

Chemostats with random inputs and wall growth

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Dedicated to Prof. Dr. Messoud Efendiyev on occasion of his 60th birthday

Chemostat refers to a laboratory device used for growing microorganisms in a cultured environment, and has been regarded as an idealization of nature to study competition modeling of mathematical biology. The simple form of chemostat model assumes that the availability of nutrient and its supply rate are both fixed. In addition the tendency of microorganism to adhere to surfaces is neglected by assuming the flow rate is fast enough. However, these assumptions largely limit the applicability of chemostat models to realistic competition systems. In this paper, we relax these assumptions and study the chemostat models with random nutrient supplying rate or random input nutrient concentration, with or without wall growth. This leads the models to random dynamical systems and requires the concept of random attractors developed in the theory of random dynamical systems. Our results include existence of uniformly bounded non-negative solutions, existence of random attractors and geometric details of random attractors for different value of parameters. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction

Competition modeling is one of the most challenging aspects of mathematical biology. The simplest form of competition occurs when two or more populations compete for the same resource, e.g. a common food supply of a growth-limiting nutrient. Chemostat can be regarded as a laboratory idealization of nature to study such competitions [31]. The chemostat model plays an important role in theoretical ecology [6, 10, 13, 16, 26, 27, 28, 30], waste water treatment [1, 21], and the study of recombinant problems in genetically altered organism [14, 15]. Derivation and analysis of chemostat models are well documented in [23, 24, 29] and references therein.

Two standard assumptions for simple chemostat models are: (1) the availability of the nutrient and its supply rate are fixed and (2) the tendency of microorganism to adhere to surfaces is not taken into account. Denoting by $x(t)$ the concentration of the growth-limiting nutrient and $y(t)$ the concentration of the microorganism at any specific time t , these assumptions lead to the following growth equations [22, 24]:

$$x'(t) = D(l - x) - aU(x(t))y(t), \quad (1)$$

$$y'(t) = -Dy(t) + aU(x(t))y(t), \quad (2)$$

where D is the rate at which the nutrient is supplied and also the rate at which the contents of the growth medium are removed, l is the input nutrient concentration which describes the quantity of nutrient available with the system at any time, a is the maximal consumption rate of the nutrient and also the maximum specific growth rate of microorganisms, and U is the functional

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response of the microorganism describing how the nutrient is consumed by the species. We assume in this paper that U follows the Michaelis-Menten or Holling type-II form:

$$U(x) = \frac{x}{m + x}, \quad (3)$$

where $m > 0$ is the half-saturation constant [22, 24].

However, in reality the chemostat models are neither autonomous nor deterministic; they process information provided by physical or chemical inputs with noise, which can be caused by environmental perturbations, internal variability, randomly fluctuating parameters, measurement errors, etc. This motivates our study of chemostat models with randomly fluctuated input parameters, the nutrient supplying rate D and the nutrient supplying concentration I . From biological point of view, for an experiment to be repeatable, one has to have a reasonable description of its random aspects. These aspects may change in time and thus the noise can be modeled as a time-dependent stochastic process with certain known properties.

Representing mathematically such a stochastic process starts with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where \mathcal{F} is the σ -algebra of measurable subsets of Ω and \mathbb{P} is the probability measure. To connect the state $\omega \in \Omega$ of the random environment with its state after a time t has elapsed, we define a family of time-dependent maps $\{\theta_t : \Omega \rightarrow \Omega\}$ that keeps track of noise. In this work, we formalize the two input parameters D as $D(\theta_t\omega)$ and I as $I(\theta_t\omega)$. In addition, we formalize $D(\theta_t\omega)$ and $I(\theta_t\omega)$ as bounded processes. This is a natural formalism to model the realistic stochastic fluctuations of a biological system caused by its interaction with the external world, because the parameters in dynamical systems of biological interest are inherently positive and bounded.

One other drawback of traditional chemostat model (1)-(2) is the ignorance of wall attachment of microorganisms. Most of the time microorganisms grow not only in the growth medium, but also along the walls of the container, either due to the ability of the microorganisms to stick on to the walls of the container or the flow rate is not fast enough to wash these organisms out of the system. Naturally, we can regard the consumer population $y(t)$ as an aggregate of two categories of populations, one in the growth medium, denoted by $y_1(t)$, and the other on the walls of the container, denoted by $y_2(t)$. These individuals may switch their categories at any time, i.e., the microorganisms on the walls may join those in the growth medium or the biomass in the medium may prefer walls.

Let r_1 and r_2 represent the rates at which the organisms stick on to and shear off from the walls, respectively. Assume that the nutrient is equally available to both of the categories, therefore it is assumed that both categories consume the same amount of nutrient and at the same rate. When the flow rate is low, the organisms may die naturally before being washed out and thus washout is no longer the only prime factor of death. Denote by $\nu (> 0)$ the collective death rate coefficient of $y(t)$ representing all the aforementioned factors such as diseases, aging, etc. On the other hand, when the flow rate is small, the dead biomass is not sent out of the system immediately and is subject to bacterial decomposition which in turn leads to regeneration of the nutrient. Expecting not 100% recycling of the dead material but only a fraction, we let constant $b \in (0, 1)$ describe the fraction of dead biomass that is recycled.

Our object is to study the evolution of concentrations of the nutrient and microorganism when the input parameters are random and wall growth is taken into account, which can be described by the following random system:

$$x'(t) = D(\theta_t\omega)(I(\theta_t\omega) - x(t)) - a \frac{x(t)}{m + x(t)}(y_1(t) + y_2(t)) + b\nu y_1(t), \quad (4)$$

$$y_1'(t) = -(\nu + D(\theta_t\omega))y_1(t) + c \frac{x(t)}{m + x(t)}y_1(t) - r_1y_1(t) + r_2y_2(t), \quad (5)$$

$$y_2'(t) = -\nu y_2(t) + c \frac{x(t)}{m + x(t)}y_2(t) + r_1y_1(t) - r_2y_2(t), \quad (6)$$

where $0 < c \leq a$ is the growth rate coefficient of the consumer species. In particular we assume that the inputs are perturbed by real noise, i.e. $D(\theta_t\omega)$ and $I(\theta_t\omega)$ are continuous and essentially bounded:

$$D(\theta_t\omega) \in d \cdot [1 - \varepsilon, 1 + \varepsilon], \quad I(\theta_t\omega) \in i \cdot [1 - \varepsilon, 1 + \varepsilon], \quad d > 0, \quad i > 0, \quad \varepsilon < 1.$$

Bounded noise can be modeled in various ways. For example in [4], given a stochastic process Z_t such as an Ornstein-Uhlenbeck process, the stochastic process

$$\zeta(Z_t) := \zeta_0 \left(1 - 2\varepsilon \frac{Z_t}{1 + Z_t^2}\right), \quad (7)$$

where ζ_0 and ε are positive constants with $\varepsilon \in (0, 1]$, takes values in the interval $\zeta_0[1 - \varepsilon, 1 + \varepsilon]$ and tends to peak around $\zeta_0(1 \pm \varepsilon)$. It is thus suitable for a noisy switching scenario. In another example, the stochastic process

$$\eta(Z_t) := \eta_0 \left(1 - \frac{2\varepsilon}{\pi} \arctan Z_t\right), \quad (8)$$

where η_0 and ε are positive constants with $\varepsilon \in (0, 1]$ takes values in the interval $\eta_0[1 - \varepsilon, 1 + \varepsilon]$ and is centered on η_0 . In the theory of random dynamical systems the driving noise process $Z_t(\omega)$ is replaced by a canonical driving system $\theta_t\omega$. This simplification allows a better understanding of the path-wise approach to model noise: a system influenced by stochastic processes for each single realization ω can be interpreted as wandering along a path $\theta_t\omega$ in Ω and thus may provide additional statistical/geological information to the modeler.

The rest of the paper is organized as follows. In section 2 we give a brief preliminary review of classical random dynamical system (RDS) theory, in section 3 we analyze the special case of system (4)–(6) without wall growth. In section 4 we analyze the full system of (4)–(6), with wall growth. Some closing remarks will be stated in section 5.

2. Random dynamical systems

In this section we first present some concepts (from [2]) related to general random dynamical systems (RDSs) and random attractors that we require in the sequel. Our situation is, in fact, somewhat simpler, but to facilitate the reader's access to the literature we give more general definitions here.

Let $(X, \|\cdot\|_X)$ be a separable Banach space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where \mathcal{F} is the σ -algebra of measurable subsets of Ω (called "events") and \mathbb{P} is the probability measure. To connect the state ω in the probability space Ω at time 0 with its state after a time of t elapses, we define a flow $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ on Ω with each θ_t being a mapping $\theta_t : \Omega \rightarrow \Omega$ that satisfies

- (1) $\theta_0 = \text{Id}_\Omega$,
- (2) $\theta_s \circ \theta_t = \theta_{s+t}$ for all $s, t \in \mathbb{R}$,
- (3) the mapping $(t, \omega) \mapsto \theta_t\omega$ is measurable and
- (4) the probability measure \mathbb{P} is preserved by θ_t , i.e. $\theta_t\mathbb{P} = \mathbb{P}$.

This set-up establishes a time-dependent family θ that tracks the noise, and $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a *metric dynamical system* [2].

Definition 1 A stochastic process $\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ is said to be a *continuous RDS over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$* with state space X if $S : [0, +\infty) \times \Omega \times X \rightarrow X$ is $(\mathcal{B}[0, +\infty) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ -measurable, and for each $\omega \in \Omega$,

- (i) the mapping $S(t, \omega) : X \rightarrow X, x \mapsto S(t, \omega)x$ is continuous for every $t \geq 0$;
- (ii) $S(0, \omega)$ is the identity operator on X ;
- (iii) (cocycle property) $S(t + s, \omega) = S(t, \theta_s\omega)S(s, \omega)$ for all $s, t \geq 0$.

Definition 2 (i) A set-valued mapping $B : \omega \rightarrow 2^X \setminus \emptyset$ is said to be a *random set* if the mapping $\omega \mapsto \text{dist}_X(x, B(\omega))$ is measurable for any $x \in X$.

- (ii) A random set $B(\omega)$ is said to be *bounded* if $B(\omega)$ is bounded for a.e. $\omega \in \Omega$; a random set $B(\omega)$ is said to be *compact* if $B(\omega)$ is compact for a.e. $\omega \in \Omega$; a random set is said to be *closed* if $B(\omega)$ is closed for a.e. $\omega \in \Omega$.
- (iii) A bounded random set $B(\omega) \subset X$ is said to be *tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$* if for a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup_{x \in B(\theta_{-t}\omega)} \|x\|_X = 0, \quad \text{for all } \beta > 0;$$

a random variable $\omega \mapsto r(\omega) \in \mathbb{R}$ is said to be *tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$* if for a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup_{t \in \mathbb{R}} |r(\theta_{-t}\omega)| = 0, \quad \text{for all } \beta > 0.$$

In what follows we use $\mathcal{D}(X)$ to denote the set of all tempered random sets of X .

Definition 3 A random set $K(\omega) \subset X$ is called a *random absorbing set in $\mathcal{D}(X)$* if for any $B \in \mathcal{D}(X)$ and a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$ such that

$$S(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega), \quad \forall t \geq T_B(\omega).$$

Definition 4 Let $\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ be an RDS over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ with state space X and let $\mathcal{A}(\omega) \subset X$ be a random set. Then $\mathcal{A}(\omega)$ is called a *global random \mathcal{D} attractor (or pullback \mathcal{D} attractor)* for $\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ if $\omega \mapsto \mathcal{A}(\omega)$ satisfies

- (i) (random compactness) $\mathcal{A}(\omega)$ is a compact set of X for a.e. $\omega \in \Omega$;
- (ii) (invariance) for a.e. $\omega \in \Omega$ and all $t \geq 0$, it holds

$$S(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega);$$

(iii) (attracting property) for any $B \in \mathcal{D}(X)$ and a.e. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} \text{dist}_X(S(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) = 0,$$

where

$$\text{dist}_X(G, H) = \sup_{g \in G} \inf_{h \in H} \|g - h\|_X$$

is the Hausdorff semi-metric for $G, H \subseteq X$.

Proposition 1 [5, 8, 12] Let $B \in \mathcal{D}(X)$ be an absorbing set for the continuous random dynamical system $\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ which is closed and satisfies the asymptotic compactness condition for a.e. $\omega \in \Omega$, i.e., each sequence $x_n \in S(t_n, \theta_{-t_n}, B(\theta_{-t_n}\omega))$ has a convergent subsequence in X when $t_n \rightarrow \infty$. Then the cocycle S has a unique global random attractor with component subsets

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq t_B(\omega)} \overline{\bigcup_{t \geq \tau} S(t, \theta_{-t}\omega)B(\theta_{-t}\omega)}.$$

If the pullback absorbing set is positively invariant, i.e., $S(t, \omega)B(\omega) \subset B(\theta_t\omega)$ for all $t \geq 0$, then

$$\mathcal{A}(\omega) = \bigcap_{t \geq 0} S(t, \theta_{-t}\omega)B(\theta_{-t}\omega).$$

For state space $X = \mathbb{R}^d$ as in this paper, the asymptotic compactness follows trivially. Note that the random attractor is pathwise attracting in the pullback sense, but need not be pathwise attracting in the forward sense, although it is forward attracting in probability, due to some possible large deviations, see e.g., Arnold [2].

When the cocycle mapping is strictly uniformly contracting [7, 18], i.e., there exists $K > 0$ such that

$$\|S(t, \omega)x_0 - S(t, \omega)y_0\|_X \leq e^{-Kt} \|x_0 - y_0\|_X$$

for all $t \geq 0$, $\omega \in \Omega$ and $x_0, y_0 \in X$, then the random attractor consists of singleton subsets $\mathcal{A}(\omega) = \{A(\omega)\}$. It is thus essentially a single stochastic process with sample paths $A(\theta_t\omega)$ for all $t \in \mathbb{R}$. The proof uses a Cauchy sequence rather than compactness argument. In this case the random attractor is pathwise attracting in both the pullback and forward senses.

2.1. Chemostat generates an RDS

We next prove that equations (4)-(6) generates a random dynamical system, and the random dynamical system has a random attractor. Letting

$$\mathbb{R}_+^3 = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$$

we have the following lemma.

Lemma 1 For any $\omega \in \Omega$, any $t_0 \in \mathbb{R}$, and any initial data $u_0 := (x(t_0), y_1(t_0), y_2(t_0))^T \in \mathbb{R}_+^3$, system (4)-(6) admits a unique solution $u(\cdot; t_0, \omega, u_0) \in C([t_0, \infty), \mathbb{R}_+^3)$ with $u(t_0; t_0, \omega, u_0) = u_0$ and generates a random dynamical system $S(t, \omega)(\cdot)$ defined as

$$S(t, \omega)u_0 = u(t; 0, \omega, u_0), \quad \forall t \geq 0, u_0 \in \mathbb{R}_+^3, \omega \in \Omega.$$

Moreover, $u(\cdot) = u(\cdot; t_0, \omega, u_0)$ is bounded.

Proof. With $u(t) = (x(t), y_1(t), y_2(t))^T$, system (4)-(6) can be written as

$$\frac{du}{dt} = L(\theta_t\omega) \cdot u + F(u, \theta_t\omega),$$

where

$$L(\theta_t\omega) = \begin{pmatrix} -D(\theta_t\omega) & 0 & 0 \\ 0 & -\nu - D(\theta_t\omega) - r_1 & r_2 \\ 0 & r_1 & -\nu - r_2 \end{pmatrix},$$

and $F : \mathbb{R}_+^3 \times [t_0, +\infty) \rightarrow \mathbb{R}^3$ is given by

$$F(v, \theta_t\omega) = \begin{pmatrix} D(\theta_t\omega)l(\theta_t\omega) - \frac{av_1}{m+v_1}(v_2+v_3) \\ \frac{cv_1}{m+v_1}v_2 \\ \frac{cv_1}{m+v_1}v_3 \end{pmatrix},$$

where $v = (v_1, v_2, v_3) \in \mathbb{R}_+^3$.

Since $D(\theta_t \omega)$ is bounded, the operator L generates an evolution system on \mathbb{R}^3 . Also because $D(\theta_t \omega)$ and $l(\theta_t \omega)$ are continuous with respect to t , function $F(\cdot, \theta_t \omega) \in C(\mathbb{R}_+^3 \times [t_0, \infty), \mathbb{R}^3)$ and is continuously differentiable with respect to the variables (v_1, v_2, v_3) , being the partial derivatives bounded inside \mathbb{R}_+^3 , what implies that is globally Lipschitz with respect to (v_1, v_2, v_3) in \mathbb{R}_+^3 . Therefore, thanks to the classical results from the theory of ordinary differential equations, system (4)–(6) possesses a unique global solution $u(\cdot; t_0, \omega, u_0) \in C^1([t_0, \infty), \mathbb{R}^3)$

Notice that it is straightforward to check that

$$u(t + t_0; t_0, \omega, u_0) = u(t; 0, \theta_{t_0} \omega, u_0)$$

for all $t_0 \in \mathbb{R}$, $t \geq t_0$, $\omega \in \Omega$, $u_0 \in \mathbb{R}_+^3$. This allows us to define a mapping $S(t, \omega)(\cdot)$, which will be our random dynamical system, as

$$S(t, \omega)u_0 = u(t; 0, \omega, u_0), \quad \forall t \geq 0, u_0 \in \mathbb{R}_+^3, \omega \in \Omega. \quad (9)$$

From now on, we will simply write $u(t; \omega, u_0)$ instead of $u(t; 0, \omega, u_0)$.

Now, as we will argue for each $\omega \in \Omega$ fixed, we will not mention explicitly the random parameter and will write $u(t) = u(t; \omega, u_0)$, and the same as far as its components are involved. By continuity of solutions, each solution has to take value 0 before it reaches a negative value. With $x = 0$ and $y_1, y_2 \geq 0$, equation (4) becomes

$$x'(t) = D(\theta_t \omega)l(\theta_t \omega) + b\nu y_1(t) > 0,$$

and thus $x(t)$ is strictly increasing at $x = 0$. Similarly, with $y_1 = 0$ and $x, y_2 \geq 0$, equation (5) gives $y_1'(t) = r_2 y_2 \geq 0$, and with $y_2 = 0$ and $x, y_1 \geq 0$, equation (6) gives $y_2'(t) = r_1 y_1 \geq 0$. Therefore, $y_1(t)$ is non-decreasing at $y_1 = 0$ and $y_2(t)$ is non-decreasing at $y_2 = 0$, i.e., $u(t) \in \mathbb{R}_+^3$ for any $t \geq 0$.

For any $u_0 \in \mathbb{R}_+^3$, solution $u(\cdot; \omega, u_0) \in \mathbb{R}_+^3$ for $t \in [0, \infty)$. Since function $F(u, \theta_t \omega) = F(u, t, \omega)$ is continuous in u, t , and is measurable in ω , $u : [0, \infty) \times \Omega \times \mathbb{R}_+^3 \rightarrow \mathbb{R}_+^3$, $(t; \omega, u_0) \mapsto u(t; \omega, u_0)$ is $(\mathcal{B}[0, \infty) \times \mathcal{F}_0 \times \mathcal{B}(\mathbb{R}_+^3), \mathcal{B}(\mathbb{R}_+^3))$ -measurable. It then follows directly that (4)–(6) generate the continuous random dynamical system $S(t, \omega)(\cdot)$ defined by (9).

For $u(t) \in \mathbb{R}_+^3$, define $\|u(t)\|_1 := x(t) + y_1(t) + y_2(t)$, and let $s(t) = x(t) + \frac{a}{c}(y_1(t) + y_2(t))$. Since $a \geq c$, then

$$\|u(t)\|_1 \leq s(t) \leq \frac{a}{c} \cdot \|u(t)\|_1.$$

The time derivative of $s(t)$ along solutions to (4)–(6) satisfies

$$\begin{aligned} \frac{ds(t)}{dt} &= D(\theta_t \omega) [l(\theta_t \omega) - x(t)] - \left[\frac{a}{c} (\nu + D(\theta_t \omega)) - b\nu \right] y_1(t) - \frac{a}{c} \nu y_2(t) \\ &\leq di(1 + \varepsilon)^2 - d(1 - \varepsilon)x(t) - \left[\frac{a}{c} (\nu + d(1 - \varepsilon)) - b\nu \right] y_1(t) - \frac{a}{c} \nu y_2(t). \end{aligned} \quad (10)$$

Note that, since $a \geq c$ and $0 < b < 1$,

$$\frac{a}{c} (\nu + d(1 - \varepsilon)) - b\nu = \frac{a}{c} d(1 - \varepsilon) + \left(\frac{a}{c} - b \right) \nu > \frac{a}{c} d(1 - \varepsilon).$$

Therefore by letting $\lambda := \min\{d(1 - \varepsilon), \nu\}$ we obtain

$$\frac{ds(t)}{dt} \leq di(1 + \varepsilon)^2 - \lambda s(t). \quad (11)$$

On the one hand, for $s(0) \geq di(1 + \varepsilon)^2/\lambda$, $s(t)$ will be non-increasing for all $t \geq 0$ and thus $s(t) \leq v(0)$. On the other hand, for $s(0) < di(1 + \varepsilon)^2/\lambda$, $s(t) \leq di(1 + \varepsilon)^2/\lambda$ for all $t \geq 0$. These imply that $\|u(t)\|_1$ is bounded:

$$0 \leq \|u(t)\|_1 \leq \max \left\{ di(1 + \varepsilon)^2/\lambda, x(0) + \frac{a}{c}(y_1(0) + y_2(0)) \right\}, \quad t \geq 0.$$

□

2.2. Existence of a random attractor

In this subsection we study the existence of tempered random bounded absorbing sets of the random dynamical system $\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$, followed by the existence of a random attractor for $\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$.

Lemma 2 For $\omega \in \Omega$, there exists a tempered bounded closed random absorbing set $K(\omega) \in \mathcal{D}(\mathbb{R}_+^3)$ of the random dynamical system $\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ such that for any $B \in \mathcal{D}(\mathbb{R}_+^3)$ and each $\omega \in \Omega$, there exists $T_B(\omega) > 0$ yielding

$$S(t, \theta_{-t} \omega)B(\theta_{-t} \omega) \subset K(\omega) \quad \forall t \geq T_B(\omega).$$

Proof. Recall that $u(t; \omega, u_0) = S(t, \omega)u_0$ denotes the solution of system (4)-(6) satisfying $u(0; \omega, u_0) = u_0$. Then, for $u_0 := u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$,

$$\|S(t, \theta_{-t}\omega)u_0\|_1 = \|u(t; \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|_1 \leq s(t; \theta_{-t}\omega, s_0(\theta_{-t}\omega)).$$

According to inequality (11),

$$s(t, \omega) \leq s_0 e^{-\lambda t} + di(1 + \epsilon)^2 / \lambda. \quad (12)$$

Substituting ω by $\theta_{-t}\omega$ in (12), we obtain

$$s(t; \theta_{-t}\omega, s_0(\theta_{-t}\omega)) \leq e^{-\lambda t} \sup_{(x, y_1, y_2) \in B(\theta_{-t}\omega)} \left(x + \frac{a}{c}(y_1 + y_2) \right) + \frac{di(1 + \epsilon)^2}{\lambda}. \quad (13)$$

Therefore for any $\epsilon > 0$, there exists $T_B(\omega)$ such that when $t > T_B$,

$$\begin{aligned} \|u(t; \theta_{-t}\omega, u_0)\|_1 &= x(t; \theta_{-t}\omega, u_0) + y_1(t; \theta_{-t}\omega, u_0) + y_2(t; \theta_{-t}\omega, u_0) \\ &\leq di(1 + \epsilon)^2 / \lambda + \epsilon, \end{aligned}$$

for all $u_0 \in B(\theta_{-t}\omega)$.

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$$K_\epsilon(\omega) = \overline{\{(x, y_1, y_2) \in \mathbb{R}_+^3 : x + y_1 + y_2 \leq di(1 + \epsilon)^2 / \lambda + \epsilon\}},$$

then $K_\epsilon(\omega)$ is positively invariant and absorbing in \mathbb{R}_+^3 . □.

It follows directly from Proposition 1, that the random dynamical system generated by system (4)-(6) possesses a random attractor $\mathcal{A} = \{A(\omega) : \omega \in \Omega\}$, consisting of nonempty compact random subsets of \mathbb{R}_+^3 contained in $K_\epsilon(\omega)$. In the next sections, we will study the geometric structure of the random attractor of the random dynamical system generated by (4)-(6), first without wall growth, then with wall growth.

3. Random chemostat without wall growth

In this section we first consider the special case when $y_2(t) \equiv 0, r_1 = r_2 = 0, \nu = 0$ and $a = c$, which describes the random chemostat system with no wall growth. In particular we will study the system with both random nutrient input concentration and random nutrient supplying rate.

When there is no wall growth, equations (4) - (6) can be reduced to

$$\frac{dx(t)}{dt} = D(\theta_t\omega)(I(\theta_t\omega) - x(t)) - a \frac{x(t)}{m + x(t)} y_1(t) \quad (14)$$

$$\frac{dy_1(t)}{dt} = -D(\theta_t\omega)y_1(t) + a \frac{x(t)}{m + x(t)} y_1(t) \quad (15)$$

Let $v(t) := x(t) + y_1(t)$, then $v(t)$ satisfies

$$\frac{dv(t)}{dt} = D(\theta_t\omega)[I(\theta_t\omega) - v(t)] \quad (16)$$

This has a nontrivial random solution which is both forward and pullback attracting. In fact, for any initial value v_0 , it holds

$$v(t; \omega, v_0) = v_0 e^{-\int_0^t D(\theta_s\omega) ds} + \int_0^t D(\theta_s\omega) I(\theta_s\omega) e^{-\int_s^t D(\theta_r\omega) d\tau} ds. \quad (17)$$

Replacing ω by $\theta_{-t}\omega$ in (17) we obtain

$$v(t; \theta_{-t}\omega, v_0) = v_0 e^{-\int_{-t}^0 D(\theta_s\omega) ds} + \int_{-t}^0 D(\theta_s\omega) I(\theta_s\omega) e^{-\int_0^t D(\theta_r\omega) d\tau} ds,$$

which is pullback convergent (i.e., when $t \rightarrow +\infty$) to

$$v^*(\omega) := \int_{-\infty}^0 D(\theta_s\omega) I(\theta_s\omega) e^{-\int_s^0 D(\theta_r\omega) d\tau} ds.$$

Note that since $d(1 - \epsilon) \leq D(\theta_t\omega) \leq d(1 + \epsilon)$ and $i(1 - \epsilon) \leq I(\theta_t\omega) \leq i(1 + \epsilon)$,

$$di(1 - \epsilon)^2 \int_{-\infty}^0 e^{-\int_s^0 d(1+\epsilon)d\tau} ds \leq v^*(\omega) \leq di(1 + \epsilon)^2 \int_{-\infty}^0 e^{-\int_s^0 d(1-\epsilon)d\tau} ds,$$

i.e.

$$\frac{i(1-\epsilon)^2}{1+\epsilon} \leq v^*(\omega) \leq \frac{i(1+\epsilon)^2}{1-\epsilon}, \quad \omega \in \Omega. \tag{18}$$

For a fixed and small enough $\delta > 0$, name the nonempty compact set

$$K(\omega) := \left\{ (x, y_1) \in \mathbb{R}_+^2 : \frac{i(1-\epsilon)^2}{1+\epsilon} - \delta \leq x + y_1 \leq \frac{i(1+\epsilon)^2}{1-\epsilon} + \delta \right\}.$$

Then for any $B(\omega) \in \mathcal{D}(\mathbb{R}_+^2)$ and a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$ such that

$$S(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega),$$

i.e. $K(\omega)$ is positively invariant and absorbing in \mathbb{R}_+^2 for system (16).

The random dynamical system generated by system (14)-(15) thus has a random attractor $\mathcal{A} = \{A(\omega) : \omega \in \Omega\}$ consisting of non-empty compact subsets of $K(\omega)$. We provide in the next theorem sufficient conditions for the extinction and persistence of microorganism y_1 .

Theorem 1 *The random pullback attractor $\mathcal{A} = \{A(\omega) : \omega \in \Omega\}$ for the random dynamical system generated by equations (14)-(15)*

(i) *has singleton component sets $A(\omega) = \{(v^*(\omega), 0)\}$ for every $\omega \in \Omega$, provided*

$$d(1-\epsilon) \geq a;$$

(ii) *possesses nontrivial component sets which include $(v^*(\omega), 0)$ and strictly positive points, provided*

$$d(1+\epsilon) \leq \frac{ai(1-\epsilon)^3}{m(1-\epsilon^2) + i(1+\epsilon)^3};$$

(iii) *contains a nontrivial entire solution that attracts all other strictly positive entire solutions, provided*

$$d(1-\epsilon) > a - \frac{ami(1-\epsilon)^2/(1+\epsilon)}{(m+i(1+\epsilon)^2/(1-\epsilon))^2}.$$

Proof. Note that since the pullback attractor contains all bounded entire solutions, then $(v^*(\omega), 0) \in A(\omega)$ for every $\omega \in \Omega$.

(i) Equation (15) can be written as

$$\begin{aligned} \frac{dy_1(t)}{dt} &= \left(-D(\theta_t\omega) + \frac{ax(t)}{m+x(t)} \right) y_1(t) \\ &< (-d(1-\epsilon) + a)y_1(t). \end{aligned}$$

Thus when $d(1-\epsilon) > a$, $y'(t) < 0$. This implies that $y_1(t) \rightarrow 0$ as $t \rightarrow \infty$ and $x(t, \omega) = v^*(\theta_t\omega)$, $y_1(t) = 0$ is asymptotically stable in \mathbb{R}_+^2 .

(ii) Besides $x + y_1 = i(1+\epsilon)^2/(1-\epsilon)$ and $x + y_1 = i(1-\epsilon)^2/(1+\epsilon)$, the absorbing set $K(\omega)$ has two other edges $x = 0$ and $y_1 = 0$. On $x = 0$, $x'(t)|_{x=0} = D(\theta_t\omega)I(\theta_t\omega) > 0$, i.e., $x(t)$ is increasing towards the interior of $K(\omega)$ on this edge. The edge $y_1 = 0$ is invariant, as $y'_1|_{y_1=0} = 0$. But for $y = \epsilon \ll i(1-\epsilon)^2/(1+\epsilon)$, $x(t)$ satisfies $i(1-\epsilon)^2/(1+\epsilon) \leq x \leq i(1+\epsilon)^2/(1-\epsilon)$. Hence when

$$d(1+\epsilon) \leq \frac{ai(1-\epsilon)^3}{m(1-\epsilon^2) + i(1+\epsilon)^3},$$

equation (15) gives

$$\frac{dy_1}{dt} = \left(-D(\theta_t\omega) + \frac{ax(t)}{m+x(t)} \right) y_1 > \left(\frac{ai(1-\epsilon)^3}{m(1-\epsilon^2) + i(1+\epsilon)^3} - d(1+\epsilon) \right) y_1 \geq 0.$$

This implies that the positive interior of the absorbing set also contains points of the random attractor.

(iii) We now consider equation (14) restricted to the stable manifold $x(t) + y_1(t) = v^*(\theta_t\omega)$, on which

$$\frac{dx(t)}{dt} = D(\theta_t\omega) (I(\theta_t\omega) - x(t)) - \frac{ax(t)}{m+x(t)} (v^*(\theta_t\omega) - x(t)). \tag{19}$$

For any two solutions $x_1(t)$ and $x_2(t)$ to equation (19), define $\delta(t) = x_1(t) - x_2(t)$. Then $\delta(t)$ satisfies

$$\begin{aligned} \frac{d\delta(t)}{dt} &= -D(\theta_t\omega)\delta(t) - av^*(\theta_t\omega) \left(\frac{x_1}{m+x_1} - \frac{x_2}{m+x_2} \right) + a \left(\frac{x_1^2}{m+x_1} - \frac{x_2^2}{m+x_2} \right) \\ &= -D(\theta_t\omega)\delta(t) - \frac{amv^*(\theta_t\omega)}{(m+x_1)(m+x_2)} \delta(t) + a \frac{m(x_1+x_2) + x_1x_2}{(m+x_1)(m+x_2)} \delta(t). \end{aligned}$$

Since for t large enough, $x_1(t), x_2(t) \leq v^*(\theta_t \omega) \leq i(1 + \varepsilon)^2/(1 - \varepsilon)$, and $v^*(\theta_t \omega) \geq i(1 - \varepsilon)^2/(1 + \varepsilon)$, we have

$$\frac{d\delta(t)}{dt} < -d(1 - \varepsilon)\delta(t) - \frac{ami(1 - \varepsilon)^2/(1 + \varepsilon)}{(m + i(1 + \varepsilon)^2/(1 - \varepsilon))^2}\delta(t) + a\delta(t).$$

Hence $\delta(t) \rightarrow 0$ as $t \rightarrow \infty$ when

$$d(1 - \varepsilon) + \frac{ami(1 - \varepsilon)^2/(1 + \varepsilon)}{(m + i(1 + \varepsilon)^2/(1 - \varepsilon))^2} > a.$$

This always holds if $d(1 - \varepsilon) \geq a$, which is scenario (i) of the theorem. But it can still hold if a is slightly larger. In fact,

$$\frac{mi(1 - \varepsilon)^2/(1 + \varepsilon)}{(m + i(1 + \varepsilon)^2/(1 - \varepsilon))^2} < \frac{mi}{(m + i)^2} < 1.$$

In this case the above estimates with neither $x_1(t)$ or $x_2(t)$ equals $v^*(\theta_t \omega)$, the system is strict uniformly contracting [7, 17] in the positive quadrant and thus has a unique entire solution $v^*(\theta_t \omega)$ as its pullback attractor in the positive quadrant. \square

4. Random chemostat with wall growth

In this section we consider the full random chemostat system with wall growth taken into account. In particular we will study the random dynamical system generated by (4)-(6) with both random nutrient input concentration and random nutrient supplying rate. To obtain more detailed information on the internal structure of the pullback attractor, we make the following change of variables:

$$y(t) = y_1(t) + y_2(t); \quad \gamma(t) = \frac{y_1(t)}{y(t)}. \quad (20)$$

System (4)-(6) then becomes

$$\frac{dx(t)}{dt} = D(\theta_t \omega)(I(\theta_t \omega) - x(t)) - \frac{ax(t)}{m + x(t)}y(t) + b\nu\gamma(t)y(t), \quad (21)$$

$$\frac{dy(t)}{dt} = -\nu y(t) - D(\theta_t \omega)\gamma(t)y(t) + \frac{cx(t)}{m + x(t)}y(t), \quad (22)$$

$$\frac{d\gamma(t)}{dt} = -D(\theta_t \omega)\gamma(t)(1 - \gamma(t)) - r_1\gamma(t) + r_2(1 - \gamma(t)). \quad (23)$$

By definition, $\gamma(t)$ represents the portion of microorganism that attaches to the wall. Noting that the dynamics of $\gamma(t) = \gamma(t; \omega, \gamma_0)$ are uncoupled with $x(t)$ and $y(t)$, we first study the dynamics of $\gamma(t)$.

4.1. Global dynamics of $\gamma(t)$

For any $y_1 > 0$ and $y_2 > 0$, we have $0 < \gamma(t) < 1$ for all $t \in \mathbb{R}$. Moreover, since

$$\left. \frac{d\gamma}{dt} \right|_{\gamma=0} = r_2 > 0; \quad \left. \frac{d\gamma}{dt} \right|_{\gamma=1} = -r_1 < 0,$$

the interval $(0, 1)$ is positively invariant for γ . Therefore equation (23) has a pullback attractor $\mathcal{A}_\gamma = \{A_\gamma(\omega)\}_{\omega \in \Omega}$ [19] with its component subsets given by

$$A_\gamma(\omega) = \bigcap_{t \geq 0} \gamma(t; \theta_{-t}\omega, [0, 1]).$$

These component subsets have the form

$$A_\gamma(\theta_t \omega) = [\gamma_l(\theta_t \omega), \gamma_u(\theta_t \omega)],$$

where $\gamma_l(\theta_t \omega)$ and $\gamma_u(\theta_t \omega)$ are entire bounded solutions of equation (23). The other bounded entire solutions of (23) lie between these two. We next estimate bounds of these entire solutions by using differential inequalities.

On the one hand, since $\gamma(t) \leq 1$ and $D(\theta_t \omega) > 0$, we have

$$\begin{aligned} \gamma'(t) &= D(\theta_t \omega)(\gamma^2(t) - \gamma(t)) - (r_1 + r_2)\gamma(t) + r_2 \\ &\leq -(r_1 + r_2)\gamma(t) + r_2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \gamma'(t) &= D(\theta_t \omega)\gamma^2(t) - (D(\theta_t \omega) + r_1 + r_2)\gamma(t) + r_2 \\ &\geq -(d(1 + \varepsilon) + r_1 + r_2)\gamma(t) + r_2. \end{aligned}$$

Let $\alpha(t)$ and $\beta(t)$ satisfy

$$\alpha(t) = -(d(1 + \varepsilon) + r_1 + r_2)\alpha(t) + r_2, \quad \alpha(0) = \gamma(0), \tag{24}$$

$$\beta(t) = -(r_1 + r_2)\beta(t) + r_2, \quad \beta(0) = \gamma(0). \tag{25}$$

Then $\alpha(t) \leq \gamma(t) \leq \beta(t)$ and $[\gamma_l(\theta_t\omega), \gamma_u(\theta_t\omega)] \subseteq [\alpha^*, \beta^*]$, where

$$\alpha^* = \frac{r_2}{r_1 + r_2 + d(1 + \varepsilon)}, \quad \beta^* = \frac{r_2}{r_1 + r_2} \tag{26}$$

are asymptotically stable steady states for (24) and (25) respectively. In summary,

$$A_\gamma(\omega) = [\gamma_l(\omega), \gamma_u(\omega)] \subseteq [\alpha^*, \beta^*].$$

We provide the sufficient condition for \mathcal{A} to consist of only a single entire solution in the next theorem.

Theorem 2 *The pullback attractor \mathcal{A} associated to the random dynamical system $\gamma(t, \omega, \cdot)$ generated by (23) consists of a single entire solution, denoted by $\gamma^*(\theta_t\omega)$, provided*

$$2r_2d(1 + \varepsilon) < (r_1 + r_2 + d(1 - \varepsilon))(r_1 + r_2).$$

Proof. Equivalently, we need to find conditions under which $\gamma_l(\theta_t\omega) \equiv \gamma_u(\theta_t\omega)$, $t \geq 0$. To this end, we consider the difference between $\gamma_l(\theta_t\omega)$ and $\gamma_u(\theta_t\omega)$, $h(t) := \gamma_u(\theta_t\omega) - \gamma_l(\theta_t\omega)$. Then h satisfies

$$\frac{dh(t)}{dt} = D(\theta_t\omega)(\gamma_l(\theta_t\omega) + \gamma_u(\theta_t\omega))h(t) - (D(\theta_t\omega) + r_1 + r_2)h(t). \tag{27}$$

Solving (27) and replacing ω by $\theta_{-t}\omega$ we obtain

$$\begin{aligned} h(t, \theta_{-t}\omega) &= h_0 e^{\int_{-t}^0 D(\theta_s\omega)(\gamma_l(\theta_s\omega) + \gamma_u(\theta_s\omega)) - (D(\theta_s\omega) + r_1 + r_2) ds} \\ &\leq h_0 e^{\int_{-t}^0 d(1 + \varepsilon) - 2\gamma_u(\theta_s\omega) - (d(1 - \varepsilon) + r_1 + r_2) ds} \\ &\leq e^{\left(\frac{2dr_2(1 + \varepsilon)}{r_1 + r_2} - (r_1 + r_2 + d(1 - \varepsilon))\right)t}. \end{aligned}$$

Therefore when $2r_2d(1 + \varepsilon) < (r_1 + r_2 + d(1 - \varepsilon))(r_1 + r_2)$, $h(t, \theta_{-t}\omega) \rightarrow 0$ as $t \rightarrow \infty$. This implies that the pullback attractor \mathcal{A} for the random dynamical system $\gamma(t, \omega, \cdot)$ has only one single entire solution. \square

Note that this sufficient condition is equivalent to

$$d\varepsilon < \frac{(r_1 + r_2)^2 + d(r_1 - r_2)}{r_1 + 3r_2},$$

which essentially represents the restriction on the magnitude of noise on D . We next study the dynamics of $x(t)$ and $y(t)$.

4.2. Global dynamics of $x(t)$ and $y(t)$

For convenience for readers, we restate equations for $x(t)$ and $y(t)$ here:

$$\begin{aligned} \frac{dx(t)}{dt} &= D(\theta_t\omega)(I(\theta_t\omega) - x(t)) - \frac{ax(t)}{m + x(t)}y(t) + b\nu\gamma(t)y(t), \\ \frac{dy(t)}{dt} &= -\nu y(t) - D(\theta_t\omega)\gamma(t)y(t) + \frac{cx(t)}{m + x(t)}y(t). \end{aligned}$$

According to the dynamics of $\gamma(t)$ discussed in the previous subsection, for t large enough, $\gamma(t)$ satisfies

$$\alpha^* \leq \gamma(t) \leq \beta^*,$$

where α^* and β^* are as stated in (26).

Theorem 3 *Given $a \geq c$, $0 < b < 1$, $\nu > 0$, assume that $D(\theta_t\omega)$ and $I(\theta_t\omega)$ are continuous and essentially bounded, with $d(1 - \varepsilon) \leq D(\theta_t\omega) \leq d(1 + \varepsilon)$ and $i(1 - \varepsilon) \leq I(\theta_t\omega) \leq i(1 + \varepsilon)$. Then, system (21)-(22) has a pullback attractor $\mathcal{A} = \{A(\omega) : \omega \in \Omega\}$ inside the nonnegative quadrant. Moreover, letting*

$$x^*(\omega) = \int_{-\infty}^0 D(\theta_s\omega)I(\theta_s\omega)e^{-\int_s^0 D(\theta_\tau\omega)d\tau} ds, \tag{28}$$

(i) the pullback attractor \mathcal{A} has a singleton component subset $A(\omega) = \{(x^*(\omega), 0)\}$ provided

$$\nu + d(1 - \varepsilon)\alpha^* \geq c,$$

(ii) the pullback attractor \mathcal{A} also contains points strictly inside the positive quadrant in addition to the singleton solution $\{(x^*(\omega), 0)\}$ provided

$$ac^2 di(1 - \varepsilon)^2 > (m(av + ad(1 + \varepsilon) - cbv\beta^*) + acdi(1 - \varepsilon)^2) \cdot (\nu + d(1 + \varepsilon)\beta^*). \quad (29)$$

Proof. Let $z(t) := cx(t) + ay(t)$, then $z(t)$ satisfies

$$\frac{dz(t)}{dt} = cD(\theta_t\omega)l(\theta_t\omega) - D(\theta_t\omega) \cdot cx(t) - \left[\nu + D(\theta_t\omega)\gamma(t) - \frac{cbv}{a}\gamma(t) \right] \cdot ay(t). \quad (30)$$

Note that since $b < 1$, $c \leq a$ and $\gamma(t) \leq \beta^* < 1$, then

$$\nu + D(\theta_t\omega)\gamma(t) - \frac{cbv}{a}\gamma(t) \geq D(\theta_t\omega)\gamma(t).$$

Therefore

$$\begin{aligned} \frac{dz(t)}{dt} &\leq cD(\theta_t\omega)l(\theta_t\omega) - D(\theta_t\omega)\gamma(t)z(t) \\ &\leq cdi(1 + \varepsilon)^2 - d(1 - \varepsilon)\alpha^*z(t), \end{aligned} \quad (31)$$

and

$$z(t) \leq \frac{ci(1 + \varepsilon)^2}{(1 - \varepsilon)\alpha^*} = \frac{ci(1 + \varepsilon)^2(r_1 + r_2 + d(1 + \varepsilon))}{(1 - \varepsilon)r_2}.$$

It follows that for $0 < \varepsilon < 1$, the nonempty compact set

$$K(\omega) := \left\{ (x, y) \in \mathbb{R}_+^2 : 0 \leq cx + ay \leq \frac{ci(1 + \varepsilon)^2(r_1 + r_2 + d(1 + \varepsilon))}{(1 - \varepsilon)r_2} \right\}$$

is positively invariant and absorbing in \mathbb{R}_+^2 . The random dynamical system on \mathbb{R}_+^2 generated by equations (21)-(22) thus has a pullback attractor $\mathcal{A} = \{A(\omega) : \omega \in \Omega\}$, consisting of nonempty compact subsets of \mathbb{R}_+^2 that are contained in $K(\omega)$.

