1-1-2012

Chen's conjecture and e-superbiharmonic submanifolds of Riemannian manifolds

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Abstract

B.-Y. Chen famously conjectured that every submanifold of Eu- clidean space with harmonic mean curvature vector is minimal. In this note we prove a much more general statement for a large class of submanifolds sat- isfying a growth condition at innity. We discuss in particular two popular competing natural interpretations of the conjecture when the Euclidean back- ground space is replaced by an arbitrary Riemannian manifold. Introducing the notion of "-superbiharmonic submanifolds, which contains each of the pre- vious notions as special cases, we prove that "-superbiharmonic submanifolds of a complete Riemannian manifold which satisfy a growth condition at innity are minimal.

Keywords

chen, conjecture, superbiharmonic, submanifolds, riemannian, manifolds, e

Disciplines

Engineering | Science and Technology Studies

Publication Details

Wheeler, G. (2012). Chen's conjecture and e-superbiharmonic submanifolds of Riemannian manifolds.

CHEN'S CONJECTURE AND ε-SUPERBIHARMONIC SUBMANIFOLDS OF RIEMANNIAN MANIFOLDS

GLEN WHEELER

Abstract. B.-Y. Chen famously conjectured that every submanifold of Euclidean space with harmonic mean curvature vector is minimal. In this note we prove a much more general statement for a large class of submanifolds satisfying a growth condition at infinity. We discuss in particular two popular competing natural interpretations of the conjecture when the Euclidean background space is replaced by an arbitrary Riemannian manifold. Introducing the notion of *ε-superbiharmonic submanifolds*, which contains each of the previous notions as special cases, we prove that ε-superbiharmonic submanifolds of a complete Riemannian manifold which satisfy a growth condition at infinity are minimal.

1. Introduction

Suppose M^m is a submanifold of a Riemannian manifold $(N^{m+n}, \langle\cdot,\cdot\rangle)$ immersed via $f : M^m \to N^{n+m}$ and equipped with the Riemannian metric induced via f. Throughout we assume that all manifolds and mappings are proper and locally smooth. Letting Δ denote the rough Laplacian, our goal is to determine sufficient conditions for the validity of:

Chen's Conjecture (B.-Y. Chen [2]). *Suppose $f : M^m \to \mathbb{R}^{n+m}$ satisfies*

$$\Delta\vec{H} \equiv 0. \tag{1}$$

Then $\vec{H} \equiv 0$.

One fundamental difficulty is that the conjecture is a local question. It appears at this time that understanding the local structure of submanifolds satisfying (1) to the point of determining their minimality is an intractable task. In fact, the conjecture continues to remain open with very little progress for hypersurfaces (where the normal bundle is trivial) with intrinsic dimension greater than 4 of Euclidean space. Nevertheless, the study of the conjecture is quite active, with many partial results. In [3, 7] Chen's conjecture is established for $m = 2$ and $n = 1$, i.e., surfaces lying in $\mathbb{R}^3$. Dimitrić [4] proved the conjecture for $m = 1$ and n arbitrary (curves in $\mathbb{R}^n$). He also proved that if f is additionally pseudo-umbilic then the conjecture holds for $m \neq 4$, and n, as well as that if f possesses at most two distinct principal curvatures and $n = m+1$ (hypersurfaces in $\mathbb{R}^{n+1}$). The case $m = 3$, $n = 4$ is proven in [6].

If $N^{n+m} = \mathbb{R}^{n+m}$ then submanifolds satisfying (1) are also critical points of the bi-energy $E(f) = \int_M |\tau(f)|^2 d\mu$, where $\tau(f)$ is the tension field of f. If the ambient space is curved however, then critical points of $E(f)$ satisfy instead

$$\Delta\vec{H} = R^N(e_i, \vec{H})e_i\,, \tag{2}$$

where R^N is the curvature tensor of N^{n+m} and $\{e_i\}$ is a local orthonormal frame of M. Equation (2) is of course the condition for $f : M^m \to N^{n+m}$ to be an intrinsic biharmonic map. If N has positive sectional curvature, then there are many well-known examples of non-minimal solutions of (2). The so-called *generalised Chen's conjecture* [1] states that if N has everywhere non-positive sectional curvature, then all solutions of (2) are minimal. Although this turned out to be false [9], it remains interesting to determine sufficient conditions which guarantee that solutions of (2) are minimal. A survey of results in this direction can be found in [8].

Now despite Chen's Conjecture being stated for submanifolds of Euclidean space, clearly equation (1) continues to make sense when the ambient space N^{n+m} has some curvature. It is thus not particularly clear which is the 'correct' generalisation of the conjecture for submanifolds of a Riemannian manifold N^{n+m}. Given that there are many non-harmonic biharmonic maps, we find it appropriate (as in [10] for example) to also investigate the minimality of solutions of the original equation (1) given by Chen, including considering the case where the ambient space N^{m+n} is curved.

