TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 4, pp. 1465-1477, August 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

CHEN INEQUALITIES FOR SUBMANIFOLDS OF REAL SPACE FORMS WITH A SEMI-SYMMETRIC METRIC CONNECTION

Adela Mihai and Cihan Özgür

Abstract. In this paper we prove Chen inequalities for submanifolds of real space forms endowed with a semi-symmetric metric connection, i.e., relations between the mean curvature associated with the semi-symmetric metric connection, scalar and sectional curvatures, Ricci curvatures and the sectional curvature of the ambient space. The equality cases are considered.

1. Introduction

In [9], H.A. Hayden introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. K. Yano studied in [16] some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. In [10] and [11], T. Imai found some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection. Z. Nakao [14] studied submanifolds of a Riemannian manifold with semi-symmetric connections.

On the other hand, one of the basic problems in submanifold theory is to find simple relationships between the extrinsic and intrinsic invariants of a submanifold. B. Y. Chen [4, 5, 8] established inequalities in this respect, well-known as *Chen inequalities*.

Afterwards, many geometers studied similar problems for different submanifolds in various ambient spaces, for example see [1-3, 12, 13], and [15].

2. Preliminaries

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold and $\widetilde{\widetilde{\nabla}}$ a linear connection on N^{n+p}. If the torsion tensor $\widetilde{T}$ of $\widetilde{\widetilde{\nabla}}$, defined by

Received August 2, 2008, accepted October 13, 2008.
Communicated by Bang-Yen Chen.
2000 *Mathematics Subject Classification*: 53C40, 53B05, 53B15.
Key words and phrases: Real space form, Semi-symmetric metric connection, Ricci curvature.

$$\widetilde{T}\left(\widetilde{X},\widetilde{Y}\right)=\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}\ -\ \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X}\ -\ [\widetilde{X},\widetilde{Y}],$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, satisfies

$$\widetilde{T}\left(\widetilde{X},\widetilde{Y}\right)=\phi(\widetilde{Y})\widetilde{X}-\phi(\widetilde{X})\widetilde{Y}$$

for a 1-form ϕ, then the connection $\widetilde{\nabla}$ is called a *semi-symmetric connection*.

Let g be a Riemannian metric on N^{n+p}. If $\widetilde{\nabla}g=0$, then $\widetilde{\nabla}$ is called a *semi-symmetric metric connection* on N^{n+p}.

Following [16], a semisymmetric metric connection $\widetilde{\nabla}$ on N^{n+p} is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}=\overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y}+\phi(\widetilde{Y})\widetilde{X}-g(\widetilde{X},\widetilde{Y})P,$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric g and P is a vector field defined by $g(P,\widetilde{X})=\phi(\widetilde{X})$, for any vector field $\widetilde{X}$.

We will consider a Riemannian manifold N^{n+p} endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection denoted by $\overset{\circ}{\widetilde{\nabla}}$.

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. On the submanifold M^n we consider the induced semi-symmetric metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\overset{\circ}{\nabla}$.

Let $\widetilde{R}$ be the curvature tensor of N^{n+p} with respect to $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{R}}$ the curvature tensor of N^{n+p} with respect to $\overset{\circ}{\widetilde{\nabla}}$. We also denote by R and $\overset{\circ}{R}$ the curvature tensors of ∇ and $\overset{\circ}{\nabla}$, respectively, on M^n.

The Gauss formulas with respect to ∇, respectively $\overset{\circ}{\nabla}$, can be written as:

$$\widetilde{\nabla}_XY=\nabla_XY+h(X,Y),\quad X,Y\in\chi(M^n),$$
$$\overset{\circ}{\widetilde{\nabla}}_XY=\overset{\circ}{\nabla}_XY+\overset{\circ}{h}(X,Y),\quad X,Y\in\chi(M^n),$$

where $\overset{\circ}{h}$ is the second fundamental form of M^n in N^{n+p} and h is a $(0,2)$-tensor on M^n. According to the formula (7) from [14] h is also symmetric.

One denotes by $\overset{\circ}{H}$ the mean curvature vector of M^n in N^{n+p}.

Let $N^{n+p}(c)$ be a real space form of constant sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$.

