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Charged multifluids in general relativity

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Abstract

The exact 1 + 3 covariant dynamical fluid equations for a multi-component plasma, together with Maxwell's equations are presented in such a way as to make them suitable for a gauge-invariant analysis of linear density and velocity perturbations of the Friedmann–Robertson–Walker model. In the case where the matter is described by a two-component plasma where thermal effects are neglected, a mode representing high-frequency plasma oscillations is found in addition to the standard growing and decaying gravitational instability picture. Further applications of these equations are also discussed.

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1. Introduction

Plasmas and electromagnetic fields have an established widespread presence in the universe and are known to play an important role in many astrophysical and cosmological processes. Although in most cases plasma physics can be adequately addressed within the Newtonian or the special relativistic framework, there are occasions where general relativistic considerations should be taken into account. The physics of the early universe offer a very good example in this respect. General relativistic treatments require the rigorous setup of a self-consistent set of equations to describe the plasma dynamics. Moreover, when perturbative techniques are employed, there are extra considerations, such as those related to the gauge invariance of the approach. In this paper, we will try to provide such a setup in the context of cosmological fluid dynamics, leaving the possibility of a kinetic theory based description open for the future[5].

[5] When analysing the CMB spectrum, the kinetic approach is used for the photons, while the electrons are treated as a fluid (their interaction is mediated via Thomson scattering). This is in contrast to many Newtonian applications of plasma physics, where the particle nature of the electromagnetic field is neglected, while electrons are described using a kinetic treatment.

A number of techniques can be used to analyse the equations describing general relativistic plasmas. Depending on the nature of the problem one might employ analytical, numerical and/or perturbative methods. Analytical results are usually based on severe symmetry assumptions, which unavoidably restricts their applicability. Moreover, the inherent nonlinearity of Einstein's theory means that numerical techniques are also non-trivial to apply. Thus, in many cases the most useful method is the perturbative one, possibly combined with numerical methods. In general, we may distinguish between two types of approach.

Non-gravitating plasmas on curved background spacetimes. This method is probably best applied to astrophysical situations, and effectively it comprises two sub-cases: (a) weak gravitational fields, described by a single potential, or weak gravitational waves; (b) strong gravitational fields, where one uses exact solutions to Einstein's field equations for the background. The 'membrane paradigm' (see [1]) is a good example of a formalism which has been developed for this purpose.

Self-gravitating plasmas. In this case one takes into account the plasma contribution to the total gravitational field. This approach, which is more technically demanding than cases (a) and (b) above, is applicable to early universe studies, when most of the baryonic matter was ionized. Below, we will give some examples of studies that have been based on the above described techniques.

A considerable amount of work has been done on the interaction between plasmas and gravity waves and on the use of electromagnetic fields for the detection of gravitational waves (see [2–4] and references therein). This includes studies of parametric excitation of plasma waves in the presence of gravitational radiation [5], the scattering of gravity waves on highly energetic plasmas during supernovae explosions [6] and the possible existence of radio waves due to the emission of weak gravitational waves from binary pulsars [7]. Also, in analogy with the frequency upshifting of short laser pulses observed in laboratory plasmas (e.g., see [8]), it was shown that weak gravitational waves could induce similar phenomena in magnetized multi-component plasmas [9]. Moreover, in [10] the exact plane-fronted parallel (pp) solution to Einstein's field equations (e.g., see [11]) was employed to gain a better understanding of nonlinearities in the interaction between plasmas and gravitational waves (see also [12]).

A number of papers employ the membrane paradigm [1], together with the appropriate fluid equations, to look into the plasma properties in the vicinity of compact astrophysical objects such as black holes. In [13], for example, the authors studied high frequency EM-waves in plasma outside a spherically symmetric black hole, and in [14] they show the possibility of an EM-wave outburst from black holes due to mode conversion. Studies looking at the plasma behaviour near rotating black holes can also be found in the literature [15].

Work has also been done on fluid dynamics and kinetic gas theory in the context of cosmology, notably, the book by Bernstein [16], which treats gas kinetics in the Friedmann–Lemaître–Robertson–Walker (FLRW) model. Nevertheless, there are relatively few relativistic cosmological studies that take into account plasma effects and the behaviour of matter in the presence of electromagnetic fields [17–23]. Thus, the general relativistic treatment of plasmas, both in astrophysics as well as in cosmology, looks like a field open to investigation.

When studying relativistic cosmological perturbations, Bardeen's gauge-invariant formalism is the most influential approach [24]. However, Bardeen's theory is one of some complexity and his variables do not always have a transparent physical and geometrical interpretation. Moreover, the approach is limited to linear perturbations around a FLRW background. Building on the work of Hawking [25] and Olson [26] and utilizing that of Steward and Walker [27], Ellis and Bruni [28] introduced a mathematically simple and physically transparent perturbation scheme. Their formalism, which is both covariant and gauge invariant, has the additional advantage of not being confined to perturbed FLRW universes (see [29] for a comprehensive review). The single fluid analysis of Ellis and Bruni has been extended to multi-component systems by Dunsby *et al* [31], where a number of possible cosmological applications were discussed. Here, we will apply the multi-component formalism of [31] to the case of a charged two-component fluid.

