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Chebyshev rational spectral and pseudospectral methods on a semi-infinite interval

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SUMMARY

A weighted orthogonal system on the half-line based on the Chebyshev rational functions is introduced. Basic results on Chebyshev rational approximations of several orthogonal projections and interpolations are established. To illustrate the potential of the Chebyshev rational spectral method, a model problem is considered both theoretically and numerically: error estimates for the Chebyshev rational spectral and pseudospectral methods are established; preliminary numerical results agree well with the theoretical estimates and demonstrate the effectiveness of this approach. Copyright © 2001 John Wiley & Sons, Ltd.

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1. INTRODUCTION

While the spectral-element approximations for PDEs in bounded domains have achieved great success and popularity in recent years (see e.g. References [1–5]), spectral approximations for PDEs in unbounded domains have only received less attention until recently. A number of spectral methods for treating unbounded domains have been proposed: direct approaches using Laguerre polynomials were investigated by Maday *et al.* [6], Furano [7] and Guo and Shen [8]; indirect approaches, e.g. reformulating original problems in unbounded domains to certain singular problems in bounded domains by variable transformations, using Jacobi polynomials have been considered recently by Guo [9, 10]. Note that it is often preferable to use direct

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approaches such as in exterior problems where the obstacles may become too complicated after variable transformations.

Another effective direct approach for problems in unbounded domains is based on rational approximations: Christov [11] and Boyd [12, 13] developed some spectral methods on infinite intervals by using mutually orthogonal systems of rational functions; most recently, Guo *et al.* [14] developed a Legendre rational spectral method which is based on a weighted orthogonal system consisting of rational functions built from Legendre polynomials under a rational transformation. It is shown in Reference [14] that the Legendre rational method is an attractive alternative for problems in semi-infinite intervals. However, there is no fast transform available for Legendre rational functions, nor for Laguerre polynomials/functions. The purpose of this paper is to introduce the Chebyshev rational functions, for which fast Fourier transform (FFT) is applicable, and to investigate related issues associated with the Chebyshev rational approximations.

This paper is organized as follows. In the next section, we establish basic properties of Chebyshev rational functions and recall some Jacobi approximation results presented in Reference [10]. In Section 3, we study several orthogonal projections. In Section 4, we consider a Chebyshev rational interpolation. The results in these two sections form the mathematical foundation for the Chebyshev rational spectral and pseudospectral methods. In Section 5, we consider a model problem and provide error analysis for the Chebyshev rational spectral and pseudospectral methods. In Section 6, we discuss numerical implementations and present some preliminary numerical results.

2. SOME BASIC RESULTS ON CHEBYSHEV RATIONAL FUNCTIONS AND JACOBI POLYNOMIALS

2.1. Basic properties of Chebyshev rational functions

Let $\Lambda=\{x|0<x<\infty\}$ and $\chi(x)$ be a weight function in the usual sense. We define

$$L^2_\chi(\Lambda)=\left\{v|v \text{ is measurable and } \|v\|_{L^2_\chi}=\left(\int_\Lambda |v(x)|^2\chi(x)\,\mathrm{d}x\right)^{1/2}<\infty\right\} \tag{1}$$

We denote by $(u,v)_\chi$ the inner product of the space $L^2_\chi(\Lambda)$, i.e.

$$(u,v)_\chi=\int_\Lambda u(x)v(x)\chi(x)\,\mathrm{d}x.$$

For any non-negative integer m, we set

$$H^m_\chi(\Lambda)=\left\{v|\partial^k_x v=\frac{d^k v}{dx^k}\in L^2_\chi(\Lambda), 0\leqslant k\leqslant m\right\} \tag{2}$$

equipped with the inner product, the semi-norm and the norm as follows:

$$(u,v)_{m,\chi}=\sum_{k=0}^m(\partial^k_x u,\partial^k_x v)_\chi,\quad |v|_{m,\chi}=\|\partial^m_x v\|_\chi,\quad \|v\|_{m,\chi}=(v,v)^{1/2}_{m,\chi}$$

For any real number $r>0$, we define the space $H^r_\chi(\Lambda)$ with the norm $\|v\|_{r,\chi}$ by space interpolation as in Adams [15]. As usual χ will be omitted from the notations if $\chi(x)\equiv 1$.

Let $T_l(y)$ be the Chebyshev polynomial of degree l. We recall that $T_l(y)$ is the eigenfunction of the singular Sturm–Liouville problem

$$\sqrt{1-y^2}\partial_y(\sqrt{1-y^2}\partial_y T_l(y)) + l^2 T_l(y) = 0, \quad l = 0, 1, 2, \ldots \tag{3}$$

We define the Chebyshev rational function of degree l by

$$R_l(x) = T_l\left(\frac{x-1}{x+1}\right) \tag{4}$$

Thus, $R_l(x)$ is the lth eigenfunction of the singular Sturm–Liouville problem

$$(x+1)\sqrt{x}\partial_x((x+1)\sqrt{x}\partial_x R_l(x)) + l^2 R_l(x) = 0, \quad x \in \Lambda \tag{5}$$

From the recurrence relation of the Chebyshev polynomials, we find that $R_l(x)$ satisfy the following recurrence formulae:

$$\begin{aligned} &R_0(x) = 1, \quad R_1(x) = \frac{x-1}{x+1} \\ &R_{l+1}(x) = 2\frac{x-1}{x+1}R_l(x) - R_{l-1}(x), \quad l \geqslant 1 \end{aligned} \tag{6}$$

and

$$(x+1)^2 R_l(x) = \frac{1}{l+1}\partial_x R_{l+1}(x) - \frac{1}{l-1}\partial_x R_{l-1}(x), \quad l \geqslant 2 \tag{7}$$

Let us denote

$$\omega(x) = \frac{1}{(x+1)\sqrt{x}}, \quad \rho(y) = \frac{1}{\sqrt{1-y^2}}, \quad y = \frac{x-1}{x+1} \tag{8}$$

Then, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{(x+1)^2}, \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2}{(1-y)^2}, \quad \omega(x)\frac{\mathrm{d}x}{\mathrm{d}y} = \rho(y) \tag{9}$$

Hence, the orthogonality relation of Chebyshev polynomials leads to

$$\int_\Lambda R_l(x)R_m(x)\omega(x)\,\mathrm{d}x = \frac{c_l\pi}{2}\delta_{l,m} \quad \text{with } c_l = \begin{cases} 2, & l = 0 \\ 1, & l \geqslant 1 \end{cases} \tag{10}$$

where $\delta_{l,m}$ is the Kronecker function. Thus, the Chebyshev rational functions $\mathscr{R}_l(x)$ form a set of orthogonal basis for $L^2_\omega(\Lambda)$, and the Chebyshev rational expansion of a function $v \in L^2_\omega(\Lambda)$ is

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l R_l(x) \quad \text{with } \hat{v}_l = \frac{2}{c_l\pi}\int_\Lambda v(x)R_l(x)\omega(x)\,\mathrm{d}x.$$

By virtue of (5) and (10), we find that $\{\partial_x R_l(x)\}$ are mutually orthogonal in $L^2_{\omega^{-1}}(\Lambda)$, namely,

$$\int_\Lambda \partial_x R_l(x)\partial_x R_m(x)\omega^{-1}(x)\,\mathrm{d}x = \frac{c_l\pi}{2}l^2\delta_{l,m} \tag{11}$$

Next, we derive some useful inequalities. Let N be any positive integer, and

$$\mathscr{R}_N = \text{span}\{R_0, R_1, \ldots, R_N\}$$

Hereafter, we denote by c a generic positive constant independent of any function and N.

