Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 138408, 17 pages
doi:10.1155/2010/138408

*Research Article*

# Chebyshev Wavelet Method for Numerical Solution of Fredholm Integral Equations of the First Kind

**Hojatollah Adibi and Pouria Assari**

*Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Avenue, Tehran 15914, Iran*

Correspondence should be addressed to Hojatollah Adibi, adibih@aut.ac.ir

Received 10 September 2009; Accepted 11 May 2010

Academic Editor: Victoria Vampa

Copyright © 2010 H. Adibi and P. Assari. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A computational method for solving Fredholm integral equations of the first kind is presented. The method utilizes Chebyshev wavelets constructed on the unit interval as basis in Galerkin method and reduces solving the integral equation to solving a system of algebraic equations. The properties of Chebyshev wavelets are used to make the wavelet coefficient matrices sparse which eventually leads to the sparsity of the coefficients matrix of obtained system. Finally, numerical examples are presented to show the validity and efficiency of the technique.

## 1. Introduction

Many problems of mathematical physics can be stated in the form of integral equations. These equations also occur as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations. Therefore, the study of integral equations and methods for solving them are very useful in application. In recent years, several simple and accurate methods based on orthogonal basic functions, including wavelets, have been used to approximate the solution of integral equation [1–5]. The main advantage of using orthogonal basis is that it reduces the problem into solving a system of algebraic equations. Overall, there are so many different families of orthogonal functions which can be used in this method that it is sometimes difficult to select the most suitable one. Beginning from 1991, wavelet technique has been applied to solve integral equations [6–10]. Wavelets, as very well-localized functions, are considerably useful for solving integral equations and provide accurate solutions. Also, the wavelet technique allows the creation of very fast algorithms when compared with the algorithms ordinarily used.

In various fields of science and engineering, we encounter a large class of integral equations which are called linear Fredholm integral equations of the first kind. Several methods have been proposed for numerical solution of these types of integral equation. Babolian and Delves [11] describe an augmented Galerkin technique for the numerical solution of first kind Fredholm integral equations. In [12] a numerical solution of Fredholm integral equations of the first kind via piecewise interpolation is proposed. Lewis [13] studied a computational method to solve first kind integral equations. Haar wavelets have been applied to solve Fredholm integral equations of first kind in [14]. Also, Shang and Han [15] used Legendre multiwavelets for solving first kind integral equations.

Consider the linear Fredholm integral equations of the first kind:

$$\int_0^1 K(x,y)u(y)\mathrm{d}y = f(x), \quad 0 \le x \le 1, \tag{1.1}$$

where $f \in L^2_w[0,1]$ and $K \in L^2_w([0,1] \times [0,1])$, in which $w(x) = 1/2\sqrt{x(1-x)}$, are known functions and $u$ is the unknown function to be determined. In general, these types of integral equation are ill-posed for given $K$ and $f$. Therefore (1.1) may have no solution, while if a solution exists, the response ration $\|\partial u\|/\|\partial f\|$ to small perturbations in $f$ may be arbitrary large [16].

The main purpose of this article is to present a numerical method for solving (1.1) via Chebyshev wavelets. The properties of Chebyshev wavelets are used to convert (1.1) into a linear system of algebraic equations. We will notice that these wavelets make the wavelet coefficient matrices sparse which concludes the sparsity of the coefficients matrix of obtained system. This system may be solved by using an appropriate numerical method.

The outline of the paper is as follows: in Section 2, we review some properties of Chebyshev wavelets and approximate the function $f$ and also the kernel function $K(x,y)$ by these wavelets. Convergence theorem of the Chebyshev wavelet bases is presented in Section 3. Section 4 is devoted to present a computational method for solving (1.1) utilizing Chebyshev wavelets and approximate the unknown function $u(x)$. In Section 5, the sparsity of the wavelet coefficient matrix is studied. Numerical examples are given in Section 6. Finally, we conclude the article in Section 7.

## 2. Properties of Chebyshev Wavelets

### *2.1. Wavelets and Chebyshev Wavelets*

Wavelets consist of a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter $a$ and the translation parameter $b$ vary continuously, we have the following family of continuous wavelets [17]:

$$\psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right), \quad a,b \in \mathbb{R}, \ a \neq 0. \tag{2.1}$$

If we restrict the parameters $a$ and $b$ to discrete values $a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0$ where $n$ and $k$ are positive integers, then we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{k/2}\psi\left(a_0^k t - nb_0\right), \tag{2.2}$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(t)$ forms an orthonormal basis [17, 18].

Chebyshev wavelets $\psi_{nm}(t) = \psi(k, n, m, t)$ have four arguments: $n = 1, 2, \ldots, 2^{k-1}$, $k$ is any nonnegative integer, $m$ is the degree of Chebyshev polynomial of first kind, and $t$ is the normalized time. The Chebyshev wavelets are defined on the interval $[0, 1)$ by [19]

$$\psi_{nm}(t) = \begin{cases} 2^{k/2}\widetilde{T}_m\left(2^k t - 2n + 1\right), & \dfrac{n-1}{2^{k-1}} \le t < \dfrac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

where

$$\widetilde{T}_m(t) = \begin{cases} \dfrac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\dfrac{2}{\pi}}T_m(t), & m > 0, \end{cases} \tag{2.4}$$

and $m = 0, 1, \ldots, M-1, n = 1, 2, \ldots, 2^{k-1}$. Here $T_m(t)$, $m = 0, 1, \ldots$, are Chebyshev polynomials of first kind of degree $m$, given by [20]

$$T_m(t) = \cos m\theta, \tag{2.5}$$

in which $\theta = \arccos t$. Chebyshev polynomials are orthogonal with respect to the weight function $w((t+1)/2) = 1/\sqrt{1-t^2}$, on $[-1, 1]$. We should note that Chebyshev wavelets are orthonormal set with the weight function:

$$w_k(t) = \begin{cases} w_{1,k}(t), & 0 \le t < \dfrac{1}{2^{k-1}}, \\ w_{2,k}(t), & \dfrac{1}{2^{k-1}} \le t < \dfrac{2}{2^{k-1}}, \\ \vdots & \vdots \\ w_{2^{k-1},k}(t), & \dfrac{2^{k-1}-1}{2^{k-1}} \le t < 1, \end{cases} \tag{2.6}$$

where $w_{n,k}(t) = w(2^{k-1}t - n + 1)$.

