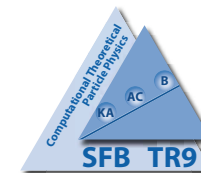


Charm and Bottom Quark Masses: An Update

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in collaboration with

K. Chetyrkin, Y. Kiyo, A. Maier, P. Maierhöfer, P. Marquard, A. Smirnov,
M. Steinhauser, C. Sturm and the HPQCD Collaboration

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EPJ C48 (2006) 107
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arXiv: 0907.2120

I. WHY precise masses?

B-decays:

$$\Gamma(B \rightarrow X_{ul}\bar{\nu}) \sim G_F^2 m_b^5 |V_{ub}|^2$$

$$\Gamma(B \rightarrow X_{cl}\bar{\nu}) \sim G_F^2 m_b^5 f(m_c^2/m_b^2) |V_{cb}|^2$$

$$B \rightarrow X_s \gamma$$

Υ -spectroscopy:

$$m(\Upsilon(1s)) = 2M_b - \left(\frac{4}{3}\alpha_s\right)^2 \frac{M_b}{4} + \dots$$

Higgs decay (ILC)

$$\Gamma(H \rightarrow b\bar{b}) = \frac{G_F M_H}{4\sqrt{2}\pi} m_b^2(M_H) \tilde{R}$$

$$\tilde{R} = 1 + 5.6667 a_S + 29.147 a_S^2 + 41.758 a_S^3 - 825.7 a_S^4 \quad \left(a_S \equiv \frac{\alpha_S}{\pi} \right)$$

a_S^4 -term = 5-loop calculation [Baikov, Chetyrkin, JK]

Yukawa Unification

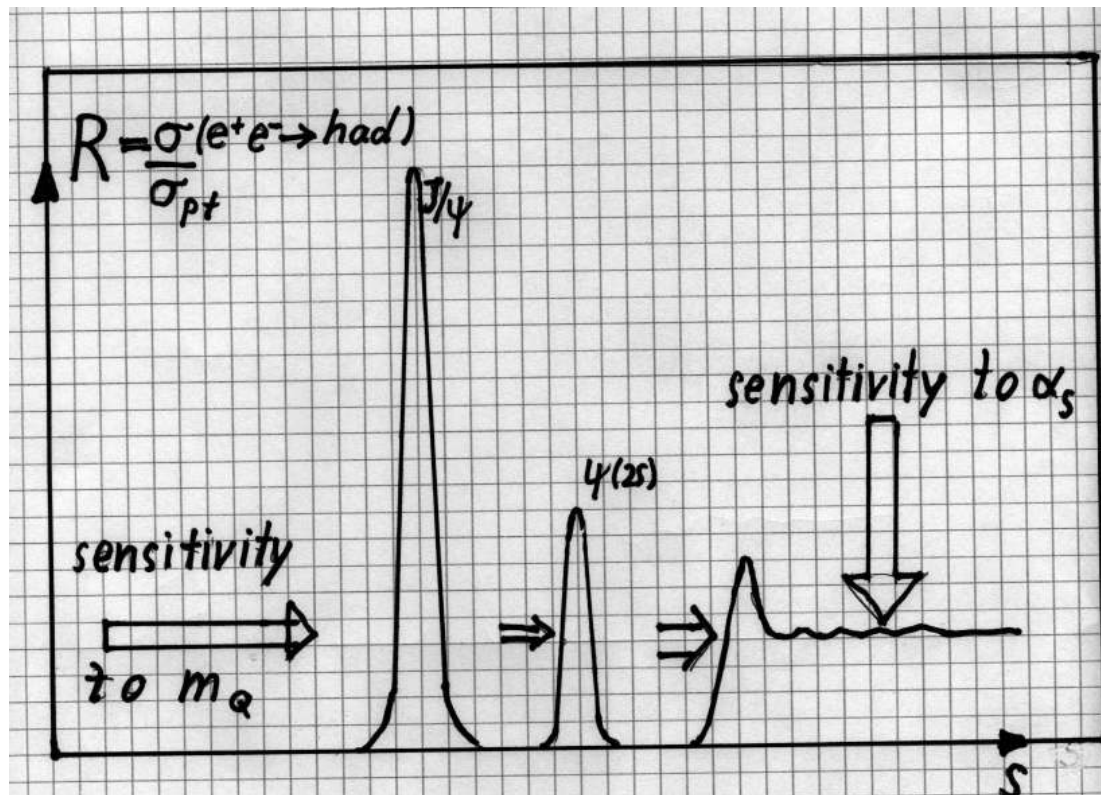
$$\lambda_\tau \sim \lambda_b \quad \text{or} \quad \lambda_\tau \sim \lambda_b \sim \lambda_t \quad \text{at GUT scale}$$

top-bottom $\rightarrow m_t/m_b \sim$ ratio of vacuum expectation values

$$\text{request } \frac{\delta m_b}{m_b} \sim \frac{\delta m_t}{m_t} \Rightarrow \delta m_t \approx 1 \text{ GeV} \Rightarrow \delta m_b \approx 25 \text{ MeV}$$

II. m_Q from SVZ Sum Rules, Moments and Tadpoles

Main Idea (SVZ)



Some definitions:

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right) \Pi(q^2) \equiv i \int d^4x e^{iqx} \langle T j_\mu(x) j_\nu(0) \rangle$$

with the electromagnetic current j_μ .

$$R(s) = 12\pi \operatorname{Im} \left[\Pi(q^2 = s + i\epsilon) \right]$$

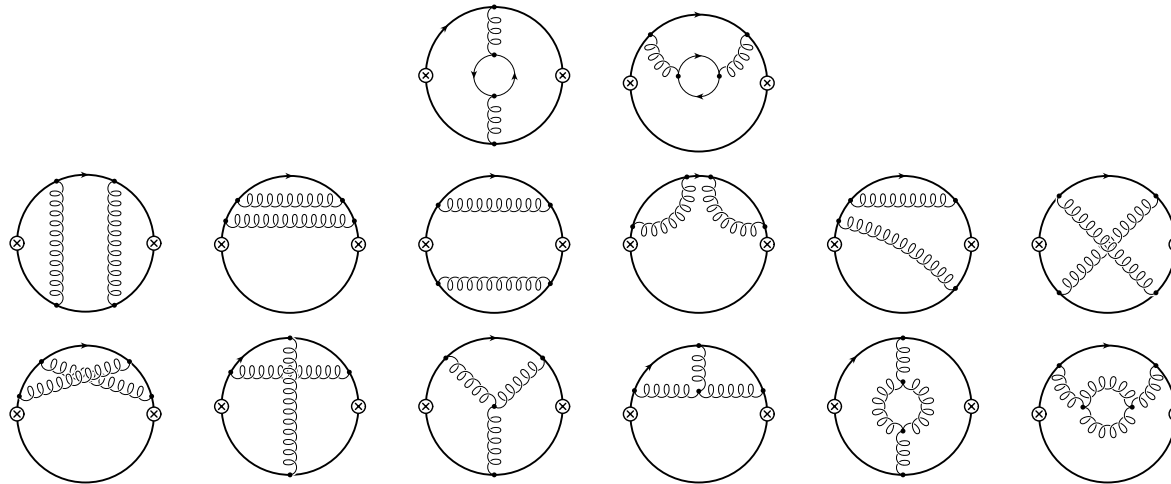
Taylor expansion: $\Pi_Q(q^2) = Q_Q^2 \frac{3}{16\pi^2} \sum_{n \geq 0} \bar{C}_n z^n$

with $z = q^2/(4m_Q^2)$ and $m_Q = m_Q(\mu)$ the $\overline{\text{MS}}$ mass.