Theorem 2.1
For any $r \geqslant 0$,

$$\|\phi\|_{r,\omega} \leqslant cN^{2r}\|\phi\|_{\omega}$$

Proof
Let $y \in I = (-1, 1)$, $x = (1+y)/(1-y)$. For any $\phi \in \mathscr{R}_N$, we set $\psi(y) = \phi((1+y)/(1-y))$. By the definition of $\mathscr{R}_N$, we have $\psi(y) \in \mathscr{P}_N$, which is the set of polynomials of degree at most N. From the the following inverse inequality for Chebyshev polynomials (cf. Reference [2]):

$$\|\partial_y^m \psi\|_{L^2_\rho(I)} \leqslant cN^{2m}\|\psi\|_{L^2_\rho(I)}$$

we derive that

$$\begin{aligned}\|\partial_x \phi\|_\omega^2 &= \frac{1}{4}\int_I \psi_y^2(y)(1-y)^4\rho(y)\,\mathrm{d}y \leqslant 4\int_I \psi_y^2(y)\rho(y)\,\mathrm{d}y \\ &\leqslant cN^4\int_I \psi^2(y)\rho(y)\,\mathrm{d}y = cN^4\|\phi\|_\omega^2\end{aligned}$$

By repeating the above procedure, we deduce that for any non-negative integer m,

$$\|\partial_x^m \phi\|_{L^2_\omega} \leqslant cN^{2m}\|\phi\|_{L^2_\omega}$$

The general result for $r > 0$ follows from the above inequality and space interpolation. □

Now, we prove a generalized Poincaré inequality which will play an essential role in the analysis of Chebyshev rational approximations.

Theorem 2.2
If $v \in H^1_\omega(\Lambda)$ and $(v^2(x)/x)\omega(x) \to 0$ as $x \to 0$, then

$$\int_\Lambda \frac{v^2}{x^2}\left(\frac{1}{2} + \frac{x}{x+1}\right)\omega(x)\,\mathrm{d}x \leqslant \int_\Lambda (\partial_x v(x))^2\omega(x)\,\mathrm{d}x$$

Proof
By the assumption $(v^2(x)/x)\omega(x) \to 0$ as $x \to 0$, we can write

$$\begin{aligned}\frac{v^2(x)}{x}\omega(x) &= \int_0^x \partial_z\left(\frac{v^2(z)}{z}\omega(z)\right)\mathrm{d}z \\ &= \int_0^x \left[2\frac{v(z)}{z}\partial_z v(z)\omega(z) - \frac{v^2(z)}{z^2}\omega(z) + \frac{v^2(z)}{z}\partial_z\omega(z)\right]\mathrm{d}z \qquad (12)\end{aligned}$$

Since $\partial_z \omega(z) = -[(3z+1)/2(z+1)z]\omega(z)$, we find by using Cauchy–Schwarz inequality that

$$\begin{aligned}\frac{v^2(x)}{x}\omega(x) + \int_0^x \frac{v^2(z)}{z^2}\left(\frac{3}{2} + \frac{z}{z+1}\right)\omega(z)\,\mathrm{d}z &\leqslant 2\int_0^x \frac{v(z)}{z}\partial_z v(z)\omega(z)\,\mathrm{d}z \\ &\leqslant \int_0^x \frac{v^2(z)}{z^2}\omega(z)\,\mathrm{d}z + \int_0^x (\partial_z v(z))^2\omega(z)\,\mathrm{d}z\end{aligned} \tag{13}$$

Letting $x \to \infty$, we obtain the desired result. □

2.2. Some basic Jacobi approximation results

As we shall see below, the Jacobi approximation results established in Reference [10] play an important role in the analysis of Chebyshev rational methods. Hence, in this subsection, we recall some basic approximation results on Jacobi polynomials presented in Reference [10].

Let us define

$$L^2_{\alpha,\beta}(I) = \left\{ u \mid \|u\|_{L^2_{\alpha,\beta}} = \left(\int_I u^2(y)(1-y)^\alpha(1+y)^\beta\,\mathrm{d}y\right)^{1/2} < +\infty \right\} \tag{14}$$

and

$$a_{\alpha,\beta,\gamma,\delta}(u,w) = \int_I \partial_y u \partial_y w (1-y)^\alpha(1+y)^\beta\,\mathrm{d}y + \int_I u(y)w(y)(1-y)^\gamma(1+y)^\delta\,\mathrm{d}y \tag{15}$$

We also denote $H^0_{\alpha,\beta,\gamma,\delta}(I) = L^2_{\gamma,\delta}(I)$ and

$$H^1_{\alpha,\beta,\gamma,\delta}(I) = \{u \mid u \text{ is measurable on } I \text{ and } \|u\|_{1,\alpha,\beta,\gamma,\delta} < +\infty\} \tag{16}$$

where $\|u\|_{1,\alpha,\beta,\gamma,\delta} = a^{1/2}_{\alpha,\beta,\gamma,\delta}(u,u)$. For $0<\mu<1$, $H^\mu_{\alpha,\beta,\gamma,\delta}(I)$ and its norm $\|u\|_{\mu,\alpha,\beta,\gamma,\delta}$ are defined by space interpolation. We also define

$$H^r_{\alpha,\beta,*}(I) = \{u \mid u \text{ is measurable on } I \text{ and } \|u\|_{r,\alpha,\beta,*} < +\infty\} \tag{17}$$

where for non-negative integer r,

$$\|u\|^2_{r,\alpha,\beta,*} = A^{(1)}_{r,\alpha,\beta}(u) + A^{(2)}_{r,\alpha,\beta}(u) \tag{18}$$

with

$$\begin{aligned}A^{(1)}_{r,\alpha,\beta}(u) &= \sum_{k=r-[r/2]+1}^{r} \int_I (\partial^k_y u(y))^2(1-y^2)^{-r+2k-1}(1-y)^\alpha(1+y)^\beta\,\mathrm{d}y \\ A^{(2)}_{r,\alpha,\beta}(u) &= \sum_{k=1}^{[(r+1)/2]} \int_I (\partial^k_y u(y))^2(1-y)^\alpha(1+y)^\beta\,\mathrm{d}y\end{aligned} \tag{19}$$

The space $H^r_{\alpha,\beta,*}(I)$ and its norm $\|u\|_{r,\alpha,\beta,*}$ for real positive r are defined by space interpolation.

Let $\tilde{P}^1_{N,\alpha,\beta,\gamma,\delta}: H^1_{\alpha,\beta,\gamma,\delta}(I) \to \mathscr{P}_N$ be the orthogonal projection operator defined by

$$a_{\alpha,\beta,\gamma,\delta}(\tilde{P}^1_{N,\alpha,\beta,\gamma,\delta}u - u, \psi) = 0 \quad \forall \psi \in \mathscr{P}_N \tag{20}$$

The following theorem is proved in Reference [10] (cf. pp. 380–381, Theorem 2.5 in Reference [10]).

