

Chebyshev Approximations for the Riemann Zeta Function

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Abstract. This paper presents well-conditioned rational Chebyshev approximations, involving at most one exponentiation, for computation of either $\zeta(s)$ or $\zeta(s) - 1$, $.5 \leq s \leq 55$, for up to 20 significant figures. The logarithmic error is required in one case. An algorithm for the Hurwitz zeta function, and an example of nearly double degeneracy are also given.

1. Introduction. The Riemann zeta function is defined by

$$(1.1) \quad \zeta(s) = \sum_{k=1}^{\infty} k^{-s} \quad (\operatorname{Re}(s) > 1)$$

or by the power series expansion

$$(1.2) \quad \zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n \quad (\operatorname{Re}(s) > 0)$$

where

$$\gamma_n = \lim_{m \rightarrow \infty} \left\{ \sum_{k=1}^m \frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1} \right\}.$$

It is an analytic function of s , regular throughout the complex plane except for a simple pole of residue 1 at $s = 1$. The zeta function satisfies the functional equation

$$(1.3) \quad \zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{1}{2}\pi s\right) \Gamma(s) \zeta(s).$$

Evaluation of the function for real s usually involves taking a partial sum of (1.1) and applying the Euler-Maclaurin summation formula to the remainder. While this procedure is theoretically valid for all $s > -2n - 1$, where n terms of the Euler-Maclaurin summation formula are used, there is serious cancellation error for $s < 1.5$. However, the reflection formula, Eq. (1.3), can be used for $s < .5$, while Thacher [7] has recently used Eq. (1.2) as a basis for expansions in Chebyshev polynomials valid both for $\frac{1}{2} \leq s \leq \frac{3}{2}$ and for $1 \leq s \leq 2$. For $s \geq 2$, it is still necessary to evaluate a partial sum of the series (1.1). The process involves an exponentiation for each new term added to the sum, and is therefore quite slow. This paper presents rational Chebyshev approximations for evaluating $\zeta(s)$ or $\zeta(s) - 1$ for up to 20S without any exponentiation for $.5 \leq s \leq 11$, and with only one exponentiation for $11 \leq s \leq 55$.

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The approximation forms and the values of s for which they are used are

$$\begin{aligned}\zeta(s) &\simeq R_{l,m}(s)/(s-1), & .5 \leq s \leq 5, \\ \zeta(s) - 1 &\simeq R_{l,m}(s), & 5 \leq s \leq 11, \\ &\simeq 2^{-s+(1/s)R_{l,m}(1/s)}, & 11 \leq s \leq 25, 25 \leq s \leq 55,\end{aligned}$$

where the $R_{l,m}(s)$ are rational functions of degree l in the numerator and m in the denominator. The maximum error was computed relative to $\zeta(s)$ for the first interval, and relative to $\zeta(s) - 1$ for the others.

2. Computational of Reference Values. Reference function values for the generation of the approximations were calculated for $.5 \leq s \leq 1.5$ from the coefficients given by Thacher [7], and for the other s from a modification of the above-described technique based on the Euler-Maclaurin summation formula. The modification involves a method for estimating the number of terms needed in the partial sum of (1.1).

The Euler-Maclaurin formula applied to the Dirichlet series for the Hurwitz zeta function,

$$(2.1) \quad \zeta(s; \alpha) = \sum_{k=0}^{\infty} (k + \alpha)^{-s}, \quad \alpha > 0, \operatorname{Re}(s) > 1,$$

gives

$$(2.2) \quad (s-1)\zeta(s; \alpha) = \frac{2\alpha + s - 1}{2\alpha^s} + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!} \frac{\Gamma(s+2k-1)}{\Gamma(s-1)\alpha^{s+2k-1}} + R_n,$$

where

$$(2.3) \quad R_n = \frac{(-1)^{n+1}\Gamma(s+n)}{n! \Gamma(s-1)} \int_0^{\infty} \frac{\hat{B}_n(-t) dt}{(\alpha+t)^{s+n}}$$

and $\hat{B}_n(x)$ is the periodic extension of the n th Bernoulli polynomial. Since $(\alpha+t)^{s+n} > 0$ for $t > -\alpha$, the mean value theorem can be used to obtain

$$(2.4) \quad R_n = (-1)^{n+1} \frac{\Gamma(s+n-1)}{n! \Gamma(s-1)} \frac{B_n(\xi)}{\alpha^{s+n-1}}, \quad 0 \leq \xi \leq 1.$$