$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi} \bar{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}_n^{(3)} + \dots$$

Analysis in NNLO

Coefficients \bar{C}_n from three-loop one-scale tadpole amplitudes with “arbitrary” power of propagators;



- **FORM program MATAD**

Coefficients \bar{C}_n up to $n = 8$ known analytically in order α_s^2

(also for axial, scalar and pseudoscalar correlators)

(Chetyrkin, JK, Steinhauser, 1996)

- up to $n = 30$ for vector correlator

(Boughezal, Czakon, Schutzmeier 2007)

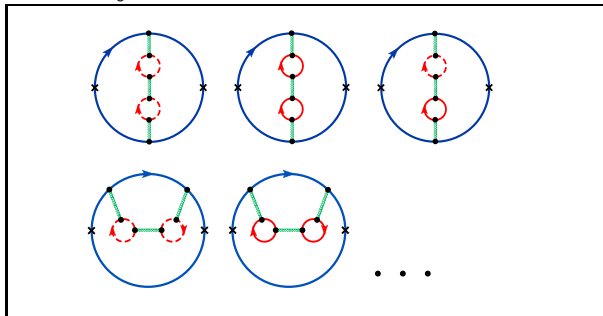
- up to $n = 30$ for vector, axial, scalar and pseudoscalar correlators

(A. Maier, P. Maierhöfer, P. Marquard, 2007)

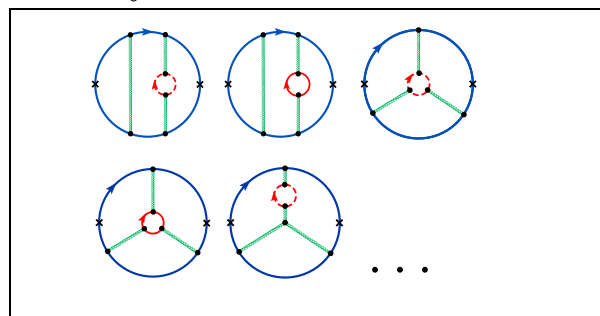
Analysis in N³LO

Algebraic reduction to 13 master integrals (Laporta algorithm);
 numerical and analytical evaluation of master integrals

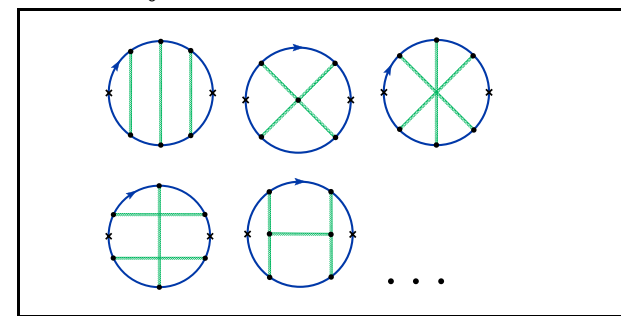
n_f^2 -contributions



n_f^1 -contributions



n_f^0 -contributions



: heavy quarks, : light quarks,

n_f : number of active quarks

⇒ About **700 Feynman-diagrams**

\bar{C}_0 and \bar{C}_1 in order α_s^3 (four loops!) (2006)

⇒ Reduction to master integrals through Laporta algorithm

(Chetyrkin, JK, Sturm; confirmed by Boughezal, Czakon, Schutzmeier)

⇒ Evaluation of master integrals numerically through difference equations

(30 digits) or Padé method or analytically in terms of transcendentals.

All master integrals known analytically and double checked.

(Schröder + Vuorinen, Chetyrkin et al., Schröder + Steinhauser,

Laporta, Broadhurst, Kniehl et al.)

Similar approach: four-loop ρ parameter

(K. G. Chetyrkin, M. Faisst, JK, P. Maierhöfer, C. Sturm)

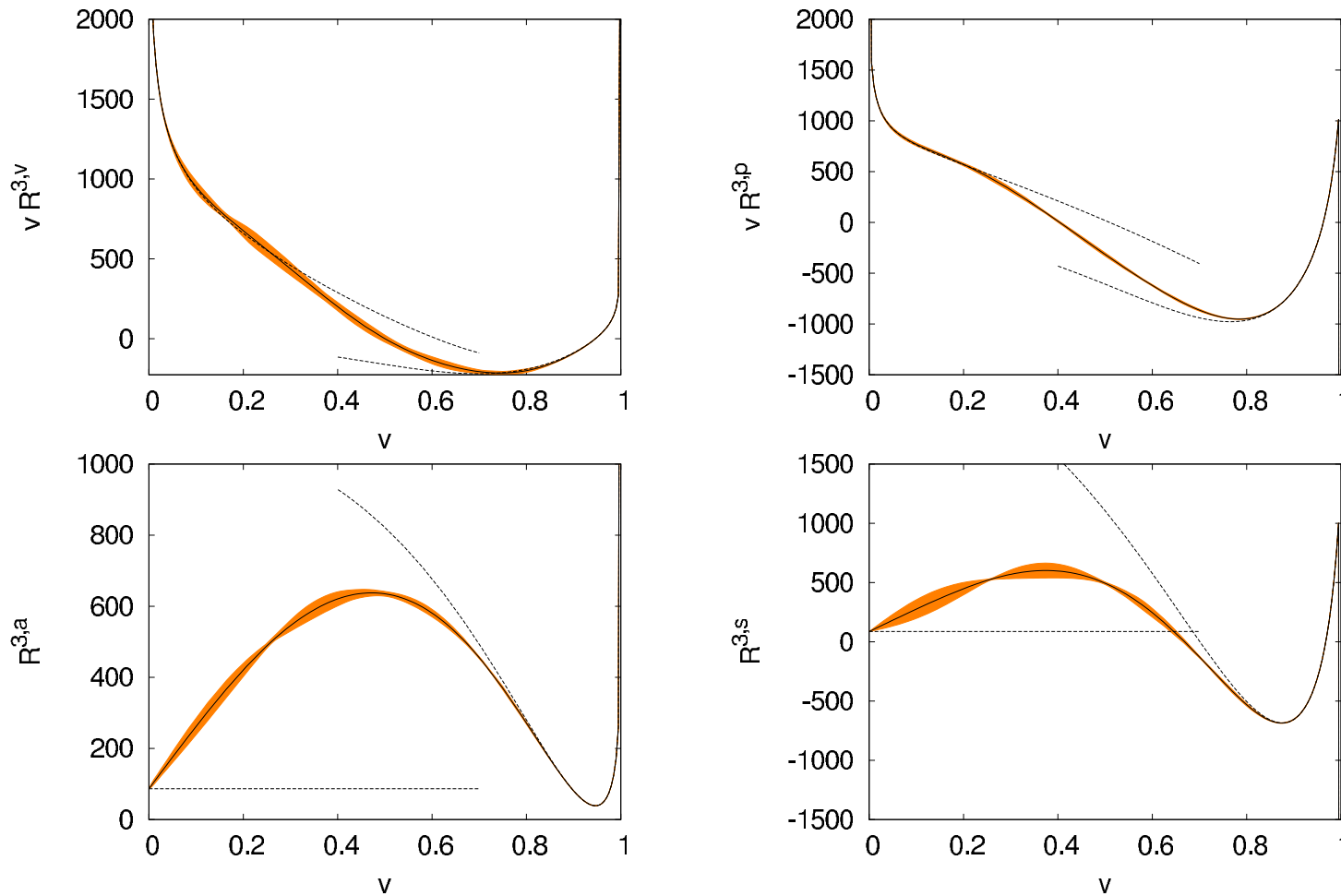
New developments

- ⇒ \bar{C}_2 : improved Laporta algorithm
(Maier, Maierhöfer, Marquard, 2008)
- ⇒ \bar{C}_3 (all correlators): special treatment of self energy insertions
(Maier, Maierhöfer, Marquard, A. Smirnov, 2008)
- ⇒ $\bar{C}_4 - \bar{C}_{10}$: extension to higher moments by Padé method, using analytic information from low energy ($q^2 = 0$), threshold ($q^2 = 4m^2$), high energy ($q^2 = -\infty$) (Kiyo, Maier, Maierhöfer, Marquard, 2009)

current	low energy	threshold	high energy	total
vector	3	2	2+2	9
axial-vector	3	0	2+2	7
scalar	3	0	2+2	7
pseudo-scalar	4	2	2+2	10