Theorem 2.3
For $\alpha \leqslant \gamma + 2$, $\beta \leqslant \delta + 2$, and for any $u \in H^r_{\alpha,\beta,\gamma,\delta}(I)$ with $r \geqslant 1$, we have

$$\|\tilde{P}^1_{N,\alpha,\beta,\gamma,\delta}u - u\|^2_{1,\alpha,\beta,\gamma,\delta} \leqslant cN^{2-2r}\|u\|^2_{r,\alpha,\beta,*} \tag{21}$$

If in addition, $\alpha \leqslant \gamma + 1$, $\beta \leqslant \delta + 1$ and $0 \leqslant \mu \leqslant 1$, then

$$\|\tilde{P}^1_{N,\alpha,\beta,\gamma,\delta}u - u\|^2_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{2\mu-2r}\|u\|^2_{r,\alpha,\beta,*} \tag{22}$$

In order to deal with the boundary condition at $x = 0$, we need another result in Reference [10]. Let us define

$$H^{1,L}_{\alpha,\beta,\gamma,\delta}(I) = \{u \in H^1_{\alpha,\beta,\gamma,\delta}(I) \mid u(-1) = 0\} \quad \mathscr{P}^L_N = \{u \in \mathscr{P}_N \mid u(-1) = 0\}$$

and the orthogonal projection $\tilde{P}^{1,L}_{N,\alpha,\beta,\gamma,\delta}: H^{1,L}_{\alpha,\beta,\gamma,\delta}(I) \to \mathscr{P}^L_N$ by

$$a_{\alpha,\beta,\gamma,\delta}(\tilde{P}^{1,L}_{N,\alpha,\beta,\gamma,\delta}u - u, \psi) = 0 \quad \forall \phi \in \mathscr{P}^L_N \tag{23}$$

The following result is from Reference [10, Theorem 2.6].

Theorem 2.4
If $\alpha \leqslant \gamma + 2$, $\beta \leqslant 0$ and $\delta \geqslant 0$, then for any $u \in H^r_{\alpha,\beta,\gamma,\delta}(I) \cap H^{1,L}_{\alpha,\beta,\gamma,\delta}(I)$, we have

$$\|\tilde{P}^{1,L}_{N,\alpha,\beta,\gamma,\delta}u - u\|^2_{1,\alpha,\beta,\gamma,\delta} \leqslant cN^{2-2r}\|u\|^2_{r,\alpha,\beta,*} \tag{24}$$

and if in addition, $\alpha \leqslant \gamma + 1$, $\beta \leqslant \delta + 1$ and $0 \leqslant \mu \leqslant 1$, then

$$\|\tilde{P}^{1,L}_{N,\alpha,\beta,\gamma,\delta}u - u\|^2_{\mu,\alpha,\beta,\gamma,\delta} \leqslant cN^{2\mu-2r}\|u\|^2_{r,\alpha,\beta,*} \tag{25}$$

3. CHEBYSHEV RATIONAL POLYNOMIAL APPROXIMATIONS

For the sake of error analysis, we need to investigate several orthogonal projections.

We define the $L^2_\omega(\Lambda)$-orthogonal projection $P_N: L^2_\omega(\Lambda) \to \mathscr{R}_N$ by

$$(P_N v - v, \phi)_\omega = 0 \quad \forall \phi \in \mathscr{R}_N$$

In order to estimate $\|P_N v - v\|_\omega$, we introduce the space

$$H^r_{\omega,A}(\Lambda) = \{v | v \text{ is measurable and } \|v\|_{r,\omega,A} < \infty\} \tag{26}$$

where for non-negative integer r,

$$\|v\|_{r,\omega,A} = \left(\sum_{k=0}^{r} \|(x+1)^{r/2+k}\partial_x^k v\|_\omega^2\right)^{1/2} \tag{27}$$

For any real $r>0$, the space $H^r_{\omega,A}(\Lambda)$ is defined by space interpolation.

Let A be the Sturm–Liouville operator in (5), namely,

$$Av(x) = -\omega^{-1}(x)\partial_x(\omega^{-1}(x)\partial_x v(x)) = -x(x+1)^2\partial_x^2 v(x) - \tfrac{1}{2}(3x+1)(x+1)\partial_x v(x) \tag{28}$$

Lemma 3.1
A^m is a continuous mapping from $H^{2m}_{\omega,A}(\Lambda)$ to $L^2_\omega(\Lambda)$.

Proof
One can easily prove by induction that

$$A^m v(x) = \sum_{k=1}^{2m} (x+1)^{m+k} p_k(x)\partial_x^k v(x) \tag{29}$$

where $p_k(x)$ are some rational functions which are bounded uniformly on the whole interval Λ. The desired result follows from (29) and (27). □

Theorem 3.1
For any $v \in H^r_{\omega,A}(\Lambda)$ and $r \geqslant 0$,

$$\|P_N v - v\|_\omega \leqslant cN^{-r}\|v\|_{r,\omega,A}$$

Proof
We first consider the case $r=2m$. By virtue of (5), (10) and integration by parts,

$$\begin{aligned}
\hat{v}_l &= \frac{2}{c_l\pi}\int_\Lambda v(x)R_l(x)\omega(x)\,\mathrm{d}x = \frac{2}{c_l\pi l^2}\int_\Lambda v(x)AR_l(x)\omega(x)\,\mathrm{d}x \\
&= -\frac{2}{c_l\pi l^2}\int_\Lambda v(x)\partial_x(\omega^{-1}(x)\partial_x R_l(x))\,\mathrm{d}x = \frac{2}{c_l\pi l^2}\int_\Lambda \omega^{-1}(x)\partial_x v(x)\partial_x R_l(x)\,\mathrm{d}x \\
&= -\frac{2}{c_l\pi l^2}\int_\Lambda \partial_x(\omega^{-1}(x)\partial_x v(x))R_l(x)\,\mathrm{d}x = \frac{2}{c_l\pi l^2}\int_\Lambda Av(x)R_l(x)\omega(x)\,\mathrm{d}x \\
&= \ldots = \frac{2}{c_l\pi l^{2m}}\int_\Lambda A^m v(x)R_l(x)\omega(x)\,\mathrm{d}x
\end{aligned} \tag{30}$$

Therefore, we derive from (29), (30) and the definition of $H^r_{\omega,A}(\Lambda)$ that

$$\begin{aligned}
\|P_N v - v\|_\omega^2 &= \sum_{l=N+1}^{\infty} \hat{v}_l^2\|R_l\|_\omega^2 \leqslant cN^{-4m}\sum_{l=N+1}^{\infty}\left(\frac{\int_\Lambda A^m v(x)R_l(x)\omega(x)\,\mathrm{d}x}{\|R_l\|_\omega^2}\right)^2\|R_l\|_\omega^2 \\
&\leqslant cN^{-4m}\|A^m v\|_\omega^2 \leqslant cN^{-4m}\|v\|_{r,\omega,A}^2
\end{aligned}$$

Next, we consider the case $r=2m+1$. By (5) and integration by parts,

$$\begin{aligned}\hat{v}_l &= \frac{2}{c_l\pi l^{2m}}\int_\Lambda A^m v(x)R_l(x)\omega(x)\,\mathrm{d}x\\ &= -\frac{2}{c_l\pi l^{2m+2}}\int_\Lambda A^m v(x)\partial_x(\omega^{-1}(x)\partial_x R_l(x))\,\mathrm{d}x\\ &= \frac{2}{c_l\pi l^{2m+2}}\int_\Lambda \partial_x(A^m v(x))\partial_x R_l(x)\omega^{-1}(x)\,\mathrm{d}x\end{aligned}$$

Owing to (11) and (29),

$$\begin{aligned}\|P_N v - v\|_\omega^2 &= \sum_{l=N+1}^{\infty}\hat{v}_l^2\|R_l\|_\omega^2 = \sum_{l=N+1}^{\infty}\frac{2}{c_l\pi l^{4m+4}}\left(\int_\Lambda \partial_x(A^m v)\partial_x R_l(x)\omega^{-1}(x)\,\mathrm{d}x\right)^2\\ &= \sum_{l=N+1}^{\infty}\frac{2\|\partial_x R_l\|_{\omega^{-1}}^2}{c_l\pi l^{4m+4}}\left(\frac{\int_\Lambda \partial_x(A^m v)\partial_x R_l(x)\omega^{-1}(x)\,\mathrm{d}x}{\|\partial_x R_l\|_{\omega^{-1}}^2}\right)^2\|\partial_x R_l\|_{\omega^{-1}}^2\\ &\leqslant cN^{-2(2m+1)}\sum_{l=N+1}^{\infty}\left(\frac{\int_\Lambda \partial_x(A^m v)\partial_x R_l(x)\omega^{-1}(x)\,\mathrm{d}x}{\|\partial_x R_l\|_{\omega^{-1}}^2}\right)^2\|\partial_x R_l\|_{\omega^{-1}}^2\\ &\leqslant cN^{-2(2m+1)}\|\partial_x(A^m v)\|_{\omega^{-1}}^2 \leqslant cN^{-2(2m+1)}\|\partial_x(A^m v)(x+1)^{3/2}\|_\omega^2\\ &\leqslant cN^{-2(2m+1)}\|v\|_{r,\omega,A}^2\end{aligned}$$

