scale transformation.

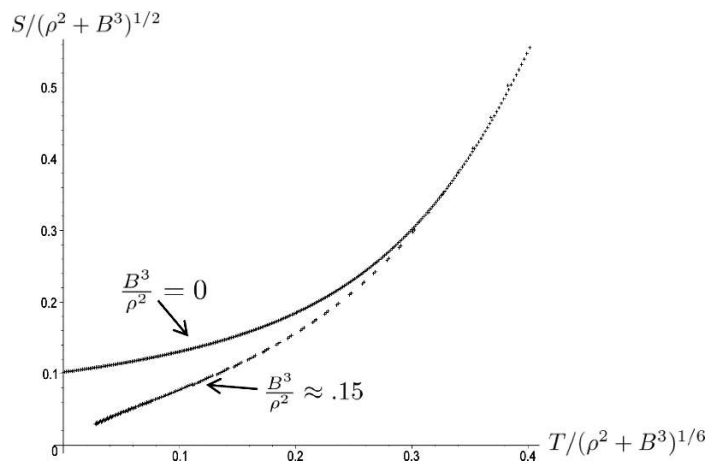


Figure 3. Plot of the entropy versus temperature at fixed $B^3/\rho^2 = 0$ and $B^3/\rho^2 = .15 \pm .002$, for $k = 2/\sqrt{3}$ (supersymmetric value). The numerical results show that a small B field causes a large drop in the entropy at low temperature.

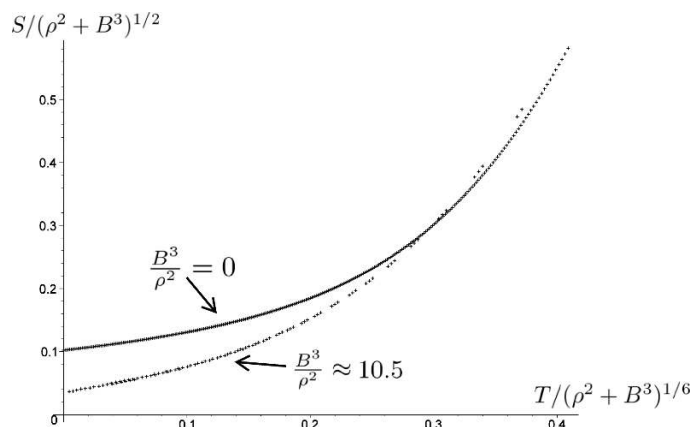


Figure 4. Plot of the entropy versus temperature at fixed $B^3/\rho^2 = 0$ and $B^3/\rho^2 = 10.5 \pm .6$, for $k = 0$. The entropy appears to go to a finite value (but see the text for comments on the strict zero temperature limit).

analysis. Thus, strictly speaking, in these cases there is no smooth extremal finite entropy geometry, just as there was not in the $k > 1$ case. However, it may be more physically relevant to focus on the behavior for small but nonzero temperature, in which case a residual entropy is evident.

Besides our numerical results, which are valid for arbitrary B and ρ , we have carried out a perturbative analysis of the solutions valid to order B^2 . These solutions can be obtained analytically by methods analogous to those employed in the AdS/fluid dynamics literature, starting with [32]. Insofar as they overlap, the perturbative results corroborate our numerical findings.

The organization of the remainder of this paper is as follows. In section 2, we briefly review the $D = 5$ Einstein/Maxwell theory with Chern-Simons term, including the definition of the boundary current and stress tensor for asymptotically AdS₅ solutions. In section 3

we present the Ansatz for uniform electric charge density and constant magnetic field, and derive the reduced field equations. In section 4, regularity and boundary conditions are discussed both at the horizon and in the asymptotic AdS_5 region. Standard AdS/CFT formalism is used to express physical quantities such as the entropy, chemical potential, currents and stress tensor in terms of these asymptotic data. In section 5, the near-horizon geometries of our solutions are constructed analytically, and we discuss the existence of solutions that interpolate between these and AdS_5 . In section 6, a perturbative expansion in powers of B is shown to be smooth, except for extremal solutions. In section 7, numerical results are presented, and a picture of the phase diagram is assembled. In section 8 we conclude with a discussion. A number of appendices include details of computations omitted in the main text, a general discussion on the existence of factorized solutions, and the construction of solutions with a near-horizon extremal BTZ factor.

2 Action, current, and stress tensor

The action for 5-dimensional Einstein-Maxwell theory with negative cosmological constant, and Chern-Simons term, is given by³

$$S_{EM} = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left(R + F^{MN} F_{MN} - \frac{12}{L^2} \right) + S_{CS} + S_{\text{bdy}} \quad (2.1)$$

where the Chern-Simons action is given by,⁴

$$S_{CS} = \frac{k}{12\pi G_5} \int A \wedge F \wedge F \quad (2.2)$$

For the value $k = k_s = 2/\sqrt{3}$, the action coincides with the bosonic part of $D = 5$ minimal gauged supergravity. In this paper, however, k will often be kept general, thus allowing for values different from k_s as well. Boundary terms in the action are required for the proper renormalization of various physical quantities [33–35]. In a coordinate system (r, x^μ) where $g_{r\mu} = 0$ asymptotically for $\mu = 0, 1, 2, 3$, the boundary action S_{bdy} is given by,

$$S_{\text{bdy}} = \frac{1}{8\pi G_5} \int_{\partial M} d^4x \sqrt{-\gamma} \left(K - \frac{3}{L} + \frac{L}{4} R(\gamma) + \frac{L}{2} \left(\ln \frac{r}{L} \right) F^{\mu\nu} F_{\mu\nu} \right) \quad (2.3)$$

Here, $\gamma_{\mu\nu}$ is the metric induced by g_{MN} on the boundary, and K is the trace (with respect to γ) of the extrinsic curvature of the boundary given by $K_{\mu\nu} = (\partial_r \gamma_{\mu\nu}) / (2\sqrt{g_{rr}})$. Henceforth we set the AdS radius to unity: $L = 1$. The non-diffeomorphism invariant $\ln r$ term in the boundary action is needed to remove the divergence associated with the trace anomaly $T_\mu^\mu \sim F^{\mu\nu} F_{\mu\nu}$.

The Bianchi identity is $dF = 0$, while the field equations are given by,

$$\begin{aligned} 0 &= d * F + k F \wedge F \\ R_{MN} &= 4g_{MN} + \frac{1}{3} F^{PQ} F_{PQ} g_{MN} - 2F_{MP} F_N^P \end{aligned} \quad (2.4)$$

³Conventions: $R^\lambda_{\mu\nu\kappa} = \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\nu\eta}$ and $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$.

⁴Note that our convention for k differs from that in [23]: $k_{\text{here}} = \frac{3}{4} k_{\text{there}}$. The present convention has been chosen to simplify the Maxwell equation. By sign-reversal of A , we are free to choose $k \geq 0$.

2.1 Boundary current and stress tensor

For large r the boundary metric of asymptotically AdS₅ solutions will behave as

$$\gamma_{\mu\nu} = r^2 \gamma_{\mu\nu}^{(0)} + \dots \quad (2.5)$$

where $\gamma_{\mu\nu}^{(0)}$ is the conformal boundary metric, given here by the flat Minkowski metric. Similarly, the components of the gauge field A_μ tangent to the boundary will go to a constant at large r , representing a constant magnetic field pointing in the x_3 direction.

By considering on-shell variations of the boundary metric and gauge field, we can define a boundary current and stress tensor in the familiar fashion:

$$\delta S = \int d^4x \sqrt{-\gamma^{(0)}} \left(J^\mu \delta A_\mu + \frac{1}{2} T^{\mu\nu} \delta \gamma_{\mu\nu}^{(0)} \right) \quad (2.6)$$

Specializing to the case of a constant field strength on a flat boundary metric, we have

$$\begin{aligned} -4\pi G_5 J^\mu &= \left(r^3 \gamma_{(0)}^{\mu\nu} F_{r\nu} + \frac{k}{3} \epsilon^{\alpha\beta\gamma\mu} A_\alpha F_{\beta\gamma} \right) \\ 8\pi G_5 T^{\mu\nu} &= r^6 \left(-K^{\mu\nu} + K \gamma^{\mu\nu} - 3\gamma^{\mu\nu} - 2 \left(F^{\mu\alpha} F_\alpha^\nu - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \gamma^{\mu\nu} \right) \ln r \right) \end{aligned} \quad (2.7)$$

where the limit of large r is implied. For our solutions, the explicit $\ln r$ terms will cancel logarithmic terms in the metric functions, yielding a finite large r limit for $T^{\mu\nu}$.

3 Ansatz and reduced field equations

The presence of uniform magnetic field and electric charge density in the boundary CFT may be achieved by an Ansatz for the bulk fields which is invariant under translations in x^μ , and space-rotations around the magnetic field, which we shall take to be pointing in the x_3 direction. The Ansatz consistent with these symmetries is given by⁵

$$F = E(r) dr \wedge dt + B dx_1 \wedge dx_2 + P(r) dx_3 \wedge dr \quad (3.1)$$

for the Maxwell field strength, and by

$$ds^2 = \frac{dr^2}{U(r)} - U(r) dt^2 + e^{2V(r)} (dx_1^2 + dx_2^2) + e^{2W(r)} (dx_3 + C(r) dt)^2 \quad (3.2)$$

for the metric. The magnetic field B is forced to be constant by the Bianchi identity, and the functions E, P, U, V, W , and C depend only on r . Reparametrization invariance in r has been used to select a coordinate r for which the same function $U(r)$ appears in both g_{rr} and the first factor of dt^2 . Rescaling x_1, x_2 can be compensated by a constant shift of V , while rescaling x_3 can be compensated by scaling C and shifting W .

⁵An additional term of the form $N(r) dx^3 \wedge dt$ must have constant N in view of the Bianchi identities, and $N = 0$ in view of the field equations, and has thus been omitted.

In this coordinate system the event horizon is located at $U(r_+) = 0$, and the Hawking temperature is given by

$$T = \frac{U'(r_+)}{4\pi} \quad (3.3)$$

The above Ansatz is covariant under the following transformation,

$$\begin{aligned} x_3 &\rightarrow x_3 - \alpha t \\ C(r) &\rightarrow C(r) + \alpha \\ E(r) &\rightarrow E(r) - \alpha P(r) \end{aligned} \quad (3.4)$$

with all other coordinates and fields left unchanged, and for any real parameter α . We shall refer to this transformation as α -*symmetry*. It may be used, for example, to set $C(\infty)$ to zero. Note that the combinations $E + CP$ and C' are invariant under α -symmetry.

The Ansatz (3.1), (3.2) is also covariant under boosts in the direction of the magnetic field. We can use these boosts to put the solution in the rest frame, defined by $C(r_+) = 0$. In the dual CFT this corresponds to setting to zero the chemical potential conjugate to momentum. See appendix A for the details.