The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ on $N^{n+p}(c)$ is expressed by

$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W)=c\left\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\right\}. \tag{2.1}$$

Then the curvature tensor $\widetilde{R}$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ on $N^{n+p}(c)$ can be written as [11]

$$\widetilde{R}(X,Y,Z,W) = \overset{\circ}{\widetilde{R}}(X,Y,Z,W) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z), \tag{2.2}$$

for any vector fields $X, Y, Z, W \in \chi(M^n)$, where α is a $(0,2)$-tensor field defined by

$$\alpha(X,Y) = \left(\overset{\circ}{\widetilde{\nabla}}_X \phi\right) Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(P)g(X,Y), \quad \forall X, Y \in \chi(M^n).$$

From (2.1) and (2.2) it follows that the curvature tensor $\widetilde{R}$ can be expressed as

$$\widetilde{R}(X,Y,Z,W) = c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z). \tag{2.3}$$

Denote by λ the trace of α.

The Gauss equation for the submanifold M^n into the real space form $N^{n+p}(c)$ is

$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = \overset{\circ}{R}(X,Y,Z,W) + g(\overset{\circ}{h}(X,Z), \overset{\circ}{h}(Y,W)) - g(\overset{\circ}{h}(X,W), \overset{\circ}{h}(Y,Z)). \tag{2.4}$$

Let $\pi \subset T_x M^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric metric connection ∇. For any orthonormal basis $\{e_1, ..., e_m\}$ of the tangent space $T_x M^n$, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

We recall the following algebraic Lemma:

Lemma 2.1. [4]. *Let $a_1, a_2, ..., a_n, b$ be $(n+1)$ $(n \ge 2)$ real numbers such that*

$$\left(\sum_{i=1}^n a_i\right)^2 = (n-1)\left(\sum_{i=1}^n a_i^2 + b\right).$$

Then $2a_1 a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3 = ... = a_n$.

Let M^n be an n-dimensional Riemannian manifold, L a k-plane section of T_xM^n, $x \in M^n$, and X a unit vector in L.

We choose an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$.

One defines [6] the *Ricci curvature* (or *k-Ricci curvature*) of L at X by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j. For each integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k on M^n is defined by:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad x \in M^n,$$

where L runs over all k-plane sections in T_xM^n and X runs over all unit vectors in L.

3. Chen First Inequality

Recall that the *Chen first invariant* is given by

$$\delta_M(x) = \tau(x) - \inf \{K(\pi) \mid \pi \subset T_xM^n, x \in M^n, \dim \pi = 2\},$$

(see for example [8]), where M^n is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M^n associated with a 2-plane section, $\pi \subset T_xM^n, x \in M^n$ and τ is the scalar curvature at x.

For submanifolds of real space forms endowed with a semi-symmetric metric connection we establish the following optimal inequality, which will call *Chen first inequality*:

Theorem 3.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. We have:*

$$\tau(x) - K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c}{2} - \lambda\right] - trace\left(\alpha_{|_{\pi^\perp}}\right), \tag{3.1}$$

where π is a 2-plane section of $T_xM^n, x \in M^n$.

Proof. From [14], the Gauss equation with respect to the semi-symmetric metric connection is

$$\begin{aligned} \widetilde{R}(X, Y, Z, W) = {} & R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) \\ & - g(h(Y, Z), h(X, W)). \end{aligned} \tag{3.2}$$

Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and $\{e_{n+1}, ..., e_{n+p}\}$ be orthonormal basis of T_xM^n and $T_x^{\perp}M^n$, respectively. For $X = W = e_i, Y = Z = e_j$, $i \neq j$, from the equation (2.3) it follows that:

$$\tilde{R}(e_i, e_j, e_j, e_i) = c - \alpha(e_i, e_i) - \alpha(e_j, e_j). \tag{3.3}$$

From (3.2) and (3.3) we get

$$c - \alpha(e_i, e_i) - \alpha(e_j, e_j) = R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)).$$

By summation over $1 \leq i, j \leq n$, it follows from the previous relation that

$$2\tau + \|h\|^2 - n^2 \|H\|^2 = -2(n-1)\lambda + (n^2 - n)c, \tag{3.4}$$

where we recall that λ is the trace of α and denote by

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)),$$
$$H = \frac{1}{n}\text{trace}h.$$