The general result follows from the previous results and space interpolation. □

Now, we consider the $H_\omega^1(\Lambda)$-orthogonal projection $P_N^1: H_\omega^1(\Lambda)\to\mathscr{R}_N$ which is defined by

$$(P_N^1 v - v, \phi)_{1,\omega} = 0 \quad \forall\phi\in\mathscr{R}_N$$

In order to estimate $\|P_N^1 v - v\|_{1,\omega}$, we introduce

$$H_{\omega,B}^r(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{r,\omega,B} < +\infty\} \tag{31}$$

for any non-negative integer r with

$$\|v\|_{r,\omega,B} = \left(\sum_{k=1}^{r}\|(x+1)^{r/2+k-1/2}\partial_x^k v\|_\omega^2\right)^{1/2} \tag{32}$$

As usual, for any $r>0$, the space $H_{\omega,B}^r(\Lambda)$ and its norm are defined by space interpolation.

Theorem 3.2
For any $v\in H_{\omega,B}^r(\Lambda)$ with $r\geqslant 1$,

$$\|P_N^1 v - v\|_{1,\omega} \leqslant cN^{1-r}\|v\|_{r,\omega,B}$$

Proof
By definition, $\|P_N^1 v - v\|_{1,\omega} \leqslant \|\phi - v\|_{1,\omega}$ for any $\phi \in \mathscr{R}_N$. Let $y=(x-1)/(x+1)$, $u(y)=v((1+y)/(1-y))$. By taking $\phi = \tilde{P}^1_{N,3/2,-1/2,-1/2,-1/2}u(y)|_{y=(x-1)/(x+1)}$ (cf. (20)), a direct computation together with (21) ($\alpha=\frac{3}{2}$, $\beta=\gamma=\delta=-\frac{1}{2}$) leads to

$$\begin{aligned}\|\phi - v\|^2_{1,\omega} &= \frac{1}{4}\int_I (\partial_y \tilde{P}^1_{N,3/2,-1/2,-1/2,-1/2}u(y) - \partial_y u(y))^2 (y-1)^4 \rho(y)\,\mathrm{d}y \\ &\quad + \int_I (\tilde{P}^1_{N,3/2,-1/2,-1/2,-1/2}u(y) - u(y))^2 \rho(y)\,\mathrm{d}y \\ &\leqslant \int_I (\partial_y \tilde{P}^1_{N,3/2,-1/2,-1/2,-1/2}u(y) - \partial_y u(y))^2 (y-1)^2 \rho(y)\,\mathrm{d}y \\ &\quad + \int_I (\tilde{P}^1_{N,3/2,-1/2,-1/2,-1/2}u(y) - u(y))^2 \rho(y)\,\mathrm{d}y \\ &= \|\tilde{P}^1_{N,3/2,-1/2,-1/2,-1/2}u - u\|^2_{1,3/2,-1/2,-1/2,-1/2} \leqslant cN^{2-2r}\|u\|^2_{r,3/2,-1/2,*}\end{aligned}$$

Note that $1-y=2/(x+1)$, $1-y^2=4x/(x+1)^2$ and one can show easily by induction that

$$\partial_y^k u(y) = \sum_{j=1}^{k} p_j(x)(x+1)^{k+j}\partial_x^j v(x) \tag{33}$$

where $p_j(x)$ are some rational polynomials which are uniformly bounded on Λ. Thus, for any non-negative integer r, using (33) and the fact that $1+y \leqslant 2$ for all $y\in[-1,1]$, we find

$$\begin{aligned}A^{(1)}_{r,\alpha,\beta}(u) &\leqslant c \sum_{k=r-[r/2]+1}^{r} \sum_{j=1}^{k} \int_\Lambda (x+1)^{r+2j-\alpha-1}(\partial_x^j v(x))^2\,\mathrm{d}x \\ &\leqslant c \sum_{k=r-[r/2]+1}^{r} \sum_{j=1}^{k} \int_\Lambda (x+1)^{r+2j-\alpha+1/2}(\partial_x^j v(x))^2 \omega(x)\,\mathrm{d}x \\ &\leqslant c \sum_{j=1}^{r} \|(x+1)^{r/2+j-\alpha/2+1/4}\partial_x^j v\|^2_\omega\end{aligned} \tag{34}$$

Similarly, we have

$$\begin{aligned}A^{(2)}_{r,\alpha,\beta}(u) &\leqslant c \sum_{k=1}^{[(r+1)/2]} \sum_{j=1}^{k} \int_\Lambda (x+1)^{2k+2j-\alpha-2}(\partial_x^j v(x))^2\,\mathrm{d}x \\ &\leqslant c \sum_{j=1}^{r} \int_\Lambda (x+1)^{r+2j-\alpha-1}(\partial_x^j v(x))^2\,\mathrm{d}x \\ &\leqslant c \sum_{j=1}^{r} \|(x+1)^{r/2+j-\alpha/2+1/4}\partial_x^j v\|^2_\omega\end{aligned} \tag{35}$$

Therefore,

$$A^{(1)}_{r,3/2,-1/2}(u) \leqslant c\|v\|^2_{r,\omega,B}$$
$$A^{(2)}_{r,3/2,-1/2}(u) \leqslant c\|v\|^2_{r,\omega,B}$$

This fact together with space interpolation completes the proof. □

When we apply the Chebyshev rational spectral method to partial differential equations with Dirichlet boundary conditions at $x=0$, we need another orthogonal projection. Let us denote

$$\begin{aligned} H^1_{0,\omega}(\Lambda) &= \left\{ v \mid v \in H^1_\omega(\Lambda),\ v(0)=0 \text{ and } \frac{v^2(x)}{x}\omega(x) \to 0, \text{ as } x\to\infty \right\} \\ \mathscr{R}^0_N &= \{\phi \in \mathscr{R}_N \mid \phi(0)=0\} \\ a^\nu_\omega(u,v) &= (\partial_x u, \partial_x(v\omega)) + \nu(u,v)_\omega \end{aligned} \tag{36}$$

We define the $H^1_{0,\omega}(\Lambda)$-orthogonal projection $P^{1,0}_N : H^1_{0,\omega}(\Lambda) \to \mathscr{R}^0_N$ by

$$a^\nu_\omega(P^{1,0}_N v - v, \phi) = 0 \quad \forall \phi \in \mathscr{R}^0_N$$

Lemma 3.2
For $\nu > \frac{14}{27}$,

$$a^\nu_\omega(v,v) \geqslant \min(\tfrac{1}{16}, \nu - \tfrac{14}{27})\|v\|^2_{1,\omega} \quad \forall v \in H^1_{0,\omega}(\Lambda)$$
$$|(\partial_x u, \partial_x(v\omega))| \leqslant \tfrac{3}{\sqrt{2}}\|\partial_x u\|_\omega \|\partial_x v\|_\omega \quad \forall u,v \in H^1_{0,\omega}(\Lambda)$$