- ⇒ $10^3 - 10^4$ “good approximations”; analysis in “on shell” scheme

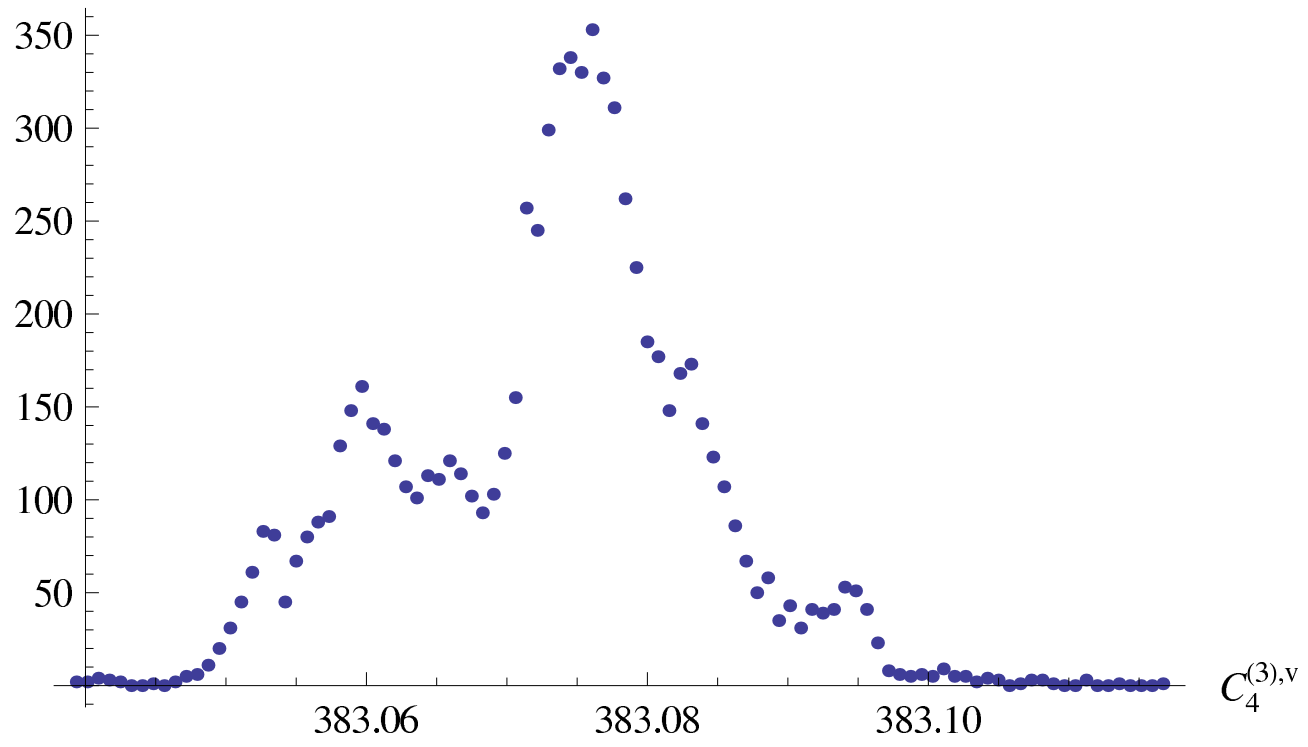
Results



Imaginary parts of the four loop contributions to the vector, pseudoscalar, axialvector and scalar correlators above the charm threshold ($v = \sqrt{1 - 1/z}$). Mean values, 3σ bands, threshold and high energy expansions (dashed).

Extraction of approximate moments

e. g. $C_4^{(3),v}$



	$n_l = 3$	$n_l = 4$	$n_l = 5$
$C_1^{(3),v}$	366.1748	308.0188	252.8399
$C_2^{(3),v}$	381.5091	330.5835	282.0129
$C_3^{(3),v}$	385.2331	338.7065	294.2224
$C_4^{(3),v}$	383.073(11)	339.913(10)	298.576(9)
$C_5^{(3),v}$	378.688(32)	338.233(32)	299.433(27)
$C_6^{(3),v}$	373.536(61)	335.320(63)	298.622(54)
$C_7^{(3),v}$	368.23(9)	331.90(10)	296.99(9)
$C_8^{(3),v}$	363.03(13)	328.33(14)	294.94(12)
$C_9^{(3),v}$	358.06(17)	324.78(18)	292.72(16)
$C_{10}^{(3),v}$	353.35(20)	321.31(22)	290.44(19)
$K_0^{(3),v}$	17(11)	17(29)	16(10)
$D_2^{(3),v}$	2.0(42)	1.2(83)	1.4(21)

⇒ translation to $\overline{\text{MS}}$ scheme

	$\bar{C}_3^{(3),v}$	$\bar{C}_4^{(3),v}$	$\bar{C}_5^{(3),v}$	$\bar{C}_6^{(3),v}$	$\bar{C}_7^{(3),v}$	$K_0^{(3),v}$
Hoang <i>et al.</i>	-3.28 ± 0.57	-4.2 ± 1.2	-5.0 ± 1.7	-5.3 ± 2.0	-5.2 ± 2.3	-10 ± 11
KMMM	-2.840 (exact)	$-3.349(11)$	$-3.737(32)$	$-3.735(61)$	$-3.39(10)$	$17(11)$

similar results for axial, scalar and pseudoscalar

n	1	2	3	4
charm	-5.6404	-3.4937	-2.8395	-3.349(11)
lower upper limits	—	-6.0 7.0	-6.0 5.2	-6.0 3.1
bottom	-7.7624	-2.6438	-1.1745	-1.386(10)
lower upper limits	—	-8.0 9.5	-8.0 8.3	-8.0 7.4

⇒ reliable predictions for the lowest four moments

⇒ consistency check:

different role of threshold vs. continuum

$$\text{recall: } \Pi_Q(q^2) = Q_Q^2 \frac{3}{16\pi^2} \sum_{n \geq 0} \bar{C}_n z^n$$

\bar{C}_n depend on the charm quark mass through $l_{m_c} \equiv \ln(m_c^2(\mu)/\mu^2)$

$$\begin{aligned} \bar{C}_n = & \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi} \left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_c} \right) \\ & + \left(\frac{\alpha_s(\mu)}{\pi} \right)^2 \left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_c} + \bar{C}_n^{(22)} l_{m_c}^2 \right) \\ & + \left(\frac{\alpha_s(\mu)}{\pi} \right)^3 \left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_c} + \bar{C}_n^{(32)} l_{m_c}^2 + \bar{C}_n^{(33)} l_{m_c}^3 \right) \end{aligned}$$

n	$\bar{C}_n^{(0)}$	$\bar{C}_n^{(10)}$	$\bar{C}_n^{(11)}$	$\bar{C}_n^{(20)}$	$\bar{C}_n^{(21)}$	$\bar{C}_n^{(22)}$	$\bar{C}_n^{(30)}$	$\bar{C}_n^{(31)}$	$\bar{C}_n^{(32)}$	$\bar{C}_n^{(33)}$
1	1.0667	2.5547	2.1333	2.4967	3.3130	-0.0889	-5.6404	4.0669	0.9590	0.0642
2	0.4571	1.1096	1.8286	2.7770	5.1489	1.7524	-3.4937	6.7216	6.4916	-0.0974
3	0.2709	0.5194	1.6254	1.6388	4.7207	3.1831	-2.8395	7.5736	13.1654	1.9452
4	0.1847	0.2031	1.4776	0.7956	3.6440	4.3713	-3.349(11)	4.9487	17.4612	5.5856

previous estimate $-6 < C_n^{(30)} < 6$ for $n = 2, 3, 4$

confirmed by exact calculation ($n = 2, 3$) and Padé estimate ($n = 4$)