Thus

$$(2.5) \quad |R_n| \leq \frac{1}{n!} \prod_{k=1}^n \frac{(s+k-2)M_n}{|\alpha|^{s+n-1}},$$

where

$$M_n = \max_{0 \leq x \leq 1} |B_n(x)|.$$

Letting G and A denote the geometric and arithmetic means of the quantities $\{(s+k-2)\}$, and using the arithmetic-geometric mean inequality, we have

$$(2.6) \quad \prod_{k=1}^n (s+k-2) = G^n \leq A^n = \left\{ \sum_{k=1}^n \frac{(s+k-2)}{n} \right\}^n = \left(s + \frac{n-3}{2} \right)^n,$$

so that

$$(2.7) \quad |R_n| \leq \frac{1}{n!} \left(s + \frac{n-3}{2} \right)^n \frac{M_n}{|\alpha|^{s+n-1}}.$$

Lehmer [4] discusses the extrema of the Bernoulli polynomials and shows that $M_n = |B_n|$ for n even. For n odd, he gives 11D values of M_n , for $n \leq 13$, as well as relatively sharp asymptotic formulas for larger n . We thus see that $[n/2]$ terms of (2.2) will approximate $(s - 1)\zeta(s; \alpha)$ with an absolute error less than 10^{-D} provided that

$$(2.8) \quad |\alpha| \geq \left\{ \left(s + \frac{n-3}{2} \right)^n \frac{M_n}{n!} 10^D \right\}^{1/(s+n-1)}.$$

In principle, then, one can obtain $(s - 1)\zeta(s; a)$ to any desired accuracy by selecting n and m so that (2.8) is satisfied for $\alpha = a + m$, evaluating $(s - 1)\zeta(s; \alpha)$ by (2.2), and finally computing $(s - 1)\zeta(s; a)$ by adding $(s - 1)$ times the appropriate partial sum of the Dirichlet series for $\zeta(s; a)$. The procedure is valid for $\text{Re}(s) > 0$.

The most efficient value of n depends upon s and D . In our calculations, we did not vary this parameter, but gave it the constant value 13, for which

$$m = [\{8.40798 \times 10^{D-11}(s + 5)^{13}\}^{1/(s+12)} + 1 - a],$$

where $[]$ denotes "the integer part of." The variable order of the partial sum prevented the use of Markman's economical method [5] of reducing the necessary number of exponentiations in (2.2).

All computations were carried out on a CDC 3600 in 25S arithmetic. Extensive checking against tables and by overlapping of methods shows that our master routines were accurate to roughly a minimum of 23S.

3. Generation of the Approximations. The various approximations were generated in 25S floating-point arithmetic on a CDC 3600 using standard versions of the Remes algorithm [2]. With two exceptions, the computations were straightforward.

The first exception was the analysis relating to the approximation form for the last two intervals. The quantity

$$(3.1) \quad \delta(s) = \frac{\zeta(s) - 1 - 2^{-s+(1/s)R_{lm}(1/s)}}{\zeta(s) - 1}$$

is the error of approximation relative to $\zeta(s) - 1$. However, the basic Remes algorithm is limited to error expressions of the form

$$(3.2) \quad \Delta(s) = \frac{f(s) - R_{lm}(\phi(x))}{g(x)}.$$

Hence, we must modify (3.1). By letting

$$(3.3) \quad \delta(s) = 1 - 2^{-d(s)},$$

we find that $d(s)$ has the form (3.2) with

$$f(s) = s[\ln(\zeta(s) - 1)/\ln 2 + s], \quad g(s) = s,$$

and

$$\phi(s) = 1/s.$$

We note that the local extrema of $d(s)$ and $\delta(s)$ occur for the same values of s , and that