3.1 The reduced Maxwell equations

The Bianchi identity is automatic on the Ansatz (3.1). The reduced Maxwell equations are,

$$\begin{aligned} ((E + CP)e^{2V+W})' + 2kBP &= 0 \\ (UPe^{2V-W} - C(E + CP)e^{2V+W})' + 2kBE &= 0 \end{aligned} \quad (3.5)$$

Both equations may be recast in terms of the α -symmetry invariant combinations C' and

$$\mathcal{E} \equiv E + PC \quad (3.6)$$

only, by eliminating the derivative of $(E + CP)e^{2V+W}$ in the second equation using the first equation. One then obtains an alternative form of the reduced Maxwell equations,

$$\begin{aligned} M1 \quad & (\mathcal{E}e^{2V+W})' + 2kBP = 0 \\ M2 \quad & (UPe^{2V-W})' - C'\mathcal{E}e^{2V+W} + 2kBE = 0 \end{aligned} \quad (3.7)$$

3.2 The reduced Einstein equations

The reduced Einstein equations may be simplified to assume the following final form,

$$\begin{aligned} E1 \quad & (C'e^{2V+3W})' = 4P\mathcal{E}e^{2V+W} \\ E2 \quad & U(V'' - W'') + (U' + U(2V' + W'))(V' - W') \\ & = \frac{1}{2}(C')^2e^{2W} - 2B^2e^{-4V} + 2UP^2e^{-2W} \\ E3 \quad & UV'' + U'V' + UV'(2V' + W') = 4 - \frac{2}{3}\mathcal{E}^2 - \frac{4}{3}B^2e^{-4V} + \frac{2}{3}UP^2e^{-2W} \end{aligned} \quad (3.8)$$

$$E4 \quad U'' + U'(2V' + W') - (C')^2 e^{2W} = 8 + \frac{8}{3}\mathcal{E}^2 + \frac{4}{3}B^2 e^{-4V} + \frac{4}{3}UP^2 e^{-2W}$$

along with the constraint equation,

$$\begin{aligned} CON \quad U'(2V' + W') + 2U(V')^2 + 4UV'W' + \frac{1}{2}(C')^2 e^{2W} \\ = 12 - 2\mathcal{E}^2 - 2B^2 e^{-4V} + 2UP^2 e^{-2W} \end{aligned} \quad (3.9)$$

The r -derivative of the constraint vanishes by the use of the other six equations, and may be enforced as an initial condition, as usual.

4 Asymptotics and initial data

The solutions we consider are asymptotically AdS_5 . Thus, $U(r)$, $e^{2V(r)}$ and $e^{2W(r)}$ behave as r^2 in the limit $r \rightarrow \infty$. The precise overall normalization depends on the normalization of the space coordinates x_1, x_2 (for V) and x_3 (for W). There are two natural ways of normalizing this behavior, by parametrizing either the initial data at the horizon, or the asymptotic behavior as $r \rightarrow \infty$. We shall consider both, and relate their behaviors. The data at the horizon is used for the numerical analysis, and for computing the entropy and temperature. The asymptotics are used to calculate the Maxwell current and the stress tensor.

4.1 Parametrizing the initial data at the horizon

In the numerical analysis it is important to choose coordinates to remove the gauge freedom. This can be done by demanding that the solution take a canonical form at the horizon. By rescaling of x_1, x_2, x_3 , and combining an α -transformation and a boost in the x_3 -direction, the field strength F_H and the metric ds_H^2 at the horizon may be arranged to take the form,

$$\begin{aligned} F_H &= q dr \wedge dt + b dx_1 \wedge dx_2 \\ ds_H^2 &= dx_1^2 + dx_2^2 + dx_3^2 \end{aligned} \quad (4.1)$$

where q and b are respectively the charge density and the magnetic field at the horizon (in the coordinates x_1, x_2, x_3). This corresponds to the following initial conditions at the horizon,

$$\mathcal{E}(r_+) = q \quad U(r_+) = V(r_+) = W(r_+) = C(r_+) = P(r_+) = 0 \quad (4.2)$$

We will refer to these coordinates as the *horizon frame*. It remains to specify $V'(r_+)$, $W'(r_+)$, and $C'(r_+)$. These quantities are generally not independent, but rather follow from the reduced equations M2, E2, and CON evaluated at the horizon, and we have,

$$\begin{aligned} q(C'(r_+) - 2kb) &= 0 \\ U'(r_+)V'(r_+) &= 4 - \frac{2}{3}q^2 - \frac{4}{3}b^2 \\ U'(r_+)W'(r_+) &= 4 - \frac{2}{3}(q^2 - b^2) - \frac{1}{2}C'(r_+)^2 \end{aligned} \quad (4.3)$$

The value of $C'(r_+)$ is specified to be $C'(r_+) = 2kb$ for $q \neq 0$, but remains an independent free parameter for $q = 0$. The quantity $U'(r_+) = 4\pi T$ is not a genuine initial datum, since the equation CON for U is of first order. Therefore, genuinely distinct solutions are specified by only two parameters, for example T and q in units of magnetic field b .

If the temperature T is nonzero, we can always rescale t to set $U'(r_+) = 1$, leaving the free parameters b and q . Furthermore, we can shift r to set the horizon at $r_+ = 1$.

4.2 Extremal solutions require $bq(k \pm 1) = 0$

It is now easy to establish a non-existence result that will play an important role in what follows. We ask under what conditions can we have an extremal horizon, $U(r_+) = U'(r_+) = 0$.

Assuming that all functions in our Ansatz are well behaved at r_+ , we can always work in the horizon frame specified in the previous subsection, in which case the conditions (4.3) apply. But then it is easy to see that the assumption of an extremal horizon with nonzero b and q is inconsistent with (4.3) unless $k^2 = 1$. To obtain an extremal horizon we must choose one (or more) of $q = 0$, $b = 0$, or $k = \pm 1$. As will be discussed in section 5, these three choices lead to near-horizon geometries of the form $\text{AdS}_2 \times R^3$, $\text{AdS}_3 \times R^2$, and warped $\text{AdS}_3 \times R^2$. But in the generic case in which none of these conditions is satisfied, finite temperature solutions cannot be smoothly brought to zero temperature, a feature that we will see explicitly from various points of view.

4.3 Asymptotic behavior as $r \rightarrow \infty$

Starting from the initial data (4.2) and integrating out to large r , we will find asymptotically AdS_5 solutions with large r behavior

$$\begin{aligned}
 U &= (r - r_0)^2 + \frac{u_2}{r^2} + u'_2 \frac{\ln r}{r^2} + \dots & C &= c_0 + \frac{c_4}{r^4} + \dots \\
 e^{2V} &= v(r - r_0)^2 + \frac{v_2}{r^2} + v'_2 \frac{\ln r}{r^2} + \dots & \mathcal{E} &= \frac{e_3}{r^3} + \dots \\
 e^{2W} &= w(r - r_0)^2 + \frac{w_2}{r^2} + w'_2 \frac{\ln r}{r^2} + \dots & P &= \frac{p_3}{r^3} + \dots
 \end{aligned} \tag{4.4}$$

where the dots stand for higher order terms in $1/r$. Some of the parameters are related to one another by the field equations,

$$\begin{aligned}
 w_2 &= -\frac{2wv_2}{v} & u'_2 &= -\frac{2b^2}{3v^2} \\
 v'_2 &= \frac{b^2}{3v} & w'_2 &= -\frac{2b^2w}{3v^2}
 \end{aligned} \tag{4.5}$$

In these coordinates the conformal boundary metric is $-dt^2 + v(dx_1^2 + dx_2^2) + w(dx_3 + c_0 dt)^2$. As we show in appendix B, we can always perform a coordinate transformation to set $v = w = 1$ and $c_0 = 0$, which brings the conformal boundary metric to the standard Minkowski form. Further, this can be done while preserving the condition that we be in the rest frame, defined by $C(r_+) = 0$. The components of the current in this frame, which

we refer to as the *asymptotic frame* since it is the relevant one for comparing with the boundary CFT, are computed to be

$$\begin{aligned}
 4\pi G_5 J^t \equiv \rho &= \gamma_c(e_3 - c_0 p_3) - \frac{2kb}{3v} A_3|_\infty \\
 J^{1,2} &= 0 \\
 4\pi G_5 J^3 &= \frac{3}{4} \gamma_c \left(\frac{p_3}{\sqrt{w}} - \sqrt{w} c_0 e_3 \right)
 \end{aligned}
 \tag{4.6}$$

Here B is the value of the magnetic field in the asymptotic frame, thus identified as the magnetic field in the CFT, and given by

$$B = \frac{b}{v} \tag{4.7}$$

γ_c is a Lorentz boost factor (we use the notation γ_c to avoid confusion with the boundary metric γ), appearing when we transform to the rest frame,

$$\gamma_c = \frac{1}{\sqrt{1 - w c_0^2}} \tag{4.8}$$

For black hole solutions in which g_{33} remains finite at the horizon, $A_3|_\infty$ is arbitrary, and can be adjusted by a constant shift of A_3 throughout spacetime. Its appearance in the expression for J^t is a consequence of the anomaly equation for the boundary current.⁶ A similar factor of $A_t|_\infty$ appears implicitly in J^3 , but since we have fixed $A_t = 0$ at the horizon, the asymptotic value of A_t is determined without ambiguity. A nonzero value of $A_3|_\infty$ corresponds in the CFT to adding a chemical potential for J^3 ; it is simplest to set it to zero, and we do that henceforth unless stated otherwise.

We similarly have expressions for the temperature, entropy density, and chemical potential in the asymptotic frame:

$$\begin{aligned}
 T &= \frac{\gamma_c U'(r_+)}{4\pi} \\
 G_5 \left(\frac{S}{\text{Vol}} \right) \equiv s &= \frac{1}{4\sqrt{v^2 w \gamma_c^2}} \\
 \mu &= \frac{3\gamma_c v}{8kb} \left(\sqrt{w} c_0 e_3 - \frac{p_3}{\sqrt{w}} \right)
 \end{aligned}
 \tag{4.9}$$

Comparing (4.6) and (4.9) we note the simple relation

$$4\pi G_5 J^3 = -\frac{1}{2} k B \mu \tag{4.10}$$

This is the chiral magnetic current [13–15]: in the presence of a nonzero anomaly coefficient k and a chemical potential, a current induced parallel to the applied magnetic field. This result also follows from the anomaly equation, as can be seen by allowing for a slow variation in x_3 , differentiating both sides, and identifying $\vec{\nabla} \mu$ with an electric field.

⁶Given $A_3(t)$ and a constant magnetic field along x^3 we have $\partial_t J^t \sim k E \cdot B \sim k B \partial_t A_3$. Integrating gives $J^t \sim k B A_3$, in accord with (2.7).

4.3.1 Physical quantities

The global AdS_5 solution is invariant under scale transformations,

$$x'^{\mu} = x^{\mu}/\ell, \quad r' = \ell r \tag{4.11}$$

Asymptotically AdS_5 solutions inherit this as an asymptotic symmetry, reflecting the CFT nature of the holographic dual theory. Individual quantities, such as B and ρ , transform under these scalings, just as the coordinates x^{μ} do, and so have no independent meaning. We should instead look at scale invariant quantities, which have physical meaning. This is the same as looking at dimensionless quantities from the boundary point of view. Let's write $O \sim \ell^p$ to denote the transformation $O' = \ell^p O$. From the asymptotic invariance of the field strength F and the metric ds^2 , we find,

$$s \sim \ell^3, \quad T \sim \ell, \quad B \sim \ell^2, \quad \rho \sim \ell^3, \quad \mu \sim \ell \tag{4.12}$$

Any combination behaving as ℓ^0 is a good physical quantity to compute.

5 Near-horizon geometries

For generic assignments of the physical parameters ρ and B , analytical solutions are not available in AdS_5 (in contrast with the AdS_4 case where an electric-magnetic duality rotation acting on the $B = 0$ solution produces a simple dyonic solution). Even the special case of $\rho = 0$ at zero temperature (and $B \neq 0$) does not lend itself to a full analytical solution [23].

Considerable qualitative and quantitative progress can be made, however, by solving for the near-horizon geometry of the solutions. This will be carried out in this section. Especially important will be the question as to whether, for given ρ, B , solutions with *extremal* near-horizon geometry exist, and whether they can support an electric field at the horizon. The existence of these extremal solutions is key to understanding the low temperature limit. One important result, which was already established in section 4.2 will be that for nonzero ρ and B , and for $k \neq 1$ there do not exist smooth, finite entropy, extremal solutions.

In the low temperature regime, the full solutions may then be viewed as interpolations between asymptotic AdS_5 and these near-horizon geometries. By numerical study, to be discussed in full in section 7, the regularity of this interpolation will be verified (except at strictly zero temperature, where singularities develop), and the physical properties of the solution, such as entropy, temperature, and mass will be evaluated.

A general discussion on the existence of factorized solutions may be found in appendix C.

5.1 General conditions for the existence of extremal solutions

At an extremal horizon r_+ we have $U(r_+) = U'(r_+) = 0$. Extremal solutions provide a natural boundary of the parameter space of all solutions.