### 2.2. Function Approximation

A function $f(x) \in L^2_w[0, 1]$ may be expanded as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{2.7}$$

where

$$c_{nm} = \langle f(x), \psi_{nm}(x) \rangle_{w_k}, \tag{2.8}$$

in which $\langle \cdot, \cdot \rangle_{w_k}$ denotes the inner product in $L^2_{w_k}[0,1]$. The series (2.7) is truncated as

$$f(x) \simeq T_{k,M}(f(x)) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = C^t\Psi(x), \tag{2.9}$$

where $C$ and $\Psi(x)$ are two vectors given by

$$\begin{aligned} C &= \left[c_{10}, c_{12}, \dots, c_{1(M-1)}, c_{20}, \dots, c_{2(M-1)}, \dots, c_{(2^{k-1})0}, \dots, c_{(2^{k-1})(M-1)}\right]^t \\ &= \left[c_1, c_2, \dots, c_{2^{k-1}M}\right]^t, \\ \Psi(x) &= \left[\psi_{10}(x), \psi_{12}(x), \dots, \psi_{1(M-1)}(x), \psi_{20}(x), \dots, \psi_{2(M-1)}(x), \dots, \psi_{(2^{k-1})0}(x), \dots, \psi_{(2^{k-1})(M-1)}(x)\right]^t \\ &= \left[\psi_1(x), \psi_2(x), \dots, \psi_{2^{k-1}M}(x)\right]^t. \end{aligned} \tag{2.10}$$

Similarly, by considering $i = M(n-1) + m + 1$ and $j = M(n'-1) + m' + 1$, we approximate $K(x,y) \in L^2_w([0,1] \times [0,1])$ as

$$K(x,y) \simeq \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{k-1}M} K_{ij}\psi_i(x)\psi_j(y), \tag{2.11}$$

or in the matrix form

$$K(x,y) \simeq \Psi^t(x)\mathbf{K}\Psi(y), \tag{2.12}$$

where $\mathbf{K} = [K_{ij}]_{1 \le i,j \le 2^{k-1}M}$ with the entries

$$K_{ij} = \left\langle \psi_i(x), \langle K(x,y), \psi_j(y) \rangle_{w_k} \right\rangle_{w_k}. \tag{2.13}$$

## 3. Convergence of the Chebyshev Wavelet Bases

In this section, we indicate that the Chebyshev wavelet expansion of a function $f(x)$, with bounded second derivative, converges uniformly to $f(x)$.

**Lemma 3.1.** *If the Chebyshev wavelet expansion of a continuous function $f(x)$ converges uniformly, then the Chebyshev wavelet expansion converges to the function $f(x)$.*

*Proof.* Let

$$g(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{3.1}$$

where $c_{nm} = \langle f(x), \psi_{nm}(x)\rangle_{w_k}$. Multiplying both sides of (3.1) by $\psi_{pq}(x)w_k(x)$, where $p$ and $q$ are fixed and then integrating termwise, justified by uniformly convergence, on $[0,1]$, we have

$$\begin{aligned}
\int_0^1 g(x)\psi_{pq}(x)w_k(x)\mathrm{d}x &= \int_0^1 \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x)\psi_{pq}(x)w_k(x)\mathrm{d}x \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\int_0^1 \psi_{nm}(x)\psi_{pq}(x)w_k(x)\mathrm{d}x \\
&= c_{pq}.
\end{aligned} \tag{3.2}$$

Thus $\langle g(x), \psi_{nm}(x)\rangle_{w_k} = c_{nm}$ for $n = 1, 2, \ldots$ and $m = 0, 1, \ldots$. Consequently $f$ and $g$ have same Fourier expansions with Chebyshev wavelet basis and therefore $f(x) = g(x); (0 \le x \le 1)$ [21]. □

**Theorem 3.2.** *A function $f(x) \in L_w^2([0,1])$, with bounded second derivative, say $|f''(x)| \le N$, can be expanded as an infinite sum of Chebyshev wavelets, and the series converges uniformly to $f(x)$, that is,*

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x). \tag{3.3}$$

*Proof.* From (2.8) it follows that

$$c_{nm} = \int_0^1 f(x)\psi_{nm}(x)w_k(x)\mathrm{d}x = \int_{(n-1)/2^{k-1}}^{n/2^{k-1}} 2^{k/2} f(x)\widetilde{T}_m\left(2^k x - 2n + 1\right)w\left(2^k x - 2n + 1\right)\mathrm{d}x. \tag{3.4}$$

If $m > 1$, by substituting $2^k x - 2n + 1 = \cos\theta$ in (3.4), it yields

$$c_{nm} = \frac{1}{2^{k/2}} \int_0^{\pi} f\left(\frac{\cos\theta + 2n - 1}{2^k}\right) \sqrt{\frac{2}{\pi}} \cos m\theta \mathrm{d}\theta$$
$$= \frac{\sqrt{2}}{2^{k/2}\sqrt{\pi}} f\left(\frac{\cos\theta + 2n - 1}{2^k}\right)\left(\frac{\sin m\theta}{m}\right)\Bigg]_0^{\pi} \tag{3.5}$$

$$+ \frac{\sqrt{2}}{2^{3k/2} m\sqrt{\pi}} \int_0^{\pi} f'\left(\frac{\cos\theta + 2n - 1}{2^k}\right) \sin m\theta \sin\theta \mathrm{d}\theta \tag{3.6}$$

$$= \frac{1}{2^{3k/2} m\sqrt{2\pi}} f'\left(\frac{\cos\theta + 2n - 1}{2^k}\right)\left(\frac{\sin(m-1)\theta}{m-1} - \frac{\sin(m+1)\theta}{m+1}\right)\Bigg]_0^{\pi}$$
$$+ \frac{1}{2^{5k/2} m\sqrt{2\pi}} \int_0^{\pi} f''\left(\frac{\cos\theta + 2n - 1}{2^k}\right) h_m(\theta) \mathrm{d}\theta, \tag{3.7}$$

where

$$h_m(\theta) = \sin\theta\left(\frac{\sin(m-1)\theta}{m-1} - \frac{\sin(m+1)\theta}{m+1}\right). \tag{3.8}$$