One takes

$$\varepsilon = 2\tau - \frac{n^2(n-2)}{n-1} \|H\|^2 + 2(n-1)\lambda - (n^2 - n)c. \tag{3.5}$$

Then, from (3.4) and (3.5) we get

$$n^2 \|H\|^2 = (n-1) \left(\|h\|^2 + \varepsilon \right). \tag{3.6}$$

Let $x \in M^n$, $\pi \subset T_xM^n$, $\dim \pi = 2$, $\pi = sp\{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$ and from the relation (3.6) we obtain:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 = (n-1) \left[\sum_{i,j=1}^{n} \sum_{r=n+1}^{n+p} (h_{ij}^r)^2 + \varepsilon \right],$$

or equivalently,

$$\begin{aligned} &\left(\sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 \\ &= (n-1) \left\{ \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{n+p} (h_{ij}^r)^2 + \varepsilon \right\}. \end{aligned} \tag{3.7}$$

By using Lemma 2.1 we have from (3.7):

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^r)^2 + \varepsilon. \tag{3.8}$$

The Gauss equation for $X = W = e_1$, $Y = Z = e_2$ gives

$$\begin{aligned}
K(\pi) &= R(e_1,e_2,e_2,e_1) = c - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \sum_{r=n+1}^{p}[h_{11}^r h_{22}^r - (h_{12}^r)^2] \\
&\geq c - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \frac{1}{2}[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^r)^2 + \varepsilon] \\
&\quad + \sum_{r=n+2}^{n+p} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+p}(h_{12}^r)^2 = c - \alpha(e_1,e_1) - \alpha(e_2,e_2) \\
&\quad + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^r)^2 + \frac{1}{2}\varepsilon + \sum_{r=n+2}^{n+p} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+p}(h_{12}^r)^2 \\
&= c - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{n+p}\sum_{i,j>2}(h_{ij}^r)^2 \\
&\quad + \frac{1}{2}\sum_{r=n+2}^{n+p}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{1}{2}\varepsilon \\
&\geq c - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \frac{\varepsilon}{2},
\end{aligned}$$

which implies

$$K(\pi) \geq c - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \frac{\varepsilon}{2}.$$

We remark that

$$\alpha(e_1,e_1) + \alpha(e_2,e_2) = \lambda - \text{trace}\left(\alpha_{|_{\pi^\perp}}\right).$$

Using (3.5) we get

$$K(\pi) \geq \tau + (n-2)\left[-\frac{n^2}{2(n-1)}\|H\|^2 - (n+1)\frac{c}{2} + \lambda\right] + \text{trace}\left(\alpha_{|_{\pi^\perp}}\right),$$

which represents the inequality to prove. ∎

Recall the following important result (Proposition 1.2) from [10].

Proposition 3.2. *The mean curvature H of M^n with respect to the semi-symmetric metric connection coincides with the mean curvature $\overset{\circ}{H}$ of M^n with respect to the Levi-Civita connection if and only if the vector field P is tangent to M^n.*

Remark 3.3. According to the formula (7) from [14] it follows that $h = \overset{\circ}{h}$ if P is tangent to M^n.

In this case inequality (3.1) becomes

Corollary 3.4. *Under the same assumptions as in the Theorem 3.1, if the vector field P is tangent to M^n then we have*

$$\tau(x)-K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\left\|\overset{\circ}{H}\right\|^2 + (n+1)\frac{c}{2} - \lambda\right] - trace\left(\alpha_{|_{\pi^\perp}}\right). \tag{3.9}$$

Theorem 3.5. *If the vector field P is tangent to M^n, then the equality case of inequality (3.1) holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of T_xM^n and an orthonormal basis $\{e_{n+1}, ..., e_{n+p}\}$ of $T_x^\perp M^n$ such that the shape operators of M^n in $N^{n+p}(c)$ at x have the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a+b=\mu,$$

$$A_{e_{n+i}} = \begin{pmatrix} h_{11}^{n+i} & h_{12}^{n+i} & 0 & \cdots & 0 \\ h_{12}^{n+i} & -h_{11}^{n+i} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 2 \leq i \leq p,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n+1 \leq r \leq n+p$.

