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Charged Pion Condensation as a Standing Wave Mode*)

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The possibility of existence of a standing wave field describing charged pion condensation in a system of nucleons with a definite charge is discussed in the framework of a modified Alternating Layer Spin model.

The dependence of the energy of the system on the total charge is presented. Using a π-N p-wave interaction it is found that, in a neutral superdense system of nucleons, the standing wave modes of the pion condensate lead to a lower energy than the running wave modes.

§ 1. Introduction

According to the Alternating Layer Spin [ALS] model,[1] when the density of a system of nucleons exceeds a critical value, the localization of nucleons in layers with a particular spin-isospin ordering acts as the source of a pion-field and π^0 condensation occurs with a remarkable energy gain.

An extension of this model was proposed later[2] which allows simultaneous condensation of π^0 and charged pions, π^c, in neutron matter. In this work the π^0 field is a standing wave, but the π^c field is taken as a running wave, the momenta of the two fields being perpendicular to each other.

The [ALS] state that provides the coexistence of both types of condensation is constructed with quasi-nucleons which are superpositions of proton and neutron of opposite spin in the same layer, without changing the source function of the π^0 field. The energy gain with this structure is approximately double of the one for π^0 condensation only. It was however suggested by the authors that a greater energy gain could perhaps be obtained if the π^c field was also taken as a standing wave mode.

In fact Sawyer and Yao[3] have pointed out that, in a system of neutron matter with densities significantly higher than the critical density, standing wave modes for π^- condensation lead to a lower energy than running wave ones, in the presence of a p-wave π-N interaction only.

The aim of the present work is to suggest an extension of the [ALS] model which allows for charged pion condensation in standing wave modes. The charged mesons are introduced through a suitable rotation in the isotopic spin space of the original [ALS] wave function for symmetric nuclear matter. The total charge is fixed by a Lagrange multiplier.

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This method not only allows the study of π^c condensation in a system of fixed charge, zero charge included, but also gives a description of the dependence of the total energy of a specific system on the charge.

§ 2. The model

2.1. *The Hamiltonian*

We restrict ourselves to a π-N p-wave interaction. The Hamiltonian can be written as a sum of nucleon and pion kinetic energy terms and an interaction term consisting of the usual $\bar{\sigma}\ \bar{k}$ coupling:

$$H = H_N + H_\pi + H_{\pi-N}\ , \tag{2·1a}$$

$$H_N = \sum_{\boldsymbol{p},\alpha,\gamma} \frac{p^2}{2M} C^+_{\boldsymbol{p},\alpha,\gamma} C_{\boldsymbol{p},\alpha,\gamma}\ , \tag{2·1b}$$

$$H_\pi = \sum_{\boldsymbol{k},\nu} W_{\boldsymbol{k}} A^+_{\boldsymbol{k},\nu} A_{\boldsymbol{k},\nu}\ , \tag{2·1c}$$

$$H_{\pi-N} = g \sum_{s} k_\nu/(2W_{\boldsymbol{k}})^{1/2} [C^+_{\boldsymbol{p}+\boldsymbol{k},\alpha,\gamma} (\tau_\mu)_{\alpha,\beta} (\sigma_\nu)_{\gamma,\delta} C_{\boldsymbol{p},\beta\delta} A_{\boldsymbol{k},\nu} + \text{c.c.}], \tag{2·1d}$$

where $C^+_{\boldsymbol{p},\alpha,\gamma}$ stands for the creation operator of a nucleon with momentum $\boldsymbol{p}$, isospin α and spin γ; $A^+_{\boldsymbol{k},\nu}$ for the creation operator of a pion with momentum $\boldsymbol{k}$ and isovector index $\nu(\nu=1,2,3)$. Moreover $g=m_\pi^{-1}$ designates the coupling constant, m_π and M the pion and nucleon mass respectively $W_{\boldsymbol{k}}=(k^2+m_\pi{}^2)^{1/2}$ and τ_μ, σ_ν are the Pauli matrices for the isospin and spin. The index s specifies the set of all quantum numbers involved, namely: $s=\{\boldsymbol{p}, \boldsymbol{k}, \alpha, \beta, \gamma, \delta, \mu, \nu\}$.