Thus, we get

$$|c_{nm}| = \left|\frac{1}{2^{5k/2} m\sqrt{2\pi}} \int_0^{\pi} f''\left(\frac{\cos\theta + 2n - 1}{2^k}\right) h_m(\theta) \mathrm{d}\theta\right|$$
$$\leq \left(\frac{1}{2^{5k/2} m\sqrt{2\pi}}\right) \int_0^{\pi} \left| f''\left(\frac{\cos\theta + 2n - 1}{2^k}\right) h_m(\theta)\right| \mathrm{d}\theta \tag{3.9}$$
$$\leq \left(\frac{N}{2^{5k/2} m\sqrt{2\pi}}\right) \int_0^{\pi} |h_m(\theta)| \mathrm{d}\theta.$$

However

$$\int_0^{\pi} |h_m(\theta)| \mathrm{d}\theta = \int_0^{\pi} \left|\sin\theta\left(\frac{\sin(m-1)\theta}{m-1} - \frac{\sin(m+1)\theta}{m+1}\right)\right| \mathrm{d}\theta$$
$$\leq \int_0^{\pi} \left|\frac{\sin\theta\sin(m-1)\theta}{m-1}\right| + \left|\frac{\sin\theta\sin(m+1)\theta}{m+1}\right| \mathrm{d}\theta \tag{3.10}$$
$$\leq \frac{2m\pi}{(m^2 - 1)}.$$

Since $n \leq 2^{k-1}$, we obtain

$$|c_{nm}| < \frac{\sqrt{2\pi} N}{(2n)^{5/2}(m^2 - 1)}. \tag{3.11}$$

Now, if $m = 1$, by using (3.6), we have

$$|c_{n1}| < \frac{\sqrt{2\pi}}{(2n)^{3/2}} \max_{0\le x\le 1}\left|f'(x)\right|. \tag{3.12}$$

Hence, the series $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}$ is absolutely convergent. It is understandable that for $m = 0$, $\{\psi_{n0}\}_{n=1}^{\infty}$ form an orthogonal system constructed by Haar scaling function with respect to the weight function $w(t)$, and thus $\sum_{n=1}^{\infty} c_{n0}\psi_{n0}(x)$ is convergence [22]. On the other hand, we have

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x)\right| &\le \left|\sum_{n=1}^{\infty} c_{n0}\psi_{n0}(x)\right| + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|c_{nm}|\left|\psi_{nm}(x)\right| \\ &\le \left|\sum_{n=1}^{\infty} c_{n0}\psi_{n0}(x)\right| + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|c_{nm}| < \infty.\end{aligned} \tag{3.13}$$

Therefore, utilizing Lemma 3.1, the series $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x)$ converges to $f(x)$ uniformly. □

## 4. Solution of First Kind Integral Equations

In this section, the Chebyshev wavelet method is used for solving (1.1) by approximating functions $f(x)$, $u(y)$, and $K(x,y)$ in the matrix forms:

$$\begin{aligned} f(x) &\simeq F^t\Psi(x), \\ u(y) &\simeq U^t\Psi(y), \\ K(x,y) &\simeq \Psi^t(x)\mathbf{K}\Psi(y). \end{aligned} \tag{4.1}$$

By substituting (4.1) into (1.1), we obtain

$$\Psi^t(x)\mathbf{K}\left(\int_0^1 \Psi(y)\Psi^t(y)\,\mathrm{d}y\right)U - \Psi^t(x)F = R_{2^{k-1}M}(x), \tag{4.2}$$

where $R_{2^{k-1}M}(x)$ is the residual. By letting

$$\int_0^1 \Psi(y)\Psi^t(y)\,\mathrm{d}y = \mathbf{L}, \tag{4.3}$$

where $\mathbf{L}$ is a $2^{k-1}M \times 2^{k-1}M$ matrix which is computed next, we have

$$\Psi^t(x)\mathbf{K}\mathbf{L}U - \Psi^t(x)F = R_{2^{k-1}M}(x). \tag{4.4}$$

Our aim is to compute $u_1, u_2, \ldots, u_{2^{k-1}M}$ such that $R_{2^{k-1}M}(x) \equiv 0$, but in general, it is not possible to choose such $u_i, i = 1, 2, \ldots, 2^{k-1}M$. In this work, $R_{2^{k-1}M}(x)$ is made as small as possible such that

$$\langle \psi_{nm}(x), R_{2^{k-1}M}(x) \rangle_{w_k} = 0, \tag{4.5}$$

where $n = 1, 2, \ldots, 2^{k-1}$ and $m = 0, 1, \ldots, M-1$. Now, by using orthonormality of Chebyshev wavelets, we obtain the following linear system of algebraic equations:

$$\mathbf{KL}U = F, \tag{4.6}$$

for unknowns $U = [u_1, u_2, \ldots, u_{2^{k-1}M}]$.