$$\hat{\delta}(s) = (-\ln 2) d(s) = \ln(1 - \delta(s)) \simeq \delta - \frac{1}{2}\delta^2,$$

where $\hat{\delta}(s)$ is the "logarithmic error" discussed by King and Phillips [3] and Sterbenz and Fike [6]. The logarithmic error has been associated primarily with obtaining starting values for various Newton iteration schemes. However, we can use $\hat{\delta}(s)$ in the Remes algorithm since it approximates the Chebyshev error $\delta(s)$ to within terms of order $\delta^2(s)$, an error that is swamped by normal roundoff in the Remes algorithm itself whenever $\delta(s)$ is small.

The second anomaly occurred in the computation of $R_{88}(1/s)$ for the interval [11, 25]. Although the error curves for R_{66} and R_{77} appear to be standard, R_{88} is nearly doubly degenerate. The method of artificial poles [2] determined the Chebyshev error for R_{88} as approximately 5.2×10^{-17} , with the error curve still not leveled. At this point, the denominator had among its zeros the values

$$s_1 = .0371111862 \quad \text{and} \quad s_2 = .13063202.$$

Corresponding zeros in the numerator were $s_1 + (1 \times 10^{-10})$ and $s_2 + (2 \times 10^{-8})$. To our knowledge, this is the first case of nearly double degeneracy that has occurred in practice.

4. Results. Table I lists the values of

$$E_{i_m} = -100 \log_{10} \delta_{i_m}$$

for selected segments of the L_∞ Walsh arrays. The minimax error δ_{i_m} of approximation by R_{i_m} is the error relative to $\zeta(s)$ for the interval [.5, 5], and relative to $\zeta(s) - 1$ for the other intervals.

Tables II-V present the approximations giving accuracies most appropriate for computers in use today. The coefficients are given to accuracy slightly greater than that justified by the approximation errors, but reasonable additional rounding should not greatly affect the overall accuracies. Each approximation listed, using the coefficients just as they appear here, was tested for random arguments against the master function routines, and the stated accuracies were all verified.

There are a few anomalies present in the Walsh array. Nonstandard error curves are flagged in Table I. Usually, a nearly degenerate case is signalled by the presence of a nonstandard error curve for the approximation that is one degree lower in both numerator and denominator. Although as previously mentioned, R_{77} for the interval [11, 25] has a standard error curve, R_{88} is nearly doubly degenerate. This troublesome approximation is not given in Table IV. Instead, the nondiagonal element R_{79} is given.

With a little care, computer subroutines returning almost full machine precision values of $\zeta(s)$ and of $\zeta(s) - 1$ can be written using these approximations. One troublesome computation is that for $\zeta(s) - 1$ for $.5 \leq s \leq 5$. If one uses

$$(4.1) \quad (s - 1)\zeta(s) \simeq R_{i_m}(s) \equiv \frac{\sum_{i=0}^i p_i s^i}{\sum_{i=0}^m q_i s^i}, \quad .5 \leq s \leq 5,$$

TABLE I

$$E_{\ell m} = -100 \log_{10} \delta_{\ell m}$$

$$.5 \leq s \leq 5.$$

$m \backslash \ell$	1	2	3	4	5	6	7	8	9
0		214	367	417				796	
1	191	322	414						
2	266	429	590	717					
3	342	536	722	782					
4	418	633	792	862*					
5					1142*				
6						1370			
7							1658		
8								1906*	
9									2193*

$$5 \leq s \leq 11$$

0	13 [†]	47 [†]	97	156	219	283	348		
1	71	146	227	316					
2	147	244	342	443					
3			456	570					
4	315	444	569	692					
5					951*				
6						1211*			
7							1571		
8								1738*	
9									1865*

[†]Nonstandard error curve.

*Coefficients for these approximations only are given in Tables II-V.