Proof. The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have the equality in the Lemma.

$$h_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$h_{ij}^{r}=0,\quad \forall i\neq j, i,j>2, r=n+1,...,n+p,$$
$$h_{11}^{r}+h_{22}^{r}=0,\quad \forall r=n+2,...,n+p,$$
$$h_{1j}^{n+1}=h_{2j}^{n+1}=0,\quad \forall j>2,$$
$$h_{11}^{n+1}+h_{22}^{n+1}=h_{33}^{n+1}=...=h_{nn}^{n+1}.$$

We may choose $\{e_1,e_2\}$ such that $h_{12}^{n+1}=0$ and we denote by $a=h_{11}^{r}, b=h_{22}^{r}, \mu=h_{33}^{n+1}=...=h_{nn}^{n+1}$.

It follows that the shape operators take the desired forms. ∎

4. Ricci Curvature in the Direction of a Unit Tangent Vector

In this section, we establish a sharp relation between the Ricci curvature in the direction of a unit tangent vector X and the mean curvature H with respect to the semi-symmetric metric connection $\widetilde{\nabla}$.

Denote by

$$N(x)=\{X\in T_xM^n \mid h(X,Y)=0,\quad \forall Y\in T_xM^n\}.$$

Theorem 4.1. *Let $M^n, n\geq 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then:*

(i) *For each unit vector X in T_xM we have*

$$\|H\|^2\geq \frac{4}{n^2}\left[\mathrm{Ric}(X)-(n-1)\,c+(2n-3)\lambda-(n-2)\alpha(X,X)\right]. \tag{4.1}$$

(ii) *If $H(x)=0$, then a unit tangent vector X at x satisfies the equality case of (4.1) if and only if $X\in N(x)$.*

Proof. (i) Let $X\in T_xM^n$ be a unit tangent vector at x. We choose an orthonormal basis $e_1,e_2,...,e_n,e_{n+1},...e_{n+p}$ such that $e_1,e_2,...,e_n$ are tangent to M^n at x, with $e_1=X$.

From (3.4) we obtain

$$\begin{aligned} n^2\|H\|^2 = {} & 2\tau+\frac{1}{2}\sum_{r=n+1}^{n+p}\left[(h_{11}^{r}+...+h_{nn}^{r})^2+(h_{11}^{r}-h_{22}^{r}-...-h_{nn}^{r})^2\right] \\ & +2\sum_{r=n+1}^{n+p}\sum_{1\leq i<j\leq n}\left(h_{ij}^{r}\right)^2-2\sum_{r=n+1}^{n+p}\sum_{2\leq i<j\leq n}\left(h_{ii}^{r}h_{jj}^{r}\right) \\ & +2(n-1)\lambda-(n^2-n)c. \end{aligned} \tag{4.2}$$

From Gauss equation (3.2) and the formula (3.3), for $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, we get

$$\begin{aligned} K_{ij} &= \widetilde{R}(e_i, e_j, e_j, e_i) + g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_i, e_j)) \\ &= c - \alpha(e_i, e_i) - \alpha(e_j, e_j) + \sum_{r=n+1}^{n+p} \left[h_{ii}^r h_{jj}^r - \left(h_{ij}^r\right)^2 \right]. \end{aligned}$$

By summation, one obtains

$$\begin{aligned} \sum_{2\leq i<j\leq n} K_{ij} &= \sum_{r=n+1}^{n+p} \sum_{2\leq i<j\leq n} \left[h_{ii}^r h_{jj}^r - \left(h_{ij}^r\right)^2 \right] \\ &\quad + \sum_{2\leq i<j\leq n} \left(c - \alpha(e_i, e_i) - \alpha(e_j, e_j)\right) \\ &= \sum_{r=n+1}^{n+p} \sum_{2\leq i<j\leq n} \left[h_{ii}^r h_{jj}^r - \left(h_{ij}^r\right)^2 \right] \\ &\quad + \frac{(n-2)(n-1)}{2} c - (n-2)\left[\lambda - \alpha(e_1, e_1)\right]. \end{aligned} \tag{4.3}$$