Relation to measurements

$$\mathcal{M}_n^{\text{th}} \equiv \frac{12\pi^2}{n!} \left(\frac{d}{dq^2} \right)^n \Pi_c(q^2) \Big|_{q^2=0} = \frac{9}{4} Q_c^2 \left(\frac{1}{4m_c^2} \right)^n \bar{C}_n$$

Perturbation theory: \bar{C}_n is function of α_s and $\ln \frac{m_c^2}{\mu^2}$

dispersion relation:

$$\Pi_c(q^2) = \frac{q^2}{12\pi^2} \int ds \frac{R_c(s)}{s(s-q^2)} + \text{subtraction}$$

$$\Leftrightarrow \mathcal{M}_n^{\text{exp}} = \int \frac{ds}{s^{n+1}} R_c(s)$$

constraint: $\mathcal{M}_n^{\text{exp}} = \mathcal{M}_n^{\text{th}}$

$$\Leftrightarrow m_c$$

Ingredients (charm)

experiment:

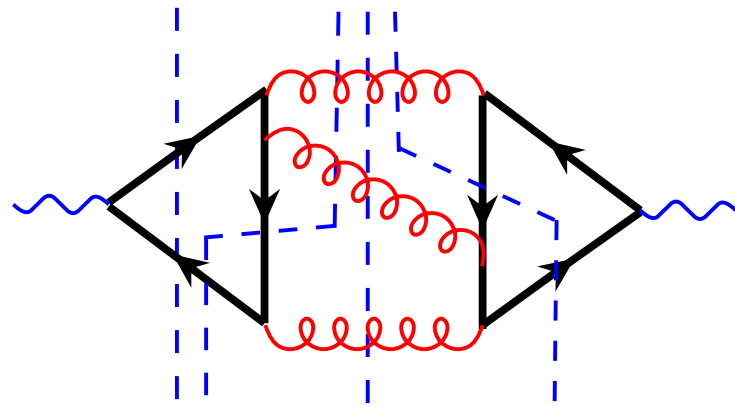
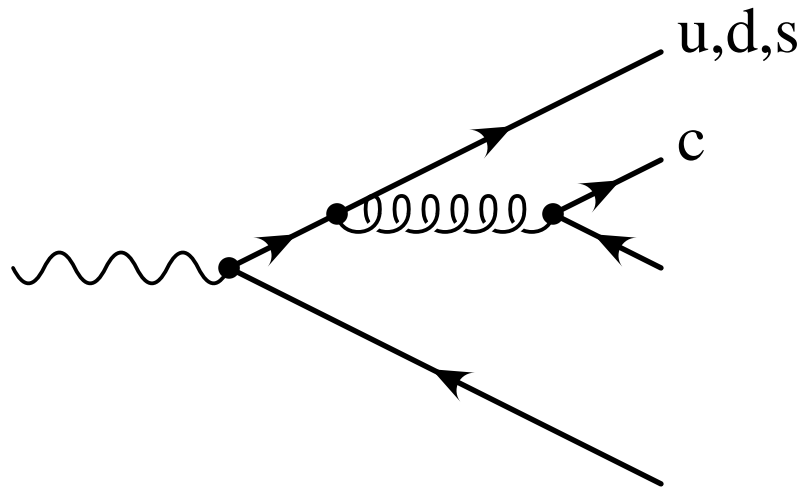
- $\Gamma_e(J/\psi, \psi')$ from BES & CLEO & Babar
- $\psi(3770)$ and $R(s)$ from BES
- $\alpha_s = 0.1187 \pm 0.0020$

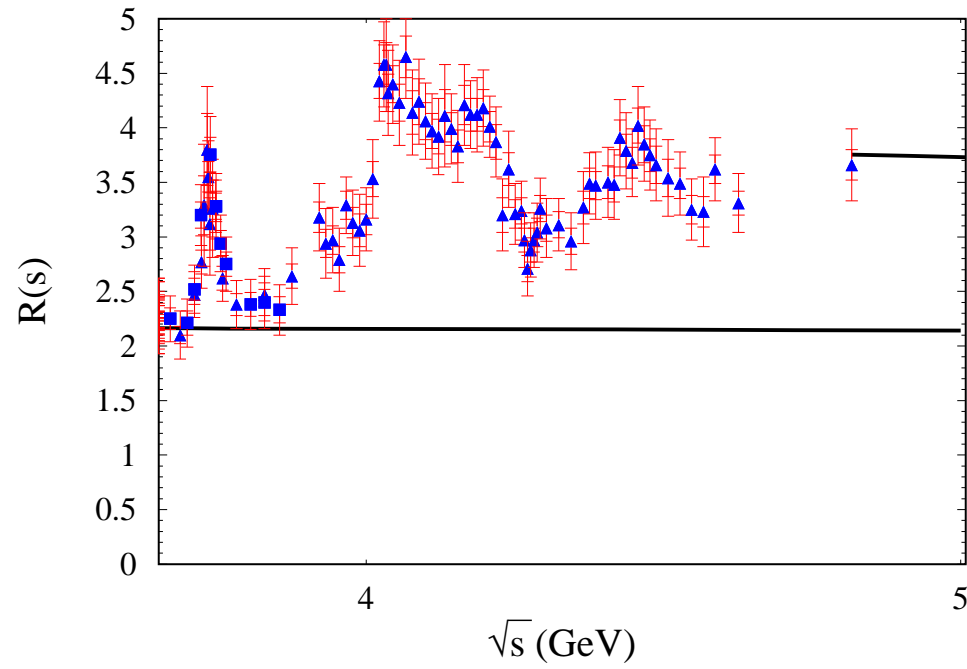
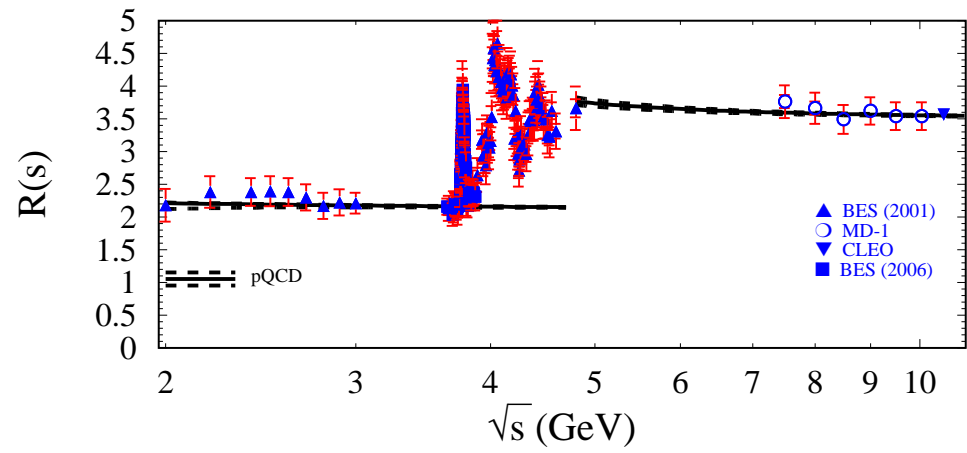
theory:

- N³LO for $n = 1, 2, 3$
- N³LO - estimate for $n = 4$
- include condensates

$$\delta\mathcal{M}_n^{\text{np}} = \frac{12\pi^2 Q_c^2}{(4m_c^2)^{(n+2)}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n \left(1 + \frac{\alpha_s}{\pi} \bar{b}_n \right)$$

- estimate of non-perturbative terms
(oscillations, based on [Shifman](#))
- careful extrapolation of R_{uds}
- careful definition of R_c





Contributions from

- narrow resonances: $R = \frac{9 \Pi M_R \Gamma_e}{\alpha^2(s)} \delta(s - M_R^2)$
- threshold region ($2 m_D - 4.8$ GeV)
- perturbative continuum ($E \geq 4.8$ GeV)

n	$\mathcal{M}_n^{\text{res}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{np}}$ $\times 10^{(n-1)}$
1	0.1201(25)	0.0318(15)	0.0646(11)	0.2166(31)	-0.0001(2)
2	0.1176(25)	0.0178(8)	0.0144(3)	0.1497(27)	0.0000(0)
3	0.1169(26)	0.0101(5)	0.0042(1)	0.1312(27)	0.0007(14)
4	0.1177(27)	0.0058(3)	0.0014(0)	0.1249(27)	0.0027(54)

Different relative importance of resonances vs. continuum for $n = 1, 2, 3, 4$.