2. Preliminaries

2.1. The multi-component fluid

We assume a family of fundamental observers moving with 4-velocity u^a and a collection of perfect fluids with individual 4-velocities given by

$$u^a_{(i)} = \gamma_{(i)}\big(u^a + v^a_{(i)}\big), \tag{1}$$

where $\gamma_{(i)} \equiv \big(1 - v^2_{(i)}\big)^{-1/2}$ is the Lorentz-boost factor and $v^a_{(i)}u_a = 0$ (i is numbering each fluid). By assumption each fluid has, in its own rest frame, an energy–momentum tensor of the form

$$T^{ab}_{(i)} = (\mu_{(i)} + p_{(i)})u^a_{(i)}u^b_{(i)} + p_{(i)}g^{ab}, \tag{2}$$

where $\mu_{(i)}$ and $p_{(i)}$ are the fluid's energy density and pressure respectively, while g_{ab} is the spacetime metric. Note that, in general, each species has its own equation of state. Relative to the fundamental frame u^a, however, the above equation reads

$$T^{ab}_{(i)} = \hat{\mu}_{(i)}u^a u^b + \hat{p}_{(i)}h^{ab} + 2u^{(a}\hat{q}^{b)}_{(i)} + \hat{\pi}^{ab}_{(i)}, \tag{3}$$

which is the stress–energy tensor of an imperfect fluid with

$$\hat{\mu}_{(i)} \equiv \gamma^2_{(i)}(\mu_{(i)} + p_{(i)}) - p_{(i)}, \tag{4a}$$

$$\hat{p}_{(i)} \equiv p_{(i)} + \tfrac{1}{3}\gamma^2_{(i)}(\mu_{(i)} + p_{(i)})v^2_{(i)}, \tag{4b}$$

$$\hat{q}^a_{(i)} \equiv \gamma^2_{(i)}(\mu_{(i)} + p_{(i)})v^a_{(i)}, \tag{4c}$$

$$\hat{\pi}^{ab}_{(i)} \equiv \gamma^2_{(i)}(\mu_{(i)} + p_{(i)})\left(v^a_{(i)}v^b_{(i)} - \tfrac{1}{3}v^2_{(i)}h^{ab}\right), \tag{4d}$$

and $h^{ab} \equiv g^{ab} + u^a u^b$ is the projection tensor orthogonal to u^a. Note that $\hat{q}^a_{(i)}$ is the heat flow and $\hat{\pi}^{ab}_{(i)}$ is the anisotropic pressure of each fluid component relative to u^a. Clearly, both quantities depend entirely on the motion of the species relative to u^a.

2.2. The electromagnetic field

Charged fluids will interact with each other in the presence of an electromagnetic field. Thus, we also assume the presence of an electromagnetic field described by the Faraday tensor

$$F^{ab} = 2u^{[a}E^{b]} + \epsilon^{abc}B_c, \tag{5}$$

where $E^a = F^{ab}u_b$ and $B^a = \frac{1}{2}\epsilon^{abc}F_{bc}$ are, respectively, the electric and magnetic fields as measured by the fundamental observers (ϵ_{abc} is the spatial permutation tensor). The electromagnetic field contributes to the total energy–momentum tensor by

$$T^{ab}_{(\mathrm{em})} = \tfrac{1}{2}(E^2 + B^2)u^a u^b + \tfrac{1}{6}(E^2 + B^2)h^{ab} + 2u^{(a}\epsilon^{b)cd}E_c B_d - (E^{\langle a}E^{b\rangle} + B^{\langle a}B^{b\rangle}), \tag{6}$$

where angular brackets indicate the projected, symmetric and trace-free part of spacelike vectors and tensors. Finally, the field obeys Maxwell's equations

$$\nabla_b F^{ab} = \mu_0 j^a, \tag{7a}$$

$$\nabla_{[a}F_{bc]} = 0. \tag{7b}$$

2.3. The gravitational field

The dynamics of the gravitational field is determined by Einstein's equations, forming a closed system once the equation of state for the individual fluid components has been established. Of course, in the presence of other physical fields (e.g., anisotropic stresses or spinor fields) we need to supplement the system with the corresponding evolution and constraint equations (e.g., see [32, 33] and references therein). In the presence of an electromagnetic field, the conservation laws for the individual charged species are

$$\nabla_b T^{ab}_{(i)} = \frac{1}{\epsilon_0} F^a{}_b j^b_{(i)} + J^a_{(i)}, \tag{8}$$

with $j^a_{(i)} = \rho_{\mathrm{c}(i)} u^a_{(i)}$ being the 4-current, $\rho_{\mathrm{c}(i)} \equiv -u_a j^a_{(i)}$ the charge density in the rest frame of the fluid and ϵ_0 is the permittivity of vacuum. The term $J^a_{(i)}$ represents interactions other than electromagnetic between the fluids and splits as

$$J^a_{(i)} = \varepsilon_{(i)} u^a + f^a_{(i)}, \tag{9}$$

where $\varepsilon_{(i)}$ is the work per unit volume due to the interaction and $f^a_{(i)}$ is the force density orthogonal to u^a. Because of overall energy–momentum conservation we require that $\sum_i J^a_{(i)} = 0$ and write the total fluid equations as

$$\sum_i \nabla_b T^{ab}_{(i)} = \frac{1}{\epsilon_0} F^a{}_b \sum_i j^b_{(i)}. \tag{10}$$

Moreover, particle conservation ensures that

$$\nabla_a\big(n_{(i)} u^a_{(i)}\big) = 0, \tag{11}$$

where $n_{(i)}$ is the number density of the individual species in their own rest frame. Finally, we point out that the current density in equation (8) can be written as $j^a_{(i)} = q_{(i)} n_{(i)} u^a_{(i)}$, where $q_{(i)}$ is the individual charge of the particles that make up the fluid[6].