Here, we define two operator equations $\mathcal{K}$ and $\mathcal{H}$ as follows:

$$\mathcal{K}(u(x)) = \int_0^1 K(x, y) u(y) \mathrm{d}y, \tag{4.7}$$

$$\mathcal{H}(u(x)) = \int_0^1 \Psi^t(x) \mathbf{K} \Psi(y) u(y) \mathrm{d}y, \tag{4.8}$$

for all $u \in L^2_w[0,1]$ and $x \in [0,1]$. We assume that integral operator $\mathcal{K}$ as defined in (4.7) is compact, one-to-one, onto, and $\|\mathcal{K}^{-1}\| < \infty$. We rewrite (1.1) and (4.2) in the operator form to obtain

$$\begin{gathered} \mathcal{K}u = f, \\ \mathcal{H}T_{k,M}(u) = T_{k,M}(f). \end{gathered} \tag{4.9}$$

Combining the latter equations yields

$$\mathcal{K}e_{k,M} = (\mathcal{K} - \mathcal{H})T_{k,M}(u) + \left(f - T_{k,M}(f)\right), \tag{4.10}$$

where $e_{k,M} = u - T_{k,M}(u)$. Provided that $\mathcal{K}^{-1}$ exists, we obtain the error bound:

$$\|e_{kM}\| \leq \left\|\mathcal{K}^{-1}\right\| \left\|(\mathcal{K} - \mathcal{H})T_{kM}(u) + \left(f - T_{kM}(f)\right)\right\|. \tag{4.11}$$

The error depends, therefore, on the conditioning of the original integral equation, as is apparent from the term $\|\mathcal{K}^{-1}\|$, on the fidelity of the finite-dimensional operator $\mathcal{H}$ to the integral operator $\mathcal{K}$, and on the approximation of $T_{kM}(f)$ to $f$.

Suppose that the function $f(x)$, defined on $[0,1]$, is $M$ times continuously differentiable, $f \in C^M([0,1])$; by using properties of Chebyshev wavelets and similar to [17], we have

$$\begin{aligned}\|f(x) - T_{k,M}(f(x))\|_2 &= \left(\int_0^1 [f(x) - T_{k,M}(f(x))]^2 \mathrm{d}x\right)^{1/2} \\ &\leq \left(\sum_{n=1}^{2^{k-1}} \int_{(n-1)/2^{k-1}}^{n/2^{k-1}} \int_0^1 [f(x) - C_{k,M}^n(f(x))]^2 \mathrm{d}x\right)^{1/2} \\ &\leq 2^{-kM} Q,\end{aligned} \tag{4.12}$$

where $Q = (2/2^M M!)\sup_{0\leq x\leq 1}|f^{(M)}(x)|$ and $C_{k,M}^n(f(x))$ denotes the polynomial of degree $M$ which agrees with $f$ at the Chebyshev nodes of the order $M$ on $[(n-1)/2^{k-1}, n/2^{k-1}]$. Therefore, if we want to have $\|f(x) - T_{k,M}(f(x))\|_2 < \varepsilon$, we can choose $k$ as

$$k = \left[\frac{Q}{\varepsilon \ln(2) M}\right] + 1. \tag{4.13}$$

*Evaluating L*

For numerical implementation of the method explained in previous part, we need to calculate matrix $\mathbf{L} = [L_{ij}]_{1\leq i,j\leq 2^{k-1}M}$. For this purpose, by considering $i = M(n-1) + m + 1$ and $j = M(n'-1) + m' + 1$, we have

$$L_{ij} = \int_0^1 \psi_i(y)\psi_j(y)\mathrm{d}y. \tag{4.14}$$

If $n \neq n'$, then $\psi_i(y)\psi_j(y) = 0$, because their supports are disjoint, yielding $L_{ij} = 0$. Hence, let $n = n'$; by substituting $2^k x - 2n + 1 = \cos\theta$ in (4.14), we obtain

$$L_{ij} = C_{mm'} \int_0^\pi \cos m\theta \cos m'\theta \sin\theta \,\mathrm{d}\theta, \tag{4.15}$$

where

$$C_{mm'} = \begin{cases} \dfrac{-1}{\pi}, & m = m' = 0, \\ \dfrac{-2}{\pi}, & m \neq 0 \neq m', \\ \dfrac{-\sqrt{2}}{\pi}, & \text{otherwise}. \end{cases} \tag{4.16}$$

Now, if $|m+m'|=1$, then

$$\int_0^{\pi}\cos m\theta\cos m'\theta\sin\theta d\theta=0, \tag{4.17}$$

implies that $L_{ij}=0$, and if $|m+m'|\neq 1$, then

$$L_{ij}=\frac{C_{mm'}}{4}\times\left(-\frac{\cos(m-m'+1)\theta}{m-m'+1}+\frac{\cos(-m+m'+1)\theta}{-m+m'+1}-\frac{\cos(m+m'-1)\theta}{m+m'-1}+\frac{\cos(-m-m'+1)\theta}{-m-m'+1}\right)\Bigg]_0^{\pi}. \tag{4.18}$$

Consequently, $\mathbf{L}$ has the following form:

$$\mathbf{L}=\operatorname{diag}\left(\underbrace{A,A,\ldots,A}_{2^{k-1}\ \text{times}}\right), \tag{4.19}$$

where $A=[A_{mm'}]$ is an $M\times M$ matrix with the elements

$$A_{mm'}=\begin{cases}C_{mm'}\dfrac{2\left(m^2+m'^2-1\right)}{1+m^4-2m'^2m^2-2m'^2-2m^2+m'^4}, & m+m' \text{ is even},\\[2ex] 0, & m+m' \text{ is odd}.\end{cases} \tag{4.20}$$

## 5. Sparse Representation of the Matrix K

We proceed by discussing the sparsity of the matrix $\mathbf{K}$, as an important issue for increasing the computation speed.

**Theorem 5.1.** *Suppose that $K_{ij}$ is the Chebyshev wavelet coefficient of the continuous kernel $K(x,y)$, where $i=M(n-1)+m+1$ and $j=M(n'-1)+m'+1$. If mixed partial derivative is $\partial^4K(x,y)/\partial x^2\partial y^2$ bounded by $N$ and $m,m'>1$, then one has*

$$\left|K_{ij}\right|<\frac{\pi N}{2^4(nn')^{5/2}(m^2-1)\left(m'^2-1\right)}. \tag{5.1}$$

*Proof.* From (2.13), we obtain

$$\left|K_{ij}\right| = 2^k \left| \int_{(n-1)/2^{k-1}}^{n/2^{k-1}} \int_{(n'-1)/2^{k-1}}^{n'/2^{k-1}} K(x,y)\widetilde{T}_m\left(2^k x - 2n + 1\right) w\left(2^k x - 2n + 1\right) \right.$$
$$\left. \times \widetilde{T}_{m'}\left(2^k y - 2n' + 1\right) w\left(2^k y - 2n' + 1\right) \mathrm{d}y\, \mathrm{d}x \right|. \tag{5.2}$$