After substituting (4.3) into (4.2) we find

$$n^2 \|H\|^2 \geq \frac{1}{2} n^2 \|H\|^2 + 2\left(\tau - \sum_{2\leq i<j\leq n} K_{ij}\right) + 2 \sum_{r=n+1}^{n+p} \sum_{j=2}^{n} \left(h_{1j}^r\right)^2$$

$$-2(n-1)\,c + 2(2n-3)\lambda - 2(n-2)\alpha(e_1, e_1),$$

which gives us

$$\frac{1}{2} n^2 \|H\|^2 \geq 2\,\mathrm{Ric}(X) - 2(n-1)\,c + 2(2n-3)\lambda - 2(n-2)\alpha(X, X).$$

This proves the inequality (4.1).

(ii) Assume $H(x) = 0$. Equality holds in (4.1) if and only if

$$h_{12}^r = ... = h_{1n}^r = 0,$$

$$h_{11}^r = h_{22}^r + ... + h_{nn}^r, \quad r \in \{n+1, ..., n+p\}.$$

Then $h_{1j}^r = 0$, $\forall j \in \{1, ..., n\}$, $r \in \{n+1, ..., n+p\}$, i.e. $X \in N(x)$. ∎

Corollary 4.2. *If the vector field P is tangent to M^n, then the equality case of inequality (4.1) holds identically for all unit tangent vectors at x if and only if either x is a totally geodesic point, or $n = 2$ and x is a totally umbilical point.*

Proof. The equality case of (4.1) holds for all unit tangent vectors at x if and only if

$$h^r_{ij} = 0, \quad i \neq j, \quad r \in \{n+1, ..., n+p\},$$

$$h^r_{11} + ... + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1, ..., n\}, \quad r \in \{n+1, ..., n+p\}.$$

We distinguish two cases:

(a) $n \neq 2$, then x is a totally geodesic point;

(b) $n = 2$, it follows that x is a totally umbilical point.

The converse is trivial. ■

5. k-Ricci Curvature

We first state a relationship between the sectional curvature of a submanifold M^n of a real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the associated squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the k-Ricci curvature of M^n (intrinsic invariant) and the squared mean curvature $\|H\|^2$ (extrinsic invariant).

In this section we suppose that the vector field P is tangent to M^n.

Theorem 5.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field P is tangent to M^n. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - c + \frac{2}{n}\lambda. \tag{5.1}$$

Proof. Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and orthonormal basis of T_xM^n. The relation (3.4) is equivalent with

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 + 2(n-1)\lambda - n(n-1)c. \tag{5.2}$$

We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{n+p}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and $e_1, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \tag{5.3}$$

(5.4) $$A_{e_r} = (h^r_{ij}),\ i,j = 1,...,n; r = n+2,...,n+p, \text{trace}\, A_r = 0.$$

From (5.2), we get

(5.5) $$\begin{aligned} n^2 \|H\|^2 &= 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{n+p} \sum_{i,j=1}^{n} (h^r_{ij})^2 \\ &\quad +2(n-1)\lambda - n(n-1)c. \end{aligned}$$

On the other hand, since

$$0 \le \sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_i a_i^2 - 2 \sum_{i<j} a_i a_j,$$

we obtain

(5.6) $$n^2 \|H\|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i<j} a_i a_j \le n \sum_{i=1}^{n} a_i^2,$$

which implies

$$\sum_{i=1}^{n} a_i^2 \ge n \|H\|^2 .$$

We have from (5.5)

(5.7) $$n^2 \|H\|^2 \ge 2\tau + n \|H\|^2 + 2(n-1)\lambda - n(n-1)c$$

or, equivalently,

$$\|H\|^2 \ge \frac{2\tau}{n(n-1)} - c + \frac{2}{n}\lambda. \qquad \blacksquare$$

Using Theorem 5.1, we obtain the following

Theorem 5.2. *Let $M^n, n \ge 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$, such that the vector field P is tangent to M^n. Then, for any integer k, $2 \le k \le n$, and any point $x \in M^n$, we have*