3. The fluid equations

3.1. The nonlinear equations

The conservation laws of the individual fluid components, relative to the u^a frame, are obtained by inserting decompositions (4*a*)–(4*d*) into equation (8). In particular, by projecting (8) onto u^a we arrive at the energy density conservation equation

$$\begin{aligned}\dot{\mu}_{(i)} = {} & -(\mu_{(i)} + p_{(i)})\left(\Theta + \tilde{\nabla}_a v^a_{(i)}\right) - \gamma^{-1}_{(i)}(\mu_{(i)} + p_{(i)})\left(\dot{\gamma}_{(i)} + \gamma_{(i)}\dot{u}_a v^a_{(i)} + v^a_{(i)}\tilde{\nabla}_a\gamma_{(i)}\right) \\ & - v^a_{(i)}\tilde{\nabla}_a\mu_{(i)} + \gamma^{-1}_{(i)}\varepsilon_{(i)}.\end{aligned} \tag{12}$$

On the other hand, we derive the momentum density conservation equation

$$\begin{aligned}(\mu_{(i)} + p_{(i)})\left(\dot{u}^a + \dot{v}^{\langle a\rangle}_{(i)}\right) = {} & -\gamma^{-2}_{(i)}\tilde{\nabla}^a p_{(i)} - \tfrac{1}{3}\Theta(\mu_{(i)} + p_{(i)})v^a_{(i)} - \dot{p}_{(i)} v^a_{(i)} \\ & - (\mu_{(i)} + p_{(i)})\left(v^b_{(i)}\tilde{\nabla}_b v^a_{(i)} + \sigma^a{}_b v^b_{(i)} + \epsilon^{abc}\omega_b v_{(i)c}\right) \\ & + \gamma^{-1}_{(i)}(\mu_{(i)} + p_{(i)})\left(v^a_{(i)}\dot{\gamma}_{(i)} + v^a_{(i)} v^b_{(i)}\tilde{\nabla}_b\gamma_{(i)}\right) \\ & - v^a_{(i)} v^b_{(i)}\tilde{\nabla}_b p_{(i)} + \gamma^{-1}_{(i)}\rho_{\mathrm{c}(i)}(E^a + \epsilon^{abc} v_{(i)b} B_c) + \gamma^{-1}_{(i)} f^a_{(i)},\end{aligned} \tag{13}$$

[6] In general, we need to employ the second law of thermodynamics $\nabla_a S^a \geqslant 0$, supply an equation of state for the species and use the covariant equations given in the appendix.

by projecting (8) orthogonal to u^a. Furthermore, the particle number conservation, expressed by equation (11), takes the form

$$\dot{n}_{(i)} = -\Theta n_{(i)} - n_{(i)}\dot{u}_a v^a_{(i)} - \gamma^{-1}_{(i)}\left[\dot{\gamma}_{(i)}n_{(i)} + \tilde{\nabla}_a\left(\gamma_{(i)}n_{(i)}v^a_{(i)}\right)\right]. \tag{14}$$

Similarly, the total fluid equations (see equation (10)) provide the total energy density conservation,

$$\dot{\mu} = -\Theta(\mu + p) - \tilde{\nabla}_a q^a - 2\dot{u}_a q^a - \sigma^a{}_b \pi^b{}_a, \tag{15}$$

and the total momentum density conservation

$$(\mu + p)\dot{u}^a = -\tilde{\nabla}^a p - \tfrac{4}{3}\Theta q^a - \dot{q}^{\langle a\rangle} - \sigma^a{}_b q^b - \epsilon^{abc}\omega_b q_c - \tilde{\nabla}_b \pi^{ab} - \dot{u}_b \pi^{ab} + \rho_c E^a + \epsilon^{abc} j_b B_c, \tag{16}$$

where $\rho_c = \sum_i \rho_{c(i)}$, $j^{\langle b\rangle} = \sum_i j^{\langle b\rangle}_{(i)}$ are the total charge and current density, respectively. Also, $\mu = \sum_i \hat{\mu}_{(i)}$, $p = \sum_i \hat{p}_{(i)}$, $q^a = \sum_i \hat{q}^a_{(i)}$, $\pi^{ab} = \sum_i \hat{\pi}^{ab}_{(i)}$ by definition and the hatted quantities are given by (4*a*)–(4*d*).

The covariant form of Maxwell's equations is obtained by substituting the Faraday tensor, given by (5), into equations (7*a*) and (7*b*). They comprise a set of two propagation and two constraint equations given by [19–22]

$$\dot{E}^{\langle a\rangle} = -\tfrac{2}{3}\Theta E^a + \sigma^a{}_b E^b + \epsilon^{abc}\omega_b E_c + \epsilon^{abc}\dot{u}_b B_c + \mathrm{curl}\, B^a - \frac{1}{\epsilon_0} j^{\langle a\rangle}, \tag{17a}$$

$$\dot{B}^{\langle a\rangle} = -\tfrac{2}{3}\Theta B^a + \sigma^a{}_b B^b + \epsilon^{abc}\omega_b B_c - \epsilon^{abc}\dot{u}_b E_c - \mathrm{curl}\, E^a, \tag{17b}$$

$$\tilde{\nabla}_a E^a = \frac{1}{\epsilon_0}\rho_c + 2\omega_a B^a, \tag{17c}$$

$$\tilde{\nabla}_a B^a = -2\omega_a E^a, \tag{17d}$$

where $\mathrm{curl} B^a \equiv \epsilon^{abc}\tilde{\nabla}_b B_c$, and analogously for $\mathrm{curl} E^a$.

3.2. The linear equations

We will now linearize the equations of the previous section about a FLRW model. Given the homogeneity and the isotropy of the FLRW spacetime, all spatial gradients and velocity components orthogonal to u^a must vanish in the background. This implies that spatial inhomogeneities are first-order quantities and that $\gamma_{(i)} = 1$ to first order. In addition, the symmetries of the FLRW background require that the electromagnetic field vanish to zero order as well. This in turn implies, through equation (17*c*), that ρ_c has zero background value. Similarly, the shear σ^{ab}, vorticity ω^a and the acceleration $\dot{u}^a$ also vanish to zeroth order. As a result, equations (12)–(14) linearize as follows

$$\dot{\mu}_{(i)} = -\left(\Theta + \tilde{\nabla}_a v^a_{(i)}\right)(\mu_{(i)} + p_{(i)}), \tag{18a}$$

$$(\mu_{(i)} + p_{(i)})\left(\dot{u}^a + \dot{v}^a_{(i)}\right) = -\tilde{\nabla}^a p_{(i)} - v^a_{(i)}\dot{p}_{(i)} - \tfrac{1}{3}\Theta(\mu_{(i)} + p_{(i)})v^a_{(i)} + \rho_{c(i)}E^a, \tag{18b}$$

$$\dot{n}_{(i)} = -\left(\Theta + \tilde{\nabla}_a v^a_{(i)}\right)n_{(i)}, \tag{18c}$$

where we have ignored non-electromagnetic interactions between the species (i.e. $\varepsilon_{(i)} = 0 = f^a_{(i)}$). Similarly, the total fluid equations (15) and (16) reduce to

$$\dot{\mu} = -\Theta(\mu + p) - \tilde{\nabla}_a q^a, \tag{19a}$$

$$(\mu + p)\dot{u}^a = -\tilde{\nabla}^a p - \tfrac{4}{3}\Theta q^a - \dot{q}^a. \tag{19b}$$

Finally, Maxwell's equations give

$$\dot{E}^a = -\tfrac{2}{3}\Theta E^a + \mathrm{curl} B^a - \frac{1}{\epsilon_0} j^a, \tag{20a}$$

$$\dot{B}^a = -\tfrac{2}{3}\Theta B^a - \mathrm{curl} E^a, \tag{20b}$$

$$\tilde{\nabla}_a E^a = \frac{1}{\epsilon_0}\rho_c, \tag{20c}$$

$$\tilde{\nabla}_a B^a = 0. \tag{20d}$$

In many applications, it has proved advantageous to adopt the energy frame, defined by the vanishing of the energy flux[7],

$$q^a = \sum_i \hat{q}^a_{(i)} = 0.$$

In this frame equation (16) reduces to

$$(\mu + p)\dot{u}^a = -\tilde{\nabla}^a p,$$

which means that for a dust background the acceleration vanishes to first order.

4. Applications

4.1. Electrically induced velocity perturbations

Consider an Einstein–de Sitter (EdS) background and a two-fluid system, with each component having a dust-like energy–momentum tensor relative to its own frame. In the background, the only non-zero scalars are the total density $\mu = \mu_1 + \mu_2$ and the expansion Θ. Note that to zero order the total charge vanishes (i.e. $\rho_c = -e(n_1 - n_2) = 0$), since both species have equal but opposite charges $q_1 = -e = -q_2$. It follows that ρ_c is a first-order gauge-invariant variable [27]. Furthermore, $\mu_i = m_i n_i$ since no thermal effects are included. In this environment, it is useful to introduce the variables

$$N = n_1 + n_2, \qquad n = n_1 - n_2, \qquad V^a = \tfrac{1}{2}\big(v_1^a + v_2^a\big), \qquad v^a = \tfrac{1}{2}\big(v_1^a - v_2^a\big).$$

Given our frame choice (i.e. $q^a = 0$), equation (4*c*) leads to the first-order result $\mu_1 v_1^a = -\mu_2 v_2^a$ and subsequently to the following relation

$$V^a = -\frac{\delta\mu}{\mu} v^a, \tag{21}$$

between V^a and v^a, where $\delta\mu = \mu_1 - \mu_2$ and $\delta\mu/\mu$ is independent of time. Then, employing equations (18*b*) and (18*c*) we obtain the propagation formulae for N, n and v^a

$$\dot{N} = -(\Theta + \tilde{\nabla}_a V^a)N, \tag{22a}$$

$$\dot{n} = -\Theta n - N\tilde{\nabla}_a v^a, \tag{22b}$$

$$\dot{v}^a = -\tfrac{1}{3}\Theta v^a - \frac{e}{2}\frac{(m_1 + m_2)}{m_1 m_2} E^a. \tag{22c}$$

As expected, equations (22*a*) and (22*b*) show how velocity perturbations, depending on the sign of their 3-divergence, can increase or decrease the number density dilution caused by the expansion. More importantly, equation (22*c*) shows that the presence of the electric field acts as a source of linear velocity perturbations in the charged plasma, even when such perturbations are originally absent (i.e. when $v_a = 0$ initially). In what follows we will see that a non-zero initial velocity perturbation can give rise to density fluctuations (cf (22*b*)), which through equation (20*c*) may seed electric fields.

[7] The electromagnetic contribution to the total heat flux through the Poynting vector $\epsilon^{abc} E_b B_c$ vanishes to first order.

4.2. Velocity induced density perturbations

Consider the dimensionless, first-order gauge-invariant variable

$$\Delta = \frac{a^2}{N}\tilde{\nabla}^2 N, \tag{23}$$

where a is the background scale factor and $\tilde{\nabla}^2 = h^{ab}\tilde{\nabla}_a\tilde{\nabla}_b$ is the covariant Laplacian operator normal to u^a. Equation (23) describes inhomogeneities in the total number density of the particles and, consequently, it also describes inhomogeneities in the total energy density. To linear order the evolution of Δ is determined by the system

$$\dot{\Delta} = -\mathscr{Z} + \frac{\delta\mu}{\mu}a\tilde{\nabla}^2\mathscr{V}, \tag{24a}$$

$$\dot{\mathscr{Z}} = -\tfrac{2}{3}\Theta\mathscr{Z} - \tfrac{1}{4}N[(m_1+m_2)\Delta + (m_1-m_2)a^2\tilde{\nabla}^2 Y], \tag{24b}$$

$$\dot{\mathscr{V}} = -\tfrac{1}{3}\Theta\mathscr{V} + \tfrac{3}{4}\alpha^2\mu a Y, \tag{24c}$$

$$\dot{Y} = -\frac{1}{a}\mathscr{V}, \tag{24d}$$

where $\alpha^2 = 4e^2/3\epsilon_0 m_1 m_2$. In deriving the above system we have employed the first-order gauge-invariant variables

$$\mathscr{Z} = a^2\tilde{\nabla}^2\Theta, \qquad \mathscr{V} = a\tilde{\nabla}_a v^a, \qquad Y = n/N, \tag{25}$$

and used Maxwell's equation (17c). Note that $\mathscr{Z}$ and $\mathscr{V}$ describe scalar inhomogeneities in the expansion and the relative velocity of the species respectively, while Y determines the net charge of the total fluid. Given that equations (24c) and (24d) have decoupled from the rest of the system we can obtain the following propagation equation for Y:

$$\ddot{Y} + \tfrac{2}{3}\Theta\dot{Y} + \tfrac{3}{4}\alpha^2\mu Y = 0. \tag{26}$$

The solution to equation (26) will act as an inhomogeneous driving term in the corresponding propagation equation for Δ:

$$\ddot{\Delta} + \tfrac{2}{3}\Theta\dot{\Delta} - \tfrac{1}{2}\mu\Delta = \left(\tfrac{3}{4}\alpha^2 + \tfrac{1}{2}\right)\frac{\delta\mu}{\mu}\mu a^2\tilde{\nabla}^2 Y, \tag{27}$$

obtained by taking the derivative of equation (24a) and using (24b). According to equations (24d) and (27), velocity inhomogeneities act as sources of density fluctuations. Note that the right-hand side of (27) is a pure multifluid effect, where the part containing α^2 stems from the plasma description.

In order to solve equations (26) and (27) it is standard to decompose the physical (perturbed) fields into a spatial and temporal part, using as eigenfunctions $Q_{(k)}$, solutions of the scalar Helmholtz equation [34]. In particular, we write

$$\Delta = \Delta_{(k)}Q^{(k)}, \qquad Y = Y_{(k)}Q^{(k)}, \tag{28}$$

where $\tilde{\nabla}_a Y_{(k)} = 0 = \tilde{\nabla}_a\Delta_{(k)}$, $\dot{Q}_{(k)} = 0$ and $\tilde{\nabla}^2 Q^{(k)} = -(k^2/a^2)Q^{(k)}$. For an EdS background, the expansion and energy density evolve as $\Theta = 2/t$ and $\mu = 4/3t^2$. Hence, applying the harmonic splitting given above, equations (27) and (26) become

$$\ddot{\Delta}_{(k)} + \frac{4}{3t}\dot{\Delta}_{(k)} - \frac{2}{3t^2}\Delta_{(k)} = -\tfrac{1}{3}k^2(3\alpha^2+2)\frac{\delta\mu}{\mu}\frac{1}{t^2}Y_{(k)}, \tag{29}$$

and

$$\ddot{Y}_{(k)} + \frac{4}{3t}\dot{Y}_{(k)} + \frac{\alpha^2}{t^2}Y_{(k)} = 0, \tag{30}$$

respectively. In order to estimate the value of the parameter α we substitute back for the gravitational constant and write

$$\alpha^2 = \frac{4}{3}\left(\frac{m_e}{m_1}\right)\left(\frac{m_e}{m_2}\right)\left(\frac{e^2}{\epsilon_0}\right)\left(\frac{1}{8\pi G m_e^2}\right) \sim \left(\frac{m_e}{m_1}\right)\left(\frac{m_e}{m_2}\right) \times 10^{42}. \tag{31}$$

Since $\alpha \gg 1$ the solutions to the above equations are

$$\Delta_{(k)} = \mathcal{C}_1 \tau^{2/3} + \mathcal{C}_2 \tau^{-1} + k^2 \frac{\delta\mu}{\mu} Y_{(k)}, \tag{32a}$$

$$Y_{(k)} = [\mathrm{C}_1 \cos(\alpha \ln \tau) + \mathrm{C}_2 \sin(\alpha \ln \tau)]\tau^{-\frac{1}{6}}, \tag{32b}$$

where we have introduced the dimensionless time-coordinate $\tau \equiv t/t_i$, with t_i corresponding to some arbitrary initial time. Hence, in addition to the usual growing and decaying modes of the standard gravitational instability picture, we have obtained a mode representing high-frequency plasma oscillations with a weak damping envelope. This mode is triggered by velocity distortions in the charged plasma and, as expected, has negligible large-scale effect. However, the extra plasma modes become increasingly important as we move on to progressively smaller scales (i.e. for $k \gg 1$).

It should also be pointed out that a *finite temperature* will, in general, cause Landau damping of the plasma oscillations. The effect (requiring kinetic treatment) is small for wavelengths much larger than the Debye length (which is proportional to the thermal velocity of the plasma particles) and in this case the dust fluid approximation is well justified.

4.3. Velocity induced electromagnetic fields

In this section, we will derive the wave equations for the electromagnetic field, seeded by velocity perturbations.

For a cold plasma, the currents for each fluid species may be written as

$$j^a_{(i)} = q_{(i)} n_{(i)} u^a_{(i)} = q_{(i)} n_{(i)} \left(u^a + v^a_{(i)}\right), \tag{33}$$

where $q_{(i)}$ is the charge and $v^a_{(i)}$ is the velocity of the species under consideration. Since we require the plasma to be neutral on the whole, the species are of opposite charge. Hence, the total current j^a appearing in Maxwell's equations reads to first order

$$j^a = j^a_1 + j^a_2 = -eNv^a. \tag{34}$$

From Maxwell's equations (20*a*)–(20*d*), using (34) and (22*c*), one can then deduce wave equations for the induced electromagnetic fields:

$$\ddot{E}_{\langle a\rangle} - \tilde{\nabla}^2 E_a + \tfrac{5}{3}\Theta \dot{E}_{\langle a\rangle} + \left[\tfrac{2}{9}\Theta^2 + \left(\tfrac{3}{4}\alpha^2 + \tfrac{1}{3}\right)\mu\right] E_a = 2\beta^2 \mu \left(\tilde{\nabla}_a Y - \tfrac{1}{3}\Theta v_a\right), \tag{35a}$$

$$\ddot{B}_{\langle a\rangle} - \tilde{\nabla}^2 B_a + \tfrac{5}{3}\Theta \dot{B}_{\langle a\rangle} + \left[\tfrac{2}{9}\Theta^2 + \tfrac{1}{3}\mu\right] B_a = -2\beta^2 \mu \,\mathrm{curl}\, v_a, \tag{35b}$$

where $\beta^2 \equiv e/\epsilon_0(m_1 + m_2)$. These equations govern the interaction of density/velocity perturbations with electromagnetic waves in the plasma and show, in particular, that density/velocity perturbations induce wave phenomena. Observe that B_a and curl v_a are both purely solenoidal, whereas $\tilde{\nabla}_a Y$ has no solenoidal part. It is worthwhile to note that the magnetic field is solely sourced by inhomogeneities in the velocity in contrast to the electric field which is sourced by inhomogeneities in the number density and velocity perturbations. Clearly, the velocity perturbation is non-zero even if $E^a = 0$, as long as $v^a \neq 0$ initially (cf (22*c*)). Both equations (35*a*) and (35*b*) look strikingly similar, the differences originating either from the total current or from a gradient in the charge density (in the case of $\tilde{\nabla}_a Y$). The additional $3\alpha^2/4$-term in the electric wave equation comes from the non-stationarity of the total current, and its large magnitude—$\alpha^2 \sim 10^{42}$ for an e^+e^--plasma—leads directly to the high-frequency behaviour of plasma effects, as will be shown below (see also the preceding section).

It will be useful to introduce expansion normalized variables,

$$\mathscr{E}_a \equiv \frac{E_a}{\Theta}, \qquad \mathscr{B}_a \equiv \frac{B_a}{\Theta}, \qquad \mathscr{K}_a \equiv \frac{\mathrm{curl}\, v_a}{\Theta}. \tag{36}$$

Equations (35a) and (35b), together with equations for the driving terms, then read

$$\ddot{\mathscr{E}}_{\langle a\rangle} - \tilde{\nabla}^2\mathscr{E}_a + \left(\Theta - \tfrac{\mu}{\Theta}\right)\dot{\mathscr{E}}_{\langle a\rangle} + \left[-\tfrac{1}{9}\Theta^2 + \left(\tfrac{3}{4}\alpha^2 + \tfrac{1}{3}\right)\mu\right]\mathscr{E}_a = 2\beta^2\tfrac{\mu}{\Theta}(\tilde{\nabla}_a Y - \tfrac{1}{3}\Theta v_a), \tag{37a}$$

$$\dot{v}_{\langle a\rangle} + \tfrac{1}{3}\Theta v_a = -\tfrac{3}{8}\tfrac{\alpha^2}{\beta^2}\Theta\mathscr{E}_a, \tag{37b}$$

$$\ddot{\mathscr{B}}_{\langle a\rangle} - \tilde{\nabla}^2\mathscr{B}_a + \left(\Theta - \tfrac{\mu}{\Theta}\right)\dot{\mathscr{B}}_{\langle a\rangle} + \left[-\tfrac{1}{9}\Theta^2 + \tfrac{1}{3}\mu\right]\mathscr{B}_a = -2\beta^2\mu\mathscr{K}_a, \tag{37c}$$

$$\dot{\mathscr{K}}_{\langle a\rangle} + \left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\tfrac{\mu}{\Theta}\right)\mathscr{K}_a = \tfrac{3}{8}\tfrac{\alpha^2}{\beta^2}\left[\dot{\mathscr{B}}_{\langle a\rangle} + \left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\tfrac{\mu}{\Theta}\right)\mathscr{B}_a\right]. \tag{37d}$$

Equation (37d) follows from (36) using (37b) and Maxwell's equation (20b).

Restricting ourselves to *scalar* perturbations, we take the divergence of the above equations to extract the scalar part of the system. Of course, there is no contribution from the magnetic field in this case. Using (25) and defining $\mathscr{E} \equiv a\tilde{\nabla}^a\mathscr{E}_a$, equation (37$b$) then transforms into (cf (24c))

$$\dot{\mathscr{V}} + \tfrac{1}{3}\Theta\mathscr{V} = -\tfrac{3}{8}\tfrac{\alpha^2}{\beta^2}\Theta\mathscr{E} = \tfrac{3}{4}\alpha^2\mu a Y, \tag{38}$$

where the last equality is a direct consequence of Maxwell's equation (20c). Combining equation (24d) with (38) and using (19a) together with the commutator expression

$$a\tilde{\nabla}^a\tilde{\nabla}^2\mathscr{E}_a = \tilde{\nabla}^2\mathscr{E} + \left(-\tfrac{2}{9}\Theta^2 + \tfrac{2}{3}\mu\right)\mathscr{E}, \tag{39}$$

one can show that the scalar part of the electric wave equation (37a) reduces to[8]

$$\ddot{\mathscr{E}} + \left(\tfrac{4}{3}\Theta - \tfrac{\mu}{\Theta}\right)\dot{\mathscr{E}} + \left[\tfrac{2}{9}\Theta^2 + \left(\tfrac{3}{4}\alpha^2 - \tfrac{1}{2}\right)\mu\right]\mathscr{E} = 0. \tag{40}$$

In addition, equation (38) gives rise to propagation equations for $\mathscr{V}$ and Y, as discussed earlier:

$$\ddot{\mathscr{V}} + \tfrac{1}{3}\Theta\dot{\mathscr{V}} + \left[-\tfrac{1}{9}\Theta^2 + \left(\tfrac{3}{4}\alpha^2 - \tfrac{1}{6}\right)\mu\right]\mathscr{V} = 0, \tag{41a}$$

$$\ddot{Y} + \tfrac{2}{3}\Theta\dot{Y} + \tfrac{3}{4}\alpha^2\mu Y = 0. \tag{41b}$$

Hence, equations (40)–(41b) all stem from (38).

Specializing to a flat FLRW model with a zero cosmological constant, for which $\mu = 1/3\Theta^2$ and $\Theta = 2/t$ always hold, solutions to these equations can easily be obtained:

$$\mathscr{V}(\tau) = \tfrac{1}{\sqrt{\tau}}\left\{A\cos(\omega\ln\tau) + \tfrac{1}{\omega}\left(\tfrac{1}{2}A + B\right)\sin(\omega\ln\tau)\right\}, \tag{42a}$$

$$\mathscr{E}(\tau) = -\tfrac{9}{4}\tfrac{\beta^2}{\alpha^2}\tfrac{1}{\sqrt{\tau}}\left\{(2A+3B)\cos(\omega\ln\tau) + \tfrac{(2-18\alpha^2)A+3B}{6\omega}\sin(\omega\ln\tau)\right\}, \tag{42b}$$

$$Y(\tau) = \tfrac{t_i}{3a_i}\tfrac{1}{\alpha^2}\tfrac{1}{\tau^{1/6}}\left\{(2A+3B)\cos(\omega\ln\tau) + \tfrac{(2-18\alpha^2)A+3B}{6\omega}\sin(\omega\ln\tau)\right\}. \tag{42c}$$

Here, we used again the dimensionless time-coordinate $\tau \equiv t/t_i$, where t_i denotes some arbitrary initial time. Initial conditions of the velocity perturbation are chosen to be $A = \mathscr{V}(1)$ and $B = \mathscr{V}'(1)$ (a prime stands for ∂_τ). The frequency of the solutions is proportional to

[8] Note that in deriving equation (40), the Laplacian terms cancel, and a harmonic decomposition is, therefore, not needed. Thus, the electric field will not contain a particular length scale, due to its Coulomb-like nature.

$\omega \equiv \sqrt{\alpha^2 - 1/36}$ and grows logarithmically in time. The solutions exhibit high-frequency plasma behaviour. Observe that although the solutions decay with time, their magnitude changes only very slowly over time, particularly if the velocity perturbations are taken to start at the onset of recombination.

We have restricted our attention to scalar perturbations, with the implication that magnetic field effects vanish. From the point of view of generating magnetic seed fields, for, e.g., the dynamo mechanism or the Biermann-battery effect (see [30] and references therein), it is of interest to analyse vector perturbations in a similar way. This is reserved for future research.

5. Discussion

In this paper we generalized the multi-component fluid equations derived by Dunsby *et al* [31] to the case of charged fluids in the presence of electromagnetic fields. The equations are given in a covariant form, relative to an observer moving with velocity u^a that is taken to coincide with the average velocity of the cosmic medium. We linearized these equations about a FRW universe and then applied them to an EdS universe. Our matter field is an ion–electron plasma with zero average pressure (which made the EdS model a suitable background). We showed how, when there is a residual net charge, the presence of an electric field can lead to velocity perturbations even when the latter are originally absent. We also found that velocity distortions can source inhomogeneities in the number density, and therefore in the energy density, of the fluid. In fact, our linear equations reveal the presence of an extra mode, representing high-frequency plasma oscillations, in addition to the standard growing and decaying modes. This mode is likely to be important on scales considerably smaller than the Hubble radius and, therefore, is of little importance as far as structure formation is concerned. It does illustrate, however, interesting small-scale physics that could play a role during the latter stages of galaxy formation.

We also applied our covariant equations to look into the generation of electromagnetic fields due to velocity perturbations in a plasma. The corresponding wave equations, with the velocity distortions playing the role of a source, were given, and they were solved in the case of scalar perturbations. The solutions show high-frequency behaviour typical of a plasma. We restricted our attention to scalar perturbations, thus obtaining an evolution equation for the electric field. However, magnetic field effects were absent since these are related to vector modes. Because magnetic seed fields play a crucial role in, e.g., the dynamo mechanism, it is of great interest to pursue the analysis of the presented equations in the context of vector perturbations. Results in this direction will be presented elsewhere.

There are a number of ways to generalize the discussion presented in this paper. One possibility is to include thermal effects which occur in a photon–baryon plasma giving a non-zero acceleration to first order. This may lead to possible coupling between acoustic and plasma oscillations. In addition, one could apply the ponderomotive force concept between neutrinos and electrons (see [32, 35] and references therein) to cosmology in a covariant context. In this picture, derived from the theory of electroweak interactions, there is an effective interaction between electrons and neutrinos due to density gradients in either species. For instance, the (non-relativistic) force density exerted by neutrinos on the electrons is given by [35]

$$f^a_{(e)} = -\frac{1}{\sqrt{2}}(1 + 4\sin^2\theta_W)G_F n_e \tilde{\nabla}^a n_\nu, \tag{43}$$

where θ_W is the Weinberg angle and G_F is the Fermi constant. Expression (43), together with its neutrino counterpart, could act as a driving force for density fluctuations in the early universe, possibly giving a neutrino signature in the CMB, having an alternating structure as compared to the regular CMB spectrum. The neutrino-driven instability discussed by Silva *et al* [36] (see also [37] for the covariant relativistic form of the same equations), using kinetic theory, could in principle be transferred to a gauge-invariant covariant formalism, suitable for cosmological applications (see also [38]), but this is left for future studies.