Results (m_c)

arXiv: 0907:2110

n	$m_c(3 \text{ GeV})$	exp	α_s	μ	np	total
1	986	9	9	2	1	13
2	976	6	14	5	0	16
3	978	5	15	7	2	17
4	1004	3	9	31	7	33

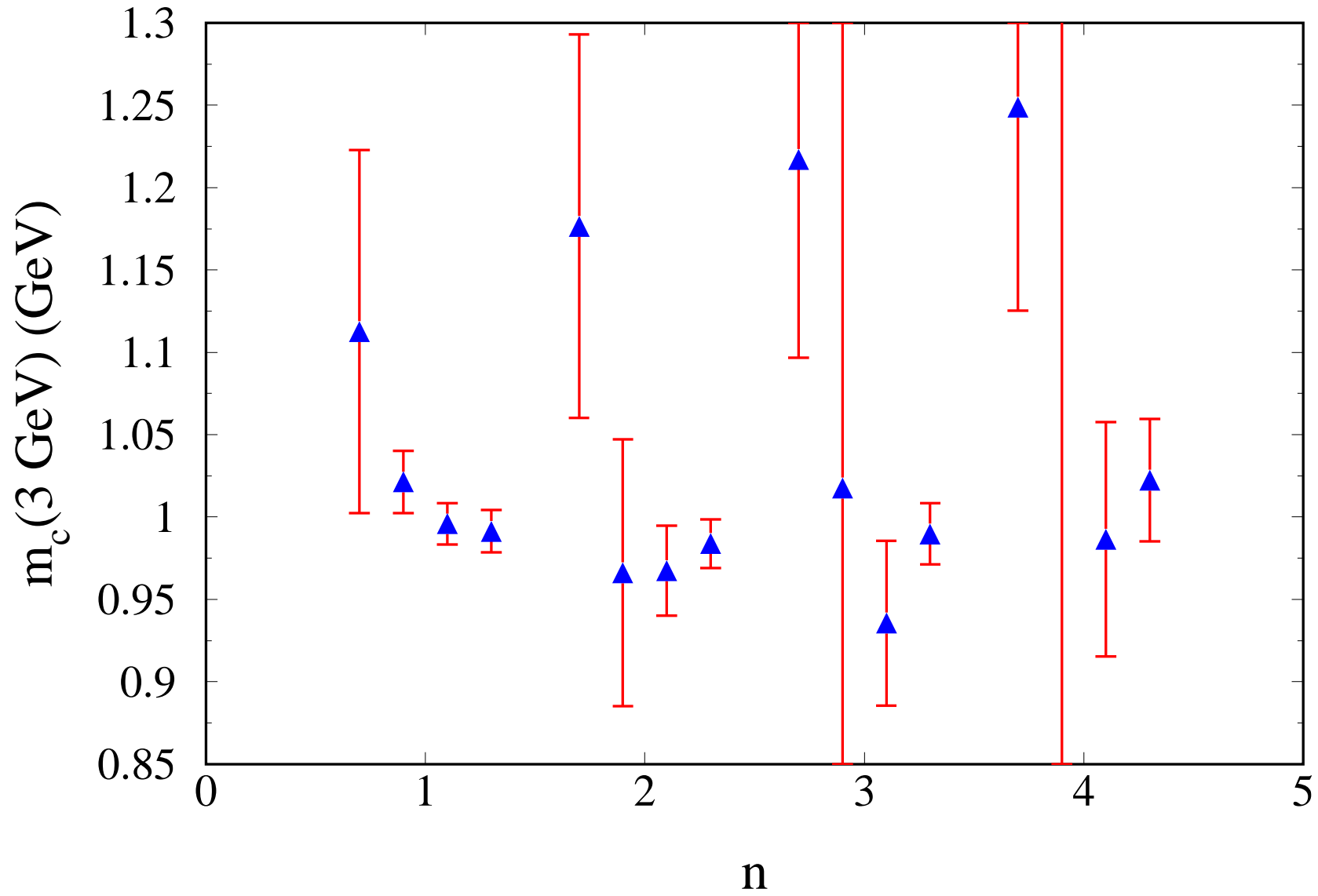
Remarkable consistency between $n = 1, 2, 3, 4$

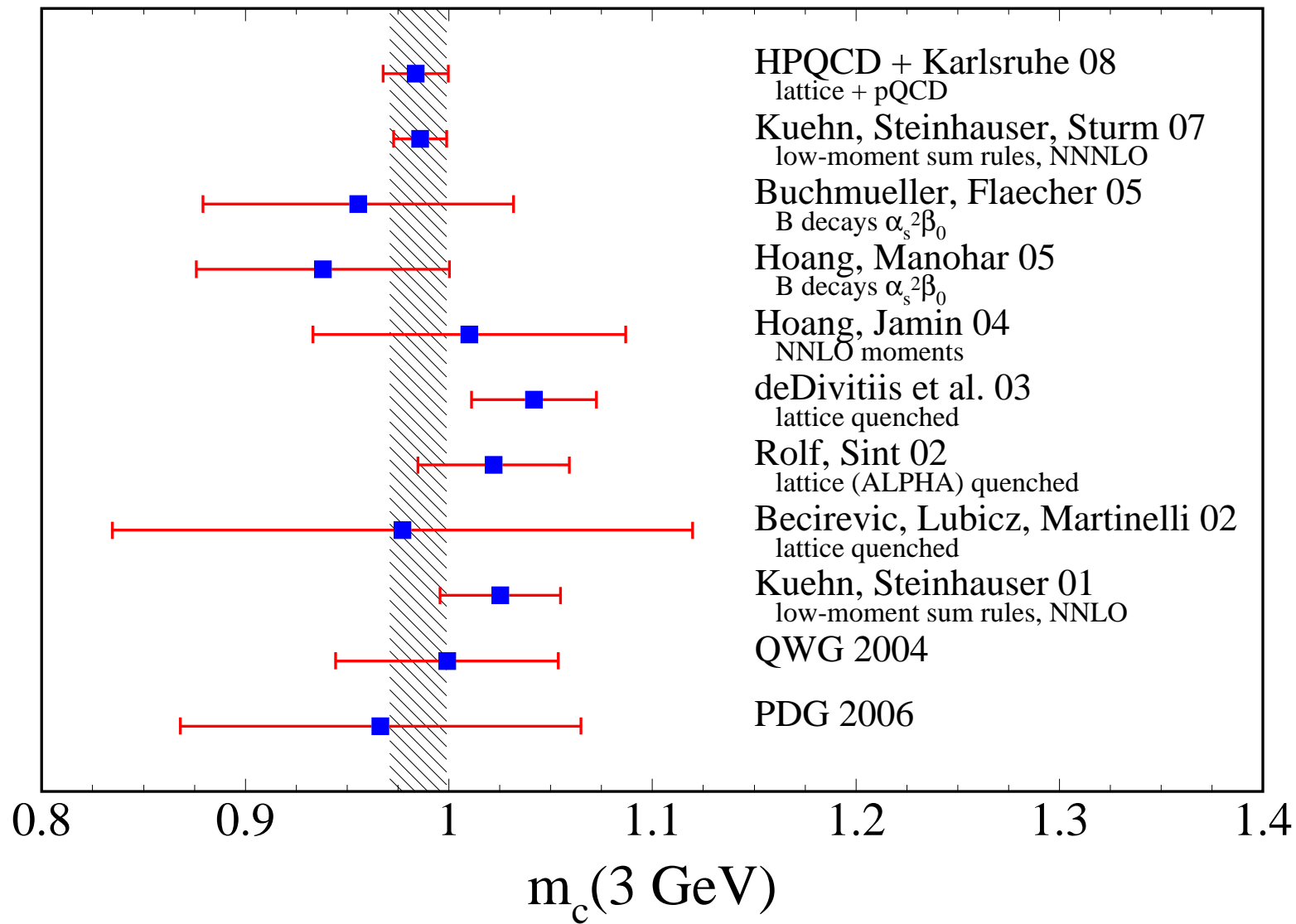
and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);