(i) When $\nu + d(1 - \varepsilon)\alpha^* \geq c$, $y(t)$ satisfies

$$\begin{aligned} \frac{dy(t)}{dt} &= - \left(\nu + D(\theta_t\omega)\gamma(t) - \frac{cx(t)}{m + x(t)} \right) y(t) \\ &< - (\nu + d(1 - \varepsilon)\alpha^* - c) y(t) \leq 0. \end{aligned}$$

Thus $y(t)$ decreases to 0 as t approaches ∞ . Consequently, $x(t)$ satisfies

$$\frac{dx(t)}{dt} = D(\theta_t\omega)(l(\theta_t\omega) - x(t)). \quad (32)$$

The solution to (32) is

$$x(t) = x(0)e^{-\int_0^t D(\theta_s\omega)ds} + \int_0^t D(\theta_s\omega)l(\theta_s\omega)e^{-\int_s^t D(\theta_\tau\omega)d\tau} ds.$$

Replacing ω by $\theta_{-t}\omega$, we obtain

$$x(t, \theta_{-t}\omega) = x(0)e^{-\int_{-t}^0 D(\theta_s\omega)ds} + \int_{-t}^0 D(\theta_s\omega)l(\theta_s\omega)e^{-\int_s^0 D(\theta_\tau\omega)d\tau} ds,$$

which converges to $x^*(\omega)$ as $t \rightarrow \infty$. According to the definition of $\gamma(t) = y_1(t)/y(t)$, $y(t)$ cannot take value 0. However, $x(t) = x^*(\omega), y(t) = 0$ corresponds to $x(t) = x^*(\omega), y_1(t) = y_2(t) = 0$ in original system (4)-(6), representing the extinction of microorganisms.

(ii) Again let $z(t) := cx(t) + ay(t)$. On the one hand by (31),

$$\frac{dz(t)}{dt} \leq cdi(1 + \varepsilon)^2 - d(1 - \varepsilon)\alpha^*z(t).$$

On the other hand since $\gamma(t) < 1$ and $\nu - cb\nu\gamma(t)/a > 0$, we derive from (30) that

$$\begin{aligned} \frac{dz(t)}{dt} &\geq cD(\theta_t\omega)I(\theta_t\omega) - D(\theta_t\omega) \cdot cx(t) - \left(\nu + D(\theta_t\omega) - \frac{cb\nu}{a}\gamma(t)\right) \cdot ay(t) \\ &\geq cD(\theta_t\omega)I(\theta_t\omega) - \left(\nu + D(\theta_t\omega) - \frac{cb\nu}{a}\gamma(t)\right) \cdot z(t) \\ &\geq cdi(1 - \varepsilon)^2 - \left(\nu + d(1 + \varepsilon) - \frac{cb\nu}{a}\beta^*\right) z(t). \end{aligned}$$

Denote by

$$l_z := \frac{acdi(1 - \varepsilon)^2}{a\nu + ad(1 + \varepsilon) - cb\nu\beta^*}; \quad u_z := \frac{ci(1 + \varepsilon)^2}{(1 - \varepsilon)\alpha^*},$$

we have the upper and lower bounds for $z(t)$ as

$$l_z \leq z(t) \leq u_z.$$

For $0 < \varepsilon < 1$, define Q_ε to be the quadrilateral

$$Q_\varepsilon := \{(x, y) \in \mathbb{R}_+^2 : x \geq 0, y \geq 0, l_z \leq cx + ay \leq u_z\}.$$

On each of the four edges of Q_ε we have:

$$\begin{aligned} \left. \frac{dx(t)}{dt} \right|_{x=0} &= D(\theta_t\omega)I(\theta_t\omega) + b\nu\gamma(t)y(t) > 0; \\ \left. \frac{dz(t)}{dt} \right|_{z=l_z} &> cdi(1 - \varepsilon)^2 - \left(\nu + d(1 + \varepsilon) - \frac{cb\nu}{a}\beta^*\right) l_z = 0; \\ \left. \frac{dz(t)}{dt} \right|_{z=u_z} &< cdi(1 + \varepsilon)^2 - d(1 - \varepsilon)\alpha^* l_u = 0; \\ \left. \frac{dy(t)}{dt} \right|_{y=0} &= 0. \end{aligned}$$

It shows that $y = 0$ is invariant. But for $y = \varepsilon \ll 1$, $x(t)$ satisfies $l_z/c \leq x(t) \leq u_z/c$. When condition (29) is satisfied, since function $\frac{x}{m+x}$ is increasing with respect to x , we have

$$\begin{aligned} \left. \frac{dy(t)}{dt} \right|_{y=\varepsilon} &= \left(-\nu - D(\theta_t\omega)\gamma(t) + \frac{cx}{m+x}\right) \varepsilon \\ &\geq \left(-\nu - d(1 + \varepsilon)\beta^* + \frac{cl_z}{m+u_z}\right) \varepsilon > 0. \end{aligned}$$

This implies that the pullback attractor \mathcal{A} also contains points strictly inside the positive quadrant in addition to the set $(x^*(\omega), 0)$. □

5. Closing remarks

During the past two decades, the theory of random dynamical system (RDS) has made substantial progress in describing the asymptotic behavior of systems with stochasticity, e.g. [25, 11, 9, 3, 5, 8] and references therein. The basic concept for RDS theory is to view an RDS as consisting of two ingredient – a stochastic but autonomous “noise process”, and a classical dynamical system driven by this process. The RDS theory thus provides an integration of classical ergodic theory with modern dynamical systems, giving a theoretical framework parallel to classical smooth and topological dynamics (stability, attractors, bifurcation theory, etc.), while allowing one to treat in a unified way the most important classes of dynamical systems with randomness – random (or stochastic) differential (or partial differential) equations [20].

Our motivation for studying chemostat models with random inputs arises from the need to provide foundations for a constructive theory of competition models exposed to external random forces. The resulting system is a system of coupled random ordinary differential equations with randomly varying input parameters, and generates a random dynamical system. Comparing to existing literatures on chemostat models, our major contributions include (1) varying randomly both the supply rate and the input concentration of the microorganism, and (2) taking into account the tendency of microorganisms to attach onto the wall.

In summary, we proved the existence of a unique random attractor to the random chemostat models. In addition, we provided geological/biological/statistical insight of the random attractor by constructing sufficient conditions for extinction and persistence

of the microorganism. More precisely, we provided conditions on the parameters under which the random attractor contains only one “stochastic equilibrium”, or contains other points strictly inside the positive quadrant besides the “stochastic equilibrium”. The stochastic equilibrium here refers to a singleton set which is essentially a stochastic process, and can be regarded as the analogue of a steady state solution in deterministic systems.

Finally, we would like to mention that our objective has been to illustrate our results here in relatively basic chemostat models to place emphasis on the new features that arise due to the noise. These will also be present in more complicated and maybe more realistic models, but perhaps not so transparent.

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