Now, let $2^k x - 2n + 1 = \cos\theta$ and $2^k y - 2n' + 1 = \cos\alpha$; then

$$\left|K_{ij}\right| = \frac{2}{2^k \pi} \left| \iint_0^\pi K\left(\frac{\cos\theta + 2n - 1}{2^k}, \frac{\cos\alpha + 2n' - 1}{2^k}\right) \cos m\theta \cos m'\alpha\, \mathrm{d}\alpha\, \mathrm{d}\theta \right|. \tag{5.3}$$

Similar to the proof of Theorem 3.2, since $m, m' > 1$, we obtain

$$\begin{aligned}
\left|K_{ij}\right| &\le \frac{1}{2^{5k+1}\pi m m'} \left| \iint_0^\pi \frac{\partial^4 K\left((\cos\theta + 2n - 1)/2^k, (\cos\alpha + 2n' - 1)/2^k\right)}{\partial t^2 \partial s^2} h_m(\theta) h_{m'}(\alpha)\, \mathrm{d}\alpha\, \mathrm{d}\theta \right| \\
&\le \frac{N}{2^{5k+1}\pi m m'} \int_0^\pi |h_m(\theta)| \mathrm{d}\theta \int_0^\pi |h_{m'}(\alpha)| \mathrm{d}\alpha \\
&< \frac{\pi N}{2^4 (n n')^{5/2} (m^2 - 1)\left(m'^2 - 1\right)}.
\end{aligned} \tag{5.4}$$

□

*Remark 5.2.* As an immediate conclusion from Theorem 5.1, when $i$ or $j \to \infty$, it follows that $|K_{ij}| \to 0$ and accordingly by increasing $k$ or $M$, we can make $\mathbf{K}$ sparse which concludes the sparsity of the coefficient matrix of system (4.6). For this purpose, we choose a threshold $\varepsilon_0$ and get the following system of linear equations whose matrix is sparse:

$$\overline{\mathbf{K}}\mathbf{L}U = F, \tag{5.5}$$

where $\overline{\mathbf{K}} = [\overline{K}_{ij}]_{2^{k-1}M \times 2^{k-1}M}$ with the entries

$$\overline{K}_{ij} = \begin{cases} K_{ij}, & \left|K_{ij}\right| \ge \varepsilon_0, \\ 0, & \text{otherwise}. \end{cases} \tag{5.6}$$

Now, we can solve (5.5) instead of (4.6).

## 6. Numerical Examples

In order to test the validity of the present method, three examples are solved and the numerical results are compared with their exact solution [11, 14, 15]. In addition, in Examples

**Table 1:** Some numerical results for Example 6.1.

| $x$ | Exact solution | Approximate solution $k = 2, M = 2, \varepsilon_0 = 10^{-5}$ | Approximate solution $k = 2, M = 4, \varepsilon_0 = 10^{-4}$ | Legendre wavelets [15] |
|---|---|---|---|---|
| 0.0 | 0.0000000000 | 0.0000000002 | −0.0000080915 | −0.0000623203 |
| 0.1 | 0.1000000000 | 0.0999467145 | 0.0999919084 | 0.0999399803 |
| 0.2 | 0.2000000000 | 0.1998628369 | 0.1999919083 | 0.1999422810 |
| 0.3 | 0.3000000000 | 0.2997789593 | 0.2999919083 | 0.2999445816 |
| 0.4 | 0.4000000000 | 0.3996950817 | 0.3999919082 | 0.3999468823 |
| 0.5 | 0.5000000000 | 0.4994017104 | 0.4999821126 | 0.5000836748 |
| 0.6 | 0.6000000000 | 0.5996197220 | 0.5999908426 | 0.6000583321 |
| 0.7 | 0.7000000000 | 0.6994489377 | 0.6999914808 | 0.7000329894 |
| 0.8 | 0.8000000000 | 0.7992781533 | 0.7999921190 | 0.8000076466 |
| 0.9 | 0.9000000000 | 0.8991073690 | 0.8999927572 | 0.8999823039 |
| 1.0 | 1.0000000000 | 0.9989365847 | 0.9999933955 | 0.9999569612 |

6.1 and 6.2, our results are compared with numerical results in [14, 15]. It is seen that good agreements are achieved, as dilation parameter $a = 2^{-k}$ decreases.

*Example 6.1.* As the first example, let

$$\int_0^1 \sin(xy)u(y)\mathrm{d}y = \frac{\sin(x) - x\cos(x)}{x^2}, \quad 0 \le x \le 1, \tag{6.1}$$

with the exact solution $u_{ex}(x) = x$ [15].

Table 1 shows the numerical results for this example with $k = 2, M = 2, \varepsilon_0 = 10^{-5}$ and $k = 2, M = 4, \varepsilon_0 = 10^{-4}$. Also, the approximate solution for $k = 2$, $M = 4$, $\varepsilon_0 = 10^{-4}$ is graphically shown in Figure 1, which agrees with exact solution and results are compared with those of [15].

*Example 6.2.* In this example we solve integral equation

$$u(x) - \int_0^1 \mathrm{e}^{xy}u(y)\mathrm{d}y = \frac{\mathrm{e}^{x+1} - 1}{x + 1}, \quad 0 \le x \le 1, \tag{6.2}$$

by the present method, where the exact solution is $u_{ex}(x) = \mathrm{e}^x$ [11].

Table 2 gives the absolute error for this example with $k = 2, M = 3, \varepsilon_0 = 10^{-5}$ and $k = 3, M = 4, \varepsilon_0 = 10^{-4}$ where $\tilde{u}$ denote the approximation of $u_{ex}$. The approximate solution for $k = 3, M = 4, \varepsilon_0 = 10^{-4}$ in collocation points $x_j = (j-1/2)/50,\ j = 1, 2, \ldots, 50$, is graphically shown in Figure 2 . It is seen that the numerical results are improved, as parameter $k$ increases. Also, results are compared with those of [14].