preferred scale: $\mu = 3 \text{ GeV}$,

conversion to $m_c(m_c)$:

- $m_c(3 \text{ GeV}) = 986 \pm 13 \text{ MeV}$
- $m_c(m_c) = 1279 \pm 13 \text{ MeV}$





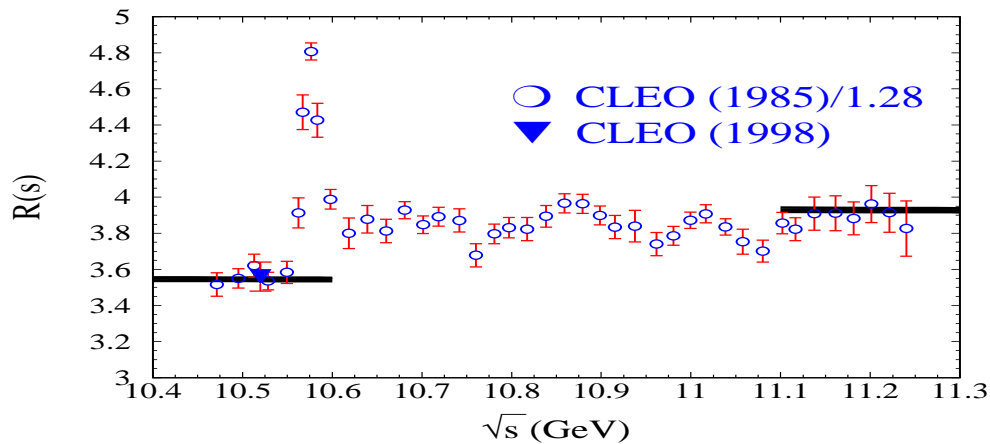
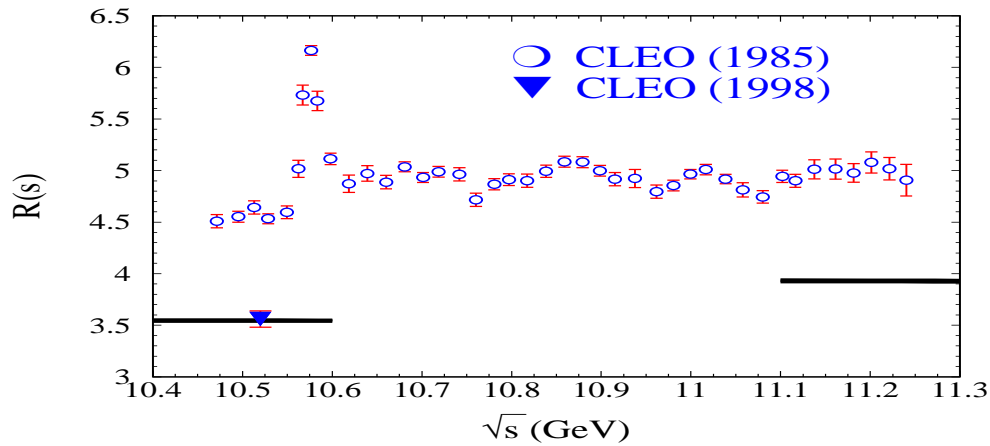
Experimental Ingredients for m_b

Contributions from

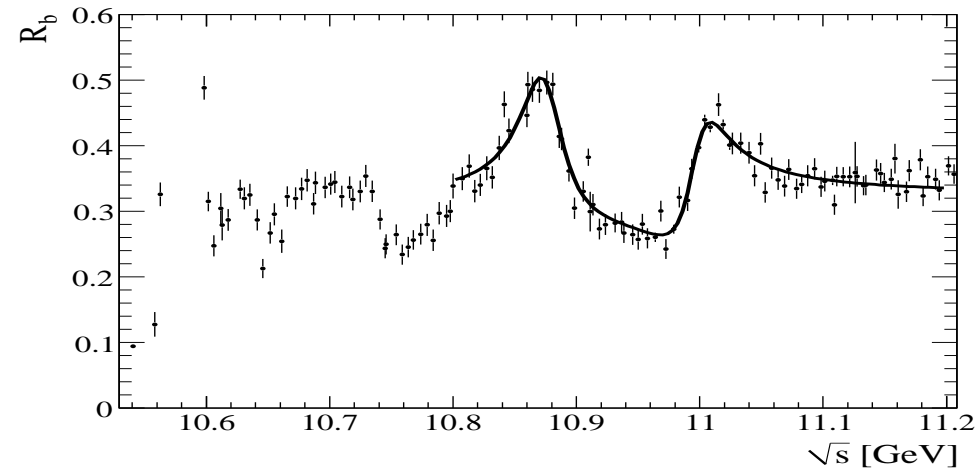
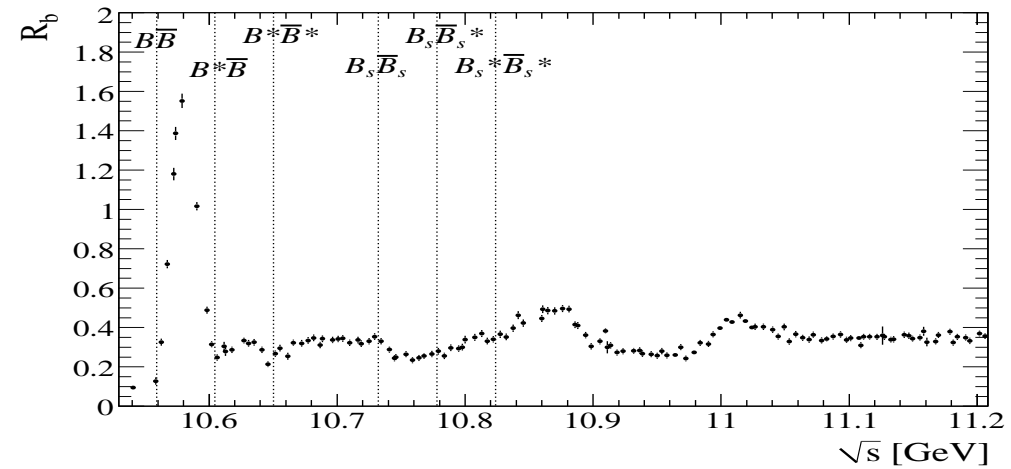
- narrow resonances ($\Upsilon(1S) - \Upsilon(4S)$)
- threshold region (10.618 GeV – 11.2 GeV)
- perturbative continuum ($E \geq 11.2$ GeV)
- different relative importance of resonances vs. continuum for $n = 1, 2, 3, 4$

n	$\mathcal{M}_n^{\text{res},(1S-4S)}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(2n+1)}$
1	1.394(23)	0.287(12)	2.911(18)	4.592(31)
2	1.459(23)	0.240(10)	1.173(11)	2.872(28)
3	1.538(24)	0.200(8)	0.624(7)	2.362(26)
4	1.630(25)	0.168(7)	0.372(5)	2.170(26)