2.2. *The nucleon and meson states*

Similarly to the [ALS] model[1)] the nucleons are localized along the z-direction giving rise to a layer structure, spaced by $z_{i+1}-z_i=b$. The spatial part of the localized functions is a Gaussian depending on a parameter λ. The total wave function for the nucleons is then a Slater determinant of single particle wave functions $|\phi_i\rangle$, given as:

$$|\phi_i\rangle = n \sum_{\boldsymbol{p},\alpha,\gamma} \exp(-\lambda P_z{}^2) \exp(iP_z z_i) \delta_{P_x P_{x_i}} \delta_{P_y P_{y_i}} \chi_\alpha{}^i \eta_\gamma{}^i C^+_{\boldsymbol{p},\alpha,\gamma} |\rangle, \tag{2·2}$$

where $\chi_\alpha{}^i \eta_\gamma{}^i$ are the isospin and spin wave functions, n is a normalization constant and $|\rangle$ the vacuum of the nucleons. The momenta along the x and y directions are subjected to the restriction imposed by the Fermi level momentum P_F, $P^2_{x_i} + P^2_{y_i} \leqq P_F{}^2$.

Not all of these wave functions are orthogonal to each other. However the overlap between two of them, with the same quantum numbers, will vanish if the distance between two layers is greater than $2\sqrt{2\lambda}$,. which greatly simplifies the calculations. The significant region corresponds to layers sufficiently spaced to satisfy the above condition.

The condensed pion state is described by a coherent state $|\xi\rangle$,[4)]

$$|\xi\rangle=\mathcal{N}\exp\sum_{\boldsymbol{k},\nu}(\xi_{\boldsymbol{k},\nu}A^{+}_{\boldsymbol{k},\nu})|\rangle, \tag{2·3}$$

where $\xi_{\boldsymbol{k},\nu}$ are variational parameters associated with the number of pions, $\mathcal{N}$ is a normalization constant and $|\rangle$ the vacuum for the pions.

Several options to specify the spin and isospin part of the nucleon functions are possible. The adequate choice for the study of π^0 condensation, as shown in Ref. 1), is made by considering states that are eigenstates of the isospin operator τ_z. The spin changes from layer to layer and in the same layer it is equal if the nucleons are of the same type and different otherwise. The expectation value of the interaction Hamiltonian in such a class of wave functions depends only on terms depending on ξ_{kz}, π^0. The pion field present is therefore the π^0 field and the total charge of the system remains constant and equal to the number of protons present. The dominant component of this field is a standing wave mode with momentum $k_c=\pi/b$ where b is the distance between two layers. The energies of two systems with different ratio of protons and neutrons differ only in the kinetic energy term, through its dependence on the Fermi level. Therefore a system made up of equal number of protons and neutrons has lower energy than a system where only neutrons are present (cf., Eq. (2·18) of Ref. 1)).

In order to study charged pion condensation and keep at the same time the total charge neutral we are lead to construct a wave function of quasi-nucleons. Such state can be obtained through a rotation of an angle θ in the isotopic spin space, around the y axis, of the [ALS] functions for symmetric nuclear matter. Mixtures of proton and neutron states are obtained for the nucleon part and π^+ and π^- fields contribute now to the total energy. The charge of the π^- cloud may compensate the charge of the protons and the one of the π^+ cloud, leading to a zero net charge. Restrictions on the charge can be made by considering a Lagrange multiplier μ in a constrained variational principle. This charge restriction gives rise to an increase in energy. However we expect this effect to be compensated by the low Fermi level that the new description allows, leading to a significant energy gain for a specific density region.

2.3. *The energy*

The energy of the system is now studied in terms of the quantities defined in the last sections, under the restriction that the total charge has a given value. This can be imposed by using a variational principle for $H+\mu Q$ with the rotated wave function, where Q is the total charge operator and μ the Lagrange multiplier that fixes the charge. The same result is obtained, of course, if one rotates the operator $H+\mu Q$, leaving the wave function unrotated. However, since H is invariant under rotations in the isotopic spin space, one has to rotate only Q. Therefore we minimize the expectation value of $H+\mu Q_\theta$, with the unrotated wave function, where

$$Q_\theta=(1/2)N+T_z\cos\theta+T_x\sin\theta \tag{2·4}$$

and T_l $(l=x, y, z)$ is the total isospin operator:

$$T_l=(\tau_l)_{\alpha,\beta}C^+_{p,\alpha,\gamma}C_{p,\beta,\delta}+(S_l)_{\mu,\nu}A^+_{k,\mu}A_{k,\nu} \tag{2·5}$$

with τ_l and S_l the isospin matrices for the nucleons and pions respectively and N the total number of nucleons. All repeated indices are summed over in (2·5).

Minimizing with respect to the pion amplitudes, $\xi_{k,\nu}$, one obtains:

$$\langle H+\mu Q_\theta\rangle_{\xi_{\min}}=\frac{N}{2M}\left[\pi\rho b+\frac{1}{4\lambda}\right]+\mu\frac{N}{2}$$

$$-g^2N\rho e^{-\lambda k_c^2}\frac{w_c^2-\mu^2\cos\theta}{w_c^2(w_c^2-\mu^2)}k_c^2\,. \tag{2·6}$$

Here ρ is the nucleon density.

The energy and the charge per nucleon are respectively:

$$E(\theta)/N=\frac{1}{2M}\left[\pi\rho b+\frac{1}{4\lambda}\right]-g^2\rho e^{-\lambda k_c^2}\frac{k_c^2}{w_c^2}\frac{w_c^2+\mu^2\cos\theta}{w_c^2-\mu^2}$$

$$+2g^2\rho e^{-\lambda k_c^2}\frac{k_c^2}{w_c^2}\mu^2\frac{w_c^2-\mu^2\cos\theta}{(w_c^2-\mu^2)^2} \tag{2·7}$$

and

$$Q(\theta)/N=\frac{1}{2}-2g^2\rho e^{-\lambda k_c^2}\frac{k_c^2}{w_c^2}\frac{\mu\sin^2\theta}{(w_c^2-\mu^2)^2}\,, \tag{2·8}$$

where the subscript c designates the lowest mode available.

The number of π^- and π^+ per nucleon is easily found to be

$$\chi^+(\theta)=\frac{g^2\rho k_c^2}{2w_c(w_c+\mu)^2}e^{-\lambda k_c^2}\frac{(\mu w_c\sin\theta-w_c^2+\mu^2\cos^2\theta)^2}{w_c^2(w_c-\mu)^2}\,, \tag{2·9}$$

$$\chi^-(\theta)=\frac{g^2\rho k_c^2}{2w_c(w_c-\mu)^2}e^{-\lambda k_c^2}\frac{(\mu w_c\sin\theta+w_c^2-\mu^2\cos^2\theta)^2}{w_c^2(w_c+\mu)^2}\,. \tag{2·10}$$

We can ask now which is the best angle of rotation in the sense of giving minimal energy for a definite charge. We easily find $\theta=\pi/2$ for values of $\mu\neq 0$. This implies that the total wave function is an eigenfunction of $T_x=\tau_x+S_x$ and that π^+ and π^- are present in the field as a π^x condensate, but π^0 is excluded. The source for π^+ and π^- are therefore alternating layers of quasi-nucleons that are eigenstates of $\tau_x\sigma_z$.