TABLE I (cont'd)

		$11 \leq s \leq 25$								
m	l	1	2	3	4	5	6	7	8	9
0			447 [†]	457	561	626	676	772		
1		361	457		615					
2		416	525	683	721					
3		466	584	719	778					
4		514	640	781	939*					
5						1165*				
6							1438			
7								1572	1585	
8								1585	1630	
9								1803*		1961*

		$25 \leq s \leq 55$								
m	l	1	2	3	4	5	6	7	8	9
0			555	624	741	795	882	948		
1		584	638	707	788					
2		646	705	779	867					
3		699	763	843*	935					
4			817	902	998					
5						1169*				
6							1354			
7								1552		
8									1760*	
9										1979*

[†] Nonstandard error curve.

* Coefficients for these approximations only are given in Tables II-V.

TABLE II

$$(s-1)\zeta(s) = \frac{\sum_{j=0}^n p_j s^j}{\sum_{j=0}^n q_j s^j}, \quad .5 \leq s \leq 5$$

n	j	p _j				q _j							
4	0	-1.32899	37437			(04)	-2.65799	37266			(04)		
	1	-1.70341	74205			(04)	-1.17972	24222			(04)		
	2	-7.70056	02483			(03)	-1.03940	13777			(03)		
	3	-1.42561	64640			(03)	-1.18419	54886			(02)		
	4	-8.36940	23543			(01)	1.00000	00000			(00)		
5	0	-3.44793	47840	721		(06)	-6.89586	96520	340		(06)		
	1	-3.25983	44394	057		(06)	-7.41777	18287	314		(05)		
	2	-9.63006	98255	869		(05)	-1.42686	20411	978		(05)		
	3	-1.08872	52505	125		(05)	-6.21717	54627	536		(03)		
	4	-6.95142	48803	854		(03)	-7.03261	35254	848		(02)		
5	-6.52319	89744	728		(02)	1.00000	00000	000		(00)			
8	0	1.28716	81214	82446	39280	9	(10)	2.57433	62429	64846	24466	7	(10)
	1	1.37539	69320	37025	11182	5	(10)	5.93816	56486	79590	16000	3	(09)
	2	5.10665	59183	64406	10368	3	(09)	9.00633	03732	61233	43908	9	(08)
	3	8.56147	10024	33314	86246	9	(08)	8.04253	66342	83289	88858	7	(07)
	4	7.48361	81243	80232	98482	4	(07)	5.60971	17595	41920	06281	4	(06)
	5	4.86010	65854	61882	51153	5	(06)	2.24743	12028	99137	52354	3	(05)
	6	2.73957	49902	21406	08772	8	(05)	7.57457	89093	41537	56011	5	(03)
	7	4.63171	08431	83427	12306	1	(03)	-2.37383	57813	73772	62308	6	(01)
8	5.78758	10040	96660	65910	9	(01)	1.00000	00000	00000	00000	0	(00)	
9	0	9.53904	31383	75296	85073	071	(11)	1.90780	86276	75059	16848	384	(12)
	1	1.09086	19179	76949	25970	267	(12)	5.83214	73972	70833	55022	737	(11)
	2	4.57302	66750	30644	70069	336	(11)	1.04516	64692	48187	07276	116	(11)
	3	9.44640	69371	30822	59763	084	(10)	1.22272	73758	61580	59174	256	(10)
	4	1.13392	40859	96679	77378	270	(10)	1.07248	18832	75479	65542	864	(09)
	5	9.69200	64774	06284	28141	723	(08)	6.86833	53570	00418	28875	818	(07)
	6	6.71953	10507	96081	75927	677	(07)	3.42258	54978	70618	48187	183	(06)
	7	3.27425	31821	83494	05394	001	(06)	1.13430	57824	29430	01348	947	(05)
	8	1.09513	05859	34055	20435	868	(05)	2.77837	18457	74528	05958	463	(03)
9	2.84456	26751	69802	16448	194	(03)	1.00000	00000	00000	00000	000	(00)	

TABLE III

$$\zeta(s) = 1 + \frac{\sum_{i=0}^{n_i} p_j T_j(t)}{\sum_{i=0}^{n_i} q_j T_j(t)}, \quad t = \frac{s-8}{3}, \quad 5 \leq s \leq 11$$

n	J	P _J					q _J				
5	0	1.84900	99918	7	(03)	4.58161	73101	5	(05)		
	1	-8.39681	24248	2	(02)	2.17592	06213	4	(05)		
	2	1.84197	92969	4	(02)	4.75387	98338	2	(04)		
	3	-2.32222	78218	6	(01)	6.01111	09082	7	(03)		
	4	1.68560	34490	2	(00)	4.36801	89581	7	(02)		
5	-5.72149	34762	5	(-02)	1.51875	00000	0	(01)			
6	0	7.65350	39697	5206	(04)	1.87400	70944	9945	(07)		
	1	-3.51659	51754	0597	(04)	8.84015	00526	2110	(06)		
	2	8.00110	63791	5310	(03)	1.94507	16063	8845	(06)		
	3	-1.09338	72046	0591	(03)	2.57600	76199	8960	(05)		
	4	9.43060	49292	2886	(01)	2.12069	28052	1806	(04)		
	5	-4.90452	46402	1741	(00)	1.04683	24143	9803	(03)		
6	1.21767	64406	2363	(-01)	2.27812	50000	0000	(01)			
8	0	-1.37639	45864	32697	9078	(07)	-2.59451	24986	97831	0818	(09)
	1	7.48218	91630	53159	7222	(06)	-9.48715	40757	99078	1663	(08)
	2	-2.07584	50481	02110	1368	(06)	-1.05496	19347	40052	0329	(08)
	3	3.55302	55709	62142	9466	(05)	4.67774	48821	19930	4847	(06)
	4	-4.06706	44955	18548	8897	(04)	3.12936	04057	38135	3370	(06)
	5	3.19804	86402	71469	1139	(03)	4.59581	80383	93050	6974	(05)
	6	-1.69820	93703	37228	5303	(02)	3.88176	10961	03968	3366	(04)
	7	5.61485	84239	42890	4752	(00)	1.92561	54483	44914	2325	(03)
8	-8.93888	70592	61549	4375	(-02)	5.12578	12500	00000	0000	(01)	
9	0	1.31731	00778	41255	95241	(08)	2.32735	50216	37673	34965	(10)
	1	-7.49969	32583	63955	41082	(07)	8.06195	84801	09344	64523	(09)
	2	2.38862	72960	74183	05175	(07)	1.20398	93324	79128	14480	(09)
	3	-4.95116	97536	05156	15979	(06)	1.93082	33109	74192	31735	(08)
	4	7.10728	21386	88157	86508	(05)	3.66527	98303	89832	19232	(07)
	5	-7.25277	77641	34448	98223	(04)	6.22570	65496	50081	68318	(06)
	6	5.26209	82645	20380	54349	(03)	7.20159	25663	83949	26869	(05)
	7	-2.63045	60896	79592	12803	(02)	5.71179	75640	89075	79564	(04)
	8	8.28410	63650	33952	88601	(00)	2.80036	61256	59096	99239	(03)
9	-1.26433	19164	40679	13509	(-01)	7.68867	18750	00000	00000	(01)	

TABLE IV

$$\zeta(s) = 1 + 2^{-s} + (1/s) \cdot \frac{\sum_{j=0}^n p_j s^{-j}}{\sum_{j=0}^n q_j s^{-j}}, \quad 11 \leq s \leq 25$$