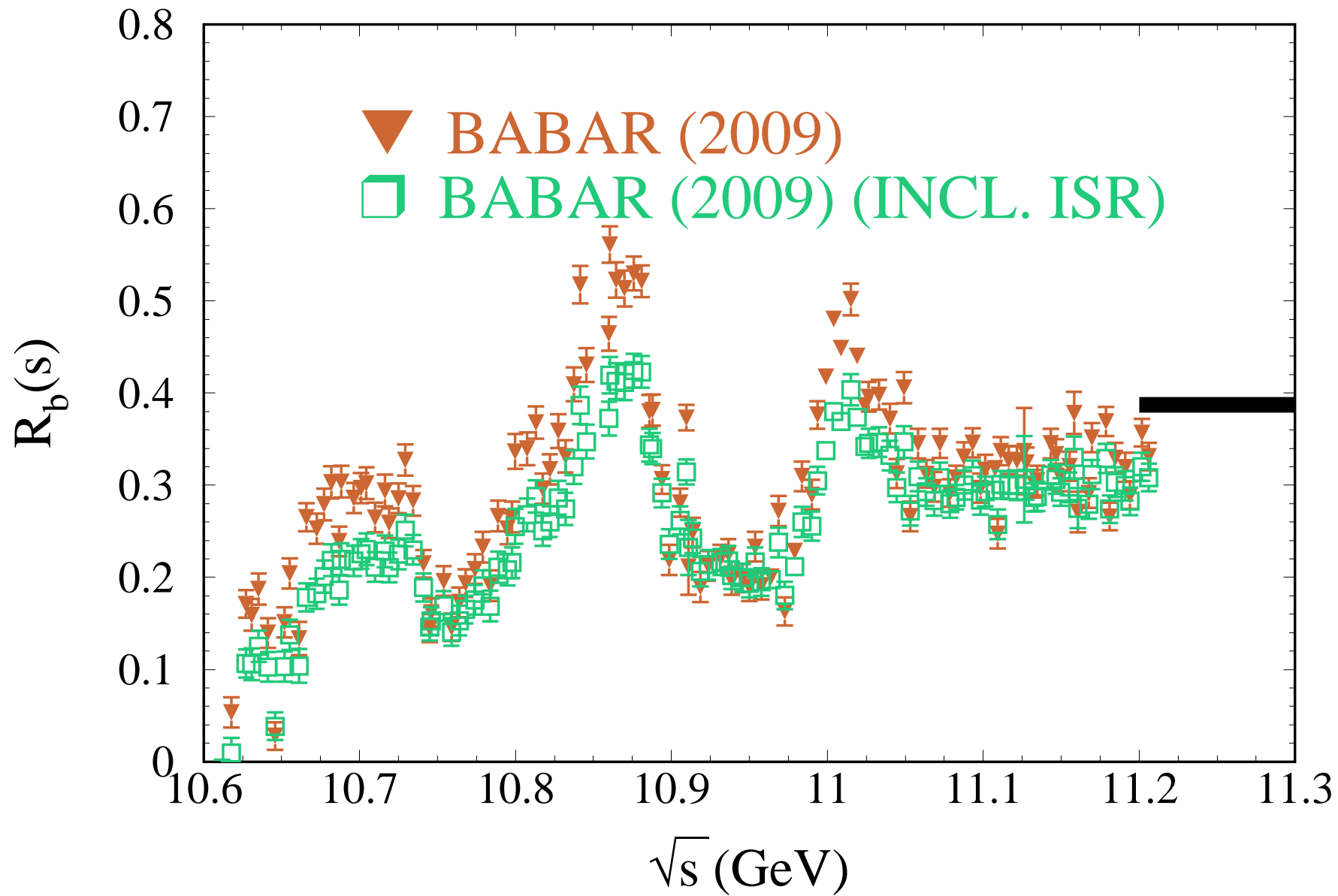
Improvements Based on Recent Babar Results

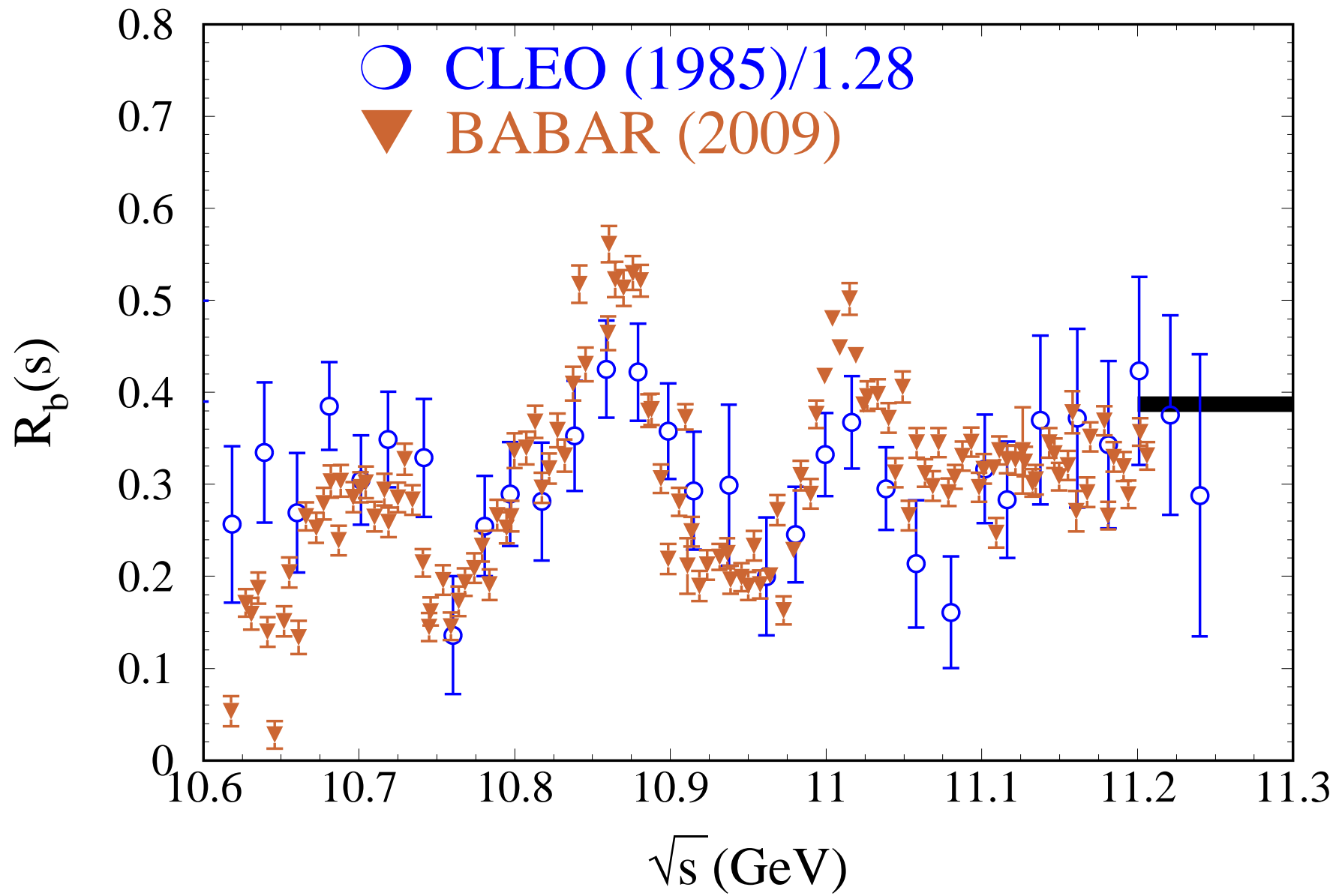


uncertainties after “renormalization”
 estimated to be 10%
 \Rightarrow dominant contribution to error