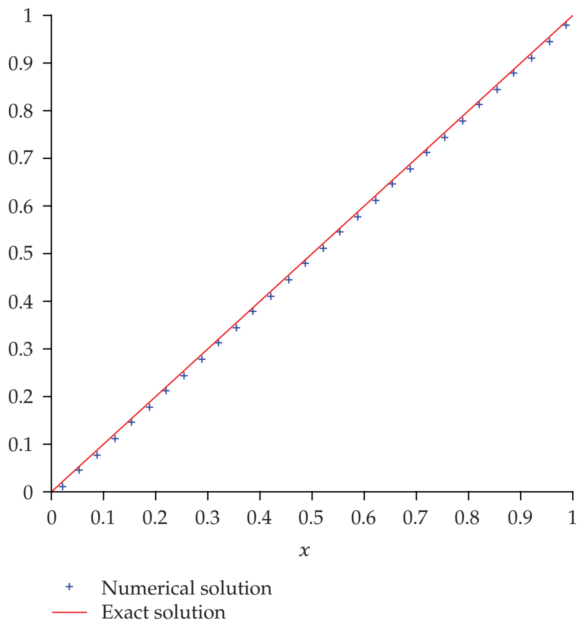



**Figure 1:** Approximate solution for Example 6.1 with $k = 2, M = 4, \varepsilon_0 = 10^{-4}$.

**Table 2:** Absolute error of exact and approximated solution of Example 6.2.

| $x$ | $\lvert\tilde{u}(x) - u_{ex}(x)\rvert$ $k = 2, M = 3, \varepsilon_0 = 10^{-5}$ | $\lvert\tilde{u}(x) - u_{ex}(x)\rvert$ $k = 3, M = 4, \varepsilon_0 = 10^{-4}$ | $\lvert\tilde{u}(x) - u_{ex}(x)\rvert$ Haar wavelets [14] |
|---|---|---|---|
| 0.0 | $0.488296e - 3$ | $0.146932e - 4$ | $0.785334e - 2$ |
| 0.1 | $0.937569e - 3$ | $0.150830e - 4$ | $0.173943e - 2$ |
| 0.2 | $0.265918e - 4$ | $0.173487e - 4$ | $0.569956e - 2$ |
| 0.3 | $0.108134e - 2$ | $0.186327e - 4$ | $0.635611e - 2$ |
| 0.4 | $0.110062e - 2$ | $0.157720e - 4$ | $0.231400e - 2$ |
| 0.5 | $0.125395e - 3$ | $0.658547e - 5$ | $0.129479e - 1$ |
| 0.6 | $0.211862e - 2$ | $0.130256e - 5$ | $0.286785e - 2$ |
| 0.7 | $0.226888e - 2$ | $0.232601e - 5$ | $0.939698e - 2$ |
| 0.8 | $0.728295e - 3$ | $0.152448e - 4$ | $0.104794e - 1$ |
| 0.9 | $0.383486e - 3$ | $0.983848e - 5$ | $0.381514e - 2$ |
| 1.0 | $0.127610e - 2$ | $0.104492e - 4$ | $0.498723e - 2$ |

*Example 6.3.* As our final example let

$$\int_0^1 \frac{(y - x)^2}{1 + y^2} u(y)\,\mathrm{d}y = 0.179171 - 0.532108x + 0.487495x^2, \quad 0 \le x \le 1, \tag{6.3}$$

with the exact solution $u(x)_{ex} = \sqrt{x}$ [14].

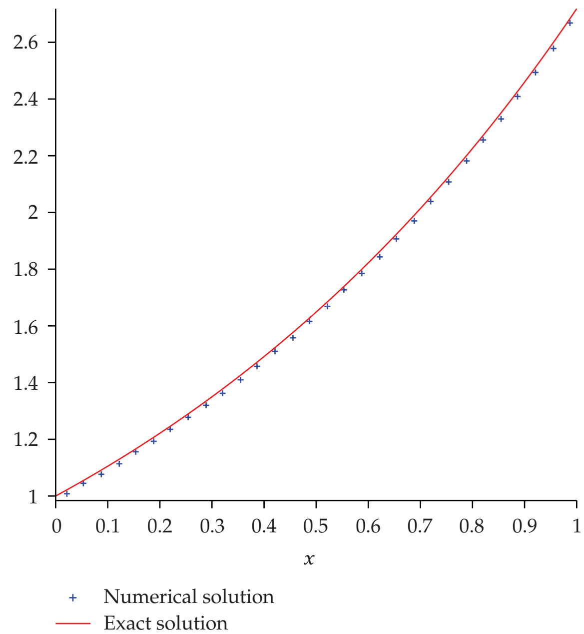



**Figure 2:** Approximate solution for Example 6.2 with $k = 3, M = 4, \varepsilon_0 = 10^{-4}$.

**Table 3:** Some error estimates for Example 6.3.

| $k$ | $M$ | $\|u - \widehat{u}\|_\infty$ | $\|u - \widehat{u}\|_2$ |
|---|---|---|---|
| 2 | 2 | $< 0.1500527192$ | $0.2898279425e - 1$ |
| 2 | 3 | $< 0.9003163152e - 1$ | $0.1298595687e - 1$ |
| 2 | 4 | $< 0.6430830822e - 1$ | $0.7461435276e - 2$ |
| 3 | 3 | $< 0.6366197694e - 1$ | $0.6493091346e - 2$ |
| 3 | 4 | $< 0.4547284061e - 1$ | $0.3730718373e - 2$ |
| 3 | 5 | $< 0.3536776479e - 1$ | $0.2438633959e - 2$ |

The proposed method was applied to approximate the solution of Fredholm integral equation (6.3) with some values of $k$ and $M$. Table 3 represents the error estimate for the result obtained of $\|.\|_\infty$ and $\|.\|_2$. The following norms are used for the errors of the approximation $\widehat{u}(x)$ of $u(x)$:

$$\|u - \widehat{u}\|_\infty = \max\{|u(x) - \widehat{u}(x)|,\ 0 \le x \le 1\},$$
$$\|u - \widehat{u}\|_2 = \left(\int_0^1 |u(x) - \widehat{u}(x)|^2 \mathrm{d}x\right)^{1/2}. \tag{6.4}$$

Also, the error $e(x) = u(x) - \widehat{u}(x)$for $k = 2$, $M = 3$, and $k = 3$, $M = 4$ is graphically shown in Figures 3 and 4 for $[0, 1/2]$ and $[1/2, 1]$, respectively.