n	j	p _j				q _j	q _j				
4	0	2.88915	56312	2	(-06)	8.72078	04947	2	(-05)		
	1	-3.60673	21549	6	(-04)	-9.58659	06458	3	(-04)		
	2	1.74441	55297	5	(-02)	2.61239	70760	9	(-02)		
	3	-3.88297	10518	9	(-01)	-1.39152	31614	9	(-02)		
	4	3.36332	63994	4	(00)	1.00000	00000	0	(00)		
5	0	-6.95395	38811	340	(-08)	1.61059	22487	913	(-06)		
	1	1.16861	20068	219	(-05)	9.59892	63880	291	(-06)		
	2	-8.12298	42080	718	(-04)	6.07669	74338	932	(-04)		
	3	2.92431	24478	121	(-02)	1.07138	06527	427	(-02)		
	4	-5.46064	90480	737	(-01)	8.47868	52767	957	(-02)		
	5	4.23648	33879	757	(00)	1.00000	00000	000	(00)		
*9	0	1.66156	48051	57746	75916	(-11)	-6.99562	63351	91916	54964	(-10)
	1	-4.68068	82766	06545	26862	(-09)	-1.77757	96189	51492	56941	(-08)
	2	5.83519	72731	91470	47318	(-07)	-9.82231	82573	40780	36442	(-07)
	3	-4.17644	01264	31456	02124	(-05)	-2.84927	28275	90964	87594	(-05)
	4	1.85468	42284	35979	59483	(-03)	-5.81727	90938	80480	93531	(-04)
	5	-5.11288	80022	04902	40591	(-02)	-1.15848	74916	97665	85807	(-02)
	6	8.10450	23175	11003	53193	(-01)	-1.28149	12405	19781	95742	(-01)
	7	-5.69951	94876	84789	22618	(00)	-1.11913	05734	90977	09324	(00)
	8						-7.67928	76160	46288	12537	(-01)
	9					1.00000	00000	00000	00000	(00)	
9	0	6.54074	87262	07601	13319	6 (-13)	-9.41483	96988	23587	07316	4 (-11)
	1	-1.91182	33182	41692	93750	0 (-10)	8.05123	81296	05720	22212	8 (-10)
	2	2.45909	27987	80779	03147	0 (-08)	-1.48342	86157	21743	05569	6 (-07)
	3	-1.80047	53535	42409	75813	5 (-06)	-2.21243	85714	34347	20342	1 (-07)
	4	8.06024	99146	89416	87391	8 (-05)	-7.12213	00202	26622	79049	2 (-05)
	5	-2.17335	07154	33717	13164	9 (-03)	-4.25218	58071	71304	01282	1 (-04)
	6	3.09201	82991	73503	66849	2 (-02)	-1.27791	21921	98850	98331	8 (-02)
	7	-1.10135	65806	72497	58178	8 (-01)	-5.86508	77210	99441	18425	8 (-02)
	8	-1.73589	22656	22043	62653	2 (00)	-3.00031	31173	95834	98139	1 (-01)
	9	3.80279	09938	36744	53166	3 (00)	1.00000	00000	00000	00000	0 (00)

* Denominator is of degree 2 greater than numerator.

TABLE V

$$\zeta(s) \approx 1 + 2^{-s + (1/s)} \frac{\prod_{j=0}^n p_j s^{-j}}{\prod_{j=0}^n q_j s^{-j}}, \quad 2.5 \leq s \leq 5.5$$

n	j	P _j				q _j								
3	0	4.45020	40561			(-09)	-9.08360	84927			(-05)			
	1	-6.39481	80674			(-07)	5.83123	29118			(-03)			
	2	3.06819	07594			(-05)	-1.30562	75204			(-01)			
	3	-4.92460	44511			(-04)	1.00000	00000			(00)			
5	0	6.30581	82031	926			(-12)	-2.45066	98983	435		(-07)		
	1	-1.69754	12776	425			(-09)	2.27263	46889	318		(-05)		
	2	1.84569	49027	429			(-07)	-9.29701	78224	265		(-04)		
	3	-1.01420	20734	917			(-05)	1.98757	34180	426		(-02)		
	4	2.81957	70988	437			(-04)	-2.20338	90078	903		(-01)		
	5	-3.17620	79765	651			(-03)	1.00000	00000	000		(00)		
8	0	1.03144	87718	88597	1168			(-15)	5.93959	41728	84190	5020	(-11)	
	1	-5.12584	61396	46882	4062			(-13)	-6.04755	35907	99918	0572	(-09)	
	2	1.12948	79419	48735	4786			(-10)	3.64680	20866	83885	6275	(-07)	
	3	-1.44234	66537	31309	5228			(-08)	-1.29456	90556	80118	1241	(-05)	
	4	1.16824	67698	44580	9766			(-06)	3.20189	49847	02292	5001	(-04)	
	5	-6.14975	16799	03148	0614			(-05)	-5.07801	55709	99940	7748	(-03)	
	6	2.05594	67798	88303	2750			(-03)	5.49628	90788	15872	6560	(-02)	
	7	-3.99339	42939	46688	6853			(-02)	-3.24517	61115	59724	1852	(-01)	
8	3.45234	97673	61784	5708			(-01)	1.00000	00000	00000	0000	(00)		
9	0	2.32320	68054	88716	51963	3		(-16)	-1.41965	98040	97653	26071	0	(-11)
	1	-1.35449	79553	19349	35076	5		(-13)	1.34852	03552	59192	68792	1	(-09)
	2	3.55929	73750	95400	68585	2		(-11)	-8.97874	17847	37323	82249	4	(-08)
	3	-5.53602	70696	83907	98449	3		(-09)	3.35733	07823	35633	51346	7	(-06)
	4	5.61986	24870	48476	40107	6		(-07)	-9.80017	32529	00364	84028	5	(-05)
	5	-3.86350	56607	15340	11238	7		(-05)	1.76658	33281	80539	21937	9	(-03)
	6	1.79968	78444	35254	07412	3		(-03)	-2.58993	67889	18623	48909	1	(-02)
	7	-5.48022	94365	91701	99314	2		(-02)	1.97161	45596	61601	18167	1	(-01)
	8	9.90367	46668	06121	96654	9		(-01)	-1.36202	98660	70096	90749	2	(00)
9	-8.09626	06414	79518	69042	8		(00)	1.00000	00000	00000	00000	0	(00)	

the computation

$$\zeta(s) - 1 = \frac{R_{lm}(s)}{s - 1} - 1$$

can lead to considerable subtraction error. Instead, one should use the form

$$(4.2) \quad \zeta(s) - 1 \approx \frac{\sum_{i=0}^M \hat{p}_i s^i}{(s - 1) \sum_{i=0}^m q_i s^i}, \quad .5 \leq s \leq 5,$$

where

$$M = \max(l, m + 1), \quad \sum_{i=0}^M \hat{p}_i s^i \equiv \sum_{i=0}^l p_i s^i - (s - 1) \sum_{i=0}^m q_i s^i,$$

and the \hat{p}_i are determined explicitly. It is not difficult to show that if δ_{lm} is the relative error in using (4.1) as an approximation to $\zeta(s)$, then the relative error in using (4.2) as an approximation to $\zeta(s) - 1$ is bounded by $30\delta_{lm}$. Thus, if the relative error in the machine is bounded by 10^{-D} , one should choose an $R_{lm}(s)$, $.5 \leq s \leq 5$, such that $30\delta_{lm} < 10^{-D}$.

In the second interval, the $R_{lm}(s)$ are poorly conditioned when expressed in the

usual way. They are therefore presented instead as well-conditioned ratios of sums of Chebyshev polynomials. In the last two intervals, better accuracy in $\zeta(s) - 1$ will be obtained by setting $n = [s]$ and computing $2^{-s + (1/s)R(1/s)}$ as $2^{-n} \times 2^{-(s-n) + (1/s)R(1/s)}$, where 2^{-n} can be done exactly on most computers by modifying the floating point exponent. Use of the new self-contained exponentiation routines [1] would also help.

Subroutines for both the CDC 3600 and the IBM 360 have been written using these techniques. In each case, essentially machine precision was achieved for $\zeta(s)$ and $\zeta(s) - 1$ for s in the respective ranges.

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