To study their existence systematically, it will be convenient to adopt the horizon frame specified in section 4.1. We scale the coordinates x_i so that $V(r_+) = W(r_+) = 0$, and

denote the magnetic field in these coordinates by b . Reduced Einstein/Maxwell equations M2, E2, E3, and CON, in which only U and U' enter on the left hand side, produce a set of non-trivial constraints,

$$\begin{aligned}
 M2 & & q(C'(r_+) - 2kb) &= 0 \\
 E2 & & C'(r_+)^2 - 4b^2 &= 0 \\
 E3 & & 6 - q^2 - 2b^2 &= 0
 \end{aligned} \tag{5.1}$$

where q is defined to be the electric field at the horizon $q = \mathcal{E}(r_+)$. The constraint equation CON is a consequence of E2 and E3, and thus has been omitted from the above list. Eliminating $C'(r_+)$ using the second equation reduces the system to $qb(k \pm 1) = 0$ and $q^2 + 2b^2 = 6$. The solutions are as follows,

1. If $k \neq \pm 1$, then we have $qb = 0$, so that either $q = 0$ or $b = 0$, leading to the solutions,
 - (a) The case $b = 0$ and $q = \pm\sqrt{6}$, corresponds to the well-known extremal electrically charged black brane (without magnetic field, and arbitrary value of k). Its near-horizon geometry is $AdS_2 \times R^3$.
 - (b) The case $q = 0$ and $b = \pm\sqrt{3}$, corresponds to the extremal purely magnetic brane (without electric charge, and arbitrary value of k), obtained in [23]. Its near-horizon geometry is $AdS_3 \times R^2$.
2. If $k = \pm 1$, we shall show below that there is in fact a regular solution for every assignment satisfying $q^2 + 2b^2 = 6$, whose near-horizon geometry smoothly interpolates between $AdS_2 \times R^3$ (at $b = 0$) and $AdS_3 \times R^2$ (at $q = 0$). These solutions can be generalized to include finite temperature and momentum, and correspond precisely to one family of warped black holes considered in [30].

5.2 Vanishing magnetic field: $AdS_2 \times R^3$

We begin by briefly reviewing the well-known black brane solution in AdS_5 for $B = 0$, charge density ρ and mass $M > 0$ (the actual charge and mass densities are proportional to these), given by the functions $P = C = 0$, and

$$\mathcal{E} = \frac{\rho}{r^3} \quad V = W = \ln r \quad U = r^2 + \frac{\rho^2}{3r^4} - \frac{M}{r^2} \tag{5.2}$$

In terms of the radii $r_- \leq r_+$ of the inner and outer horizons, we obtain a convenient parametrization of ρ, M and U ,

$$\begin{aligned}
 \rho^2 &= 3r_+^2 r_-^2 (r_+^2 + r_-^2) \\
 M &= r_+^4 + r_-^4 + r_+^2 r_-^2 \\
 U &= \frac{1}{r^4} (r^2 - r_+^2)(r^2 - r_-^2)(r^2 + r_+^2 + r_-^2)
 \end{aligned} \tag{5.3}$$

As long as $M^3/\rho^4 \geq 3/4$, the singularity at $r = 0$ is protected by a horizon. The near-horizon metric reduces to

$$ds_H^2 = \frac{dr^2}{U_H(r)} - U_H(r) dt^2 + r_+^2 (dx_1^2 + dx_2^2 + dx_3^2) \tag{5.4}$$

with $U_H(r) \sim (r^2 - r_+^2)$ for the non-extremal case, and $U_H(r) \sim (r^2 - r_+^2)^2$ for the extremal case. The near-horizon geometry is factorized into the space-part which is flat, and an AdS_2 factor. The temperature T and entropy density s of the black brane are given by

$$T = \frac{(r_+^2 - r_-^2)(2r_+^2 + r_-^2)}{2\pi r_+^3} \quad s = \frac{r_+^3}{4} \quad (5.5)$$

The black brane becomes extremal as $r_- \rightarrow r_+$, so that the temperature goes to zero, but the charge density and entropy density remain finite, and related by $s/\rho = 1/(4\sqrt{6})$.

5.3 Vanishing charge density: $AdS_3 \times R^2$

With vanishing charge density, the Maxwell field strength F reduces to the B -term only. The solutions in this case were obtained in [23]. The Maxwell-Einstein equations have an analytical solution, given by $\mathcal{E} = P = C = 0$, and

$$U = 3(r^2 - r_+^2) \quad e^{2V} = \frac{B}{\sqrt{3}} \quad e^{2W} = r^2 \quad (5.6)$$

which represents the product of a (non-rotating) BTZ black hole with R^2 . It was confirmed numerically in [23] that there exists a family of regular solutions, parametrized by T/\sqrt{B} , which interpolate between the BTZ black hole of (5.6) at the horizon, and AdS_5 at $r = \infty$. The entropy of these solutions tends to zero as $T \rightarrow 0$, while the physical magnetic field B is kept fixed.

More generally, we can have nonextremal rotating BTZ black holes, whose metric is given by $e^{2V} = B/\sqrt{3}$, and

$$U = 3 \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} \quad e^{2W} = r^2 \quad C = -\sqrt{3} \frac{r_+ r_-}{r^2} \quad (5.7)$$

with $\mathcal{E} = P = 0$. A useful alternative parametrization of the rotating BTZ solution is given by,

$$U = 12(r - r_+)(r - r_-) \quad C = 2\sqrt{3}(r - r_+) \quad (5.8)$$

and $V = W = \mathcal{E} = P = 0$. Note that both forms admit a smooth extremal limit.

5.4 $k = 1$: warped AdS_3 black holes

As shown above, there is a special value of the Chern-Simons coupling, namely $k = \pm 1$ (recall that this is less than the value required for supersymmetry), for which there exist extremal solutions for any q, b satisfying the relation,

$$q^2 + 2b^2 = 6 \quad (5.9)$$

Furthermore, there is a simple nonextremal generalization. The solution is given by $V = W = P = 0$, and

$$\mathcal{E} = q \quad (5.10)$$

$$\begin{aligned}
 U &= 12(r - r_+)(r - r_-) \\
 C &= 2b(r - r_+)
 \end{aligned}
 \tag{5.11}$$

We have used α -symmetry to set $C(r_+) = 0$. The metric and field strength are then given by,

$$\begin{aligned}
 ds^2 &= \frac{dr^2}{U(r)} - U(r)dt^2 + \left(dx_3 + 2b(r - r_+)dt\right)^2 + dx_1^2 + dx_2^2 \\
 F &= q dr \wedge dt + b dx_1 \wedge dx_2
 \end{aligned}
 \tag{5.12}$$

The extremal limit is given by taking $r_- = r_+$ as usual.

These solutions can be identified with the “self-dual” solutions described in section 6.1.1 of [30], where we make the identification

$$\nu^2 = \frac{3b^2}{12 - b^2}
 \tag{5.13}$$

Note that as b^2 ranges over its allowed values between 0 and 3, ν^2 ranges between 0 and 1. The equivalence can be seen most easily by comparing our metric with eqtn. 1.2 of [36]. Under the identifications

$$\Phi = \frac{12l^2}{3 + \nu^2}t, \quad T = \sqrt{\frac{12l^2}{3 + \nu^2}}x_3, \quad R = r
 \tag{5.14}$$

the metrics are seen to be proportional. Unlike in [36], we do not compactify $\Phi \sim t$. Our metric has no closed timelike curves or other pathologies.

These solutions also arise in a context closely related to ours, namely 2 + 1 dimensional Einstein-Maxwell-Chern-Simons theory (see [37] for solutions which are the analytic continuation of these). This can be understood from the fact that if we reduce our theory down to three dimensions along $x_{1,2}$, then we recover the equations of 2 + 1 dimensional Einstein-Maxwell-Chern-Simons theory coupled to a massless scalar field. The condition for the scalar field to take a constant value, representing a constant value of V , is precisely the condition $q^2 + 2b^2 = 6$ that we found above. It is curious that these solutions exist only at the special value $k = \pm 1$.

5.5 Existence of interpolating solutions

As we have seen, for nonzero values of ρ and B , and $k \neq \pm 1$, there do not exist smooth zero temperature solutions under the assumptions of our Ansatz (it is possible that such solutions do exist if one, for example, relaxes the condition of translation invariance). So in these cases, if we start from an asymptotically AdS₅ solution at finite temperature, as we lower the temperature some of the functions in our solution will start to diverge; we will see this as a breakdown of our numerics.

This leaves the question of what happens in the zero temperature limit in the cases for which there do exist candidate extremal horizons. In the case of $B = 0$ and nonzero ρ (the value of k is immaterial in this case), the answer is that we end up at the usual AdS₅

extremal Reissner-Nordstrom solution. This will turn out to be the only case in which we find a truly nonsingular extremal solution.

For $\rho = 0$ and nonzero B (k again drops out of the discussion if one notes that for $q = 0$ the equation M2 appearing in (5.1) becomes trivial) interpolating solutions were constructed numerically in [23]. At low temperatures the near-horizon geometry approaches $\text{AdS}_3 \times R^2$, but as will be discussed momentarily a singularity develops in the full interpolating solution at strictly zero temperature.

For nonzero ρ and B , but $k = \pm 1$, we have candidate near-horizon geometries corresponding to warped AdS_3 black holes times R^2 , and these have a smooth extremal limit. At any nonzero temperature, our numerics will establish the existence of solutions smoothly interpolating between these near-horizon geometries and AdS_5 . As the temperature is taken to zero the entropy remains finite, but nevertheless a singularity develops at the horizon, for reasons that can be seen as follows.

To construct a candidate extremal interpolating solution, we can start with the exact near horizon extremal warped black hole geometry, and then introduce a perturbation that grows near the boundary, representing the change in asymptotic boundary conditions taking us towards AdS_5 . This perturbation is obtained by solving the equations obtained by linearizing around the near-horizon solution.

For our near-horizon solution we have

$$\begin{aligned}
 U &= 12(r-1)^2 & (5.15) \\
 C &= 2b(r-1) \\
 \mathcal{E} &= q \\
 V &= W = P = 0
 \end{aligned}$$

with $q^2 + 2b^2 = 6$. Note that we have used the freedom to rescale r to set the horizon at $r_+ = 1$.

Now we linearize around this solution, denoting the perturbations by lower case letters (ϵ denotes the perturbation of \mathcal{E}). Plugging in we find the following equations to linear order

$$\begin{aligned}
 M1 : \quad & \delta\epsilon' + q(2v' + w') + 2bp = 0 & (5.16) \\
 M2 : \quad & (12(r-1)^2 p)' - qc' - 2qb(2v + w) = 0 \\
 E1 : \quad & c'' + 2b(3w' + 2v') - 4qp = 0 \\
 E2 : \quad & (12(r-1)^2(v' - w'))' - 8b^2v - 2bc' - 4b^2w = 0 \\
 E3 : \quad & (12(r-1)^2 v')' - \frac{16}{3}b^2v + \frac{4}{3}q\epsilon = 0 \\
 E4 : \quad & 3u'' + 72(r-1)(2v' - w') + 8b^2(2v - 3w) - 12bc' - 16q\epsilon = 0 \\
 CON : \quad & 24(r-1)(2v' + w') + 4b^2(w - 2v) + 2bc' + 4q\epsilon = 0
 \end{aligned}$$

We impose the boundary conditions

$$u(1) = u'(1) = v(1) = w(1) = c(1) = c'(1) = p(1) = \epsilon(1) = 0 \quad (5.17)$$

It is fairly straightforward to solve these equations iteratively. Of most relevance are the resulting expressions for v , ϵ and p , which are

$$\begin{aligned} v &= a_1(r-1)^\alpha & (5.18) \\ \epsilon &= 3qa_1(r-1)^\alpha \\ p &= -\frac{3q\alpha a_1}{2b}(r-1)^{\alpha-1} - \frac{q}{2b}a_2 \end{aligned}$$

where $a_{1,2}$ are integration constants, and

$$\alpha = -\frac{1}{2} + \frac{\sqrt{81-8b^2}}{6} \tag{5.19}$$

If $q^2 \neq 6$ then $b^2 > 0$, in which case $\alpha < 1$, and then we see that p diverges at the horizon. Thus to have a smooth solution we are forced to set $a_1 = 0$. But this means that $v = \epsilon = 0$, and from here it follows rapidly that the entire solution is just the original near-horizon geometry we started from.

That the divergence in p represents a physical singularity can be seen by considering an infalling observer. One finds that such an observer sees a diverging physical field strength at the horizon.

We therefore do not expect to find a smooth extremal interpolating solution when $q \neq 0$, even for $k = \pm 1$ when smooth near-horizon geometries do exist. In our numerics, as we lower the temperature we indeed find that P begins to diverge at the horizon in precisely the manner described above. Nevertheless, since the metric components have well defined limits (though not their derivatives) we find that the entropy appears to smoothly approach a finite value.

This leaves the case of nonzero B but $\rho = 0$, which was studied in [23]. There solutions were found numerically that interpolated between near-horizon $BTZ \times R^2$ and AdS_5 . At any finite temperature these interpolating solutions are smooth, but it follows from the above that a mild singularity develops at strictly zero temperature. For these solutions k does not appear in the field equations, and so we can freely set $k = \pm 1$, in which case we can compare with the linearized analysis just described by setting $q = 0$. For $q = 0$ we have that $p = 0$, and so we avoid the divergence in that quantity. However, we have that $v \sim (r-1)^\alpha$, $\alpha = -1/2 + \sqrt{57}/6$. Since $\alpha < 1$, the first derivative of V will diverge at the horizon, presumably indicating a singularity.

Actually, one special case remains. When $\rho = 0$, rather than considering the zero temperature limit of BTZ corresponding to pure AdS_3 , we can look for solutions with near-horizon geometry given by a finite entropy extremal BTZ solution. These can be constructed by a fairly standard construction, as we describe in appendix D. The resulting interpolating solutions exhibit the same singularity at extremality as above; namely, derivatives of V diverge at the horizon.

6 Perturbation theory in powers of B

In this section, we shall construct solutions perturbatively in powers of B around the analytically known solution for $B = 0$ and arbitrary charge density ρ and mass M . We

expect this expansion to be reliable for $T^2 \gg B$. For small T , however, the $B = 0$ solution is close to extremal, and we know from the conditions of extremality of (5.1) that no extremal solutions exist with $B \neq 0$ and $\rho \neq 0$, unless we also have $k = \pm 1$. Thus, for $k \neq \pm 1$, we expect perturbation theory in B to break down near $B \sim T^2$. For $k = \pm 1$ the behavior is better, but we find that a mild singularity still results. The structure of the computation is very similar to the long wavelength fluid dynamics from gravity analysis in [32]. This is because a weak magnetic field corresponds to a slowly varying gauge field.

The functions $E(r), U(r), V(r), W(r)$ are even in B , while the functions $P(r)$ and $C(r)$ are odd in B . Here, we shall expand up to order B^2 included, so that,

$$\begin{aligned} U &= U_0 + B^2 U_2 & E &= E_0 + B^2 E_2 \\ V &= V_0 + B^2 V_2 & C &= B C_1 \\ W &= W_0 + B^2 W_2 & P &= B P_1 \end{aligned} \tag{6.1}$$

The zeroth order solution coincides with (5.2), and is given by $P_0 = C_0 = 0$, and

$$E_0 = \frac{\rho}{r^3} \quad V_0 = W_0 = \ln r \quad U_0 = r^2 + \frac{\rho^2}{3r^4} - \frac{M}{r^2} \tag{6.2}$$

The horizons of U_0 will be denoted by r_{\pm} . The boundary conditions include $C_1(r_+) = C_1(\infty) = 0$, together with the requirement that V_2 and W_2 fall off faster than r^{-2} as $r \rightarrow \infty$. To separate out the spin zero (scalar) and spin two (tensor) perturbative corrections, it will be convenient to introduce the combinations,

$$S_2 = 2V_2 + W_2 \quad T_2 = V_2 - W_2 \tag{6.3}$$

Equations M2 and E1 are odd in B , and thus have contributions only to first order in B , while all other equations, M1, E2, E3, E4, and CON are even in B and thus have only second order contributions. We begin by solving M2 and E1 first.

6.1 Spin one sector

To order B , equation M2 is given by,

$$(U_0 r P_1 - \rho C_1)' = -2 \frac{k\rho}{r^3} \tag{6.4}$$

It may be readily integrated, and the integration constant is fixed by the boundary conditions at the horizon, $U_0(r_+) = C_1(r_+) = 0$. As a result, P_1 is given in terms of C_1 by,

$$P_1(r) = \frac{\rho}{r U_0(r)} \left(C_1(r) + \frac{k}{r^2} - \frac{k}{r_+^2} \right) \tag{6.5}$$

Note that the function P_1 is automatically smooth at the horizon. To obtain C_1 , we substitute the solution P_1 into equation E1. Using the special form of the function U_0 , this equation may be recast as follows,

$$\left(r U_0^2 \left(\frac{r^2 C_1}{U_0} \right)' \right)' = \frac{4k\rho^2}{r^5} - \frac{4k\rho^2}{r^3 r_+^2} \tag{6.6}$$

In this form, the equation may be solved by two successive integrations, producing two integration constants. These constants are fixed uniquely by the requirements that $C_1(r_+) = C_1(\infty) = 0$, and we find,

$$C_1(r) = -k\rho^2 \frac{U_0(r)}{r^2} \int_{\infty}^r \frac{dr'}{r' U_0^2(r')} \left(\frac{1}{r'^2} - \frac{1}{r_+^2} \right)^2 \quad (6.7)$$

Thus, the functions P_1 and C_1 are uniquely determined by the boundary conditions.

6.2 Spin two sector

Equation E2 to order B^2 gives,

$$(r^3 U_0 T_2')' = \frac{1}{2} r^5 (C_1')^2 + 2r U_0 (P_1)^2 - \frac{2}{r} \quad (6.8)$$

The terms on the right hand side are known from the solution in the spin one sector. This equation may be solved by two successive integrations. The two resulting integration constants may be fixed by demanding smoothness of T_2' at the horizon, and the vanishing of T_2 at infinity, so that T_2 is uniquely given by,

$$T_2(r) = \int_{\infty}^r dr'' \frac{1}{r''^3 U_0(r'')} \int_{r_+}^{r''} dr' \left(\frac{1}{2} r'^5 (C_1')^2 + 2r' U_0 (P_1)^2 - \frac{2}{r'} \right) \quad (6.9)$$

6.3 Spin zero sector

The functions E_2 , U_2 and S_2 all correspond to scalar perturbations. The linear combination $3 \times E3 - CON$ gives an equation for S_2 in terms of C_1 and T_2 ,

$$U_0 S_2'' + \frac{2}{r} U_0 S_2' + U_0 T_2'' + \frac{3}{r} U_0 T_2' + 3U_0' T_2' = \frac{1}{2} r^2 (C_1')^2 - \frac{2}{r^4} \quad (6.10)$$

Eliminating T_2 using (6.9), the remaining equation can be recast in the form, $(r^2 S_2')' = -2P_1^2$, which may be solved by two successive integrations. We need S_2 to fall off faster than $1/r^2$ at infinity to preserve the boundary. The solution is therefore uniquely fixed to be,

$$S_2(r) = 2 \int_{\infty}^r dr' \left(\frac{1}{r} - \frac{1}{r'} \right) P_1(r')^2 \quad (6.11)$$

Next, equation M1 determines E_2 ,

$$(r^3 E_2)' + (\rho S_2)' + (r^3 P_1 C_1)' + 2k P_1 = 0 \quad (6.12)$$

The integration constant can be reabsorbed into ρ , and so we have

$$E_2(r) = -\frac{\rho}{r^3} S_2 - P_1 C_1 - \frac{2k}{r^3} \int_{\infty}^r dr' P_1(r') \quad (6.13)$$

Equation E4 determines U_2 . It may be expressed as $(r^3 U_2')' = X$, with X given by,

$$X(r) = -r^3 U_0' S_2' + \frac{16\rho}{3} (E_2 + C_1 P_1) + r^5 (C_1')^2 + \frac{4r U_0 P_1^2}{3} + \frac{4}{3r} \quad (6.14)$$

The solution which goes to zero at infinity is

$$U_2(r) = \int_{\infty}^r \frac{dr''}{r''^3} \int_{r_+}^{r''} dr' X(r') - \frac{a_3}{2r^2} \quad (6.15)$$

where a_3 is an integration constant. Finally, the constraint equation may be checked to hold at $r \rightarrow \infty$ to leading order. This will guarantee that it is obeyed throughout.

6.4 Asymptotic behavior of the perturbative solution

The full perturbative solution is now fixed. The free parameters are: B, ρ, M , where we're not counting a_3 since it can be absorbed into M , nor r_+ since it is a function of M and ρ . As a result, the asymptotic behavior of these functions can now be computed, and we find,

$$\begin{aligned}
 C_1(r) &= \frac{k\rho^2}{4r_+^2 r^4} & c_4 &= \frac{k\rho^2 B}{4r_+^2} \\
 P_1(r) &= -\frac{k\rho}{r_+^2 r^3} & p_3 &= -\frac{k\rho B}{r_+^2} \\
 T_2(r) &= \frac{\ln r}{2r^4} & v &= 1 \\
 S_2(r) &= -\frac{k^2 \rho^2}{15r_+^4 r^6} & w &= 1 \\
 E_2(r) &= -\frac{k^2 \rho}{r_+^2 r^5} & e_3 &= \rho \\
 U_2(r) &= -\frac{2 \ln r}{3r^2} - \frac{1}{3r^2} - \frac{a_3}{2r^2} & u_2 &= -M + \left(\frac{2}{3} \ln r_+ - \frac{1}{3} - \frac{a_3}{2} \right)
 \end{aligned} \tag{6.16}$$

6.5 Regularity of the perturbative expansion

For the perturbative expansion around the non-extremal black brane with $r_- < r_+$, the functions C_1 and P_1 fall off fast as $r \rightarrow \infty$, and are smooth at the outer horizon r_+ , as well as at all other values of $r > r_+$. As a result, the integrals giving S_2, T_2, E_2 and U_2 are rapidly convergent, and define regular functions throughout.

The perturbative expansion around the extremal black brane with $r_- = r_+$, however, is not, generally, well-behaved. For $r_- = r_+$, the function $C_1(r)$ is given analytically by

$$C_1(r) = -\frac{k}{3r_+^2} \frac{U_0(r)}{r^2} \left[\frac{2r_+^2}{r^2 + 2r_+^2} - \frac{r_+^2}{r^2 - r_+^2} + \frac{1}{3} \ln \left(\frac{r^2 - r_+^2}{r^2 + 2r_+^2} \right) \right] \tag{6.17}$$

The functions C_1 and C_1' are smooth throughout. The function P_1 is given analytically by

$$P_1(r) = -\frac{k\rho}{r_+^2 r^3} \left[\frac{2r_+^2}{r^2 + 2r_+^2} + \frac{3r^2 + 2r_+^2}{3(r^2 + 2r_+^2)} + \frac{1}{3} \ln \left(\frac{r^2 - r_+^2}{r^2 + 2r_+^2} \right) \right] \tag{6.18}$$

and exhibits a logarithmic singularity as $r \rightarrow r_+$. This singularity is integrable in the formulas giving the perturbation functions S_2, T_2, U_2 and E_2 . As a result, the functions S_2, U_2, E_2 are smooth, while the function T_2 has a logarithmic singularity as $r \rightarrow r_+$, given by

$$T_2(r) \sim \left(\frac{k^2 - 1}{6r_+^4} \right) \ln(r - r_+) \tag{6.19}$$

To summarize, in the extremal limit, the electric current density P , and the tensor perturbation of the metric T_2 both diverge logarithmically at the horizon, and the solution is not globally smooth.