Proof
Since

$$\partial_x\omega(x) = -\frac{3x+1}{2(x+1)x}\omega(x), \quad \partial^2_x\omega(x) = \frac{15x^2+10x+3}{4x^2(x+1)^2}\omega(x) \tag{37}$$

using integration by parts and that $(v^2(x)/x)\omega(x) \to 0$, as $x \to \infty$, we find

$$\begin{aligned} (\partial_x v, \partial_x(v\omega)) &= \int_\Lambda ((\partial_x v(x))^2\omega(x) + \frac{1}{2}\partial_x(v^2(x))\partial_x\omega(x))\,\mathrm{d}x \\ &= \int_\Lambda (\partial_x v(x))^2\omega(x)\,\mathrm{d}x - \int_\Lambda \frac{1}{2}v^2(x)\partial^2_x\omega(x)\,\mathrm{d}x \\ &= \int_\Lambda (\partial_x v(x))^2\omega(x)\,\mathrm{d}x - \frac{1}{8}\int_\Lambda \frac{v^2(x)}{x^2}\frac{15x^2+10x+3}{(x+1)^2}\omega(x)\,\mathrm{d}x \end{aligned}$$

Let us first prove the following elementary inequality:

$$\frac{1}{8}\frac{15x^2+10x+3}{(x+1)^2} \leqslant \frac{14}{27}x^2 + \frac{15}{32} \quad \forall x \geqslant 0 \tag{38}$$

We denote

$$f(x)=\frac{1}{8}\frac{15x^2+10x+3}{(x+1)^2}-\frac{14}{27}x^2-\frac{15}{32}$$

Then by direct computation, we have

$$f'(x)=\frac{5x+1}{2(x+1)^3}-\frac{28}{27}x, \quad f'\left(\frac{1}{2}\right)=0$$
$$f''(x)=-\frac{5x+1}{(x+1)^4}-\frac{28}{27}<0 \quad \forall x\geqslant 0$$

Hence, $\frac{1}{2}$ is the only root of $f'(x)$ in $[0,+\infty)$. Thus,

$$f(x)\leqslant f\left(\frac{1}{2}\right)<0 \quad \forall x\geqslant 0$$

which implies (38).

Owing to (38) and Theorem 2.2, we have

$$\begin{aligned} a_\omega^\nu(v,v) &\geqslant \int_\Lambda (\partial_x v)^2\omega(x)\,\mathrm{d}x-\frac{15}{32}\int_\Lambda \frac{v^2(x)}{x^2}\omega(x)\,\mathrm{d}x+\left(\nu-\frac{14}{27}\right)\int_\Lambda v^2(x)\omega(x)\,\mathrm{d}x \\ &\geqslant \frac{1}{16}\int_\Lambda (\partial_x v)^2\omega(x)\,\mathrm{d}x+\left(\nu-\frac{14}{27}\right)\int_\Lambda v^2(x)\omega(x)\,\mathrm{d}x \end{aligned} \tag{39}$$

Using the Cauchy–Schwartz inequality, we find

$$\begin{aligned} \left|\int_\Lambda \partial_x u(x)v(x)\partial_x\omega(x)\,\mathrm{d}x\right| &= \int_\Lambda \partial_x u(x)\frac{v(x)}{x}\frac{3x+1}{2(x+1)}\omega(x)\,\mathrm{d}x \\ &\leqslant \frac{3}{2}\left(\int_\Lambda (\partial_x u(x))^2\omega(x)\,\mathrm{d}x\right)^{1/2}\left(\int_\Lambda \frac{v^2(x)}{x^2}\omega(x)\,\mathrm{d}x\right)^{1/2} \end{aligned}$$

The second result is then a direct consequence of the above inequality and Theorem 2.2. □

Theorem 3.3

For any $v\in H^r_{\omega,B}(\Lambda)\cap H^1_{0,\omega}(\Lambda)$, $\nu>\frac{14}{27}$ and $r\geqslant 1$,

$$\|P_N^{1,0}v-v\|_{1,\omega}\leqslant cN^{1-r}\|v\|_{r,\omega,B}$$

Proof

By Lemma 3.2, for any $\phi\in\mathscr{R}_N^0$,

$$\begin{aligned} \|P_N^{1,0}v-v\|_{1,\omega}^2 &\leqslant ca_\omega^\nu(P_N^{1,0}v-v,P_N^{1,0}v-v) \\ &= ca_\omega^\nu(P_N^{1,0}v-v,\phi-v) \\ &\leqslant c\|P_N^{1,0}v-v\|_{1,\omega}\|\phi-v\|_{1,\omega} \end{aligned}$$

Therefore,

$$\|P_N^{1,0}v - v\|_{1,\omega} \leqslant c \inf_{\phi \in \mathscr{R}_N^0} \|\phi - v\|_{1,\omega} \tag{40}$$

Next, let $x=(1+y)/(1-y)$, $u(y)=v((1+y)/(1-y))$ and take $\phi = \tilde{P}^{1,L}_{N,3/2,-1/2,-1/2,-1/2} u(y)|_{y=(x-1)/(x+1)}$ in (40) (cf. (23)). Then, the desired result follows from (24) and a similar argument as in the proof of Theorem 3.2. □

In order to study the Chebyshev rational interpolation approximations, we need to consider another orthogonal projection. Let

$$\hat{a}_\omega(u,v) = \int_\Lambda \partial_x u(x)\partial_x v(x)(x+1)\sqrt{x}\,\mathrm{d}x + \int_\Lambda u(x)v(x)\omega(x)\,\mathrm{d}x \tag{41}$$

The orthogonal projection $\hat{P}_N^1 : H^1_{\omega,A}(\Lambda) \to \mathscr{R}_N$ is a mapping such that for any $v \in H^1_{\omega,A}(\Lambda)$,

$$\hat{a}_\omega(\hat{P}_N^1 v - v, \phi) = 0 \quad \forall \phi \in \mathscr{R}_N \tag{42}$$

Theorem 3.4
For any $v \in H^r_{\omega,A}(\Lambda)$ and $r \geqslant 1$,

$$\|\hat{P}_N^1 v - v\|_\omega \leqslant cN^{-r}\|v\|_{r,\omega,A}$$
$$\|x^{1/4}(x+1)^{1/2}\partial_x(\hat{P}_N^1 v - v)\| \leqslant cN^{1-r}\|v\|_{r,\omega,A}$$

Proof
Let us denote

$$u(y) = v((1+y)/(1-y)), \quad u_N^*(y) = \hat{P}_N^1 v(x)|_{x=(1+y)/(1-y)}$$

By definition, we have

$$\int_I \partial_y(u_N^*(y) - u(y))\partial_y\psi(y)\sqrt{1-y^2}\,\mathrm{d}y + \int_I (u_N^*(y) - u(y))\psi(y)\rho(y)\,\mathrm{d}y = 0 \quad \forall \psi \in \mathscr{P}_N \tag{43}$$

Thus, $u_N^*(y) = \tilde{P}^1_{N,1/2,1/2,-1/2,-1/2}u(y)$ (cf. (20)). Under the transform $x=(1+y)/(1-y)$, we have

$$\int_\Lambda (\hat{P}_N^1 v - v)^2\omega(x)\,\mathrm{d}x = \frac{1}{2}\int_{\tilde{\Lambda}} (u_N^* - u)^2(1-y^2)^{-1/2}\,\mathrm{d}y$$
$$\int_\Lambda x(x+1)^2[\partial_x(\hat{P}_N^1 v - v)]^2\omega(x)\,\mathrm{d}x = \int_{\tilde{\Lambda}} [\partial_y(u_N^* - u)]^2(1-y^2)^{1/2}\,\mathrm{d}y$$

Therefore, we derive from (22) with $\alpha=\beta=\frac{1}{2}$, $\gamma=\delta=-\frac{1}{2}$ that

$$\|\hat{P}_N^1 v - v\|^2_\omega \leqslant \|u_N^* - u\|^2_{0,1/2,1/2,-1/2,-1/2} \leqslant cN^{-2r}\|v\|^2_{r,1/2,1/2,*} \tag{44}$$

and

$$||x^{1/4}(x+1)^{1/2}\partial_x(\hat{P}_N^1 v - v)||^2 \leqslant \|u_N^* - u\|^2_{1,1/2,1/2,-1/2,-1/2} \leqslant cN^{2-2r}\|v\|^2_{r,1/2,1/2,*} \tag{45}$$

A direct computation using (34)–(35) leads to

$$A^{(1)}_{r,1/2,1/2}(u) \leqslant c\sum_{j=1}^{r}\|(x+1)^{r/2+j}\partial_x^j v\|^2_\omega \leqslant c\|v\|^2_{r,\omega,A}$$

Similarly,

$$A^{(2)}_{r,1/2,1/2}(u) \leqslant c\sum_{j=1}^{r}\|(x+1)^{r/2+j}\partial_x^j v\|^2_\omega \leqslant c\|v\|^2_{r,\omega,A}$$

The above estimates imply the desired results. □

4. CHEBYSHEV RATIONAL INTERPOLATION APPROXIMATION

In actual computations, it is convenient to use interpolations. We will only deal with the Chebyshev–Gauss–Radau rational interpolation. The interpolation using rational Chebyshev–Gauss points can be considered in a similar fashion. We denote $\sigma_{N,j} = \cos 2j\pi/(2N+1)$ which are the $N+1$ Chebyshev–Gauss–Radau points, and

$$\zeta_{N,j} = (1+\sigma_{N,j})(1-\sigma_{N,j})^{-1} \tag{46}$$

The Chebyshev–Gauss–Radau formula implies that

$$\int_\Lambda \phi(x)\omega(x)\,\mathrm{d}x = \int_I \phi\left(\frac{1+y}{1-y}\right)\rho(y)\,\mathrm{d}y = \sum_{j=0}^{N}\phi(\zeta_{N,j})\omega_j \quad \forall\phi\in\mathscr{R}_{2N} \tag{47}$$

where $\omega_0 = \pi/(2N+1)$ and $\omega_j = \pi/(N+1)$ for $1\leqslant j\leqslant N$. The discrete inner product and the discrete norm associated with the Chebyshev–Gauss–Radau rational interpolation points are

$$(u,v)_{\omega,N} = \sum_{j=0}^{N-1} u(\zeta_{N,j})v(\zeta_{N,j})\omega_j, \quad \|v\|_{\omega,N} = (v,v)^{1/2}_{\omega,N}$$

Owing to (47),

$$(\phi,\psi)_{\omega,N} = (\phi,\psi)_\omega \quad \forall\phi\cdot\psi\in\mathscr{R}_{2N} \tag{48}$$

The Chebyshev–Gauss–Radau rational interpolation operator $I_N v(x): C(\bar{\Lambda}) \to \mathscr{R}_N$ is such that

$$I_N v(\zeta_{N,j}) = v(\zeta_{N,j}), \quad 0\leqslant j\leqslant N$$

The following theorem is related to the stability of the Chebyshev–Gauss–Radau rational interpolation.

Theorem 4.1
For any $v\in H^1_{\omega,A}(\Lambda)$,

$$\|I_N v\|_\omega \leqslant c(\|v\|_\omega + N^{-1}\|(x+1)^{1/2}x^{1/4}\partial_x v\|)$$

Proof
Let $x=(1+\cos\theta)/(1-\cos\theta)$ and $\hat{v}(\theta)=v((1+\cos\theta)/(1-\cos\theta))$. Let us also denote $K_j=[2j\pi/(2N+1),(2j+1)\pi/(2N+1)]$ for $j=0,\ldots,N$. Then, by (48),

$$\begin{aligned}\|I_N v\|_\omega^2 &= \|I_N v\|_{\omega,N}^2=\sum_{j=0}^{N} v^2(\zeta_{N,j})\omega_j\\ &= \sum_{j=0}^{N}\hat{v}^2(\theta_{N,j})\omega_j \leqslant \frac{\pi}{N+1}\sum_{j=0}^{N-1}\sup_{\theta\in K_j}\hat{v}^2(\theta) \end{aligned}\tag{49}$$

We recall the following inequality (see (13.7) in Reference [4]):

$$\max_{a\leqslant x\leqslant b}|f(x)|\leqslant c\left(\frac{1}{\sqrt{b-a}}\|f\|_{L^2(a,b)}+\sqrt{b-a}\|\partial_x f\|_{L^2(a,b)}\right)\quad \forall f\in H^1(a,b)\tag{50}$$

and apply it for each of the interval K_j, using the following relations:

$$\cos\theta=\frac{x-1}{x+1},\quad \frac{\mathrm{d}x}{\mathrm{d}\theta}=-\frac{2\sin\theta}{(1-\cos\theta)^2}=\frac{1}{2}(x+1)\sqrt{x},\tag{51}$$

we find that

$$\begin{aligned}\|I_N v\|_\omega^2 &\leqslant c\sum_{j=0}^{N}(\|\hat{v}(\theta)\|_{L^2(K_j)}^2+N^{-2}\|\partial_\theta\hat{v}(\theta)\|_{L^2(K_j)}^2)\\ &\leqslant c(\|\hat{v}(\theta)\|_{L^2(0,\pi)}^2+N^{-2}\|\partial_\theta\hat{v}(\theta)\|_{L^2(0,\pi)}^2)\\ &\leqslant c(\|v(x)\|_{L^2_\omega(\Lambda)}^2+N^{-2}\|(x+1)^{1/2}x^{1/4}\partial_x v(x)\|_{L^2(\Lambda)}^2\end{aligned}\tag{52}$$

which implies the desired result. □

Theorem 4.2
For any $v\in H^r_{\omega,A}(\Lambda)$ and $0\leqslant\mu\leqslant 1\leqslant r$,

$$\|I_N v-v\|_{\mu,\omega}\leqslant cN^{2\mu-r}\|v\|_{r,\omega,A}$$

Proof
Since $I_N(\hat{P}^1_N v)=\hat{P}^1_N v$, we have from Theorems 3.4 and 4.1 that

$$\begin{aligned}\|I_N v-\hat{P}^1_N v\|_\omega &\leqslant c(\|\hat{P}^1_N v-v\|_\omega+N^{-1}\|(x+1)^{1/2}x^{1/4}\partial_x(\hat{P}^1_N v-v)\|)\\ &\leqslant cN^{-r}\|v\|_{r,\omega,A}\end{aligned}\tag{53}$$

Using Theorem 3.4 again,

$$\begin{aligned}\|I_N v-v\|_\omega &\leqslant \|\hat{P}^1_N v-v\|_\omega+\|I_N v-\hat{P}^1_N v\|_\omega\\ &\leqslant cN^{-r}\|v\|_{r,\omega,A}\end{aligned}\tag{54}$$

Furthermore, by (53), and Theorems 2.1 and 3.4,

$$\begin{aligned} |I_N v - v|_{1,\omega} &\leqslant |\hat{P}_N^1 v - v|_{1,\omega} + |I_N v - \hat{P}_N^1 v|_{1,\omega} \\ &\leqslant \|\hat{P}_N^1 v - v\|_{1,\omega} + cN^2 \|I_N v - \hat{P}_N^1 v\|_{\omega} \\ &\leqslant c\|\hat{P}_N^1 v - v\|_{1,\omega} + cN^{2-r}\|v\|_{r,\omega,A} \\ &\leqslant cN^{2-r}\|v\|_{r,\omega,A} \end{aligned} \tag{55}$$

Finally, we get the desired result by (53), (55) and space interpolation. □

5. ERROR ESTIMATES FOR A MODEL PROBLEM

We consider the following model problem:

$$\begin{aligned} -\partial_x^2 U(x) + vU(x) = f(x), \quad 0 < x < \infty \\ U(0) = 0, \quad \frac{U^2(x)}{x}\omega(x) \to 0, \quad \text{as } x \to \infty \end{aligned} \tag{56}$$

where $f(x)$ is a given function, and for the sake of simplicity, we assume $v > \frac{14}{27}$. For $0 < v \leqslant \frac{14}{27}$, we may use the variable transformation $t = \alpha x$ with $\alpha < \sqrt{27v/14}$ to rescale Equation (56).

A weak formulation of (56) with $v > \frac{14}{27}$ is to find $U \in H^1_{0,\omega}(\Lambda)$ such that

$$a^v_\omega(U, v) = (f, v)_\omega \quad \forall v \in H^1_{0,\omega}(\Lambda) \tag{57}$$

If $f \in (H^1_{0,\omega}(\Lambda))'$, then by Lemma 3.2 and the Lax–Milgram Lemma, (5.7) with $v > \frac{14}{27}$ has a unique solution in $H^1_{0,\omega}(\Lambda)$.

The Chebyshev rational spectral scheme for (56) is to find $u_N \in \mathscr{R}^0_N$ such that

$$a^v_\omega(u_N, \phi) = (f, \phi)_\omega \quad \forall \phi \in \mathscr{R}^0_N \tag{58}$$

Theorem 5.1
If $U \in H^r_{\omega,B}(\Lambda) \cap H^1_{0,\omega}(\Lambda)$, $v > \frac{14}{27}$ *and* $r \geqslant 1$, *then*

$$\|u_N - U\|_{1,\omega} \leqslant cN^{1-r}\|U\|_{r,\omega,B}$$

Proof
Let $U_N = P^{1,0}_N U$. By (57),

$$a^v_\omega(U_N, \phi) = (f, \phi)_\omega \quad \forall \phi \in \mathscr{R}^0_N \tag{59}$$

Thus,

$$a^v_\omega(u_n - U_N, \phi) = 0 \quad \forall \phi \in \mathscr{R}^0_N \tag{60}$$

Therefore, $u_N = U_N$ and the desired result follows from Theorem 3.3. □

We now consider the Chebyshev–Guass–Radau rational pseudospectral scheme for (56). Let

$$a^{v}_{\omega,N}(v,\phi) = -(\partial_x^2 v,\phi)_{\omega,N} + v(v,\phi)_{\omega,N}$$

Note that

$$\partial_x R_l(x) = \frac{2}{(x+1)^2} T_l'\left(\frac{x-1}{x+1}\right) = \frac{1}{2}(1-y)^2 T_l'(y)|_{y=(x-1)/(x+1)} \in \mathscr{R}_{l+1}$$

Hence, for any $v \in \mathscr{R}_l$, we have $\partial_x^2 v \in \mathscr{R}_{l+2}$. Therefore, owing to (48),

$$a^{v}_{\omega,N}(\phi,\psi) = a^{v}_{\omega}(\phi,\psi) \quad \forall \phi,\psi \in \mathscr{R}_{N-1} \tag{61}$$

A Chebyshev rational pseudospectral method for (56) is to find $u_N \in \mathscr{R}^0_{N-1}$ such that

$$a^{v}_{\omega,N}(u_N,\phi) = (f,\phi)_{\omega,N} \quad \forall \phi \in \mathscr{R}^0_{N-1} \tag{62}$$

Theorem 5.2
If $U \in H^r_{\omega,B}(\Lambda) \cap H^1_{0,\omega}(\Lambda)$, $f \in H^{r-1}_{\omega,A}(\Lambda)$, $v > \frac{14}{27}$ and $r \geqslant 1$, then

$$\|u_N - U\|_{1,\omega} \leqslant cN^{1-r}(\|U\|_{r,\omega,B} + \|f\|_{r-1,\omega,A}) \tag{63}$$

Proof
We derive from (48) and Theorem 2.2 that for any $\phi \in \mathscr{R}_{N-1}$,

$$\begin{aligned}|(f,\phi)_{\omega,N}| &= \left|\left(I_N((x+1)f), \frac{\phi}{x+1}\right)_{\omega,N}\right| = \left|\left(I_N((x+1)f), \frac{\phi}{x+1}\right)_{\omega}\right| \\ &\leqslant \|I_N((x+1)f)\|_\omega \left\|\frac{\phi}{x+1}\right\|_\omega \leqslant c\|I_N((x+1)f)\|_\omega \|\phi\|_{1,\omega}\end{aligned}$$

Thanks to the Lax–Milgram Lemma and Lemma 3.2, we assert that (62) has a unique solution such that

$$\|u_N\|_{1,\omega} \leqslant c\|I_N((x+1)f)\|_\omega$$

Let $U_N = P^{1,0}_{N-1}U$. Then by (61) and (62), we have for any $\phi \in \mathscr{R}^0_{N-1}$,

$$\begin{aligned} a^{v}_{\omega}(U_N,\phi) &= (f,\phi)_\omega \\ a^{v}_{\omega}(u_N,\phi) &= (I_N f,\phi)_\omega \end{aligned} \tag{64}$$

Therefore,

$$a^{v}_{\omega}(U_N - u_N,\phi) = (f - I_N f,\phi)_\omega \quad \phi \in \mathscr{R}^0_{N-1}$$

Let $\delta = v - \frac{14}{27}$. Taking $\phi = U_N - u_N$ and using Lemma 3.2, we obtain

$$\begin{aligned}\delta\|u_N - U_N\|^2_\omega + \tfrac{1}{16}|u_N - U_N|^2_{1,\omega} &\leqslant a^{v}_{\omega,N}(u_N - U_N, u_N - U_N) \\ &= (f - I_N f, U_N - u_N)_\omega \leqslant \|f - I_N f\|_\omega \|U_N - u_N\|_\omega \\ &\leqslant \frac{2}{\delta}\|f - I_N f\|^2_\omega + \frac{\delta}{2}\|U_N - u_N\|^2_\omega \end{aligned} \tag{65}$$

Hence, by Theorems 3.3 and 4.2,

$$\begin{aligned}\|u_N - U\|_{1,\omega} &\leqslant \|U_N - U\|_{1,\omega} + \|u_N - U_N\|_{1,\omega}\\ &\leqslant cN^{1-r}(\|U\|_{r,\omega,B} + \|f\|_{r-1,\omega,A})\end{aligned} \tag{66}$$

□

6. NUMERICAL IMPLEMENTATIONS AND DISCUSSIONS

We now present some implementation details for solving (56) by using the rational pseudospectral scheme (62). As is shown in Reference [16] (see also Reference [17]), it is advantageous to use *compact combinations* of rational Chebyshev polynomials as basis functions. Indeed, setting $\psi_j(x) = R_j(x) + R_{j+1}(x)$, we have $\psi(0) = 0$. Hence,

$$\mathscr{R}^0_{N-1} = \text{span}\{\psi_j\colon\ j = 0, 1, \ldots, N-2\} \tag{67}$$