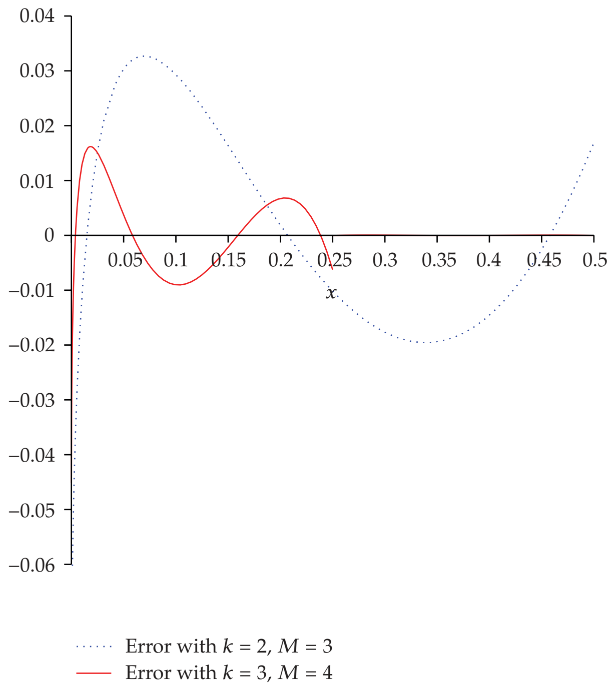



**Figure 3:** Error distributions for Example 6.3 with $k = 2$, $M = 3$ and $k = 3$, $M = 4$ on $[0, 1/2]$.

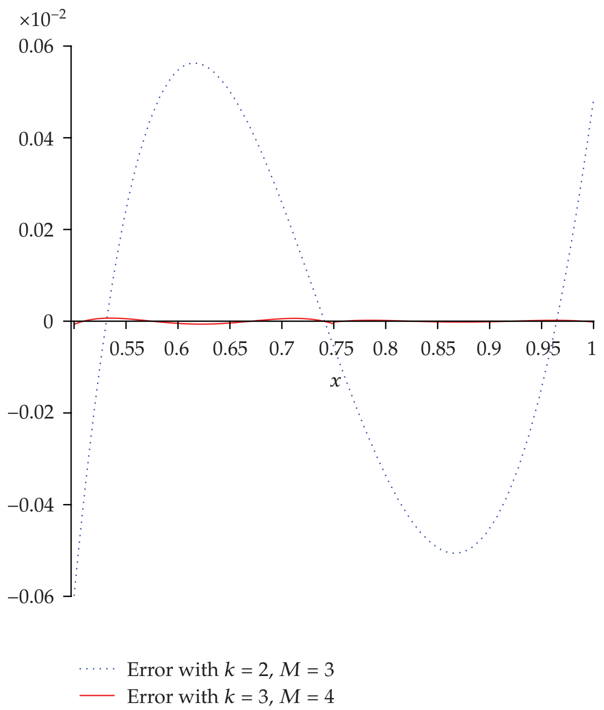



**Figure 4:** Error distributions for Example 6.3 with $k = 2$, $M = 3$ and $k = 3$, $M = 4$ on $[1/2, 1]$.

## 7. Conclusion

Integral equations are usually difficult to solve analytically, and therefore, it is required to obtain the approximate solutions. In this study we develop an efficient and accurate method for solving Fredholm integral equation of the first kind. The properties of Chebyshev wavelets are used to reduce the problem into solution of a system of algebraic equations whose matrix is sparse. However, to obtain better results, using the larger parameter $k$ is recommended. The convergence accuracy of this method was examined for several numerical examples.

## References


[1] M. Razzaghi and Y. Ordokhani, "Solution of nonlinear Volterra-Hammerstein integral equations via rationalized Haar functions," *Mathematical Problems in Engineering*, vol. 7, no. 2, pp. 205–219, 2001.

[2] M. Razzaghi and Y. Ordokhani, "A rationalized Haar functions method for nonlinear Fredholm-Hammerstein integral equations," *International Journal of Computer Mathematics*, vol. 79, no. 3, pp. 333–343, 2002.

[3] A. Alipanah and M. Dehghan, "Numerical solution of the nonlinear Fredholm integral equations by positive definite functions," *Applied Mathematics and Computation*, vol. 190, no. 2, pp. 1754–1761, 2007.

[4] C. Hsiao, "Hybrid function method for solving Fredholm and Volterra integral equations of the second kind," *Journal of Computational and Applied Mathematics*, vol. 230, no. 1, pp. 59–68, 2009.

[5] A. Akyüz-Daşcıoğlu, "Chebyshev polynomial solutions of systems of linear integral equations," *Applied Mathematics and Computation*, vol. 151, no. 1, pp. 221–232, 2004.

[6] S. Yousefi and A. Banifatemi, "Numerical solution of Fredholm integral equations by using CAS wavelets," *Applied Mathematics and Computation*, vol. 183, no. 1, pp. 458–463, 2006.

[7] S. Yousefi and M. Razzaghi, "Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations," *Mathematics and Computers in Simulation*, vol. 70, no. 1, pp. 1–8, 2005.

[8] M. Lakestani, M. Razzaghi, and M. Dehghan, "Solution of nonlinear Fredholm-Hammerstein integral equations by using semiorthogonal spline wavelets," *Mathematical Problems in Engineering*, vol. 2005, no. 1, pp. 113–121, 2005.

[9] H. Adibi and P. Assari, "Using CAS wavelets for numerical solution of Volterra integral equations of the second kind," *Dynamics of Continuous, Discrete & Impulsive Systems. Series A*, vol. 16, no. 5, pp. 673–685, 2009.