6.6 Perturbative calculation of entropy, temperature, and mass

Having normalized the metric at $r = \infty$ to be the conformally standard Minkowski metric, (B.1) with $v = w = 1$, the same metric at the horizon then reads,

$$ds_H^2 = r_+^2 e^{2V(r_+)} (d\tilde{x}_1^2 + d\tilde{x}_2^2) + r_+^2 e^{2W(r_+)} d\tilde{x}_3^2 \quad (6.20)$$

The perturbative corrections to the entropy density, temperature, and mass to order B^2 may then be deduced from the metric functions as follows. The entropy density s is given by

$$s = s_0 (1 + B^2 S_2(r_+)) \quad (6.21)$$

Similarly, the temperature T and mass M are given by

$$\begin{aligned} T &= T_0 + \frac{1}{4\pi} B^2 U_2'(r_+) \\ M &= M_0 + B^2 \left(\frac{1}{3} - \frac{2}{3} \ln r_+ + \frac{1}{2} \int_{r_+}^{\infty} dr' \left(X(r') - \frac{4}{3r'} \right) \right) \end{aligned} \quad (6.22)$$

Here, s_0 , T_0 , and M_0 are respectively the entropy density, temperature and mass of the $B = 0$ black brane. Note that it follows from the form of (6.11), that $S_2(r_+) < 0$, so that the correction to the entropy density is always negative.

6.6.1 Eliminating r_+ -dependence

The position of the outer horizon, r_+ , has dimension and hence no direct physical meaning. It may be thought of as setting the overall scale, and plays the role of ℓ above. Therefore, any physical quantity must be independent of r_+ . Thus, we shall introduce dimensionless coordinates and quantities, such as

$$x \equiv r/r_+ \quad \lambda \equiv r_-/r_+ \quad (6.23)$$

The r_+ -dependence may now be isolated in each one of the functions that enter perturbation theory. For C_1 and P_1 we define the dimensionless functions $\hat{C}_1(x)$ and $\hat{P}_1(x)$ by,

$$C_1(r) = k\rho^2 \hat{C}_1(x)/r_+^8 \quad P_1(r) = k\rho \hat{P}_1(x)/r_+^5 \quad (6.24)$$

We also define the dimensionless functions $\sigma(\lambda)$ and $\tau(\lambda)$ by,

$$S_2(r_+) \equiv \sigma(\lambda)/r_+^4 \quad U_2'(r_+) \equiv \tau(\lambda)/r_+^3 \quad (6.25)$$

and we use ρ as a physical quantity that sets the scale for r_+ ,

$$\rho^2 = r_+^6 \nu(\lambda)^6 \quad \nu(\lambda)^6 = 3\lambda^2(1 + \lambda^2) \quad (6.26)$$

so that $r_+ = \rho^{1/3}/\nu(\lambda)$. The expressions for entropy density and temperature normalized to the physical dimensionful quantity ρ are as follows,

$$\begin{aligned} \frac{s}{\rho} &= \frac{4\pi}{3\nu(\lambda)^3} \left(1 + \frac{B^2}{\rho^{4/3}} \sigma(\lambda) \nu(\lambda)^4 \right) \\ \frac{T}{\rho^{1/3}} &= \frac{1}{2\pi\nu(\lambda)} \left((1 - \lambda^2)(2 + \lambda^2) + \frac{B^2}{2\rho^{4/3}} \tau(\lambda) \nu(\lambda)^4 \right) \end{aligned} \quad (6.27)$$

6.6.2 Calculating the dimensionless functions

The dimensionless functions $\hat{C}_1(x)$ and $\hat{P}_1(x)$ may be readily computed (analytically) from (6.7) and (6.5), and used to evaluate the dimensionless functions $\sigma(\lambda)$ and $\tau(\lambda)$, given by

$$\begin{aligned}\sigma(\lambda) &= -2k^2\nu(\lambda)^6 \int_1^\infty dx \left(1 - \frac{1}{x}\right) \hat{P}_1(x)^2 \\ \tau(\lambda) &= -\frac{4}{3} + k^2\lambda^2(1 + \lambda^2)\hat{\tau}(\lambda)\end{aligned}\tag{6.28}$$

where $\hat{\tau}(\lambda)$ is given by

$$\begin{aligned}\hat{\tau}(\lambda) &= -\frac{9}{2}\lambda^2(1 + \lambda^2)\hat{C}'_1(1)^2 - 9\lambda^2(1 + \lambda^2) \int_1^\infty dx x^3 \hat{C}'_1(x)^2 - 24 \int_1^\infty dx \hat{P}_1(x) \\ &+ 8 \int_1^\infty \frac{dx}{x^4} \hat{P}_1(x) + \int_1^\infty dx \hat{P}_1(x)^2 \left(8x - 4\lambda^2(1 + \lambda^2)\frac{1}{x^5}\right) \\ &+ 2(1 - \lambda^2)(2 + \lambda^2) \int_1^\infty dx \hat{P}_1(x)^2 - 72\lambda^2(1 + \lambda^2) \int_1^\infty dx \left(1 - \frac{1}{x}\right) \hat{P}_1(x)^2\end{aligned}\tag{6.29}$$

The functions $\sigma(\lambda)$ and $\hat{\tau}(\lambda)$ may be evaluated numerically, using the analytic expression for $\hat{C}_1(x)$ and $\hat{P}_1(x)$. The results are as follows.

Numerical evaluation of the integral entering $\sigma(\lambda)$ shows very little dependence on λ throughout the interval $\lambda \in [0, 1]$, and may be well approximated there by the average value of 0.015 (specifically, its value drops uniformly from 0.01505 at $\lambda = 0$ to 0.01490 at $\lambda = 1$). As a result, we have the approximate formula,

$$\sigma(\lambda) \sim -0.090 \times k^2\lambda^2(1 + \lambda^2)\tag{6.30}$$

for λ throughout the interval $[0, 1]$.

Numerical evaluation of the function $\hat{\tau}(\lambda)$ produces a dependence given in figure 5 below. We record the end point values,

$$\begin{aligned}\hat{\tau}(0) &= 8.605 & \tau(0) &= -1.333 \\ \hat{\tau}(1) &= 4.527 & \tau(1) &= -1.333 + 9.054 \times k^2\end{aligned}\tag{6.31}$$

and for the supersymmetric value $k^2 = 4/3$, we have $\tau(1) = 10.739$.

6.7 Physical interpretation of the perturbative corrections

- If $\tau(1) > 0$, then T is non-zero and positive at the extremal value $\lambda = 1$. We can extract an estimate for the minimum temperature under the assumption that $B^3/\rho^2 \ll 1$, by simply estimating the temperature at $\lambda = 1$, and we get,

$$T_{\min} = \frac{\tau(1)\nu(1)^3 B^2}{4\pi \rho}\tag{6.32}$$

which for the supersymmetric value $k = 2/\sqrt{3}$ gives approximately $T_{\min} = 2.0932B^2/\rho$. Of course, at $\lambda = 1$ our perturbative analysis breaks down, and so higher order terms could well invalidate this result.

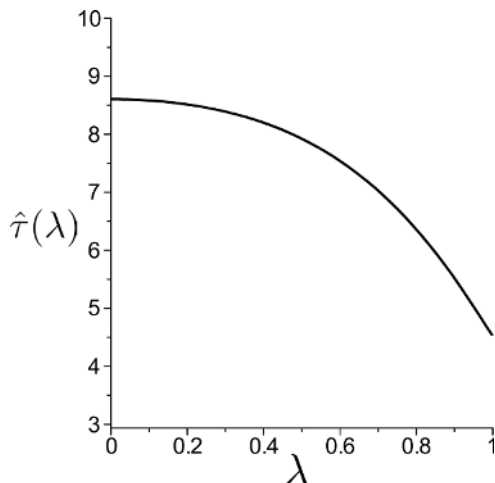


Figure 5. The function $\hat{\tau}(\lambda)$.

- If $\tau(1) < 0$, then $T = 0$ is attained for $\lambda = \lambda_c < 1$, and the geometry must have a naked singularity whenever $\lambda_c < \lambda \leq 1$, as it would correspond to negative temperature.

7 Numerical analysis

7.1 Setup

We turn now to a discussion of our results obtained by numerical integration of the equations of motion. The first step is to specify our coordinate system. We impose the conditions corresponding to the horizon frame described in section 4.1, including choosing $r_+ = 1$ and $U'(1) = 1$. This coordinate system will inevitably break down in the limit of vanishing temperature, since in that case we would have $U'(r_+) = 0$, and no rescaling of the time coordinate can bring us to our chosen gauge. We will see this breakdown occurring explicitly in the numerics.

Solutions in this gauge are parameterized by the values of b and q , both of which we take to be non-negative without loss of generality. Choosing a value for the pair (b, q) fixes initial data at the horizon, and then we can integrate out to the asymptotically AdS₅ region at large r (we used Maple to do this).

From the large r form of the obtained solution we can then compute the numerical coefficients (v, w , etc.) appearing in (4.4). We then convert these into physical quantities using the formulas given in section 4.3. The expressions we will be using in the following are

$$\begin{aligned}
 B &= \frac{b}{v} & (7.1) \\
 T &= \frac{\gamma_c}{4\pi} \\
 s &= \frac{1}{4\sqrt{v^2 w \gamma_c^2}} \\
 \rho &= \gamma_c(e_3 - c_0 p_3)
 \end{aligned}$$

with $\gamma_c = 1/\sqrt{1 - wc_0^2}$.

It is most illuminating to provide plots of entropy density versus temperature with the magnetic field and charge density held fixed. However, it only makes sense to keep fixed the dimensionless ratio B^3/ρ^2 . Similarly, it is only meaningful to compute dimensionless versions of the entropy and temperature, and for these we choose

$$\frac{s}{(\rho^2 + B^3)^{1/2}}, \quad \frac{T}{(\rho^2 + B^3)^{1/6}} \tag{7.2}$$

An instructive special case is the Reissner-Nordstrom solution with $B = 0$ reviewed in section 5.2. The solution is originally given in terms of r_+ and ρ . Transforming this solution into the gauge used here, we find $q = \rho/r_+^3$, along with

$$\frac{s}{\rho} = \frac{1}{4q}, \quad \frac{T}{\rho^{1/3}} = \frac{1}{4\pi} \left(\frac{4 - \frac{2}{3}q^2}{q^{1/3}} \right) \tag{7.3}$$

Note that the extremal limit in this parametrization is $q = \sqrt{6}$ with $s/\rho = 1/(4\sqrt{6}) \approx .102$. A consistency check on our numerics is that we recover the curve described by (7.3) along with the extremal endpoint.

The next step is to determine the region of the (b, q) parameter space that gives rise to smooth solutions. The boundary of this region depends on the value of k . Numerical integration shows that as we move out radially from the origin in the (b, q) plane we eventually find that some of the parameters v, w , etc. start to diverge or go to zero as we approach a curve in the (b, q) plane, depicted by the red lines in figure 2. The analytic form of this curve is only known at $k = 1$, where it is given by $q^2 + 2b^2 = 6$. Points on the $k = 1$ critical curve correspond to extremal warped AdS₃ black holes, as was described in section 5.4. For the other values of k that we consider, the curve acquires some bulges, but continues to look roughly like that for $k = 1$. In the generic case, we determined the critical curve numerically by evolving outward until γ_c exceeds some specified value, which we take to be roughly 12. This requires fine tuning (b, q) to the critical curve to roughly four decimal places. The $k = 0$ case is special since here $\gamma_c = 1$ exactly for all solutions; here we locate the critical curve by looking for a divergence in $1/v$. In all cases, the dimensionless temperature tends to zero as we approach the critical curve.

Once the critical curve has been identified, we can set up a grid in the (b, q) plane, and scan over the gridpoints. We took about 12,000 roughly evenly spaced gridpoints. Once data at the gridpoints has been obtained we can search for curves along which B^3/ρ^2 is approximately constant. These curves were illustrated schematically by the flow diagrams in figure 2. Finally, we plot the entropy versus temperature for points along such a curve.

