

1996

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Kachroo, P., Tomizuka, M. (1996). Chattering Reduction and Error Convergence in the Sliding-mode Control of a Class of Nonlinear Systems. *IEEE Transactions on Automatic Control*, 41(7), 1063-1068. [https://digitalscholarship.unlv.edu/ece\\_fac\\_articles/35](https://digitalscholarship.unlv.edu/ece_fac_articles/35)

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## Chattering Reduction and Error Convergence in the Sliding-Mode Control of a Class of Nonlinear Systems

Pushkin Kachroo and Masayoshi Tomizuka

**Abstract**—To reduce chattering in sliding-mode control, a boundary layer around the switching surface is used, and a continuous control is applied within the boundary. The effects of various control laws within the boundary layer on chattering and error convergence in different systems are studied. New functions for chattering reduction and error convergence inside the boundary layer are proposed which are discontinuous in magnitude but not in sign. The internal model principle has been used to generalize the design for the class of nonlinear systems being considered.

### I. INTRODUCTION

Sliding-mode control is a robust nonlinear feedback control technique [4], [5] but with the drawback of chattering. One approach for chattering reduction involves introducing a boundary layer around the switching surface and using a continuous control within the boundary layer [2], [3]. The robustness term in the control law is  $k(\mathbf{x}, t) \operatorname{sgn}(s(t))$ ,  $s(t)$  is the sliding variable,  $\operatorname{sgn}(s(t))$  is defined as

$$\operatorname{sgn}(s(t)) = \begin{cases} +1 & \text{if } s(t) \geq 0 \\ -1 & \text{if } s(t) < 0. \end{cases} \quad (1)$$

In the method proposed in [2] and [3], this term is replaced by  $k(\mathbf{x}, t) \operatorname{sat}(s(t), \phi)$ , where  $\phi$  is the boundary layer thickness which is made varying to take advantage of the system bandwidth. The function  $\operatorname{sat}(s(t), \phi)$  is defined as

$$\operatorname{sat}(s(t), \phi) = \begin{cases} s(t)/\phi & \text{if } |s(t)| < \phi \\ \operatorname{sgn}(s(t)) & \text{otherwise.} \end{cases} \quad (2)$$

However, it is not necessary to vary  $\phi$ . One drawback of varying the boundary layer is that for some systems the boundary width can become large. An alternate method for chattering reduction is proposed which achieves the same results without varying the boundary layer. In that case the system is still represented by discontinuous right-hand side, but the solution does not have chattering. For some systems, as shown in this note, the  $\operatorname{sat}$  function does not give satisfactory results, and hence some other functions should be used. New functions, determined by the system dynamics, are proposed for use inside the boundary layer to reduce chattering and cause error convergence.

### II. BACKGROUND

Let a single-input nonlinear system be defined as

$$\dot{x}^{(n)} = f(\mathbf{x}, t) + b(\mathbf{x}, t)u(t). \quad (3)$$

Manuscript received November 10, 1994; revised September 8, 1995 and December 14, 1995.

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Publisher Item Identifier S 0018-9286(96)04376-0.

Here,  $\mathbf{x}(t) = [x(t) \dot{x}(t) \cdots x^{(n-1)}(t)]^T$  is the state vector,  $u$  is the control input, and  $x$  is the output state. The superscript  $n$  on  $x(t)$  signifies the order of differentiation.

A time-varying surface  $S(t)$  is defined by equating variable  $s(t)$  to zero, where

$$s(t) = \left( \frac{d}{dt} + \gamma \right)^{n-1} \tilde{x}(t). \quad (4)$$

Here,  $\gamma$  is a design constant and  $\tilde{x}(t) = x(t) - x_d(t)$  is the error in the output state where  $x_d(t)$  is the desired output state. The switching condition

$$\frac{1}{2} \frac{d}{dt} (s(t)^2) \leq -\eta |s(t)|, \quad \eta > 0 \quad (5)$$

makes the surface  $S(t)$  an invariant set. All trajectories outside  $S(t)$  point toward the surface, and trajectories on the surface remain there. It takes finite time to reach the surface  $S(t)$  from outside. Moreover, (4) implies that once the surface is reached, the convergence to zero error is exponential. Chattering is caused by nonideal switching around the switching surface. Delay in digital implementation causes  $s(t)$  to pass to the other side of the surface which in turn produces chattering.

Consider a second-order system

$$\ddot{x}(t) = f(\mathbf{x}, t) + u(t) \quad (6)$$

where  $f(\mathbf{x}, t)$  is generally nonlinear and/or time varying and is estimated as  $\hat{f}(\mathbf{x}, t)$ ,  $u(t)$  is the control input, and  $x(t)$  is the output, desired to follow trajectory  $x_d(t)$ . The estimation error on  $f(\mathbf{x}, t)$  is assumed to be bounded by some known function  $F = F(\mathbf{x}, t)$  so that

$$|\hat{f}(\mathbf{x}, t) - f(\mathbf{x}, t)| \leq F(\mathbf{x}, t). \quad (7)$$

We define a sliding variable according to (4)

$$s(t) = \left( \frac{d}{dt} + \gamma \right) \tilde{x}(t) = \dot{\tilde{x}}(t) + \gamma \tilde{x}(t). \quad (8)$$

The next two theorems give controls that guarantee the satisfaction of (5).

**Theorem 1 [5]:** For a single-input second-order nonlinear lumped parameter system, affine in control, given by (6), where  $\mathbf{x} \in R^2$ ,  $u \in R$ ,  $x \in R$ , and  $f: R^2 \times R^+ \rightarrow R$ , choosing control law as

$$u(t) = \hat{u}(t) - k(\mathbf{x}, t) \operatorname{sgn}(s(t)) \text{ with } k(\mathbf{x}, t) = F(\mathbf{x}, t) + \eta, \text{ and } \hat{u}(t) = -\hat{f} + \ddot{x}_d - \gamma \dot{\tilde{x}} \quad (9)$$

satisfies the invariant condition of (5).

Results for a second-order system with uncertain control gain are given by the following theorem [2].

**Theorem 2:** For a single-input second-order nonlinear lumped parameter system, affine in control, given by

$$\ddot{x}(t) = f(\mathbf{x}, t) + b(\mathbf{x}, t)u(t) \text{ where } 0 \leq b_{\min}(\mathbf{x}, t) \leq b(\mathbf{x}, t) \leq b_{\max}(\mathbf{x}, t) \quad (10)$$

where  $\mathbf{x} \in R^2$ ,  $u \in R$ ,  $x \in R$ ,  $b: R^2 \times R^+ \rightarrow R$ , and  $f: R^2 \times R^+ \rightarrow R$ , control law

$$u(t) = \hat{b}(\mathbf{x}, t)^{-1} [\hat{u}(t) - k(\mathbf{x}, t) \operatorname{sgn}(s(t))] \quad (11)$$

where

$$k(\mathbf{x}, t) \geq \alpha(\mathbf{x}, t)(F(\mathbf{x}, t) + \eta) + (\alpha(\mathbf{x}, t) - 1)|\hat{u}(t)| \quad (12)$$

$$\hat{b}(\mathbf{x}, t) = \sqrt{b_{\min}(\mathbf{x}, t)b_{\max}(\mathbf{x}, t)} \quad (13)$$

and

$$\alpha(\mathbf{x}, t) = \sqrt{b_{\max}(\mathbf{x}, t)/b_{\min}(\mathbf{x}, t)} \quad (14)$$

ensures the invariant condition of (5).

### III. METHODOLOGY FOR CHATTERING REDUCTION

To remove chattering, a thin boundary of thickness  $\phi$  around the switching surface is defined as

$$B(t) = \{\mathbf{x}(t), |s(t)| < \phi\}. \quad (15)$$

We can guarantee that all the trajectories outside the boundary layer are attracted toward the boundary by imposing the following condition [2]:

$$|s(t)| \geq \phi(t) \Rightarrow \frac{1}{2} \frac{d}{dt} s(t)^2 \leq (\dot{\phi}(t) - \eta)|s(t)|. \quad (16)$$

The following theorem gives the result for chattering reduction for system (6) while satisfying (16).

*Theorem 3:* For a single-input second-order nonlinear lumped parameter system, affine in control, given by (6), where  $\mathbf{x} \in R^2$ ,  $u \in R$ ,  $x \in R$ , and  $f: R^2 \times R^+ \rightarrow R$ , control law

$$\begin{aligned} u(t) &= \hat{u}(t) - \bar{k}(\mathbf{x}, t) \operatorname{sat}(s(t), \phi(t)) \text{ with } \bar{k}(\mathbf{x}, t) \\ &= k(\mathbf{x}, t) - \dot{\phi}(t) \end{aligned} \quad (17)$$

with  $k(\mathbf{x}, t) = F(\mathbf{x}, t) + \eta + \dot{\phi}$  ensures the invariant condition of (16). Moreover, when  $|s(t)| \leq \phi(t)$ , the variable  $s(t)$  passes through a first-order low-pass filter

$$\dot{s}(t) = -\gamma s(t) + (-\Delta f(\mathbf{x}_d, t) + o(\xi)) \quad (18)$$

where  $\Delta f(\mathbf{x}, t) = \hat{f}(\mathbf{x}, t) - f(\mathbf{x}, t)$ ,  $o(\xi)$  represents a term of relatively small magnitude caused by using a desired state instead of actual state vector in (18), and the boundary layer width varies as

$$\dot{\phi}(t) = -\gamma \phi(t) + k(\mathbf{x}_d, t). \quad (19)$$

*Proof:* When  $|s(t)| > \phi(t)$ , (17) reduces to (9) which ensures (16). When  $|s(t)| \leq \phi(t)$ , the system trajectories can be expressed in terms of the variable  $s$  as

$$\dot{s}(t) = -\bar{k}(\mathbf{x}, t)s(t)/\phi(t) - \Delta f(\mathbf{x}, t). \quad (20)$$

Since  $\Delta f(\mathbf{x}, t)$  and  $\bar{k}(\mathbf{x}, t)$  are continuous, (20) can be written as (18) by using  $\bar{k}(\mathbf{x}_d, t)/\phi(t) = \gamma$ , where  $o(\xi)$  represents the error terms introduced by replacing  $\mathbf{x}(t)$  by  $\mathbf{x}_d(t)$  in the first two terms in

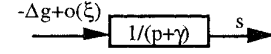


Fig. 1. Filter for chattering reduction.

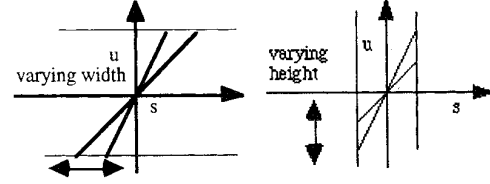


Fig. 2. Interpolation in the boundary layer.

(20). Using (17) with  $\gamma$ , we obtain the variation of  $\phi(t)$  with time in terms of the differential equation of (19).

This filter removes the high-frequency chattering to give a smooth  $s(t)$ . The expression for sliding gain is obtained by using (17) and (19)

$$\bar{k}(\mathbf{x}, t) = \gamma \phi(t) + k(\mathbf{x}, t) - k(\mathbf{x}_d, t). \quad (21)$$

Fig. 1 shows the first-order low-pass filter for  $s(t)$ , where  $p$  is the Laplace operator  $d/dt$ . The same filter can be obtained by using a constant width boundary. Note that we could have used  $\bar{k}(\mathbf{x}, t)/\phi = \gamma$  instead of  $\bar{k}(\mathbf{x}_d, t)/\phi(t) = \gamma$  and would not then have the  $o(\xi)$  term, but by doing so we keep the gain calculation off-line, saving precious on-line computation time. The results for a constant boundary width can be summarized by following Theorem 4. First we define a new function  $msat(a(\mathbf{x}, t), s(t), \phi)$  as

$$msat(a(\mathbf{x}, t), s(t), \phi) = \begin{cases} a(\mathbf{x}, t)s(t)/\phi & \text{for } |s(t)| < \phi \\ \operatorname{sgn}(s(t)) & \text{otherwise.} \end{cases} \quad (22)$$

*Theorem 4:* For a single-input second-order nonlinear lumped parameter system, affine in control, given by (6), where  $\mathbf{x} \in R^2$ ,  $u \in R$ ,  $x \in R$ , and  $f: R^2 \times R^+ \rightarrow R$ , control law

$$u(t) = \hat{u}(t) - k(\mathbf{x}, t) msat(a(\mathbf{x}, t), s(t), \phi) \quad (24)$$

with  $k(\mathbf{x}, t) = F(\mathbf{x}, t) + \eta$  ensures the invariant condition of (16). Moreover, when  $|s(t)| \leq \phi$ , the variable  $s(t)$  passes through the first-order low-pass filter (18).

*Proof:* When  $|s(t)| > \phi$ , (24) reduces to (9) which ensures (16). When  $|s(t)| \leq \phi$ , the system trajectories can be expressed in terms of the variable  $s$  as

$$\dot{s}(t) = -k(\mathbf{x}_d, t) as(t)/\phi + (-\Delta f(\mathbf{x}_d, t) + o(\xi)). \quad (25)$$

To obtain the desired filter, we use the substitution

$$a(\mathbf{x}, t) = \frac{\gamma \phi}{k(\mathbf{x}_d, t)}. \quad (26)$$

Theorem 4 shows that the same filter can be obtained by using the  $msat$  function as the one obtained by using the  $sat$  function. The advantage of using the  $msat$  function is that the boundary width is kept fixed so that the area in which the system trajectories are attracted toward the boundary is not changed. The boundary width, however, can become large by using the  $sat$  function as is shown in Section IV. The  $msat$  function produces the same filter as the  $sat$  function by changing the variation of width of the boundary layer into a variation of height as shown in Fig. 2. Notice that the  $msat$

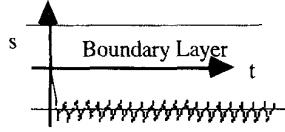


Fig. 3. Chattering across the boundary.

function is discontinuous at  $s(t) = \phi$ . If the trajectories on both sides of the boundary face inwards, i.e., toward  $S(t)$ , the discontinuity does not produce any problems. This is the case when the input  $-\Delta f(\mathbf{x}_d) + o(\xi)$  to the first-order filter is an impulse input.

Now if the input  $-\Delta f(\mathbf{x}_d) + o(\xi)$  to the first-order filter is a step input, then the variable  $s(t)$  has a steady-state value. Similarly, if the input term is a ramp, then  $s(t)$  keeps increasing. In that case, if the sat function with a varying boundary is being used, the boundary might keep increasing too. When a fixed boundary width is used, the variation of  $s$  with respect to time may increase until it hits the boundary layer, and once it is out of the boundary, it is forced back inwards because of the attractiveness of the boundary layer. This effect causes chattering on the boundary as shown in Fig. 3. This chattering is caused due to the discontinuity in the  $msat$  function at  $s(t) = \phi$ . The amount of discontinuity is governed by the variable  $a$  which in turn fixes the bandwidth of the  $s$ -filter inside the boundary. Therefore, the amount of discontinuity limits the achievable bandwidth of the filter. This problem is solved by forcing the trajectories on both sides of the boundary to face inwards for which an integral action is needed as explained next by Theorem 5.

**Theorem 5:** For a single-input second-order nonlinear lumped parameter system, affine in control, given by (6), where  $\mathbf{x} \in R^2$ ,  $u \in R$ ,  $x \in R$ , and  $f: R^2 \times R^+ \rightarrow R$ , and when  $-\Delta f(\mathbf{x}_d, t) + o(\xi)$  is time invariant, control law

$$u(t) = \hat{u}(t) - k(\mathbf{x}, t) \text{int}(a(\mathbf{x}, t), j(\mathbf{x}, t), s(t), \phi) \quad (27)$$

where

$$\text{int}(a(\mathbf{x}, t), j(\mathbf{x}, t), s(t), \phi) = \begin{cases} a(\mathbf{x}, t)s(t)/\phi + \frac{j(\mathbf{x}, t)}{\phi} \int_0^t s(t) dt & \text{for } |s(t)| \leq \phi \\ \text{sgn}(s(t)) & \text{otherwise} \end{cases} \quad (28)$$

with  $k(\mathbf{x}, t) = F(\mathbf{x}, t) + \eta$  ensures the invariant condition of (16). Moreover, if  $\exists t' \text{ s.t. } |s(t')| \leq \phi$ , then we can choose  $a(\mathbf{x}, t)$  and  $j(\mathbf{x}, t)$  such that  $s(t) \rightarrow 0$ .

*Proof:* When  $|s(t)| > \phi$ , (27) reduces to (9) which ensures (16). When  $|s(t)| \leq \phi$ , the system trajectories can be expressed in terms of the variable  $s$  as

$$\begin{aligned} \dot{s}(t) &= -\bar{k}(\mathbf{x}_d, t)a(\mathbf{x}, t)s(t)/\phi \\ &\quad - \bar{k}(\mathbf{x}_d, t)\frac{j(\mathbf{x}, t)}{\phi} \int_0^t s(t) dt \\ &\quad + (-\Delta f(\mathbf{x}_d, t) + o(\xi)). \end{aligned} \quad (29)$$

To obtain  $s(t) \rightarrow 0$ , we use  $a(\mathbf{x}, t) = 2\gamma\phi/k(\mathbf{x}_d, t)$  and  $j(\mathbf{x}, t) = \gamma^2\phi/k(\mathbf{x}_d, t)$ . Note that we could also take  $a(\mathbf{x}, t) = 1$ ,  $\bar{k}(\mathbf{x}_d, t)/\phi(t) = 2\gamma$  and  $j(\mathbf{x}, t) = \gamma^2\phi(t)/k(\mathbf{x}_d, t)$  with the boundary

$$\dot{\phi}(t) + 2\gamma\phi(t) = k(\mathbf{x}_d, t). \quad (30)$$

This filter is shown in Fig. 4. Extending the argument in the same vein, if the filter input has a term with Laplace transform  $m/p^n$ , we

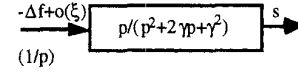


Fig. 4. Integral filter for chattering reduction.

introduce  $n$  integral terms, e.g., for a system with

$$\Delta f(\mathbf{x}) = -2.5x^2(t) \quad \text{and} \quad x_d(t) = 0.23t. \quad (31)$$

The control law inside the boundary layer should have integrals up to the third order.

Now, for (10) to satisfy (16) in the presence of uncertainty  $\alpha$  on the control gain for a variable width saturation function implementation, we let

$$\begin{aligned} \dot{\phi}(t) > 0 &\Rightarrow \bar{k}(\mathbf{x}, t) = k(\mathbf{x}, t) - \dot{\phi}(t)/\alpha(\mathbf{x}, t) \\ \dot{\phi}(t) < 0 &\Rightarrow \bar{k}(\mathbf{x}, t) = k(\mathbf{x}, t) - \dot{\phi}(t)\alpha(\mathbf{x}, t). \end{aligned} \quad (32)$$

The balance condition for this system can be written as

$$\begin{aligned} \left[ \frac{\bar{k}(\mathbf{x}_d, t)}{\phi(t)} \right] \left[ \frac{b(\mathbf{x}_d, t)}{\hat{b}(\mathbf{x}_d, t)} \right]_{\max} &= \gamma \quad \text{or} \quad \bar{k}(\mathbf{x}_d, t) \\ &= \gamma\phi(t)/\alpha(\mathbf{x}_d, t). \end{aligned} \quad (33)$$

Applying relation (33) to (32) yields

$$\begin{aligned} \dot{\phi}(t) > 0 &\Rightarrow \gamma\phi(t)/\alpha(\mathbf{x}_d, t) = k(\mathbf{x}_d, t) - \dot{\phi}(t)/\alpha(\mathbf{x}_d, t) \\ \dot{\phi}(t) < 0 &\Rightarrow \gamma\phi(t)/\alpha(\mathbf{x}_d, t) = k(\mathbf{x}_d, t) - \dot{\phi}(t)\alpha(\mathbf{x}_d, t) \end{aligned} \quad (34)$$

that is

$$\begin{aligned} k(\mathbf{x}_d, t) \geq \gamma\phi(t)/\alpha(\mathbf{x}_d, t) &\Rightarrow \dot{\phi}(t) + \gamma\phi(t) \\ &= \alpha(\mathbf{x}_d, t)k(\mathbf{x}_d, t) \\ k(\mathbf{x}_d, t) < \gamma\phi(t)/\alpha(\mathbf{x}_d, t) &\Rightarrow \dot{\phi}(t) + \gamma\phi(t)/[\alpha(\mathbf{x}_d, t)]^2 \\ &= k(\mathbf{x}_d, t)/\alpha(\mathbf{x}_d, t) \end{aligned} \quad (35)$$

with initial condition

$$\phi(0) = \alpha(\mathbf{x}_d, 0)k(\mathbf{x}_d(0), 0)/\gamma. \quad (36)$$

The sliding gain for the sat function can be obtained as

$$\begin{aligned} \bar{k}(\mathbf{x}, t) &= (\bar{k}(\mathbf{x}, t) - \bar{k}(\mathbf{x}_d, t)) + \bar{k}(\mathbf{x}_d, t) \\ &= (k(\mathbf{x}, t) - k(\mathbf{x}_d, t)) + \phi(t)\gamma/\alpha(\mathbf{x}_d, t). \end{aligned} \quad (37)$$

We can eliminate the variation of  $\phi(t)$  and use the  $msat$  function. Since, for this case,  $\dot{\phi}(t) = 0$ , we obtain  $\bar{k}(\mathbf{x}, t) = k(\mathbf{x}, t)$ . The balance condition is changed to

$$\left[ \frac{k(\mathbf{x}_d, t)a(\mathbf{x}, t)}{\phi} \right] \left[ \frac{b(\mathbf{x}_d, t)}{\hat{b}(\mathbf{x}_d, t)} \right]_{\max} = \gamma. \quad (38)$$

To obtain a low-pass filter of bandwidth  $\gamma$ , the variable  $a$  will be taken as

$$a(\mathbf{x}, t) = \frac{\gamma\phi}{\alpha(\mathbf{x}_d, t)k(\mathbf{x}_d, t)}. \quad (39)$$

Implementation of this scheme is simpler because  $\phi$  is constant and the design of  $a(\mathbf{x}, t)$  is straightforward. For (10), the system trajectories inside the boundary layer can be expressed as

$$\begin{aligned} \dot{s}(t) &= (f(\mathbf{x}, t) - b(\mathbf{x}, t)\hat{b}(\mathbf{x}, t)^{-1}\hat{f}(\mathbf{x}, t)) \\ &\quad + (1 - b(\mathbf{x}, t)\hat{b}(\mathbf{x}, t)^{-1})(-\ddot{x}_d(t) + \gamma\dot{x}(t)) \\ &\quad - b(\mathbf{x}, t)\hat{b}(\mathbf{x}, t)^{-1}k(\mathbf{x}, t) \text{sgn}(s(t)) \end{aligned} \quad (40)$$

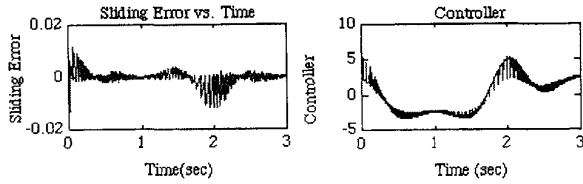


Fig. 5. Simulation results using signum function.

or

$$\dot{s}(t) = -b(x,t)\hat{b}(x,t)^{-1}k(x,t) \operatorname{sgn}(s(t)) - i(x,t) \quad (41)$$

with

$$i(x,t) = -(f(x,t) - b(x,t)\hat{b}(x,t)^{-1}\hat{f}(x,t) - (1 - b(x,t)\hat{b}(x,t)^{-1})(-\ddot{x}_d(t) + \gamma\dot{x}(t))). \quad (42)$$

The input to the filter for  $s(t)$  variable is  $-i(x_d, t)$ , and therefore this term should be analyzed as explained for (6) to ascertain which function should replace the  $\operatorname{sgn}$  function for chattering reduction and error convergence.

#### IV. NUMERICAL EXAMPLES

*Example 1:* Consider a second-order system of the form of (6) with

$$\begin{aligned} f &= -h(t)\dot{x}^2 \cos 3x; & h(t) &= |\sin t| + 1 \\ \hat{f} &= -1.5\dot{x}^2 \cos 3x \\ F &= 0.5\dot{x}^2 |\cos 3x|; & \eta &= 0.1; & \gamma &= 20 \\ x_d &= \sin(\pi t/2) \end{aligned} \quad (43)$$

and the sampling frequency of 0.5 KHz. Control law (9) for this example is

$$u = 1.5\dot{x}^2 \cos 3x - (\pi^2/4) \sin(\pi t/2) - 20\dot{x} - (0.1 + 0.5\dot{x}^2 |\cos 3x|) \operatorname{sgn}(s) \quad (44)$$

where the  $s$  variable is defined in (8). Fig. 5 shows that tracking performance using this control law is excellent but at a price of high-control chattering. To remove chattering, we define a constant width boundary  $\phi = 0.1$  and use the control law

$$u = 1.5\dot{x}^2 \cos 3x - (\pi^2/4) \sin(\pi t/2) - 20\dot{x} - (0.1 + 0.5\dot{x}^2 |\cos 3x|) \operatorname{sat}(s/0.1). \quad (45)$$

As is evident from Fig. 6, although the error has increased, the performance is still acceptable and chattering has been removed. Now, the boundary  $\phi$  is made varying and the control law (17) is used which for this example is

$$u = 1.5\dot{x}^2 \cos 3x - (\pi^2/4) \sin(\pi t/2) - 20\dot{x} - (0.1 + 0.5\dot{x}^2 |\cos 3x| - \dot{\phi}) \operatorname{sat}(s/\phi) \quad (46)$$

with

$$\dot{\phi} = -20\phi + 0.5\dot{x}_d^2 |\cos 3x_d| + 0.1. \quad (47)$$

Fig. 7 shows that by using a varying boundary, the error has improved and there is no chattering. Now, the boundary is kept constant at 0.1 which is greater than the maximum value of the varying boundary for the previous case. Control law (22) is used as

$$u = 1.5\dot{x}^2 \cos 3x - (\pi^2/4) \sin(\pi t/2) - 20\dot{x} - (0.1 + 0.5\dot{x}^2 |\cos 3x|) \operatorname{msat}(a, s, 0.1) \quad (48)$$

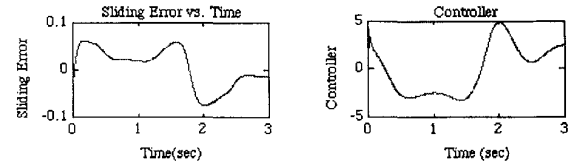


Fig. 6. Simulation results using sat function.

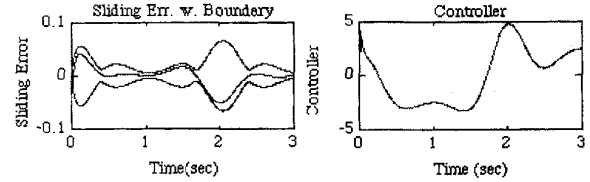
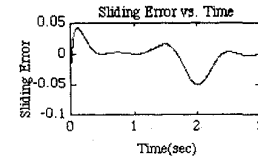
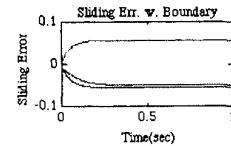


Fig. 7. Simulation results using variable width sat function.

Fig. 8. Result using  $\operatorname{msat}(\cdot)$ .Fig. 9. Form-1 system using variable width  $\operatorname{sat}(\cdot)$ .

where

$$a = 20(0.1)/(0.5\dot{x}_d^2 |\cos 3x_d|). \quad (49)$$

We obtain the same results as were obtained by using the control (46) and (47), as is shown in Fig. 8. Now, to show the advantage of using the new interpolated functions with fixed boundaries, consider the second-order system of the form of (6)

$$\begin{aligned} f &= -2.00; & \hat{f} &= -1.0; & F &= 1.01; & \eta &= 0.1 \\ \gamma &= 20; & x_d &= \sin(\pi t/2). \end{aligned} \quad (50)$$

Define a term form- $n$  to indicate that the Laplace transform of the input to the filter inside the boundary layer is  $m/p^n$ . For example, for a step input, the system will be of form-1, and for a ramp input it will be of form-2, and so on. Control law (17) with (50) is applied

$$u = 1 - (\pi^2/4) \sin(\pi t/2) - 20\dot{x} - (1.11 - \dot{\phi}) \operatorname{sat}(s/\phi) \quad (51)$$

with

$$\dot{\phi} = -20\phi + 1.11. \quad (52)$$

A steady-state error is obtained in the value of  $s$  and the output error, as shown in Fig. 9.

*Example 2:* Consider a form-2 system with the following parameters and desired output:

$$\begin{aligned} f &= -2.00t; & \hat{f} &= -1.0t; & F &= 1.01t; & \eta &= 0.1 \\ \gamma &= 20; & x_d &= \sin(\pi t/2). \end{aligned} \quad (53)$$

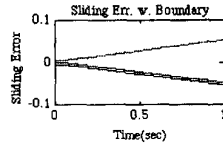


Fig. 10. Form-2 using variable width sat(.).

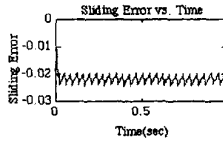


Fig. 11. Form-1 system using msat(.).

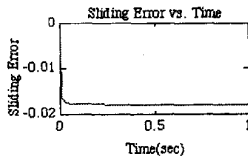


Fig. 12. Form-1 using sat(.).

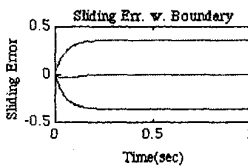


Fig. 13. SForm-1 system using int(.).

When the control law given by

$$u = t - (\pi^2/4) \sin(\pi t/2) - 20\dot{x} - (1.11t - \dot{\phi}) \text{sat}(s/\phi) \quad (54)$$

with

$$\dot{\phi} = -20\phi + 1.11t \quad (55)$$

is applied, the error as well as the boundary keep increasing. This phenomenon can be seen in Fig. 10. The chattering surface is shifted from zero to the boundary ( $\phi = 0.02$  here) when the control law given by

$$u = 1 - (\pi^2/4) \sin(\pi t/2) - 20\dot{x} - 1.11 \text{msat}(0.0018, s, 0.02) \quad (56)$$

is applied to the form-1 system with (50). This phenomenon is seen (Fig. 11) because the trajectories inside the boundary as well as outside are attracted toward the lower boundary. When the control law with the sat function, and a fixed boundary is used for the system with (50), for a low value of sliding gain  $k$ ,  $s$  tends to hit the boundary and stay there as shown in Fig. 12. For a higher value of  $k$  it would either start chattering across the lower boundary or across the whole boundary width depending on the value.

Finally, the result of using the control law

$$u = 1 - (\pi^2/4) \sin(\pi t/2) - 20\dot{x} - (7.5 - \dot{\phi}) \text{int}(1, 10, s, \phi) \quad (57)$$

with

$$\dot{\phi} = -20\phi + 7.5 \quad (58)$$

to (6) with form-1 parameters of (50) is shown in Fig. 13. Here, the output error and  $s$  go to zero. We could have also used a fixed boundary here.

## V. GENERALIZATION

The claim made in Section III can be generalized to a class of nonlinear systems by using the internal model principle approach [6]. Consider the nonlinear systems (6) and (10). Denote the input to the filters described in Section III, for both nonlinear systems, by  $d(x_d, t)$ . Note that  $x_d(t)$  is a function of time so that we can write the input as  $d(t)$ . The disturbance  $d(t)$  satisfies

$$A(p)d(t) = 0. \quad (59)$$

Examples of such  $d(t)s$  are: a)  $pd(t) = 0$  for  $d(t) = \text{constant}$ ; b)  $p^2d(t) = 0$  for  $d(t) = t$ ; c)  $(p^2 + \omega^2)d(t) = 0$  for  $d(t) = \sin(\omega t)$  or  $\cos(\omega t)$ .

*Corollary 1:* For a single-input second-order nonlinear lumped parameter system, affine in control, given by (6), where  $x \in R^2$ ,  $u \in R$ ,  $x \in R$ , and  $f: R^2 \times R_+ \rightarrow R$ , and when  $-\Delta f(x_d, t) + o(\xi)$  is a disturbance  $d(t)$  and the dynamics inside the boundary layer are  $\dot{s} = -g(s) + d(t)$ , where the Laplace transform of  $g(s)$  is  $[R(p) + T(p)/A(p)]S(p)$ ;  $R(p)$  and  $T(p)$  are polynomials in  $p$ ;  $S(p)$  is the Laplace transform of  $s(t)$ , control law

$$u = \hat{u} - k \text{gen}(s) \quad (60)$$

where

$$\begin{aligned} \text{gen}(s) &= \text{sgn}(s) \quad \text{for } |s| \geq \phi \\ \text{gen}(s) &= g(s)/k \quad \text{for } |s| < \phi \end{aligned} \quad (61)$$

with  $k(x, t) = F(x, t) + \eta$  ensures the invariant condition of (16). Moreover, if  $\exists t' \text{ s.t. } |s(t')| \leq \phi$ , then we can choose  $a(x, t)$  and  $j(x, t)$  such that  $s(t) \rightarrow 0$ .

*Proof:* When  $|s(t)| > \phi$ , (60) reduces to (9) which ensures (16). Taking the Laplace transform of equation  $\dot{s} = -g(s) + d(t)$  and rearranging terms, we obtain

$$S(p) = \left[ \frac{A(p)}{(p + R(p))A(p) + T(p)} \right] D(p). \quad (62)$$

In this equation,  $R(p)$  and  $T(p)$  can be chosen by using the Diophantine equation to place the poles in the left half-plane of the complex variable  $p$ . Hence,  $s(t) \rightarrow 0$ .

As an example, let us take a constant  $d(t)$ . Obviously, we have  $A(p) = p$ . We can obtain the following characteristic equation by choosing  $R(p) = 2\gamma$ , and  $T(p) = \gamma^2$ :

$$p^2 + 2\gamma p + \gamma^2 = 0. \quad (63)$$

The Laplace transform of the dynamic equation for  $s$  inside the boundary is

$$ps(p) = [-2\gamma s(p) - (\gamma^2/p)s(p)] + D(p) \quad (64)$$

which verifies control law (27).

## VI. CONCLUSIONS

Various control laws were proposed within the boundary layer for chattering reduction and error convergence in sliding-mode control for a class of nonlinear systems, and example simulation results are shown to illustrate their effect.

## ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their contribution to this paper.

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### Reduced-Order Observer Design for Descriptor Systems with Unknown Inputs

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**Abstract**—A new method for the design of reduced-order observers for descriptor systems with unknown inputs is presented. The approach is based on the generalized constrained Sylvester equation. Sufficient conditions for the existence of the observer are given.

## I. INTRODUCTION

The problem of observer design for standard systems with unknown inputs has received considerable attention in the last two decades (see [1]–[4] and references therein). This problem is of great importance in theory and practice, since there are many situations where disturbances or partial inputs are inaccessible. In [5] a technique for computing an efficient solution for the unknown input observer design is given. This solution uses the constrained Sylvester equation. The usage of constrained and coupled Sylvester equation in automatic control is well known [6]–[8]. Recently, a great deal of work has been devoted to the observer design for descriptor systems, and many approaches to design such observers exist [9]–[16]. In [9] a method based on the singular value decomposition and the concept of a matrix generalized inverse to design a reduced-order observer has been proposed. In [11] the generalized Sylvester equation was used to develop a procedure for designing reduced-order observers. In [12] a method based on the generalized inverse was presented. Observers for continuous descriptor under less restrictive conditions and using

Manuscript received August 9, 1994; revised April 19, 1995.

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Publisher Item Identifier S 0018-9286(96)04377-2.

only a straightforward matrix manipulation have been presented in [17] and [18]. Observers for discrete-time descriptor systems have been developed in [14] and [16].

However, few results have been presented to design observers for descriptor systems with unknown inputs [19]–[21]. Descriptor systems are very sensitive to slight input changes, and the presence of immeasurable disturbances or unknown inputs is very detrimental to the design of observers. This fact justifies the importance of the observers design for descriptor systems in the presence of unknown inputs. On the other hand, many practical systems can be described by descriptor models, and the fault diagnosis of these systems may be based on the unknown input observer design.

In [19] and [20] only square singular systems have been considered under the regularity condition. In addition, the strong observability [19] and the modal observability [20] have been assumed. In [21], a coordinate transformation is used to design a reduced-order observer.

In this paper, we present a new method to design a reduced-order observer for continuous-time descriptor systems subject to unknown inputs and unknown measurement disturbances.

As in [21], systems considered are in a general form and less restrictive conditions are required.

## II. STATEMENT OF THE PROBLEM

Consider the linear time-invariant descriptor system

$$E^* \dot{x} = A^* x + B^* u + F^* w \quad (1a)$$

$$y^* = C^* x + G^* w \quad (1b)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^k$ ,  $w \in \mathbb{R}^q$ , and  $y^* \in \mathbb{R}^p$  are the state vector, the control input vector, the immeasurable input vector, and the output vector, respectively.  $E^* \in \mathbb{R}^{m \times n}$ ,  $A^* \in \mathbb{R}^{m \times n}$ ,  $B^* \in \mathbb{R}^{m \times k}$ ,  $F^* \in \mathbb{R}^{m \times q}$ ,  $G^* \in \mathbb{R}^{p \times q}$ , and  $C^* \in \mathbb{R}^{p \times n}$  are known constant matrices. We assume that  $\text{rank } E^* = r < n$ , and without loss of generality  $\text{rank } [C^* \ G^*] = p$ .

*Assumptions:* In the sequel we assume that

$$\begin{aligned} \text{i) } & \text{rank} \begin{bmatrix} F^* \\ G^* \end{bmatrix} = q \leq p \\ \text{ii) } & \text{rank} \begin{bmatrix} E^* & A^* & F^* & 0 \\ 0 & E^* & 0 & F^* \\ 0 & C^* & G^* & 0 \\ 0 & 0 & 0 & G^* \end{bmatrix} = \text{rank} \begin{bmatrix} E^* & F^* \\ 0 & G^* \end{bmatrix} \\ & = n + q. \end{aligned}$$

These conditions are not restrictive. Condition i) can always be met by redefining the unknown input. If  $\text{rank} \begin{bmatrix} F^* \\ G^* \end{bmatrix} = s < q$ , then we have  $\begin{bmatrix} F^* \\ G^* \end{bmatrix} w = \begin{bmatrix} F_1^* \\ G_1^* \end{bmatrix} v$ , where  $\begin{bmatrix} F_1^* \\ G_1^* \end{bmatrix}$  is of full column rank, and  $v$  can be considered as a new unknown input. Condition ii) generalizes the condition of the impulse observability of singular square systems (i.e.,  $m = n$  and  $\det E^* = 0$ ) when  $F^* = G^* = 0$ .

For  $m = n$ ,  $E^* = I$ , and  $G^* = 0$ , (1) becomes a standard one with unknown inputs; in this case condition ii) can be written as

$$\begin{aligned} \text{rank} \begin{bmatrix} I & F^* \\ C^* & 0 \end{bmatrix} &= n + q \text{ or equivalently} \\ \text{rank } C^* F^* &= \text{rank } F^* = q \end{aligned}$$

which is the condition generally assumed in the standard observer for unknown input systems [1]–[4].