Therefore, setting

$$\begin{aligned}&b_{kj} = (\psi_j, \psi_k)_{\omega,N} = (\psi_j, \psi_k)_\omega, \quad a_{kj} = a^{v}_{\omega,N}(\psi_j, \psi_k) = -(\partial_x^2\psi_j, \psi_k)_\omega\\ &u_N = \sum_{j=0}^{N-2} x_j\psi_j(x), \quad \bar{x} = (x_0, x_1, \ldots, x_{N-2})^{\mathrm{t}}\\ &\bar{f} = (f_0, f_1, \ldots, f_{N-2})^{\mathrm{t}} \quad \text{with } f_k = (f, \psi_j)_{\omega,N}\end{aligned} \tag{68}$$

the Rational Chebyshev pseudospectral approximation (62) is reduced to

$$(vB + A)\bar{x} = \bar{f} \tag{69}$$

Setting $x = (1+y)/(1-y)$ and $\phi_j(y) = \psi_j(x)$, one verify easily that

$$\begin{aligned}b_{kj} &= \int_{-1}^{1} (T_j(y) + T_{j+1}(y))(T_k(y) + T_{k+1}(y))\rho(y)\,\mathrm{d}y\\ a_{kj} &= -\frac{1}{4}\int_{-1}^{1} (1-y)^2\partial_y\{(1-y)^2\partial_y\phi_j(y)\}\phi_k(y)\rho(y)\,\mathrm{d}y\\ &= \frac{1}{4}\int_{-1}^{1} (1-y)^2\partial_y\phi_j(y)[(1-y)^2\partial_y\phi_k(y) - \frac{(1-y)(2+y)}{1+y}\phi_k(y)]\rho(y)\,\mathrm{d}y\end{aligned} \tag{70}$$

By using the orthogonality of Chebyshev polynomials, one find immediately that $B = (b_{kj})$ is a symmetric tridiagonal matrix, and that $a_{kj} = 0$ for $k < j-3$. By using the orthogonal relation of the Chebyshev polynomials, it is possible to derive explicit formulae for a_{kj}. However, this process is rather tedious. Alternatively, one can easily compute a_{kj} numerically by using a suitable Gaussian integration formula. Note that for problems with variable coefficients, a preconditioned iterative method should be used with a preconditioner obtained from a suitable problem with constant coefficients, see, for instance, References [16, 17].

We now present some numerical experiments using the above scheme to solve (56) with $v = 1$. Three illustrative examples involving three typical decaying behaviours are considered.

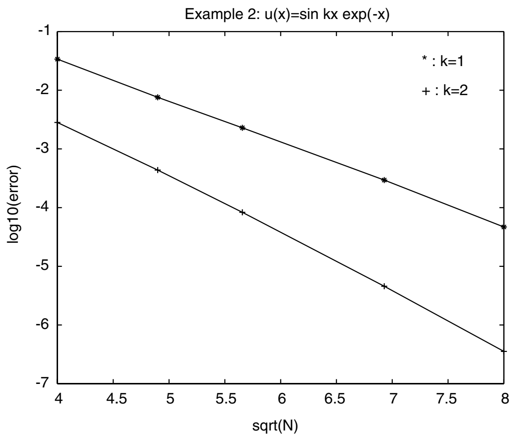

Figure 1. Convergence rates of the rational pseudospectral approximation: Example 1.

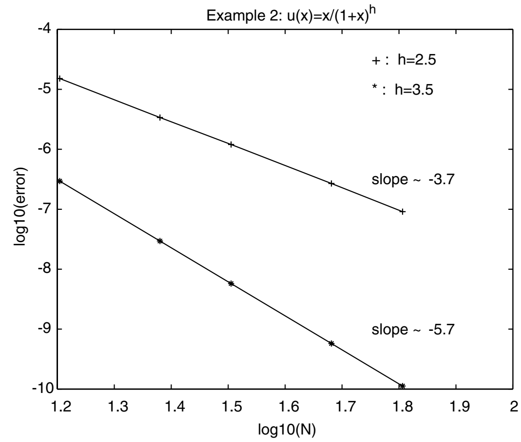

Figure 2. Convergence rates of the rational pseudospectral approximation: Example 2.

Example 1
$U(x) = \sin kx\mathrm{e}^{-x}$.

Here, the function decays exponentially at infinity, so Theorem 5.2 predicts that H^1_ω-errors of rational pseudospectral approximation will decrease faster than any algebraic rate. In Figure 1, we plot the $\log_{10}$ of H^1_ω-errors vs $\sqrt{N}$. The two near straight lines corresponding to $k=1$, 2 indicate that the errors decay like $\mathrm{e}^{-c\sqrt{N}}$.

Example 2
$U(x) = x/(1+x)^h$.

The second example decays algebraicly at infinity without essential singularity. One can check directly that $\|U\|_{r,\omega,B} + \|f\|_{r-1,\omega,A}$ is finite for $r < 2h - \frac{1}{2}$. Hence, according to Theorem 5.2, we can expect a convergence rate for the H^1_ω-norm to be of the order $2h - \frac{3}{2} - \varepsilon$ for any $\varepsilon > 0$. The observed convergence rate for the H^1_ω-norm plotted in Figure 2 is about $2h - 4/3$. Note that when h is a positive integer, the exact solution will be a rational polynomial so its pseudospectral approximation with $N \geqslant h+2$ will be exact.

Example 3
$U(x) = \sin 2x/(1+x)^h$.

The third example decays algebraicly at infinity but with an essential singularity at infinity. One can check directly that $\|U\|_{r,\omega,B} + \|f\|_{r-1,\omega,A}$ is finite for $r < \frac{2}{3}h + \frac{1}{2}$. Hence, according to Theorem 5.2, we can expect a convergence rate of order $\frac{2}{3}h - \frac{1}{2} - \varepsilon$ for any $\varepsilon > 0$, which agrees well with the observed convergence rate for the H^1_ω-errors plotted in Figure 3.

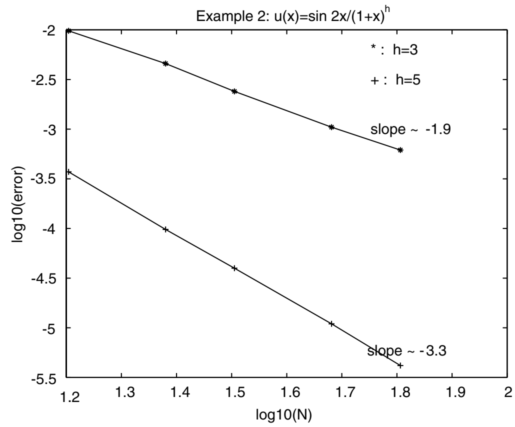

Figure 3. Convergence rates of the Chebyshev rational approximation: Example 3.

7. CONCLUDING REMARKS

For the above three examples, it is observed that the behaviours of H^1_ω-errors are essentially the same as that of the Legendre rational approximations presented in Reference [14]. More precisely, as measured in H^1_ω norm (with different $\omega(x)$!), the convergence order of the Chebyshev rational method is slightly smaller than that of the Legendre rational method. This slightly smaller order of convergence in H^1_ω norm is due to the fact that the weight function $1/(x+1)\sqrt{x}$ in the Chebyshev rational case is slight stronger than the weight function $1/(x+1)^2$ in the Legendre rational case.

As for the implementation, the Chebyshev rational method presented here allows the use of fast Fourier transforms, so it could result in substantial savings in CPU times for multidimensional applications; on the other hand, the linear system arising from Legendre rational method for PDEs with constant-coefficients is sparse while that from Chebyshev rational method is essentially full. However, this advantage of Legendre rational method disappears for problems with variable coefficients.

In order to apply the Legendre or Chebyshev rational spectral method to realistic engineering problems, it is often necessary to couple it with a spectral element or finite element method. There is already an important body of work in the finite element community on the so-called infinite element method (see, e.g. References [18–20]). It can be expected that the new Legendre or Chebyshev rational method coupled with a spectral element method will be an effective tool for solving problems on unbounded domains. In addition, the nice convergence and resolution properties of the Legendre or Chebyshev rational polynomials can also be exploited to within the framework of the infinite element method.

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