(5.8) $$\|H\|^2 (p) \ge \Theta_k(p) - c + \frac{2}{n}\lambda.$$

Proof. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_xM^n. Denote by $L_{i_1...i_k}$ the k-plane section spanned by $e_{i_1}, ..., e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1...i_k}) = \frac{1}{2} \sum_{i\in\{i_1,...,i_k\}} Ric_{L_{i_1\cdots i_k}}(e_i),$$
$$\tau(x) = \frac{1}{C_{n-2}^{k-2}} \sum_{1\leq i_1<...<i_k\leq n} \tau(L_{i_1...i_k}).$$

From (5.1) and the above relations, one derives

$$\tau(x) \geq \frac{n(n-1)}{2}\Theta_k(p),$$

which implies (5.8). ∎

Acknowledgments

This paper was prepared during the visit of the first author to Balikesir University, Turkey, in July-August 2008. The first author was supported by the Scientific and Technical Research Council of Turkey (TÜBITAK) for Advanced Fellowships Programme.

References

1. K. Arslan, R. Ezentaş, I. Mihai, C. Murathan, C. Özgür and B. Y. Chen, inequalities for submanifolds in locally conformal almost cosymplectic manifolds, *Bull. Inst. Math., Acad. Sin.*, **29** (2001), 231-242.
2. K. Arslan, R. Ezentaş, I. Mihai, C. Murathan and C. Özgür, Certain inequalities for submanifolds in (k, μ)-contact space forms, *Bull. Aust. Math. Soc.*, **64** (2001), 201-212.
3. K. Arslan, R. Ezentaş, I. Mihai, C. Murathan and C. Özgür, Ricci curvature of submanifolds in locally conformal almost cosymplectic manifolds, *Math. J. Toyama Univ.*, **26** (2003), 13-24.
4. B. Y. Chen, Some pinching and classification theorems for minimal submanifolds, *Arch. Math.* (*Basel*), **60(6)** (1993), 568-578.
5. B. Y. Chen, *Strings of Riemannian invariants, inequalities, ideal immersions and their applications*, The Third Pacific Rim Geometry Conference (Seoul, 1996), 7-60, Monogr. Geom. Topology, 25, Int. Press, Cambridge, MA, 1998.
6. B. Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions, *Glasg. Math. J.*, **41(1)** (1999), 33-41.
7. B. Y. Chen, Some new obstructions to minimal and Lagrangian isometric immersions, *Japanese J. Math.*, **26** (2000), 105-127.

8. B. Y. Chen, *δ-invariants, Inequalities of Submanifolds and Their Applications, in Topics in Differential Geometry*, Eds. A. Mihai, I. Mihai, R. Miron, Editura Academiei Romane, Bucuresti, 2008, pp. 29-156.

9. H. A. Hayden, Subspaces of a space with torsion, *Proc. London Math. Soc.*, **34** (1932), 27-50.

10. T. Imai, Hypersurfaces of a Riemannian manifold with semi-symmetric metric connection, *Tensor* (*N.S.*), **23** (1972), 300-306.

11. T. Imai, Notes on semi-symmetric metric connections, Vol. I. *Tensor* (*N.S.*), **24** (1972), 29-296.

12. K. Matsumoto, I. Mihai and A. Oiaga, Ricci curvature of submanifolds in complex space forms, *Rev. Roumaine Math. Pures Appl.*, **46(6)** (2001), 775-782.

13. A. Mihai, *Modern Topics in Submanifold Theory*, Editura Universităţii Bucureşti, Bucharest, 2006.

14. Z. Nakao, Submanifolds of a Riemannian manifold with semisymmetric metric connections, *Proc. Amer. Math. Soc.*, **54** (1976), 261-266.

15. A. Oiaga and I. Mihai, B. Y. Chen inequalities for slant submanifolds in complex space forms, *Demonstratio Math.*, **32(4)** (1999), 835-846.

16. K. Yano, On semi-symmetric metric connection, *Rev. Roumaine Math. Pures Appl.*, **15** (1970), 1579-1586.

Adela Mihai
University of Bucharest,
Faculty of Mathematics,
Academiei 14,
010014 Bucharest,
Romania
E-mail: adela_mihai@fmi.unibuc.ro

Cihan Özgür
University of Balikesir,
Department of Mathematics,
10145, Cagis, Balikesir,
Turkey
E-mail: cozgur@balikesir.edu.tr