Chebyshev Approximations for Dawson's Integral

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Abstract. Rational Chebyshev approximations to Dawson's integral are presented in well-conditioned forms for $|x| \leqq 2.5$, $2.5 \leqq |x| \leqq 3.5$, $3.5 \leqq |x| \leqq 5.0$ and $5.0 \leqq |x|$. Maximal relative errors range down to between 2×10^{-20} and 7×10^{-22}.

1. Introduction. Dawson's integral,

$$F(x) \equiv e^{-x^2} \int_0^x e^{t^2} dt \,, \tag{1.1}$$

appears in a variety of applications including spectroscopy, heat conduction, and electrical oscillations in certain special vacuum tubes. It is closely related to the (modified) complex error function,

$$w(z) \equiv e^{-z^2} \operatorname{erfc}(-iz) = e^{-z^2} \left\{ 1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2} dt \right\} = e^{-z^2} + \frac{2i}{\sqrt{\pi}} F(z) \,. \tag{1.2}$$

Because of its many applications this latter function has been extensively studied and tabulated. Its more important mathematical properties are included in Gautschi's [1] summary, while Armstrong [2] reviews both its applications and some computational methods. Many schemes for computing $w(z)$ are less effective near the real axis, and algorithms for computation there frequently begin with a value of $F(x)$.

Extensive tabulations of Dawson's integral exist. Perhaps the most accurate is that of Lohmander and Rittsten [3], which gives 10D values for much of the range, and selected values to 20D. More recently, Hummer [4] has published 18D values of the coefficients in the expansion

$$F(x) = \sum_{k=0}^{33} a_k T_{2k+1}(x/5) \,, \qquad |x| \leqq 5 \,, \tag{1.3}$$

where $T_{2k+1}(s)$ is the Chebyshev polynomial of the first kind of degree $2k + 1$.

In this paper we present a set of nearly-best rational approximations to $F(x)$ for all real x and with relative accuracies up to 22S. These approximations are not only more accurate than Hummer's, but may be made the basis of significantly faster subroutines.

2. Functional Properties. Dawson's integral is a special case of the confluent hypergeometric function:

Received January 9, 1969.

AMS Subject Classifications. Primary 6520, 6525; Secondary 4117, 4140, 3317, 3345.

Key Words and Phrases. Rational Chebyshev approximations, Dawson's integral, complex error function.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

** Work supported in part by the U. S. Atomic Energy Commission and in part by the University of Notre Dame.

$$F(z) = +z_1F_1(1; 3/2; -z^2) . \tag{2.1}$$

It follows that Dawson's integral is an antisymmetric function, and that values for all real x can be obtained from computations valid for $[0, \infty)$.

Differentiating (1.1) we find that for all complex z

$$F'(z) = -2zF(z) + 1 , \tag{2.2}$$

while for $k \geqq 1$ the derivatives can be shown to obey the recurrence

$$F^{(k+1)}(z) + 2zF^{(k)}(z) + 2kF^{(k-1)}(z) = 0 . \tag{2.3}$$

These results allow the computation of the Taylor series expansion about any point at which $F(z)$ is known with sufficient accuracy. In particular, since $F(0) = 0$, the Maclaurin series

$$F(z) = z \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!} (2z)^{2k} ; \qquad |z| < \infty \tag{2.4}$$

is easily obtained.

Although (2.4) converges for all finite z (hence $F(z)$ is an entire function), the ratio of successive terms is only $-2z^2/(2k + 3)$. Practical convergence is thus delayed until k becomes greater than $|z^2| - 3/2$, and serious cancellation errors may occur in summing (2.4) for even moderately large z. A more efficient expansion of $F(z)$ in the neighborhood of the origin is the continued fraction

$$F(z) = \frac{z}{1+} \frac{2z^2/3}{1-} \frac{4z^2/15}{1+} \frac{6z^2/35}{1-} \cdots + \frac{(-1)^{k+1}(2kz^2)/(2k-1)(2k+1)}{1 + \cdots} , \tag{2.5}$$

which can be obtained either by applying the QD algorithm to (2.4), or as a special case of the continued fraction for $1/{}_1F_1(1; \gamma; x)$ given by Perron [5, p. 123, Eq. (8)]. Thacher [6] shows that (2.5) converges throughout the complex plane, is uniformly more efficient than (2.4) and is numerically stable in the first octant of the complex plane. For large $|z|$, however, convergence is slow, and some 70 convergents are needed to attain 25D accuracy at $z = 6$.

Thus, for large z, expansions about the point at ∞ are desirable. By deforming the path in the integral representation of the (modified) complex error function [1, Eq. 7.1.4], it can be shown that $F(x)$ can be represented as the Cauchy principal value

$$F(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{x - t} dt . \tag{2.6}$$

Replacing $1/(x - t)$ by the identity

$$\frac{1}{x - t} = \frac{1}{x} \left\{ \sum_{k=0}^{n-1} \left(\frac{t}{x}\right)^k + \frac{(t/x)^n}{1 - t/x} \right\} \tag{2.7}$$

and integrating, we have

$$F(x) = \frac{1}{2x} \left\{ \sum_{k=0}^{[(n-1)/2]} \frac{\Gamma(k + \frac{1}{2})}{\Gamma(\frac{1}{2})} x^{-2k} + \frac{x^{-n}}{\Gamma(\frac{1}{2})} \int_{-\infty}^{\infty} \frac{t^n e^{-t^2}}{1 - t/x} dt \right\} , \tag{2.8}$$

where $[x]$ is the greatest integer contained in x. Equation (2.8) is the familiar asymptotic series with two different expressions for the remainder, depending on whether n is even or odd. (We are indebted to the referee for pointing out that Stieltjes [7] gives a similar remainder term for n even.) The development of recurrences and converging factors does not seem worthwhile in view of the good accuracy of (2.8) (almost 29D for $x = 8$).

TABLE I. $\mathcal{E}_{\ell m} = -100 \log_{10} \max \left| \frac{F(x) - F_{\ell m}(x)}{F(x)} \right|$

$0 \leq |x| \leq 2.5$

m \ ℓ	1	2	3	4	5	6	7	8	9
0	33	73	124	182	246	316	391	471	
1	126*	185	251	323	399	480	565		
2	209	284	363	447	534	624			
3	259	378	469*	563	660				
4		493	573	677	782	889			
5		593	653	790	903*				
6						1144*			
7							1398		
8								1663*	
9									1939*

$2.5 \leq |x| \leq 3.5$

m \ ℓ	1	2	3	4	5	6	7	8	9
0	279	417†	453	596†	626	718	828	868	963
1	384*	450		624		792	862		
2		524	724†	727	883	930	1019		
3			727		928				
4				964*	1020				
5					1104*	1245			
6						1338	1551	1574	
7							1574*		
8								1839*	
9									2112*

$3.5 \leq |x| \leq 5.0$

m \ ℓ	1	2	3	4	5	6	7	8	9
0	353	481	596	767†	768	865	945	1024	1102
1	507*	652	709	768		925	997		
2		707		847	976	1040	1135		
3			860*	995	1033	1098	1195	1296	
4				1032					
5					1242*				
6						1359			
7							1576*		
8								1751*	
9									1973*

$5.0 \leq |x|$

m \ ℓ	1	2	3	4	5	6	7	8	9
0	521	659	783	893	991	1076	1154	1232	1327
1	707*	838	952	1050	1138	1229	1330	1337	
2	858	978	1080	1175	1296	1312	1337		
3	973		1197*	1302	1312				
4	1068			1311					
5					1449*				
6						1571			
7							1685*		
8								1802*	
9									1911*

† Nonstandard error curve

* Coefficients for these approximations only are given in Tables II-V.

The continued fraction

$$F(x) \sim \frac{x}{2}\left\{\frac{1}{x^2-}\;\frac{1/2}{1-}\;\frac{2/2}{x^2-}\cdots-\frac{2k/2}{x^2-}\;\frac{(2k+1)/2}{1-}\cdots\right\} \tag{2.9}$$

corresponding to (2.8) diverges for all real x, but appropriate convergents can be profitably used for computation. Gautschi [9] has recently observed that it is in fact asymptotic in the sense of Poincaré.

TABLE II. $F(x) \approx x \sum_{j=0}^{n} p_j x^{2j} \Big/ \sum_{j=0}^{n} q_j x^{2j}$ $|x| \le 2.5$

n	j	p_j		q_j	
1	0	1.145	(00)	1.085	(00)
	1	-8.426	(-02)	1.000	(00)
3	0	4.76800 8	(01)	4.76791 2	(01)
	1	-4.30297 8	(00)	2.75055 1	(01)
	2	1.36584 5	(00)	6.90076 0	(00)
	3	-3.56642 4	(-02)	1.00000 0	(00)
5	0	1.08326 55887 3	(04)	1.08326 55877 2	(04)
	1	-1.28405 83227 9	(03)	5.93771 27693 5	(03)
	2	4.19672 90228 0	(02)	1.48943 55724 2	(03)
	3	-1.51982 15242 2	(01)	2.19728 33183 3	(02)
	4	1.67795 11618 9	(00)	1.99422 33636 4	(01)
	5	-2.38594 56569 6	(-02)	1.00000 00000 0	(00)
6	0	2.31569 75201 341	(05)	2.31569 75201 425	(05)
	1	-2.91794 64300 780	(04)	1.25200 37031 851	(05)
	2	9.66963 98191 665	(03)	3.13846 20138 163	(04)
	3	-4.35011 60207 595	(02)	4.74470 98440 662	(03)
	4	5.46161 22556 699	(01)	4.66849 06545 115	(02)
	5	-8.54106 81195 954	(-01)	2.93919 95612 556	(01)
	6	2.08468 35103 886	(-02)	1.00000 00000 000	(00)
8	0	1.73971 38358 72305 762	(08)	1.73971 38358 72305 803	(08)
	1	-2.35903 54309 49078 422	(07)	9.23905 68081 99581 718	(07)
	2	7.94595 11256 26974 712	(06)	2.31472 94223 70433 794	(07)
	3	-4.49408 95997 95344 826	(05)	3.59959 82595 90670 414	(06)
	4	6.26435 22480 53304 262	(04)	3.83498 04512 71685 588	(05)
	5	-1.64294 23044 87861 388	(03)	2.90022 12938 95164 274	(04)
	6	1.10415 15859 64097 196	(02)	1.54480 44953 25198 331	(03)
	7	-1.23806 01126 69044 439	(00)	5.42357 27435 06117 292	(01)
	8	1.70141 56251 64813 150	(-02)	1.00000 00000 00000 000	(00)
9	0	5.83917 15512 36746 64696 3	(09)	5.83917 15512 36746 64671 7	(09)
	1	-8.10874 57862 86504 21438 5	(08)	3.08190 64555 29180 70960 1	(09)
	2	2.74522 24075 69207 53304 4	(08)	7.72014 13077 99080 38383 4	(08)
	3	-1.66059 10282 27997 89467 4	(07)	1.21117 71647 64931 95888 2	(08)
	4	2.37774 44786 51142 48955 9	(06)	1.32000 94110 32992 82644 7	(07)
	5	-7.23343 62824 52533 46646 8	(04)	1.04469 18186 92075 89871 5	(06)
	6	5.42029 78360 01654 65492 4	(03)	6.06500 93218 89635 57172 5	(04)
	7	-7.87402 38311 16328 77922 8	(01)	2.52420 27853 26999 70863 5	(03)
	8	2.44411 40962 76240 45913 0	(00)	6.96419 57634 28417 04830 2	(01)
	9	-1.57085 62593 09369 50290 0	(-02)	1.00000 00000 00000 00000 0	(00)

3. Generation of the Approximations. The approximation forms and corresponding intervals used are

$$F_{lm}(x) = xR_{lm}(x^2)\,, \qquad |x| \leqq 2.5$$

(3.1)
$$= \frac{1}{x} R_{lm}(1/x^2)\,, \qquad 2.5 \leqq |x| \leqq 3.5; 3.5 \leqq |x| \leqq 5.0$$

$$= \frac{1}{2x}\left[1 + \frac{1}{x^2} R(1/x^2)\right], \qquad 5.0 \leqq |x|$$

where the $R_{lm}(z)$ are rational functions of degree l in the numerator and m in the

TABLE III. $F(x) \approx \frac{1}{x}\{\alpha_0 + \frac{\beta_0}{\alpha_1 + x^2} / + \ldots + \frac{\beta_{n-1}}{\alpha_n + x^2}\}$, $2.5 \leq |x| \leq 3.5$

n	j	α_j	β_j
1	0	4.99160 (-01)	2.51472 (-01)
	1	-1.96079 (00)	
4	0	5.00652 75443 7 (-01)	2.14221 96577 8 (-01)
	1	-4.91605 36574 1 (00)	3.73902 76351 2 (01)
	2	4.07068 10166 7 (00)	1.38216 34118 2 (01)
	3	-1.26817 90159 8 (01)	8.87619 38676 4 (01)
	4	3.47393 74258 6 (00)	
5	0	5.01401 06611 704 (-01)	1.88975 53014 354 (-01)
	1	-7.44990 50579 364 (00)	7.02049 80729 194 (01)
	2	7.50778 16490 106 (00)	4.18218 06337 830 (01)
	3	-2.66290 01073 842 (01)	3.73430 84728 334 (01)
	4	3.09840 87863 402 (01)	1.25993 23546 764 (03)
	5	-4.08473 91212 716 (01)	
7	0	5.00236 89608 86678 82 (-01)	2.26064 66607 43091 60 (-01)
	1	-5.97678 08682 34888 63 (00)	1.15840 29255 18881 28 (02)
	2	1.52644 09962 36985 89 (01)	7.29177 55641 55315 00 (01)
	3	-8.89106 47974 78123 30 (00)	1.12461 66202 45754 35 (02)
	4	-7.57931 91808 93692 74 (-02)	7.21193 21760 02290 59 (00)
	5	-4.00000 89364 35497 21 (01)	1.24486 78826 22516 16 (03)
	6	2.93365 74739 54485 30 (01)	-6.73106 06974 48133 14 (-01)
	7	-1.50695 65118 71605 55 (00)	
8	0	4.99753 72322 38672 65701 (-01)	2.82505 12959 56025 34507 (-01)
	1	5.14905 19894 61839 18332 (00)	-1.71845 97911 60867 73018 (02)
	2	6.76056 09265 22734 73204 (00)	2.85094 29523 41035 37104 (02)
	3	5.31365 22629 36985 87395 (00)	5.71551 83515 55917 20071 (01)
	4	-1.46536 07407 01534 12337 (01)	2.09472 56189 26938 46157 (02)
	5	4.70341 81870 14092 00108 (00)	-2.57668 08798 49772 32129 (00)
	6	9.69230 82777 47642 74334 (01)	1.05565 30121 09847 04166 (04)
	7	-1.07998 24592 49835 68179 (02)	4.65888 43814 36620 82502 (-01)
	8	-1.66279 86292 29032 28978 (00)	
9	0	5.00260 18362 20279 67838 339(-01)	2.06522 69153 96421 05009 383(-01)
	1	-1.73717 17784 36727 91148 539(01)	9.51190 92396 03814 58746 503(02)
	2	4.65842 08794 00152 95572 731(01)	1.70641 26974 52362 27356 448(02)
	3	-7.33080 08989 64028 70749 508(00)	3.02890 11061 01226 63923 423(02)
	4	4.56604 25072 51633 10121 912(00)	4.06209 74221 89356 89922 236(01)
	5	-2.27571 82952 50758 91337 450(01)	4.53642 11110 25777 27152 835(02)
	6	1.25808 70373 89512 51884 757(01)	-7.08465 68667 65730 00364 345(00)
	7	2.61935 63126 88259 92834 528(01)	1.10067 08103 45155 32890 922(03)
	8	-3.79258 97727 10428 80785 664(01)	1.82180 09331 35144 78378 375(00)
	9	-1.70953 80470 08554 94930 087(00)	

denominator. Experiences with other choices of intervals and forms closely paralleled previous experiences in approximating

(3.2)
$$\mathrm{Ei}\,(v) = -\fint_{-\infty}^{v} \frac{e^t}{t}\,dt$$

[8], including the existence of "barriers" and entire counter-diagonals of cases in the Walsh array with nonstandard error curves. Aside from the form in the first interval, the final choice of forms and intervals was achieved by analogy with the Ei(x) case.

TABLE IV. $F(x) \approx \frac{1}{x}\{\alpha_0 + \frac{\beta_0}{\alpha_1 + x^2} / + \ldots + \frac{\beta_{n-1}}{\alpha_n + x^2}\}$, $3.5 \le |x| \le 5.0$

n	j	α_j		β_j	
1	0	5.00221 90	(-01)	2.40464 13	(-01)
	1	-1.99531 75	(00)		
3	0	5.00009 65199	(-01)	2.49246 32421	(-01)
	1	-1.58539 35006	(00)	-5.37142 72981	(-01)
	2	-1.03850 24821	(01)	1.77617 78077	(01)
	3	-4.10832 33770	(00)		
5	0	5.00001 53840 8193	(-01)	2.49811 16284 5499	(-01)
	1	-1.53672 06927 1915	(00)	-6.53419 35986 0764	(-01)
	2	-1.77068 69371 7670	(01)	2.04866 41097 6332	(02)
	3	7.49584 01627 8357	(00)	-2.29875 84192 8600	(00)
	4	4.02187 49020 5698	(01)	2.53388 00696 3558	(03)
	5	-5.93915 91850 0315	(01)		
7	0	4.99999 90270 50535 94	(-01)	2.50011 45961 18389 42	(-01)
	1	-1.49838 04203 66907 23	(00)	-1.48715 81178 71947 48	(00)
	2	-4.98544 80298 66076 69	(00)	3.30707 72467 61143 70	(01)
	3	5.06460 15374 22307 72	(00)	1.46515 16778 31092 86	(02)
	4	-1.50507 70349 66919 57	(01)	7.51701 27774 40669 33	(01)
	5	-9.16804 87981 35517 10	(00)	2.56105 72234 22263 53	(01)
	6	-2.66167 67489 63992 81	(01)	2.87776 12297 31873 57	(02)
	7	4.76405 64527 32287 81	(00)		
8	0	5.00000 24559 63383 2639	(-01)	2.49955 21697 80615 1194	(-01)
	1	-1.51367 66880 71179 3746	(00)	-9.52887 96162 11405 9108	(-01)
	2	-1.64140 11434 80851 0959	(01)	2.82166 88221 34411 2618	(02)
	3	1.25225 24765 67802 1319	(01)	2.12179 97847 65171 2123	(01)
	4	-3.07219 08168 86247 9422	(01)	6.12148 21935 89774 4415	(02)
	5	1.28215 21658 43074 6296	(01)	-1.19322 18919 12575 0535	(01)
	6	4.84015 36834 58465 7257	(00)	3.42841 52809 32221 0445	(02)
	7	-2.73122 93682 83313 8394	(01)	3.88492 00485 30068 5817	(00)
	8	-5.44601 26093 27636 7554	(00)		
9	0	4.99999 81092 48588 24981 0	(-01)	2.50041 49236 99223 81760 6	(-01)
	1	-1.48432 34182 33439 65307 5	(00)	-2.31251 57538 51451 43070 0	(00)
	2	7.50964 45983 89196 12289 4	(00)	-6.88024 95250 45122 54535 0	(01)
	3	-3.35044 14982 05924 49071 5	(01)	1.24018 50000 99171 63022 7	(03)
	4	2.69790 58673 54676 49968 7	(01)	-9.18871 38529 32158 73406 3	(00)
	5	4.84507 26508 14914 52130 0	(01)	3.48817 75882 22863 53588 2	(03)
	6	-6.68407 24033 76967 56837 9	(01)	1.40238 37312 61493 85227 7	(01)
	7	-7.36315 66912 68305 26753 7	(00)	9.98607 19803 94520 81913 3	(01)
	8	-1.86647 12333 84938 52581 7	(01)	4.47820 90802 59717 49851 5	(01)
	9	-4.55169 50325 50948 15111 5	(00)		

With a simple change of variable in (1.1) we have

$$e^{z^2}F(z) = \int_0^{\sqrt{z}} \frac{e^y}{2\sqrt{y}}\,dy\,. \tag{3.3}$$

Comparison of (3.2) and (3.3), ignoring the difference in the lower limit of integration, indicates a similarity of the integrals involved provided $v = \sqrt{z}$, i.e., $v^2 = z$. The approximating forms and intervals for Ei(x) were

$$(3.4)\qquad \begin{aligned} E_{lm}(x) &= \frac{e^x}{x} R_{lm}(1/x)\,, && 6 \leqq x \leqq 12;\, 12 \leqq x \leqq 24 \\ &= \frac{e^x}{x}\left[1 + \frac{1}{x} R_{lm}(1/x)\right], && 24 \leqq x\,. \end{aligned}$$

Replacing x by x^2 everywhere in (3.4), except in the $1/x$ factor out front (because of the relationship of the denominators of the integrands in (3.2) and (3.3)), we arrive at the forms and approximate intervals of (3.1).

TABLE V. $F(x) \approx \frac{1}{2x}\{1 + x^{-2}[\alpha_0 + \frac{\beta_0}{\alpha_1 + x^2}/\!+\ldots+ \frac{\beta_{n-1}}{\alpha_n + x^2}]\}, \qquad 5.0 \leq |x|$

n	j	α_j		β_j	
1	0	5.00038 1123	(-01)	7.44938 7745	(-01)
	1	-2.74862 7766	(00)		
3	0	5.00000 00167 450	(-01)	7.49999 19056 701	(-01)
	1	-2.50017 11668 562	(00)	-2.48787 65880 441	(00)
	2	-4.67312 02214 124	(00)	-4.12544 06560 831	(00)
	3	-1.11952 16423 662	(01)		
5	0	5.00000 00000 87358 0	(-01)	7.49999 99263 58122 3	(-01)
	1	-2.50000 27830 30495 0	(00)	-2.49963 00606 78980 2	(00)
	2	-4.51057 82777 83269 5	(00)	-6.58834 68001 31477 4	(00)
	3	-7.82636 28103 36344 1	(00)	-6.89636 11433 76130 9	(-01)
	4	-4.05239 81738 80339 4	(01)	5.20416 17289 69394 6	(02)
	5	4.12716 33274 69802 1	(00)		
7	0	5.00000 00000 00600 406	(-01)	7.49999 99991 99070 733	(-01)
	1	-2.50000 00485 79511 590	(00)	-2.49998 93551 22795 503	(00)
	2	-4.50052 01819 65761 489	(00)	-6.96290 12664 80073 481	(00)
	3	-6.72908 69322 16828 475	(00)	-8.06883 04211 17507 663	(00)
	4	-1.60863 57366 82904 117	(01)	2.34427 60077 33044 876	(01)
	5	-1.71955 24711 37642 621	(01)	2.30134 88351 50799 040	(01)
	6	-3.57614 99688 64747 149	(01)	2.63095 17219 54352 167	(02)
	7	-6.03624 87799 36117 815	(00)		
8	0	4.99999 99999 99950 09314	(-01)	7.50000 00000 81665 93551	(-01)
	1	-2.49999 99938 94251 97124	(00)	-2.50000 16639 46111 86828	(00)
	2	-4.49989 76775 67821 31861	(00)	-7.00937 73348 38185 24718	(00)
	3	-6.42407 78694 83867 90346	(00)	-1.60030 12838 36633 52091	(01)
	4	-5.74412 67565 44925 85234	(00)	-2.23978 04989 32484 55641	(01)
	5	-2.73184 46770 62325 14516	(01)	2.02858 16509 18903 12954	(02)
	6	-8.36640 14630 09488 18251	(00)	-7.72379 65289 21189 27712	(00)
	7	2.37388 67308 46445 74897	(01)	2.31971 95624 54408 63755	(03)
	8	-7.07853 08236 53407 93384	(01)		
9	0	5.00000 00000 00004 88400 1	(-01)	7.49999 99999 90270 92188 5	(-01)
	1	-2.50000 00008 89558 34951 8	(00)	-2.49999 97010 41844 64568 4	(00)
	2	-4.50002 29300 03555 85708 4	(00)	-6.99732 73504 15472 47160 7	(00)
	3	-6.52828 72752 69807 41589 8	(00)	-1.22097 01055 89348 38708 5	(01)
	4	-1.13867 36573 60661 02576 6	(01)	1.97325 36569 23161 83530 7	(01)
	5	-5.99085 54041 82220 02197 2	(00)	2.08210 24693 55645 47888 9	(02)
	6	-2.88301 99246 70561 05853 6	(01)	6.11539 67148 01158 46173 1	(01)
	7	-2.23787 66902 87518 86675 4	(01)	5.04198 95874 24657 52861 1	(01)
	8	-3.84043 88247 74544 53429 6	(01)	2.69382 30041 72388 16428 1	(02)
	9	-8.11753 64755 84326 85797 2	(00)		

The approximations were computed using standard versions of the Remes algorithm for rational Chebyshev approximation [10], [11] in 25S arithmetic on a CDC 3600. Function values were computed as needed using the continued fractions (2.5) and (2.9), and Taylor series expansions about $x = 5$ and $x = 7$ derived with the aid of (2.2) and (2.3). All expansion coefficients used were derived in 40S arithmetic. The final master function routines in 25S arithmetic were extensively checked by comparing calculations based on two different methods wherever possible, and by direct comparison against calculations in 40S arithmetic. These checks indicated an accuracy of from 23S to 25S in our master routine.

4. Results. Table I lists the values of

$$\mathcal{E}_{lm} = -100 \log_{10} \max \left| \frac{F(x) - F_{lm}(x)}{F(x)} \right| ,$$

where the maximum is taken over the appropriate interval, for the initial segments of the various L_∞ Walsh arrays. Tables II-V present coefficients for selected approximations along the main diagonals of these arrays. Each approximation listed, with the coefficients just as they appear here, was tested against the master function routines with 5000 pseudo-random arguments. In all cases the maximal error agreed in magnitude and location with the values given by the Remes algorithm.

In all intervals except the first the approximations were found to be slightly ill-conditioned when expressed as ratios of polynomials. For these intervals the approximations are presented here as well-conditioned J-fractions:

$$R_{nn}(1/x^2) = \alpha_0 + \frac{\beta_0}{\alpha_1 + x^2} \Big/ + \cdots + \frac{\beta_{n-1}}{\alpha_n + x^2} .$$

The coefficients are all presented to an accuracy slightly greater than that warranted by the maximal errors in the approximations. Reasonable additional rounding will not seriously affect the overall accuracy of the approximations. Subroutines based on these coefficients and achieving essentially machine accuracy have been written for the CDC 3600 and the IBM System/360 at Argonne National Laboratory.

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