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Appendix. Gravitational dynamics

The covariant equations for the dynamics of the gravitational field were given in [29], and we use their notation.

A.1. Covariant equation

- Evolution equations for kinematic variables:

$$\dot{\Theta} - \tilde{\nabla}_a \dot{u}^a = -\tfrac{1}{3}\Theta^2 + (\dot{u}_a \dot{u}^a) - 2\sigma^2 + 2\omega^2 - \tfrac{1}{2}(\mu + 3p) + \Lambda, \tag{A.1}$$

$$\dot{\omega}^{\langle a\rangle} - \tfrac{1}{2}\eta^{abc}\tilde{\nabla}_b \dot{u}_c = -\tfrac{2}{3}\Theta\omega^a + \sigma^a{}_b\omega^b, \tag{A.2}$$

$$\dot{\sigma}^{\langle ab\rangle} - \tilde{\nabla}^{\langle a}\dot{u}^{b\rangle} = -\tfrac{2}{3}\Theta\sigma^{ab} + \dot{u}^{\langle a}\dot{u}^{b\rangle} - \sigma^{\langle a}{}_c\sigma^{b\rangle c} - \omega^{\langle a}\omega^{b\rangle} - \left(E^{ab} - \tfrac{1}{2}\pi^{ab}\right). \tag{A.3}$$

- Constraint equations for kinematic variables:

$$0 = \tilde{\nabla}_b\sigma^{ab} - \tfrac{2}{3}\tilde{\nabla}^a\Theta + \eta^{abc}[\tilde{\nabla}_b\omega_c + 2\dot{u}_b\omega_c] + q^a, \tag{A.4}$$

$$0 = \tilde{\nabla}_a\omega^a - (\dot{u}_a\omega^a), \tag{A.5}$$

$$0 = H^{ab} + 2\dot{u}^{\langle a}\omega^{b\rangle} + \tilde{\nabla}^{\langle a}\omega^{b\rangle} - (\mathrm{curl}\,\sigma)^{ab}, \tag{A.6}$$

  where $(\mathrm{curl}\,\sigma)^{ab} = \eta^{cd\langle a}\tilde{\nabla}_c\sigma^{b\rangle}{}_d$.

- Evolution equations for the curvature variables:

$$\begin{aligned}\left(\dot{E}^{\langle ab\rangle} + \tfrac{1}{2}\dot{\pi}^{\langle ab\rangle}\right) - (\mathrm{curl}\,H)^{ab} + \tfrac{1}{2}\tilde{\nabla}^{\langle a}q^{b\rangle} &= -\tfrac{1}{2}(\mu + p)\sigma^{ab} - \Theta\left(E^{ab} + \tfrac{1}{6}\pi^{ab}\right)\\ &\quad + 3\sigma^{\langle a}{}_c\left(E^{b\rangle c} - \tfrac{1}{6}\pi^{b\rangle c}\right) - \dot{u}^{\langle a}q^{b\rangle} + \eta^{cd\langle a}\left[2\dot{u}_c H^{b\rangle}{}_d + \omega_c\left(E^{b\rangle}{}_d + \tfrac{1}{2}\pi^{b\rangle}{}_d\right)\right],\end{aligned} \tag{A.7}$$

$$\begin{aligned}\dot{H}^{\langle ab\rangle} + (\mathrm{curl}\,E)^{ab} - \tfrac{1}{2}(\mathrm{curl}\,\pi)^{ab} - \Theta H^{ab} + 3\sigma^{\langle a}{}_c H^{b\rangle c} + \tfrac{3}{2}\omega^{\langle a}q^{b\rangle}\\ - \eta^{cd\langle a}\left[2\dot{u}_c E^{b\rangle}{}_d - \tfrac{1}{2}\sigma^{b\rangle}{}_c q_d - \omega_c H^{b\rangle}{}_d\right],\end{aligned} \tag{A.8}$$

  where

$$(\mathrm{curl}\,H)^{ab} = \eta^{cd\langle a}\tilde{\nabla}_c H^{b\rangle}{}_d, \tag{A.9}$$

$$(\mathrm{curl}\,E)^{ab} = \eta^{cd\langle a}\tilde{\nabla}_c E^{b\rangle}{}_d, \tag{A.10}$$

$$(\mathrm{curl}\,\pi)^{ab} = \eta^{cd\langle a}\tilde{\nabla}_c \pi^{b\rangle}{}_d. \tag{A.11}$$

- Constraint equations for the curvature variables:

$$0 = \tilde{\nabla}_b\big(E^{ab} + \tfrac{1}{2}\pi^{ab}\big) - \tfrac{1}{3}\tilde{\nabla}^a\mu + \tfrac{1}{3}\Theta q^a - \tfrac{1}{2}\sigma^a{}_b q^b - 3\omega_b H^{ab} - \eta^{abc}\big[\sigma_{bd}H^d{}_c - \tfrac{3}{2}\omega_b q_c\big], \tag{A.12}$$

$$0 = \tilde{\nabla}_b H^{ab} + (\mu + p)\omega^a + 3\omega_b\big(E^{ab} - \tfrac{1}{6}\pi^{ab}\big) + \eta^{abc}\big[\tfrac{1}{2}\tilde{\nabla}_b q_c + \sigma_{bd}\big(E^d{}_c + \tfrac{1}{2}\pi^d{}_c\big)\big]. \tag{A.13}$$

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