If $\mu=0$ the energy, of course, depend on θ due to the symmetry of H. In this case the minimum is therefore independent of the value of θ and is equal to the minimum for π^0 condensation in symmetric nuclear matter:

$$E_{\mu=0}/N=\frac{1}{2M}\left[\pi\rho b+\frac{1}{4\lambda}\right]-g^2\rho e^{-\lambda k_c^2}\frac{k_c^2}{w_c^2} \tag{2·11}$$

the total charge being also equal to $N/2$. The π^x and π^0 condensation are then energetically equivalent in symmetric nuclear matter.

§ 3. Results and discussion

Equation (2·6) for $\theta=\pi/2$ still depends on the chemical potential, μ, the parameter of the gaussian nucleon wave function, λ, and the distance between two layers, b.

The minima with respect to the parameters b and λ are found numerically for each value of μ and consequently for each charge value.

Table I. Numerical results, for $\rho=3\rho_0$, in a system with charge per nucleon varying from .5 to .0; energies per nucleon and parameters calculated with $g=m_\pi^{-1}$, $m_\pi=.7\ \text{fm}^{-1}$.

Q/N	E/N	μ	k_c	λ
.5	−.090	.00	1.85	.11
.4	−.076	.30	1.85	.11
.3	−.038	.45	1.65	.13
.2	+.008	.55	1.60	.14
.1	+.084	.65	1.50	.16
.0	+.157	.70	1.40	.18
	(fm^{-1})	(fm^{-1})	(fm^{-1})	(fm^2)

In Table I we present the resulting values of the energy of a charged pion condensate for different values of the charge. The nucleon density is considered to be 3 ρ_0, where ρ_0 is the normal nuclear density ($\rho_0=.18\ fm^{-3}$). Similar patterns are obtained for other densities.

In Fig. 1 the energy is plotted as a function of the charge. As one can see, the lowest energy is obtained for a total charge equal to $N/2$ and corresponds to a neutral condensate. This is in agreement with the work of Migdal.[5]

Table II shows values of the energy gain, ΔE, of a system with zero charge, relatively to a free neutron Fermi gas, for densities above $\rho=2.5\rho_0$. Below this density we are entering a region where the overlap between the individual wave functions plays a significant role and our approach breaks down.

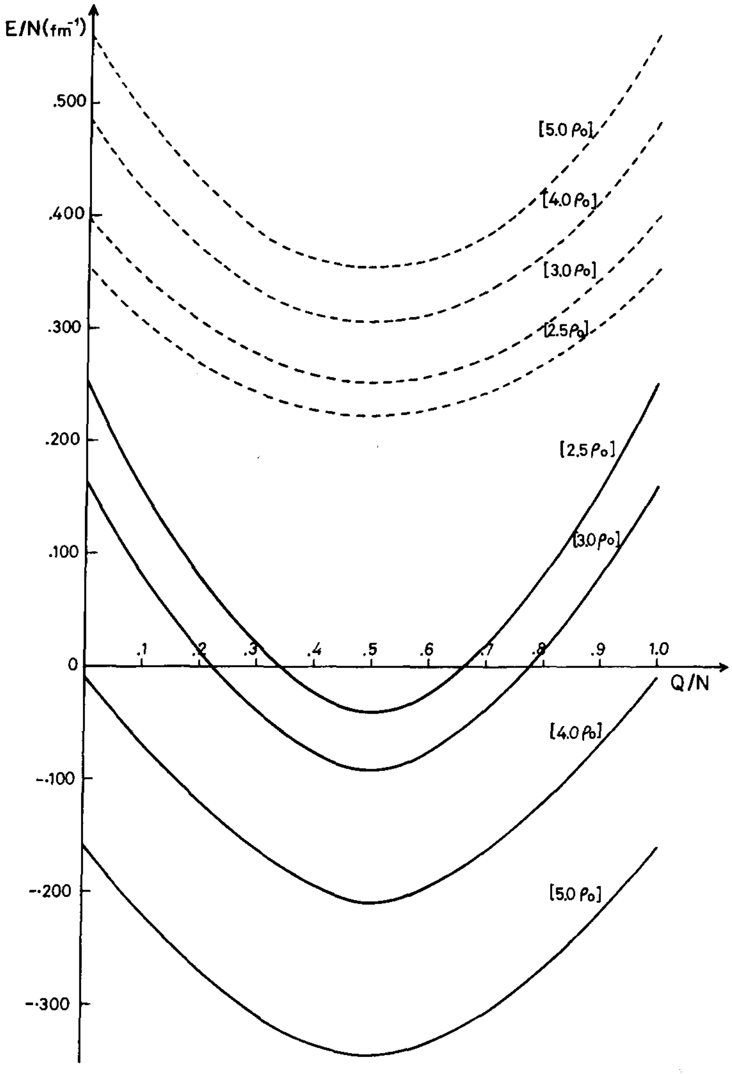

Fig. 1. Energy per nucleon as a function of the charge, for different values of the density, of a system in the π^c condensation phase (full lines); the energies of a free Fermi gas of nucleons are also shown (dashed lines).

Our results are compared in Fig. 2 with the results of Refs. 2) and 3), where running wave modes in the plane and in the three dimensional space are considered for a similar type of condensation. Our condensate is favoured for densities above $\simeq 3\rho_0$. This fact can be understood, as pointed out by Bäckman and Weise[6] by noticing that the chemical potential for the reaction $n \to p + \pi^-$ is nonvanishing in neutron matter and zero in symmetric nuclear matter. This would imply a running wave state in the first case and a static pion field in the

Table II. Numerical results, for different values of the density, of π^c condensation in a system with zero charge. The energy gains (ΔE), number of pions per nucleon (X^+ and X^-) and parameters have been calculated for g and m_π as given in Table I.

ρ	ΔE [a]	μ	k_c	λ	X^+	X^-
.45	−.105	.70	1.26	.28	.067	.561
.54	−.243	.70	1.43	.18	.089	.585
.72	−.476	.60	1.50	.13	.140	.640
.90	−.727	.60	1.75	.09	.176	.657
(fm^{-3})	(fm^{-1})	(fm^{-1})	(fm^{-1})	(fm^2)		

a) $\Delta E = E/N$ (pion condensed phase) $- E_{FG}/N$ (pure neutron phase); $E_{FG}/N = \frac{1}{2M}\frac{3}{5}(3\pi^2\rho)^{2/3}$.

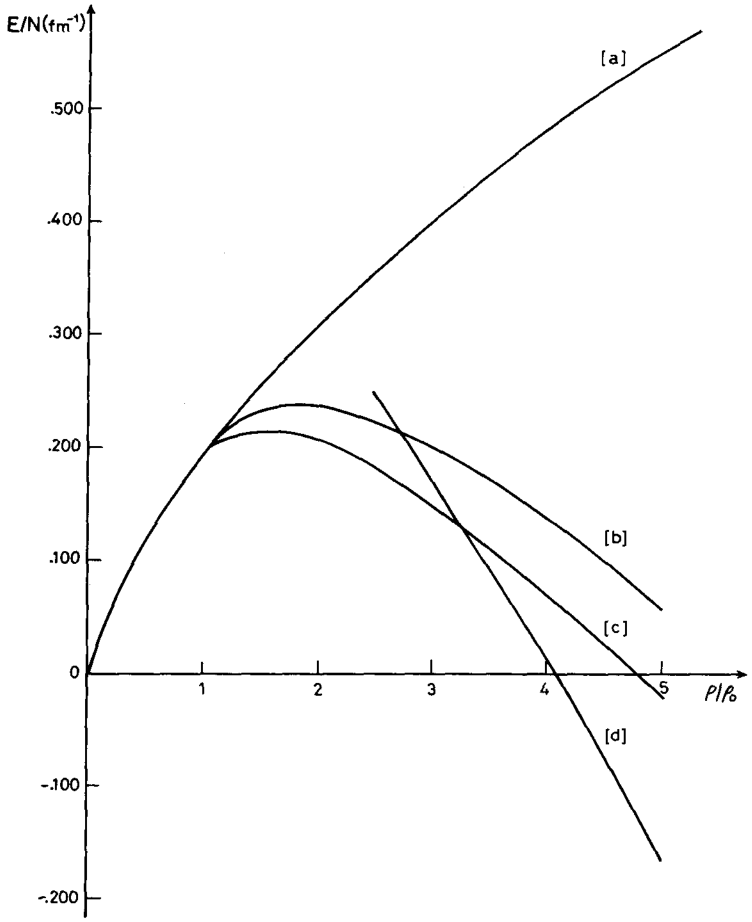

Fig. 2. Energy per nucleon, as a function of the density of a system with zero charge:
[a]. energy of a free Fermi gas of neutrons,
[b] [c] and [d]. energies of the π^c condensed phase described in terms of:
[b]. running waves (Ref. 2)),
[c]. 〃 〃 (Ref. 3)),
[d]. standing waves (present calculation).

case of nuclear matter. However in a well-developed π^c condensate in neutron matter a great number of π^- are present (see Fig. 5 where the ratio of pions to the nucleons as a function of the density is presented for a system with zero charge), giving rise to proton and neutron Fermi levels. A similar situation to the one in symmetric nuclear matter is reached and charged pion condensate in standing waves is responsible for the stability of the system.

For densities below $\simeq 3\rho_0$ our results would lead us to say that a description in terms of running waves is preferable, however a variational method is not appropriate to study this region because when the condensation is weak the pion field leads to the appearance of a potential which is better described by

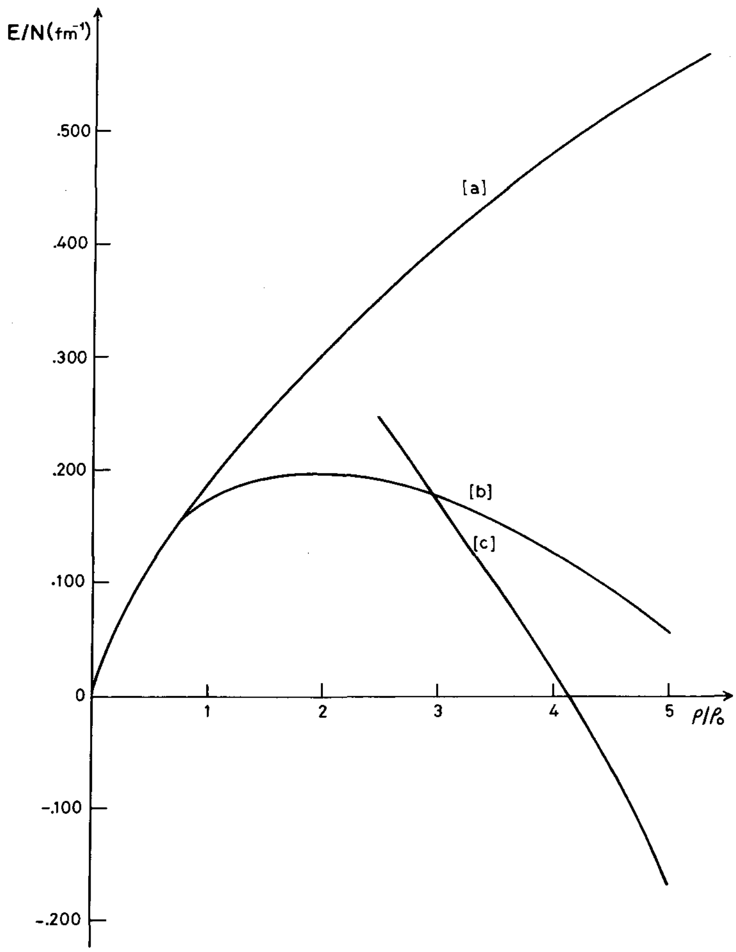

Fig. 3. Energy per nucleon as a function of the density for a system with zero charge:
[a]. Free Fermi gas of neutrons,
[b]. π^0 condensed phase (Ref. 1)),
[c]. π^c 〃 〃 (present calculation).

perturbation theory; otherwise the variational method is appropriate and convenient for the region of well-developed condensate. What probably happens is that different types of localized structures appear, both very near and very far from the critical density, as it seems to be the case in symmetric nuclear matter.[7]

The discrepancy between Fig. 2 in the low density region, $\rho \lesssim 3\rho_0$, and the calculation of Sawyer and Yao,[3] namely Eqs. (32) and (33), is to be expected. These authors consider the region of weak condensate in the framework of perturbation theory, which is probably a more adequate method at these densities. For higher densities, $\rho \gtrsim 3\rho_0$, the agreement between both results is very good, as it can be seen from Fig. 1 of Ref. 3) and Fig. 2 in this paper.

The energy obtained with the present extension of the [ALS] model for a π^c condensate in a system with neutral charge is, if $\rho \gtrsim 2.8\rho_0$, lower than the one obtained for a π^0 condensate in pure neutron matter, described by the original [ALS] model. This is shown in Fig 3. Above these densities the lowering of the Fermi level compensates for the energy raising due to the restriction on the charge. Here also the discrepancy between our results and what would be expected from the work of Sawyer and Yao, concerning the crossing region of curves [b] and [c], is due to the limitation of our model in this parameter region.

In Fig. 4 we plot the ratio of pions as a function of the charge for $\rho = 3\rho_0$. Figure 5 shows the ratio of pions, in a system of zero charge, as a function of the density.

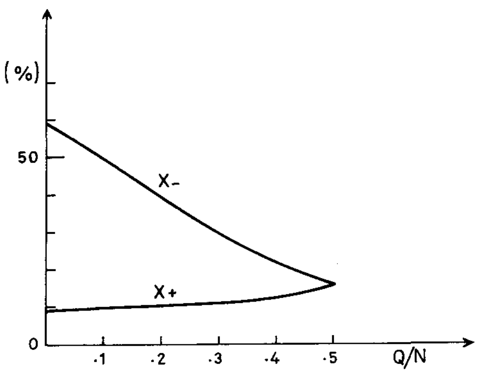

Fig. 4. Number ratios of pions to the nucleons, X^- and X^+, as a function of the charge for $\rho = 3\rho_0$.

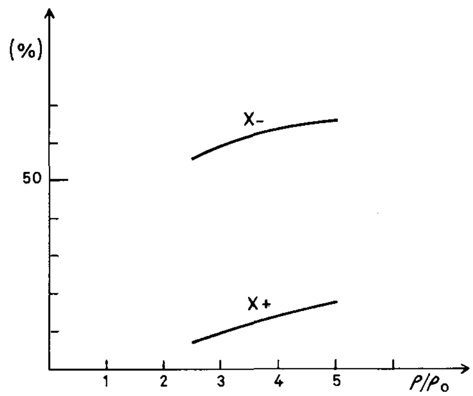

Fig. 5. Number ratios of pions to the nucleons, X^- and X^+, as a function of the density, in a system with zero charge.

§ 4. Conclusions

The present method allows the study of possible occurrence of π^c condensation in standing wave modes, associated with a solid-like structure for the nucleons, in a system with fixed charge.

The variation of the energy as a function of the charge shows that the most stable system is the one isotopically symmetric and with a neutral condensate.

We were also able to prove that, for a system with zero charge, the description of π^c condensate by standing wave modes is preferable to a description in terms of running waves, for densities above $\rho \simeq 3\rho_0$. Below this density our results indicate that running wave modes lead to a lower energy, but the inadequacy of our variational technique for this region does not allow us to reach definitive conclusions concerning this point.

Although our method can certainly be improved, we can already conclude that a well-developed condensate in neutron matter can be described in terms of standing waves, in the presence of a π-Np-wave interaction only. This comes to reinforce similar conclusions of Ref. 3), where both a different method was used and where the π^+ component of the pion field was not taken into account.

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