7.2 Results

We will discuss three cases: $k = 0$, $k = 1$, and $k = 2/\sqrt{3}$, the latter being the supersymmetric value. We expect that these are representative of the general cases $k < 0$, $k = 1$ and $k > 1$.

Curves of approximately fixed B^3/ρ^2 are shown in figure 6, which may be compared with the schematic version in figure 2. In the $k = 0$ and $k = 2/\sqrt{3}$ cases, the curves appear

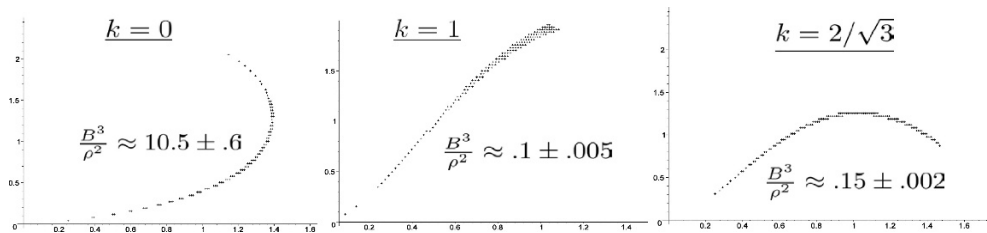


Figure 6. Points in the (b, q) plane (b on horizontal axis) obtained by scanning over a grid and keeping points with B^3/ρ^2 fixed within some interval. These numerical plots are to be compared with the schematic flows illustrated in figure 2

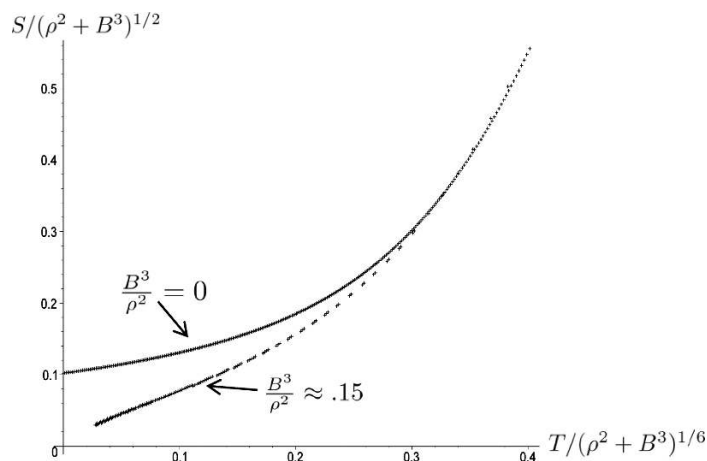


Figure 7. Entropy versus temperature for $k = 1$ and $B^3/\rho^2 \approx .1 \pm .005$

to be heading towards $b = 0$ and $q = 0$ respectively. As they do so, they begin to approach very near to the critical curve discussed above. We cannot follow them all the way there, as our limited precision prevents us from collecting data points arbitrarily close to the critical curves. It is conceivable that the true curves instead terminate at some location on the critical curves, as apparently occurs in the $k = 1$ case.

Given the points along a fixed B^3/ρ^2 curve, we can construct a plot of entropy versus temperature. Such plots for the $k = 2/\sqrt{3}$ and $k = 0$ cases are displayed in figures 3 and 4. In figure 7 we show the corresponding plot for $k = 1$. In all of these plots we compare the finite B results against those for $B = 0$. The $B = 0$ curves represent the Reissner-Nordstrom black brane solution, and reproduce numerically the curve described in (7.3).

In the $k = 0, 1$ cases, the entropy appears to go to a finite value at extremality, even though a singularity seems to be developing at the horizon in this limit, as was discussed in section 5.5. By adjusting B^3/ρ^2 , we can tune this limiting entropy to any desired value between 0 and that of the Reissner-Nordstrom solution at $B = 0$. In the plots we have chosen the value of B^3/ρ^2 such that the extremal entropy is roughly half of its maximal value. We see that this requires a much larger magnetic field in the $k = 0$ case as compared to $k = 1$. At $k = 1$ the entropy is controlled by the near-horizon warped AdS₃ black hole solution.

The situation for $k > 1$ appears to be very different, at least for the values that we

have studied. The main effect, which becomes more pronounced at larger k , is that the entropy decreases substantially at low temperatures, and appears to be headed towards zero, until our numerics break down. For the supersymmetric case $k = 2/\sqrt{3}$ shown in figure 3 the effect is relatively modest due to the fact that $2/\sqrt{3} \approx 1.15$ is not too much larger than 1. But even in this case it is evident that at low temperature a value of B^3/ρ^2 much less than 1 causes a decrease in the entropy by factor much larger than 1, and the effect grows as the temperature is decreased.

The main property driving this behavior is the apparent location of the endpoint of the flows at fixed B^3/ρ^2 . We can locate the endpoint numerically by setting $q = 0$ and increasing b until a singularity (or numerical breakdown) occurs. Calling this endpoint value b_c , we find $b_c = \sqrt{3} = 1.732$ for $k \leq 1$, but $b_c < \sqrt{3}$ for $k > 1$. For $k = 2/\sqrt{3}$ we find $b_c \approx 1.568$. On the other hand, with $q = 0$ we showed that only for $b = \sqrt{3}$ does a smooth, extremal, near-horizon geometry exist. Since there is no candidate smooth endpoint, it seems likely from this perspective that the flows terminate in a singularity at the point $(q = 0, b = b_c)$. The metric functions we compute certainly behave very badly as this point is approached, but it is difficult to completely disentangle physical divergences from a breakdown of our numerics.

The breakdown of our numerics at very low temperature is a reflection of our gauge fixing choice $U'(r_+) = 1$. Because of this, we are not able to unambiguously determine whether the $k > 1$ curves end at zero temperature and entropy, or terminate before then in a singularity. But we certainly do not see any evidence of a finite entropy endpoint at zero temperature, a conclusion which is bolstered both by our perturbative analysis and by the non-existence of any candidate near extremal geometries to match onto. The most likely scenario seems to be that finite entropy at extremality requires the magnetic field to be fine tuned to zero. One clear goal for the future is to improve the numerical treatment to allow a more refined study of the very low temperature regime.

8 Discussion

In this work we have constructed asymptotically AdS₅ black brane solutions carrying nonzero charge density and magnetic field. They were found analytically in a perturbative expansion for small B , and numerically for general values. The most interesting results centered around the low temperature regime, where we found a sensitive dependence on both the magnetic field and the value of the Chern-Simons coupling k . For $k > 1$, including the supersymmetric value of $k = 2/\sqrt{3}$, we found that a small magnetic field causes a rapid decrease of the entropy at low temperatures. We proposed an AdS/CFT version of Nernst's theorem and the third law of thermodynamics consistent with the observed behavior. More general tests were left to the future [29].

$k = 1$ emerged as a special value, for here, and only here, there exist smooth finite entropy extremal near-horizon geometries carrying charge and magnetic field. We identified these, and their nonextremal generalizations, with one class of warped AdS₃ × R² black hole solutions studied in [30] (without the R² factor) in the context of topologically massive gravity. We found that these could be connected to asymptotically AdS₅ spacetimes, although a singularity in the interpolating solution develops at the horizon in the strict extremal limit.

It is an intriguing question as to whether the value $k = 1$, which appears special from the point of view of supergravity solutions, has a special significance also on the CFT side, perhaps because it corresponds to a special embedding of the gauge field $U(1) \subset SU(4)_R$.

There are some other contexts in which sufficiently large values of the Chern-Simons coupling k causes novel effects. In the recent work [38] it was found that sufficiently large k can cause an instability in Reissner-Nordstrom black brane solutions. In [39] it was found that large k likely causes an instability of spinning black hole solutions. In both of these cases k must be larger than the supersymmetric value for an instability to occur, while we have seen here that the extremal Reissner-Nordstrom entropy is apparently destabilized even at the supersymmetric value.

There should be many applications of these solutions to the study of condensed matter and finite density QCD. Transport properties can be computed in these backgrounds; although we lack analytical solutions, a numerical treatment should be tractable.

We conclude with a curious observation concerning the frequent appearance of $3/4$ in this subject. To wit: the ratio of the high temperature entropy in gravity to that in the gauge theory is $3/4$; the corresponding ratio of the low temperature entropy in the presence of a magnetic field is $\sqrt{4/3}$; and the ratio of the “special” value of k ($k = 1$ in our conventions) to the supersymmetric value is $\sqrt{3/4}$. These three factors are not logically related in any obvious way. Perhaps this is just coincidence.

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A Asymptotic boost symmetry

The Ansatz (3.1), (3.2) is covariant under boosts in the direction of the magnetic field. Assuming the metric ds^2 to be asymptotically AdS_5 , the boundary space-time coordinates x_μ may be rescaled so that its asymptotic behavior, as $r \rightarrow \infty$, is given by,

$$ds^2 \sim \frac{dr^2}{r^2} + r^2 (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) \tag{A.1}$$

Performing a boost in the x_3 -direction

$$\begin{aligned} t &= \gamma_c(\tilde{t} + \beta\tilde{x}_3) & \gamma_c^2(1 - \beta^2) &= 1 \\ x_3 &= \gamma_c(\tilde{x}_3 + \beta\tilde{t}) & |\beta| &< 1 \end{aligned} \tag{A.2}$$

produces a field strength and a metric of the same form as (3.1) and (3.2), but with the coordinates r, t, x_3 replaced by $\tilde{r}, \tilde{t}, \tilde{x}_3$ and the functions E, P, U, V, W, C of r replaced by the functions $\tilde{E}, \tilde{P}, \tilde{U}, \tilde{V}, \tilde{W}, \tilde{C}$ of \tilde{r} respectively, leaving x_1, x_2 , and B unchanged. (A

reparametrization $r \rightarrow \tilde{r}$ is generally needed to put the U -function back into the gauge of the Ansatz.) The relation between the transformed and original Maxwell fields are as follows,

$$\begin{aligned}\tilde{E}d\tilde{r} &= \gamma_c(E - \beta P)dr \\ \tilde{P}d\tilde{r} &= \gamma_c(P - \beta E)dr\end{aligned}\tag{A.3}$$

while for the metric fields, we have $\tilde{U}^{-1}d\tilde{r} = U^{-1}dr$, and $\tilde{V} = V$, as well as,

$$\begin{aligned}\tilde{U} - e^{2\tilde{W}}\tilde{C}^2 &= \gamma_c^2 U - \gamma_c^2 (C + \beta)^2 e^{2W} \\ e^{2\tilde{W}} &= \gamma_c^2 (1 + \beta C)^2 e^{2W} - \gamma_c^2 \beta^2 U\end{aligned}\tag{A.4}$$

and the following transformation law between \tilde{C} and C ,

$$\tilde{C} = \frac{(C + \beta)(1 + \beta C)e^{2W} - \beta U}{(1 + \beta C)^2 e^{2W} - \beta^2 U}\tag{A.5}$$

If $C(\infty) = 0$, then it follows that $\tilde{C}(\infty) = 0$ for all β . At an event horizon, $r = r_+$, where $U(r_+) = 0$, we have a simplified formula,

$$\tilde{C}(\tilde{r}_+) = \frac{C(r_+) + \beta}{1 + \beta C(r_+)}\tag{A.6}$$

As a result, in these coordinates, the value of C at the horizon characterizes the velocity of the configuration. Performing a boost $\beta = -C(r_+)$ brings the solution to its rest frame.

B Relation between horizon and asymptotic frames

Our solutions being asymptotically AdS_5 implies that by rescaling t, x_1, x_2, x_3 , and performing an α -transformation and a boost, the asymptotic behavior of the field strength and of the metric as $r \rightarrow \infty$ may be put in standard form,

$$\begin{aligned}F &= Edr \wedge d\tilde{t} + Bd\tilde{x}_1 \wedge d\tilde{x}_2 + Pd\tilde{x}_3 \wedge dr \\ ds^2 &\sim \frac{dr^2}{r^2} + r^2(-d\tilde{t}^2 + d\tilde{x}_1^2 + d\tilde{x}_2^2 + d\tilde{x}_3^2)\end{aligned}\tag{B.1}$$

In general, the coordinates \tilde{t}, \tilde{x}_i will be different from the coordinates t, x_i used in specifying initial conditions at the horizon. In the coordinates t, x_i , the asymptotics of the functions take the form displayed in (4.4) and (4.5). The rescaling and α -transformation relating the coordinates t, x_i and \tilde{t}, \tilde{x}_i is given by,

$$\tilde{t} = t \qquad \tilde{x}_{1,2} = \sqrt{v}x_{1,2} \qquad \tilde{x}_3 = \sqrt{w}(x_3 + c_0 t)\tag{B.2}$$

This is combined with a constant shift in the r coordinate by an amount r_0 . Since the magnetic field term in F is r -independent, we obtain a simple relation between the magnetic field b at the horizon, and the magnetic field B at infinity,

$$b = vB\tag{B.3}$$

The functions $\tilde{U}, \tilde{V}, \tilde{W}, \tilde{C}, \tilde{\mathcal{E}}, \tilde{P}$ of the full metric in the coordinates \tilde{t}, \tilde{x}_i are then given by,

$$\begin{aligned}
 \tilde{U} &= r^2 + \frac{u_2}{r^2} - \frac{2B^2 \ln r}{3 r^2} + \dots & \tilde{C} &= \frac{\sqrt{w} c_4}{r^4} + \dots \\
 e^{2\tilde{V}} &= r^2 + \frac{v_2}{vr^2} + \frac{B^2 \ln r}{3 r^2} + \dots & \tilde{\mathcal{E}} &= \frac{e_3}{r^3} + \dots \\
 e^{2\tilde{W}} &= r^2 - \frac{2v_2}{vr^2} - \frac{2B^2 \ln r}{3 r^2} + \dots & \tilde{P} &= \frac{p_3/\sqrt{w}}{r^3} + \dots
 \end{aligned} \tag{B.4}$$

While the metric at the boundary of AdS_5 now takes the standard form of (B.1), the metric ds_H^2 at the horizon has been rescaled and α -transformed to become,

$$ds_H^2 = \frac{1}{v}(d\tilde{x}_1^2 + d\tilde{x}_2^2) + \frac{1}{w}(d\tilde{x}_3 - c_0\sqrt{w}d\tilde{t})^2 \tag{B.5}$$

corresponding to a solution moving with velocity $c_0\sqrt{w}$.

We can move to the rest frame of the solution by performing a boost in the \tilde{x}_3 -direction, thereby removing the cross term in ds_H^2 while preserving the conformal boundary metric,

$$\tilde{t} = \gamma_c(\hat{t} + \sqrt{w}c_0\hat{x}_3), \quad \tilde{x}_3 = \gamma_c(\hat{x}_3 + \sqrt{w}c_0\hat{t}), \quad \gamma_c = \frac{1}{\sqrt{1 - wc_0^2}} \tag{B.6}$$

In these coordinates the field strength is

$$F = \gamma_c E d\hat{r} \wedge d\hat{t} + \frac{b}{v} d\hat{x}_1 \wedge d\hat{x}_2 + \left(\frac{P}{\sqrt{w}\gamma_c} - \sqrt{w}c_0\gamma_c E \right) d\hat{x}_3 \wedge d\hat{r} \tag{B.7}$$

and the metric on the horizon is

$$ds_H^2 = \frac{1}{v}(d\hat{x}_1^2 + d\hat{x}_2^2) + \frac{1}{w\gamma_c^2} d\hat{x}_3^2 \tag{B.8}$$

which corresponds to a solution at rest in the hatted frame. This hatted (asymptotic) frame is the one that we use to express physical quantities in the CFT.

B.1 Evaluating entropy, temperature and chemical potential

Now we can give expressions for the various physical quantities in these coordinates. The entropy density s is read off from metric on the horizon, and is given by

$$s = \frac{1}{4} \frac{1}{\sqrt{v^2 w \gamma_c^2}} \tag{B.9}$$

The factors in the denominator ensure that s is the physical entropy density per unit volume, as measured in the CFT.

To evaluate the temperature, we first recall its behavior under a boost. Let T_0 be the temperature of some system in the rest frame. Then in a boosted frame the temperature is given by $T = T_0/\gamma_c$, as can be seen by writing the Boltzmann factor e^{-E/T_0} in terms of the boosted energy and momentum. Now in the original (unhatted) frame the temperature is $\tilde{T} = U'(r_+)/4\pi$. This represents the moving frame, while the rest frame is one in which

$C(r_+) = 0$, which is the hatted frame. Therefore, the temperature in the rest frame, which is what we're interested in, is

$$\hat{T} = T_0 = \gamma_c \tilde{T} = \frac{\gamma_c U'(r_+)}{4\pi} \tag{B.10}$$

From now on, when we write T we have in mind \hat{T} .

We also want the chemical potential, given by the asymptotic value of \hat{A}_t . Since we set $\hat{A}_t = 0$ at the horizon we have $\hat{A}_t|_\infty = \int_{r_+}^\infty d\hat{r} \hat{F}_{rt}$. This gives

$$\mu = \hat{A}_t|_\infty = \gamma_c \int_{r_+}^\infty dr E \tag{B.11}$$

If $kB \neq 0$, we may use the second Maxwell equation, in the form given in (3.5) to recast E as a total r -derivative. The chemical potential may now be evaluated in the original unprimed coordinates, where the magnetic field is b , and we find,

$$\mu = \frac{3\gamma_c v}{8kb} \left(\sqrt{w} c_0 e_3 - \frac{p_3}{\sqrt{w}} \right) \tag{B.12}$$

B.2 Evaluating the Maxwell current

We identify the Maxwell current in terms of the on-shell variation of the Maxwell-Einstein action of (2.1), with respect to the Maxwell field at the boundary of AdS_5 ,

$$\delta S = \int d^4x \sqrt{-\gamma^{(0)}} J^\mu \delta A_\mu \tag{B.13}$$

Here, $\gamma_{\mu\nu}^{(0)}$ is the asymptotic conformal metric defined by $\gamma_{\mu\nu}/r^2 \rightarrow \gamma_{\mu\nu}^{(0)}$ in the limit $r \rightarrow \infty$, where $\gamma_{\mu\nu}$ is the metric induced from $g_{\mu\nu}$ on a surface of constant r . The on-shell variation of the action with respect to the gauge field may be calculated from (2.1) and we find the following expression for the current,

$$4\pi G_5 J^\mu = - \left(r^3 F^{r\mu} + r^4 (\ln r) \partial_\nu F^{\nu\mu} + \frac{k}{3} \epsilon^{\alpha\beta\gamma\mu} A_\alpha F_{\beta\gamma} \right) \tag{B.14}$$

Here, $\epsilon^{\alpha\beta\gamma\mu}$ denotes the volume form for $\gamma_{\mu\nu}^{(0)}$. Using the leading forms $g^{rr} = r^2$ and $g^{\mu\nu} = \gamma_{(0)}^{\mu\nu}/r^2$ we can write this as

$$4\pi G_5 J^\mu = \left(r^3 \gamma_{(0)}^{\mu\nu} F_{r\nu} + (\ln r) \gamma_{(0)}^{\mu\alpha} \gamma_{(0)}^{\nu\beta} \partial_\nu F_{\alpha\beta} + \frac{k}{3} \epsilon^{\alpha\beta\gamma\mu} A_\alpha F_{\beta\gamma} \right) \tag{B.15}$$

Since we only consider solutions with constant field strength on the boundary the middle term will not contribute.

B.3 Evaluating the current in the rest frame

The current can be computed directly in the double primed coordinates system as

$$4\pi G_5 \hat{J}^t = \left(\hat{e}_3 - \frac{2k\hat{B}}{3\sqrt{\hat{v}^2 \hat{w}}} \hat{A}_3 \Big|_\infty \right) \tag{B.16}$$

$$4\pi G_5 \hat{J}^{1,2} = 0$$

$$4\pi G_5 \hat{J}^3 = \left(\frac{\hat{p}_3}{\hat{w}} + \frac{2k\hat{B}}{3\sqrt{\hat{v}^2\hat{w}}} \hat{A}_t \Big|_{\infty} \right)$$

The double primed coordinates have been defined so that $\hat{v} = \hat{w} = 1$. From the formulas given above we also have,

$$\hat{B} = \frac{b}{v}, \quad \hat{e}_3 = \gamma_c(e_3 - c_0 p_3), \quad \hat{p}_3 = \gamma_c \left(\frac{p_3}{\sqrt{w}} - \sqrt{w} c_0 e_3 \right) \quad (\text{B.17})$$

and

$$\hat{A}_t \Big|_{\infty} = -\frac{3\gamma_c v}{8kb} \left(\frac{p_3}{\sqrt{w}} - \sqrt{w} c_0 e_3 \right) = -\frac{3v}{8kb} \hat{p}_3 \quad (\text{B.18})$$

We therefore finally have

$$4\pi G_5 \hat{J}^t = \gamma_c(e_3 - c_0 p_3) - \frac{2kb}{3v} \hat{A}_3 \Big|_{\infty} \quad (\text{B.19})$$

$$4\pi G_5 \hat{J}^3 = \frac{3}{4} \gamma_c \left(\frac{p_3}{\sqrt{w}} - \sqrt{w} c_0 e_3 \right)$$

We note that $\hat{A}_3 \Big|_{\infty}$ can be chosen arbitrarily as long as g_{33} is finite at the horizon (though we should set it to zero unless we want to add a source for J^3 in the CFT partition function), while $\hat{A}_t \Big|_{\infty}$ is given in (B.11).

C Factorized solutions

In the general 5-dimensional solutions we have obtained here, the 2-dimensional x_1, x_2 -plane perpendicular to the magnetic field is warped over the remaining 3-dimensional space. The near-horizon geometry of these solutions, however, invariably reduces to a space-time in which the x_1, x_2 -plane factorizes from the solutions because the field V becomes r -independent. This raises the question as to the structure of the most general factorized solution. In this appendix we shall show that, modulo certain regularity conditions, the only factorized solutions are of the type given in section 4.

We begin by proving the following auxiliary result: if the function P vanishes, then we only have the factorized $k^2 = 1$ solution of section 4.4. Next, assuming now that $P \neq 0$, but $V = 0$, we show that again only the factorized $k^2 = 1$ solution of section 4.4 exists.

C.1 $P = 0$ leads to the near-horizon geometry

By shifting V, W by constants, we may always assume that $V(r_+) = W(r_+) = 0$. Recall that the magnetic field in these coordinates is denoted by b . We assume $kb \neq 0$ and $q \neq 0$. We begin by showing that the condition $P = 0$ leads to U'', V, W, \mathcal{E} constant. When $P = 0$, equations M1, M2, E1 may be integrated, and give,

$$\mathcal{E} = q e^{-2V-W}$$

$$\begin{aligned} C' &= 2kb e^{-2V-W} \\ C' &= 2kb e^{-2V-3W} \end{aligned} \tag{C.1}$$

Comparing the two expressions above for C' , we find that $W = 0$ identically. Taking the difference between E2 and E3 (for $W = 0$), we find,

$$0 = 24 - 4\mathcal{E}^2 + 4b^2 e^{-4V} - 3(C')^2 \tag{C.2}$$

Substituting the above expression for C' and \mathcal{E} , we find,

$$0 = 24 + e^{-4V} (-4q^2 + 4b^2 - 12k^2 b^2) \tag{C.3}$$

Since the terms in the parenthesis are constants, V must be constant, and hence $V = 0$ in view of the boundary condition. Thus, \mathcal{E} and C' are constant. Equation E2 now requires $0 = 2k^2 b^2 - 2b^2$ which requires $k^2 = 1$. Equation E3 requires

$$0 = 6 - q^2 - 2b^2 \tag{C.4}$$

which is precisely the boundary curve equation for $k^2 = 1$. CON is automatic, and E4 gives $U'' = 24$. But this gives precisely the factorized solution of section 4.4.