[10] U. Lepik and E. Tamme, "Application of the Haar wavelets for solution of linear integral equations," in *Dynamical Systems and Applications*, pp. 395–407, 2005.

[11] E. Babolian and L. M. Delves, "An augmented Galerkin method for first kind Fredholm equations," *Journal of the Institute of Mathematics and Its Applications*, vol. 24, no. 2, pp. 157–174, 1979.

[12] G. Hanna, J. Roumeliotis, and A. Kucera, "Collocation and Fredholm integral equations of the first kind," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 6, no. 5, article 131, pp. 1–8, 2005.

[13] B. A. Lewis, "On the numerical solution of Fredholm integral equations of the first kind," *Journal of the Institute of Mathematics and Its Applications*, vol. 16, no. 2, pp. 207–220, 1975.

[14] K. Maleknejad, R. Mollapourasl, and K. Nouri, "Convergence of numerical solution of the Fredholm integral equation of the first kind with degenerate kernel," *Applied Mathematics and Computation*, vol. 181, no. 2, pp. 1000–1007, 2006.

[15] X. Shang and D. Han, "Numerical solution of Fredholm integral equations of the first kind by using linear Legendre multi-wavelets," *Applied Mathematics and Computation*, vol. 191, no. 2, pp. 440–444, 2007.

[16] M. T. Rashed, "Numerical solutions of the integral equations of the first kind," *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 413–420, 2003.

[17] B. K. Alpert, "A class of bases in $L^2$ for the sparse representation of integral operators," *SIAM Journal on Mathematical Analysis*, vol. 24, no. 1, pp. 246–262, 1993.

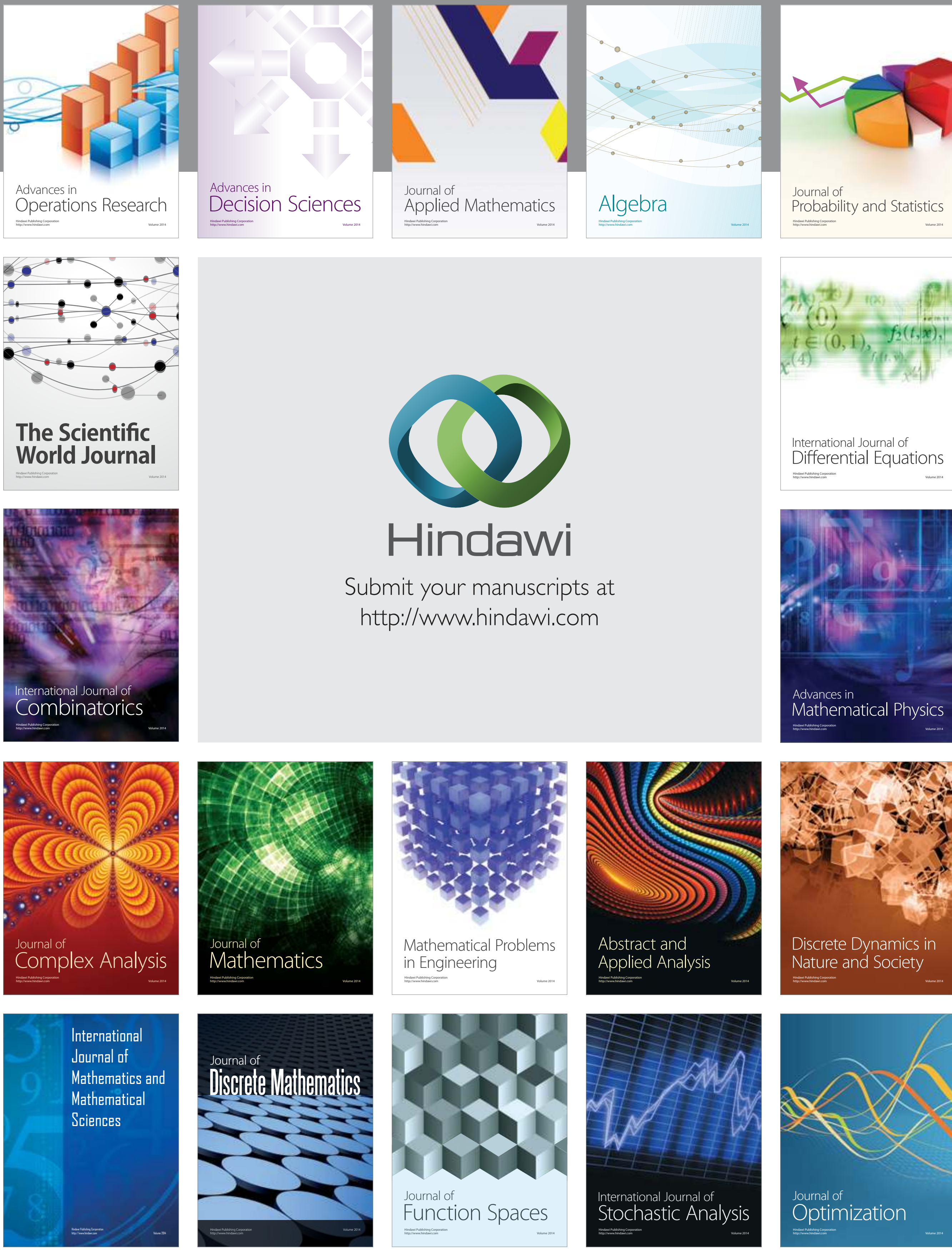
[18] M. Razzaghi and S. Yousefi, "The Legendre wavelets operational matrix of integration," *International Journal of Systems Science*, vol. 32, no. 4, pp. 495–502, 2001.

[19] E. Babolian and F. Fattahzadeh, "Numerical computation method in solving integral equations by using Chebyshev wavelet operational matrix of integration," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 1016–1022, 2007.

[20] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Chapman & Hall/CRC Press, Boca Raton, Fla, USA, 2003.

[21] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, Pure and Applied Mathematics, John Wiley & Sons, New York, NY, USA, 2nd edition, 1999.

[22] I. Daubechies, *Ten Lectures on Wavelets*, vol. 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1992.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization