

CHARLIER SPACES OF ENTIRE FUNCTIONS

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The paper is concerned with examples of Hilbert spaces whose elements are entire functions and which have these properties:

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w , the function $F(z)(z-\bar{w})/(z-w)$ belongs to the space and has the same norm as $F(z)$.

(H2) For each nonreal number w , the linear functional defined on the space by $F(z) \rightarrow F(w)$ is continuous.

(H3) The function $F^*(z) = \overline{F(\bar{z})}$ belongs to the space whenever $F(z)$ belongs to the space, and it always has the same norm as $F(z)$.

The theory of these spaces is related to the theory of entire functions $E(z)$ which satisfy the inequality

$$|E(x-iy)| < |E(x+iy)|$$

for $y > 0$. If $E(z)$ is such a function, we write $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions which are real for real z , and

$$K(w, z) = [B(z)\overline{A(w)} - A(z)\overline{B(w)}]/[\pi(z - \bar{w})].$$

Let $\mathcal{H}(E)$ be the set of entire functions $F(z)$ such that

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt < \infty$$

and such that

$$|F(z)|^2 \leq \|F\|^2 K(z, z)$$

for all complex z . Then $\mathcal{H}(E)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). For each complex number w , $K(w, z)$ belongs to $\mathcal{H}(E)$ as a function of z , and the identity

$$F(w) = \langle F(t), K(w, t) \rangle$$

holds for all elements $F(z)$ of $\mathcal{H}(E)$. A Hilbert space, whose elements are entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element is equal isometrically to a space $\mathcal{H}(E)$.

The spaces now studied are finite dimensional spaces related to Charlier's orthogonal polynomials. They are characterized by an

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identity involving a positive parameter a . The identity implies a recurrence relation for the defining functions $A(z)$ and $B(z)$ of the space.

THEOREM 1. *Let $\mathfrak{H}(E)$ be a given space such that $E(z)$ has a nonzero value at the origin, and let a be a given positive number. Assume that the functions $z[F(z) - F(z - 1)]$ and $F(z + 1) - F(z)$ belong to the space whenever $F(z)$ belongs to the space, and that the identity*

$$\begin{aligned} \langle t[F(t) - F(t - 1)] - a^2[F(t + 1) - F(t)], G(t) \rangle \\ = \langle F(t), t[G(t) - G(t - 1)] - a^2[G(t + 1) - G(t)] \rangle \end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space. Then there exist real numbers u_+, v_+, u_-, v_- such that the functions

$$S_+(z) = A(z)u_+ + B(z)v_+ \text{ and } S_-(z) = A(z)u_- + B(z)v_-$$

are linearly independent and satisfy the recurrence relations

$$\begin{aligned} \lambda_+ S_+(z) &= z[S_+(z) - S_+(z - 1)] - a^2[S_+(z + 1) - S_+(z)], \\ \lambda_- S_-(z) &= z[S_-(z) - S_-(z - 1)] - a^2[S_-(z + 1) - S_-(z)], \\ \lambda_+ S_-(z) &= aS_+(z + 1) - aS_+(z), \\ S_+(z) &= a^{-1}zS_-(z - 1) - aS_-(z), \end{aligned}$$

for some real numbers λ_+ and λ_- such that $\lambda_+ = 1 + \lambda_-$.

Kummer's confluent hypergeometric function

$$F(a; c; z) = 1 + \frac{a}{1!c} z + \frac{a(a + 1)}{2!c(c + 1)} z^2 + \dots$$

is used to construct spaces satisfying the hypotheses of Theorem 1.

THEOREM 2. *If $a > 0$ is given, then the polynomials $\Phi_n(z)$ defined by*

$$\Phi_n(z) = (-a)^{-n} [\Gamma(n - z) / \Gamma(-z)] F(-n; 1 + z - n; a^2)$$

are real for real z and satisfy the identities

$$\begin{aligned} n\Phi_n(z) &= z[\Phi_n(z) - \Phi_n(z - 1)] - a^2[\Phi_n(z + 1) - \Phi_n(z)], \\ n\Phi_{n-1}(z) &= a\Phi_n(z + 1) - a\Phi_n(z), \\ \Phi_{n+1}(z) &= a^{-1}z\Phi_n(z - 1) - a\Phi_n(z), \\ z\Phi_n(z) &= an\Phi_{n-1}(z) + (n + a^2)\Phi_n(z) + a\Phi_{n+1}(z). \end{aligned}$$

There exist spaces $\mathfrak{H}(E_n)$, $n = 1, 2, 3, \dots$, satisfying the hypotheses of Theorem 1, such that $\mathfrak{H}(E_n)$ is contained isometrically in $\mathfrak{H}(E_{n+1})$

for every n , such that $\Phi_0(z)$ spans $\mathfrak{H}(E_1)$, and such that $\Phi_n(z)$ spans the orthogonal complement of $\mathfrak{H}(E_n)$ in $\mathfrak{H}(E_{n+1})$ for every $n > 0$. The spaces can be chosen so that $\|\Phi_n(t)\|^2 = \Gamma(1+n)$ for every n . The identity

$$e^{a^2} \|F(t)\|^2 = \sum_{n=0}^{\infty} |F(n)|^2 a^{2n} / \Gamma(1+n)$$

then holds for every polynomial $F(z)$.

These are essentially all the spaces which satisfy the hypotheses of Theorem 1.

THEOREM 3. *If $\mathfrak{H}(E)$ is a space which satisfies the hypotheses of Theorem 1, then there exists an entire function $S(z)$ which is real for real z , has only real zeros, and is periodic of period one, and there exists an index r in Theorem 2 such that the transformation $F(z) \rightarrow S(z)F(z)$ is an isometry of $\mathfrak{H}(E_r)$ onto $\mathfrak{H}(E)$.*

These spaces, like those of previous work [2], are related to generalized spaces of square summable power series. Let a and c be numbers such that the coefficients of Kummer's series $F(a; c; z)$ are all positive. By $\mathfrak{C}(a; c; z)$ we mean the Hilbert space of power series $f(z) = \sum a_n z^n$ with complex coefficients such that

$$\|f(z)\|^2 = |a_0|^2 + \frac{1!c}{a} |a_1|^2 + \frac{2!c(c+1)}{a(a+1)} |a_2|^2 + \dots < \infty.$$

The series which belong to $\mathfrak{C}(a; c; z)$ converge in the complex plane and represent entire functions. The series $F(a; c; \bar{w}z)$ belongs to the space for all complex numbers w , and the identity

$$f(w) = \langle f(z), F(a; c; \bar{w}z) \rangle$$

holds for every element $f(z)$ of the space.

THEOREM 4. *In Theorem 2 if $f(z) = \sum a_n z^n$ is a polynomial of degree less than r , then its eigentransform $F(z) = \sum a_n \Phi_n(z)$ belongs to $\mathfrak{H}(E_r)$ and*

$$\int_{-\infty}^{+\infty} |F(t)/E_r(t)|^2 dt = \|f(z)\|^2$$

where the norm of $f(z)$ is taken in $\mathfrak{C}(1; 1; z)$. Every element of $\mathfrak{H}(E_r)$ is of this form. The identity

$$\Gamma(z)F(-z) = \int_0^{\infty} f(-a - t/a)e^{-t}t^{z-1} dt$$

holds for $x > 0$ whenever $f(z)$ is a polynomial and $F(z)$ is its eigentransform. Let $f(z)$ and $g(z)$ be polynomials, and let $F(z)$ and $G(z)$ be their eigentransforms. The condition

$$G(z) = z[F(z) - F(z - 1)] - a^2[F(z + 1) - F(z)]$$

is necessary and sufficient that $g(z) = zf'(z)$. The condition $G(z) = aF(z + 1) - aF(z)$ is necessary and sufficient that $g(z) = f'(z)$. The condition $G(z) = a^{-1}zF(z - 1) - aF(z)$ is necessary and sufficient that $g(z) = zf(z)$. The condition $G(z) = zF(z)$ is necessary and sufficient that $g(z) = (z + a)f'(z) + a(z + a)f(z)$.

PROOF OF THEOREM 1. Let L_+ , L_- , and D be the transformations on entire functions defined by $D: F(z) \rightarrow G(z)$ if

$$G(z) = z[F(z) - F(z - 1)] - a^2[F(z + 1) - F(z)],$$

$L_-: F(z) \rightarrow G(z)$ if

$$G(z) = aF(z + 1) - aF(z),$$

and $L_+: F(z) \rightarrow G(z)$ if

$$G(z) = a^{-1}zF(z - 1) - aF(z).$$

A straightforward calculation will show that the commutator identities

$$DL_- - L_-D = -L_-, \quad DL_+ - L_+D = L_+, \quad L_-L_+ - L_+L_- = 1,$$

are satisfied. By hypothesis the restriction of D to the space is a self-adjoint transformation in the space. Since D is everywhere defined in the space, it is bounded. The hypotheses also imply that L_- takes the space into itself. Since the restriction of L_- to the space has a closed graph, it is bounded. If $F(z)$ belongs to the domain of multiplication by z and if $D: F(z) \rightarrow G(z)$, then

$$D: zF(z) \rightarrow zG(z) + zF(z - 1) - a^2F(z + 1).$$

It follows that the identity

$$\langle tF(t - 1) - a^2F(t + 1), G(t) \rangle = - \langle F(t), tG(t - 1) - a^2G(t + 1) \rangle$$

holds whenever $F(z)$ and $G(z)$ belong to the domain of multiplication by z in the space. Since D is selfadjoint, the identity

$$\langle tF(t - 1), G(t) \rangle = a^2 \langle F(t), G(t + 1) \rangle$$

holds whenever $F(z)$ and $G(z)$ belong to the domain of multiplication by z in the space. As in the proof of Theorem 1 of [2], this implies

that L_+ acts as a bounded transformation on the domain of multiplication by z . Since the action of $D+aL_++aL_-$ coincides with multiplication by z , multiplication by z is a bounded transformation in the space. An argument in the proof of Theorem 1 of [2] will show that the space is finite dimensional.

Let r be the dimension of the space. Since we assume that $E(z)$ has a nonzero value at the origin, there exists an element of the space which has a nonzero value at the origin. Since the transformation $F(z) \rightarrow z[F(z) - F(z-1)]$ does not take the space onto itself, it has a nonzero kernel. It follows that there exists a nonzero element $S(z)$ of the space which is periodic of period one. Since $z[F(z+1) - F(z)]$ belongs to the space whenever $F(z)$ belongs to the space, and since $E(z)$ has a nonzero value at the origin, the functions $F(z+1) - F(z)$ and $F(z+1)$ belong to the space whenever $F(z)$ belongs to the space. Since the space is finite dimensional, there exists no zero w of $S(z)$ such that $S(z)/(z+n-w)$ belongs to the space for every $n=0, 1, 2, \dots$. It follows that there exists no zero w of $S(z)$ such that $S(z)/(z-w)$ belongs to the space. By Problem 88 of [1], $S(z)$ and $S^*(z)$ are linearly dependent, and the elements of the space are the entire functions $F(z)$ such that $F(z)/S(z)$ is a polynomial of degree less than r . We assume that $S(z)$ is chosen of norm one and real for real z .

Let $S_0(z), S_1(z), S_2(z), \dots$ be the entire functions defined inductively by $S_0(z) = S(z)$ and

$$L_+: S_n(z) \rightarrow S_{n+1}(z).$$

The commutator identities imply that

$$D: S_n(z) \rightarrow nS_n(z)$$

for every n and that

$$L_-: S_n(z) \rightarrow nS_{n-1}(z)$$

for every $n > 0$. It follows that the identity

$$zS_n(z) = anS_{n-1}(z) + (n + a^2)S_n(z) + aS_{n+1}(z)$$

holds for $n > 0$, and for $n=0$ with the term in $S_{n-1}(z)$ omitted. It is clear that $S_n(z)/S(z)$ is a polynomial of degree n . So $S_n(z)$ belongs to $\mathfrak{H}(E_r)$ when $n < r$. The functions $S_0(z), \dots, S_{r-1}(z)$ are orthogonal in $\mathfrak{H}(E_r)$ since they are eigenfunctions of a selfadjoint operator for distinct eigenvalues. Since multiplication by z is a symmetric transformation, the identity

$$\langle \iota S_{n-1}(t), S_n(t) \rangle = \langle S_{n-1}(t), \iota S_n(t) \rangle$$

holds when $0 < n < r$. It follows that $\|S_n(t)\|^2 = n\|S_{n-1}(t)\|^2$. Since we assume that $\|S_0(t)\| = 1$, we can conclude that $\|S_n(t)\|^2 = \Gamma(1+n)$.

As in the proof of Theorem 1 of [2], there exists a space $\mathcal{H}(E_{r+1})$, satisfying the hypotheses of Theorem 1, which contains $\mathcal{H}(E_r)$ isometrically, such that $S_r(z)$ spans the orthogonal complement of $\mathcal{H}(E_r)$ in $\mathcal{H}(E_{r+1})$. The theorem now follows from Theorem 23 and Problem 87 of [1].

PROOF OF THEOREM 2. It is clear from the definition of Kummer's series that $\Phi_n(z)$ is a polynomial of degree n which is real for real z . The stated identities for $\Phi_n(z)$ follow from the well-known relations between contiguous hypergeometric series, Erdélyi [3]. Consider the unique inner product on polynomials with respect to which the functions $\Phi_n(z)$ are an orthogonal set and $\|\Phi_n(t)\|^2 = \Gamma(1+n)$ for every n . Define L_+ , L_- , and D as in the proof of Theorem 1. It is easily verified that the identities

$$\begin{aligned} \langle DF, G \rangle &= \langle F, DG \rangle, \\ \langle L_+F, G \rangle &= \langle F, L_-G \rangle, \\ \langle tF(t), G(t) \rangle &= \langle F(t), tG(t) \rangle \end{aligned}$$

hold for all polynomials $F(z)$ and $G(z)$. For every $r = 1, 2, 3, \dots$, the polynomials of degree less than r are a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). By Theorem 23 of [1], the space is equal isometrically to a space $\mathcal{H}(E_r)$. The function $E_r(z)$ has a nonzero value at the origin since the space contains a constant function which has a nonzero value at the origin. The polynomials $z[F(z) - F(z-1)]$ and $F(z+1) - F(z)$ belong to the space whenever $F(z)$ belongs to the space since their degrees do not exceed the degree of $F(z)$. The restriction of D to the space is selfadjoint since the space admits an orthogonal basis of eigenfunctions of D corresponding to real eigenvalues. From this we see that $\mathcal{H}(E_r)$ satisfies the hypotheses of Theorem 1 for every r .

To complete the proof of the theorem, we consider a new inner product on polynomials defined by

$$\langle F(t), G(t) \rangle_1 = \sum_{n=0}^{\infty} F(n)\overline{G(n)}a^{2n}/\Gamma(1+n).$$

It is easily verified that the identity

$$\langle tF(t-1), G(t) \rangle_1 = a^2\langle F(t), G(t+1) \rangle_1$$

holds for all polynomials $F(z)$ and $G(z)$. It follows that the identities

$$\langle DF, G \rangle_1 = \langle F, DG \rangle_1, \quad \langle L_+F, G \rangle_1 = \langle F, L_-G \rangle_1,$$

hold for all polynomials $F(z)$ and $G(z)$. The proof of Theorem 1 will show that

$$\langle F(t), G(t) \rangle_1 = \kappa \langle F(t), G(t) \rangle$$

for all polynomials $F(z)$ and $G(z)$, where κ is a constant. When $F(z) = G(z) = 1$, we obtain

$$\kappa = \sum_{n=1}^{\infty} a^{2n} / \Gamma(1+n) = e^{a^2}.$$

PROOF OF THEOREM 3. This more general result follows from the above proofs of Theorems 1 and 2.

PROOF OF THEOREM 4. The theorem follows by a routine calculation once it is known that the formula

$$\Gamma(z)\Phi_n(-z) = \int_0^{\infty} (-a - t/a)^n e^{-t} t^{z-1} dt$$

holds for every n when $x > 0$. The formula is true when $n = 0$ by the definition of the gamma function. A straightforward calculation will show that the functions defined by this integral formula satisfy the recurrence relations of Theorem 2. These functions must therefore coincide with the functions of Theorem 2.

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