C.2 $V = 0$ leads to the near-horizon geometry

Next, we assume $P \neq 0$ and $V = 0$, so that the geometry is factorized. Under this assumption, equation M2 may be traded for the constraint equation, since upon differentiation, CON will require the use of M2 when $P \neq 0$. Clearly, we must now retain E4 as an independent equation. Equations E2, E3, and CON are equivalent to the following equations,

$$\begin{aligned} \mathcal{E}^2 &= 6 - 2b^2 + UP^2 e^{-2W} \\ U'W' &= -\frac{1}{2}(C')^2 e^{2W} + 2b^2 \\ -W'' - (W')^2 &= 2P^2 e^{-2W} \end{aligned} \tag{C.5}$$

By eliminating \mathcal{E}^2 , $(C')^2$, and P^2 from E4, using the above equations, we obtain a relatively simple equation relating only U and W ,

$$U'' + 3U'W' + 2UW'' + 2U(W')^2 = 24 \tag{C.6}$$

Finally, equation M1 may be viewed as giving P in terms of \mathcal{E} and W ,

$$P = -\frac{1}{2kb} (\mathcal{E}e^W)' \tag{C.7}$$

and E1 may be integrated upon using M1, to give,

$$C' = -\frac{1}{kb} \mathcal{E}^2 e^{-W} + \alpha e^{-3W} \tag{C.8}$$

where α is an integration constant. As promised, we omit equation M2.

Equation (C.6) may be readily integrated in terms of the function τ , defined by,

$$\tau(r) \equiv \int_{r_0}^r dr' e^{W(r')} \quad (\text{C.9})$$

and we have

$$U = \frac{12}{(\tau')^2}(\tau^2 - \tau_0^2) \quad (\text{C.10})$$

where r_0 and τ_0 are integration constants. Using the relations $\tau' = e^W$ and,

$$W' = \frac{\tau''}{\tau'} \quad W'' + (W')^2 = \frac{\tau'''}{\tau'} \quad (\text{C.11})$$

and eliminating \mathcal{E}, U, P, W from (C.5) in favor of τ and $\hat{E} = \tau' \mathcal{E}$, we find the following three equivalent equations,

$$\begin{aligned} \hat{E}^2 &= (6 - 2b^2)(\tau')^2 - \frac{6\tau'''}{\tau'}(\tau^2 - \tau_0^2) \\ (\hat{E}')^2 &= -2k^2b^2\tau'\tau''' \\ 0 &= 48(\tau'')^2(\tau^2 - \tau_0^2) - 48\tau(\tau')^2\tau'' + 4b^2(\tau')^4 - \frac{1}{k^2b^2} \left(\hat{E}^2 - \alpha kb \right)^2 \end{aligned} \quad (\text{C.12})$$

Finally, \hat{E} and \hat{E}' may be eliminated from the above equations to yield two equations for τ . Eliminating \hat{E}^2 between the first and third equations yields an equation of third order in τ . Eliminating \hat{E}' requires some extra care, as we do not wish to unduly increase the order of the resulting equation, or introduce square roots. To obtain a second equation of third order, we take the product of the first two equations giving $[(\hat{E}^2)']^2/4$, and use the last equation to eliminate \hat{E}^2 from this. The results are as follows,

$$\begin{aligned} 0 &= 6(6 - 2b^2)\tau^2(\tau''')^2 + 12\tau_0^2b^2(\tau''')^2 + (6 - 2b^2)^2(\tau')^2(\tau'')^2 \\ &\quad - 12(6 - 2b^2)\tau\tau'\tau''\tau''' + 2b^2(6 - 2b^2)(\tau')^3\tau''' \\ 0 &= 48k^2b^2(\tau'')^2(\tau^2 - \tau_0^2) - 48k^2b^2\tau(\tau')^2\tau'' + 4k^2b^4(\tau')^4 \\ &\quad - \left((6 - 2b^2)(\tau')^2 - 6\frac{\tau'''}{\tau'}(\tau^2 - \tau_0^2) - q^2 - 2k^2b^2 \right)^2 \end{aligned} \quad (\text{C.13})$$

We shall now examine the existence of joint solutions to both equations, which solve the boundary conditions, $U(r_+) = W(r_+) = P(r_+) = 0$, and $C'(r_+) = 2kb$. These conditions translate as follows

$$\begin{aligned} \tau(r_+) &= \tau_0 & kb\alpha &= q^2 + 2k^2b^2 \\ \tau'(r_+) &= 1 \\ \tau'''(r_+) &= 0 & \hat{E}'(r_+) &= 0 \end{aligned} \quad (\text{C.14})$$

Evaluating the first equation of (C.12) at the horizon, using the above boundary conditions, gives the boundary curve relation,

$$q^2 + 2B^2 = 6 \quad (\text{C.15})$$

Evaluating both equations of (C.13) at the horizon, and using the boundary conditions, $\tau'(r_+) = 1$ and $\tau'''(r_+) = 0$, we find,

$$\begin{aligned} 0 &= (6 - 2b^2)\tau''(r_+)^2 \\ 0 &= -48k^2b^2\tau_0\tau''(r_+) + 4k^2b^4 - 4k^4b^4 \end{aligned} \tag{C.16}$$

where in the last equation we have used the relation (C.15). Since we assume that $q \neq 0$, relation (C.15) gives $6 - 2b^2 \neq 0$, and thus we must have $\tau''(r_+) = 0$ from the first equation in (C.16). Using this result in the second equation gives altogether

$$\tau''(r_+) = 0 \qquad k^2 = 1 \tag{C.17}$$

Note that we have now, in principle, overdetermined the boundary conditions even just on a single equation, as we have $\tau(r_+) = \tau_0$, $\tau'(r_+) = 1$, $\tau''(r_+) = \tau'''(r_+) = 0$ for a differential equation which is of third order. To proceed further, it appears necessary to make some assumption on the regularity properties of τ near the horizon. A general Ansatz consistent with the above boundary conditions is as follows,

$$\tau(r) = \tau_0 + (r - r_+) + \tau_p(r - r_+)^p + \text{higher orders} \tag{C.18}$$

for any real number $p > 3$. Substituting this Ansatz into the first equation of (C.13) shows that, as $r \rightarrow r_+$, the last term is of order $(r - r_+)^{(p-3)}$ and dominates the other 4 terms, which all vanish faster as $r \rightarrow r_+$. Thus, we must have $\tau_p = 0$, for any $p > 3$. Thus, within the class of asymptotic behaviors given by (C.18), the expression

$$\tau(r) = \tau_0 + (r - r_+) \tag{C.19}$$

is the only solution satisfying the boundary conditions. Clearly, for this solution, we have $U'' = 24$, and W constant, which is the solution of section 4.4.

D Interpolating between extremal BTZ $\times R^2$ and AdS₅

In section 5.5 we alluded to the existence of solutions that interpolate between a near-horizon extremal BTZ $\times R^2$ geometry and AdS₅. In this appendix we give the details of their construction. They can be thought of as infinitely boosted versions of the solutions studied in [23].

D.1 Extremal solutions with momentum

In [23] we found zero temperature solutions interpolating between AdS₃ $\times R^2$ (with magnetic flux on the R^2) and AdS₅. The metric and field strength are

$$\begin{aligned} ds^2 &= e^{-2W(r)}dr^2 + e^{2W(r)}(-dt^2 + dx_3^2) + e^{2V(r)}(dx_1^2 + dx_2^2) \\ F &= Bdx_1 \wedge dx_2 \end{aligned} \tag{D.1}$$

The Einstein-Maxwell equations reduce to

$$\begin{aligned} 2V'' + W'' + 2(V')^2 + (W')^2 &= 0 \\ (V')^2 + (W')^2 + 4V'W' &= 6e^{-2W} - e^{-4V-2W}B^2 \end{aligned} \tag{D.2}$$

One can (numerically) find solutions with small r behavior $e^{2V} = B/\sqrt{3}$, $e^{2W} = r^2$, and large r behavior $e^{2V} = vr^2$, $e^{2W} = r^2$, with $v \approx 1.87$. Since B can be set to unity by a coordinate rescaling, there is actually a unique such solution. The solutions are not entirely smooth, as V develops a subleading small r dependence r^α with $0 < \alpha < 1$, as was discussed in section 5.5.

Physically, these solutions represent the RG flow of $\mathcal{N} = 4$ SYM theory in the presence of a magnetic field. At low energies the theory is governed by fermions in the lowest Landau level. These fermions are free to move parallel to the magnetic field lines, and give rise to a 1 + 1 dimensional CFT at low energies, hence the appearance of AdS₃. In [23] the central charges in gravity and $\mathcal{N} = 4$ SYM at vanishing coupling were compared, and found to differ by a factor of $\sqrt{3/4}$. Note that the theory is nonsupersymmetric even at zero temperature due to the presence of the magnetic field.

On the CFT side, one should be able to excite one chirality of fermions to arrive at zero temperature configurations carrying momentum, and we expect corresponding solutions on the gravity side as well. The structure of such solutions follows, as in [40, 41], from the existence of null translational isometries.

We take as our Ansatz

$$ds^2 = e^{-2W(r)}dr^2 + e^{2W(r)}(-dt^2 + dx_3^2) + e^{2V(r)}(dx_1^2 + dx_2^2) - u(r)(dx_3 - dt)^2 \quad (\text{D.3})$$

$$F = Bdx_1 \wedge dx_2$$

with V and W obeying (D.2). Plugging in, we find that the Einstein-Maxwell equations reduce to the following linear equation

$$u'' + (2V' - W')u' - 2(W'' + (W')^2 + 2V'W')u = 0 \quad (\text{D.4})$$

This equation can be solved subject to the boundary condition that $u(r)$ should fall off as $1/r^2$ as $r \rightarrow \infty$. The solution is given by

$$u(r) = p e^{2W(r)} \int_{\infty}^r d\xi e^{-2V(\xi) - 3W(\xi)} \quad (\text{D.5})$$

where p is an integration constant. Given the asymptotics of V and W , the precise fall-off is

$$u(r) = -\frac{p}{4r^2} + \dots \quad (\text{D.6})$$

Evaluating the boundary stress tensor on these solutions we find

$$8\pi G_5 T^{tt} = -\frac{3}{2}u_2 + \frac{1}{2}p \quad (\text{D.7})$$

$$8\pi G_5 T^{t3} = \frac{1}{2}p$$

where u_2 determines the large r falloff of $U = e^{2W}$ according to $U = r^2 + \frac{u_2}{r^2} + \dots$. The form of the stress tensor is consistent with a momentum p being carried by chiral excitations. Another manifestation of this is the formula for the entropy density, which is

$$\frac{1}{G_5} s = 2\pi \sqrt{\frac{c}{6}} L_0 \quad (\text{D.8})$$

with

$$L_0 = \frac{T^3}{2\pi}, \quad c = \frac{B}{2G_5} \tag{D.9}$$

The central charge is the same as was computed in [23].

The solutions just described carry momentum but no electric charge or current. The latter can be included in a fairly trivial way by turning on constant gauge potentials subject to regularity conditions at the horizon. According to (2.7), the combination of the Chern-Simons coupling k and the magnetic field B converts such constant potentials into charges and currents. The regularity condition at the horizon is $A_t + A_3$, and so we can turn on $A_t = -A_3 = A$ for any constant A . This induces

$$4\pi G_5 J^t = 4\pi G_5 J^3 = \frac{1}{2} k B A \tag{D.10}$$

Although we have thus constructed finite entropy, extremal, solutions carrying both charge and magnetic field, we see that this requires that we have a nonzero value of A_3 at infinity, which corresponds to a nonzero chemical potential for current in the CFT. If we demand that this chemical potential is zero, corresponding to the boundary condition $A_3|_\infty = 0$, then these solutions do not appear.

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