2% systematic experimental error;
 Deconvolute ISR and apply
 radiative corrections





n	$\mathcal{M}_n^{\text{threshold}}$ $\times 10^{(2n+1)}$ CLEO	$\mathcal{M}_n^{\text{threshold}}$ $\times 10^{(2n+1)}$ BABAR
1	0.296(32)	0.287(12)
2	0.249(27)	0.240(10)
3	0.209(22)	0.200(8)
4	0.175(19)	0.168(7)

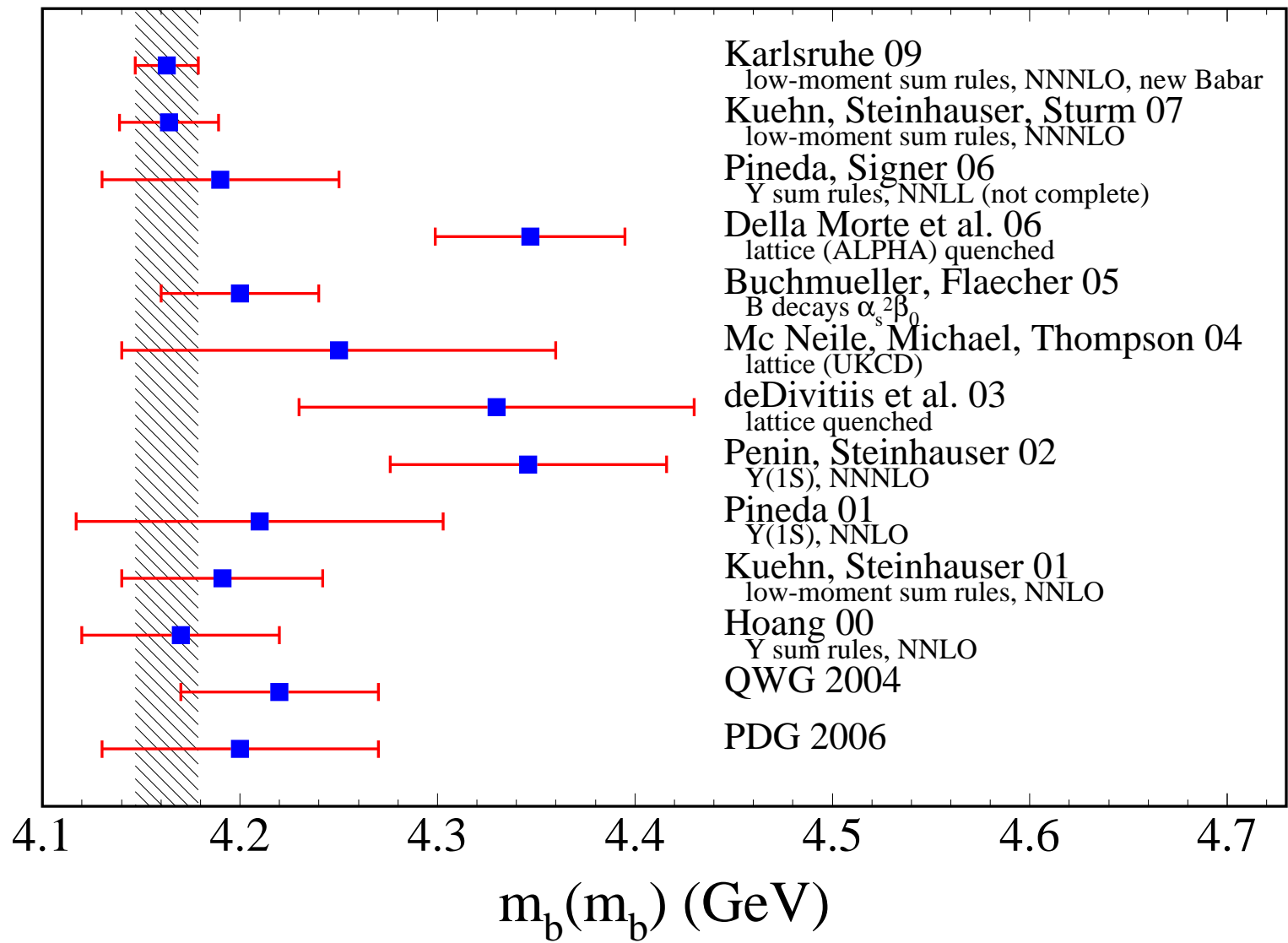
- consistency between BABAR and CLEO
- reduction of experimental error in this region by factor 3,
total error by factor 2/3

n	$m_b(10 \text{ GeV})$	exp	α_s	μ	total	$m_b(m_b)$
1	3597	14	7	2	16	4151
2	3610	10	12	3	16	4163
3	3619	8	14	6	18	4172
4	3631	6	15	20	26	4183

Consistency ($n = 1, 2, 3, 4$) and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);
(slight dependence on n could result from input into $\mathcal{M}_{\text{exp}}^n$)

- $m_b(10 \text{ GeV}) = 3610 \pm 16 \text{ MeV}$
- $m_b(m_b) = 4163 \pm 16 \text{ MeV}$

well consistent with **KSS 2007**



lattice & pQCD

(HPQCD + Karlsruhe, Phys. Rev. D78, 054513)

lattice evaluation of pseudoscalar correlator

⇒ replace experimental moments by lattice simulation

input: $M(\eta_c) \hat{=} m_c$, $M(\Upsilon(1S)) - M(\Upsilon(2S)) \hat{=} \alpha_s$

pQCD for pseudoscalar correlator available:

“all” moments in $\mathcal{O}(\alpha_s^2)$

three lowest moments in $\mathcal{O}(\alpha_s^3)$.

lowest moment: dimensionless: $\sim \left(\bar{C}^{(0)} + \frac{\alpha_s}{\pi} \bar{C}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}^{(3)} + \dots \right)$

⇒ $\alpha_s(3\text{GeV}) \Rightarrow \alpha_s(M_Z) = 0.1174(12)$

higher moments: $\sim m_c^2 \times \left(1 + \dots \frac{\alpha_s}{\pi} \dots \right)$

⇒ $m_c(3\text{GeV}) = 986(10) \text{ MeV}$

to be compared with 986(13) MeV from e^+e^- !

$$m_c(3 \text{ GeV}) = \left(986 - \frac{\alpha_s - 0.1189}{0.002} \cdot 9 \pm 10 \right) \text{ MeV}$$

$$m_b(10 \text{ GeV}) = \left(3610 - \frac{\alpha_s - 0.1189}{0.002} \cdot 12 \pm 11 \right) \text{ MeV}$$

$$m_b(m_b) = \left(4163 - \frac{\alpha_s - 0.1189}{0.002} \cdot 12 \pm 11 \right) \text{ MeV}$$

$$m_b(M_Z) = \left(2835 - \frac{\alpha_s - 0.1189}{0.002} \cdot 27 \pm 8 \right) \text{ MeV}$$

$$m_b(161.8 \text{ GeV}) = \left(2703 - \frac{\alpha_s - 0.1189}{0.002} \cdot 28 \pm 8 \right) \text{ MeV}$$

SUMMARY

$$m_c(3 \text{ GeV}) = 986(13) \text{ MeV}$$

$e^+e^- + \text{pQCD}$

$$m_c(3 \text{ GeV}) = 986(10) \text{ MeV}$$

lattice + pQCD

$$m_b(10 \text{ GeV}) = 3610(16) \text{ MeV}$$

$$m_b(m_b) = 4163(16) \text{ MeV}$$

$e^+e^- + \text{pQCD}$