One concept which generalises both (2) in the setting of a negatively curved background space and the biharmonicity condition (1) is

$$\left\langle \Delta\vec{H}, \vec{H} \right\rangle \ge (\varepsilon - 1)|\nabla H|^2\,, \tag{3}$$

for $\varepsilon \in [0, 1]$. For $\varepsilon = 0$ this implies $\Delta|\vec{H}|^2 \ge 0$, and so (somewhat respecting standard convention, see Duffin [5] for example), we term solutions f of (3) *ε-superbiharmonic submanifolds* or say that the mean curvature vector field is *ε-superharmonic.*

If f is compact, then Chen's Conjecture is simple to prove. This is a consequence of the argument used by Jiang [7]. An alternative method to obtain this result is as follows. Compute

$$0 = \int_M \left\langle \vec{H}, \Delta\vec{H} \right\rangle d\mu = -\int_M |\nabla\vec{H}|^2 d\mu$$

so that $\nabla\vec{H} \equiv 0$. Now this implies

$$0 = \int_M \Delta^2 |f|^2 d\mu = 2\int_M \Delta\Big(\left\langle \vec{H}, f \right\rangle + m\Big)\, d\mu = 2\int_M |\vec{H}|^2 d\mu$$

and so we conclude $\vec{H} \equiv 0$ and f is minimal. This simple argument does not require any special conditions on the ambient space N^{n+m}, and is readily generalised to the setting of non-compact solutions of (3).

Theorem 1. *Let N^{n+m} be a complete Riemannian manifold. Suppose $f : M^m \to N^{n+m}$ is ε-superbiharmonic in the sense that it satisfies* (3) *for $\varepsilon \ge 0$. Assume in addition that f satisfies the growth condition*

$$\liminf_{\rho\to\infty} \frac{1}{\rho^2} \int_{f^{-1}(B_\rho)} |\vec{H}|^2 d\mu = 0\,. \tag{4}$$

Then $\vec{H} \equiv 0$ and f is minimal.

Remark. Clearly $\mathbb{R}^{n+m}$ is complete and so the theorem applies in the setting of Chen's Conjecture. This resolves the conjecture up to the growth condition (4).

Remark. One must be careful in interpreting the growth condition in the case where N is closed. In this setting, the inverse image of the geodesic balls $f^{-1}(B_\rho)$ (geodesic in N^{n+m}) will cover M^m infinitely many times. The growth condition (4) thus becomes more restrictive than it first appears (although not completely vacuous).

2. Setting

In this section we briefly set our notation and conventions. We have as our main object of study an immersion $f : M^m \to (N^{n+m}, \langle\cdot,\cdot\rangle)$ and consider the m-dimensional Riemannian submanifold (M^m, g) with the metric g induced by f, that is, given a local frame $\tau_1, \ldots, \tau_m$ of the tangent bundle define the induced metric and associated induced volume form by

$$g_{ij} = \langle \partial_i f, \partial_j f\rangle \qquad d\mu = \sqrt{\det\, g_{ij}}\, d\mathcal{L}^m\,,$$

where $d\mathcal{L}^m$ denotes m-dimensional Lebesgue (or Hausdorff) measure. Associated with M^m is its (vector valued) second fundamental form, given by

$$A_{ij} = \left(D_i \partial_j f\right)^{\perp},$$

where D is the covariant derivative with respect to the Levi-Civita connection on N, and $(\cdot)^\perp$ is the projection onto the normal bundle $(TM)^\perp$, which is given by

$$X^\perp = X - X^\perp = X - g^{ij}\,\langle X, \partial_i f\rangle\, \partial_j f$$

for a vector field $X : M^m \to N^{m+n}$. The trace of A under the metric g is the mean curvature vector,

$$\vec{H} = g^{ij} A_{ij}.$$

The Levi-Civita connection ∇ for g is the unique metric connection on M with coefficients given in local coordinates by

$$\nabla_{\tau_i}\tau_j = \Gamma^k_{ij}\tau_k, \quad \text{where} \quad \Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

Tracing $\nabla\nabla = \nabla_{(2)}$ over g gives the metric or rough laplacian Δ, and (p,q)-tensor fields T over M^m are termed *harmonic* if $\Delta T \equiv 0$ and *biharmonic* if

$$\Delta^2 T := \Delta\Delta T \equiv 0$$

on M^m. In particular, the immersion $f : M^m \to N^{m+n}$ is itself called *biharmonic* if $\Delta^2 f \equiv 0$ (cf. (1)). In this case we term M^m (and $f(M^m)$) a *biharmonic submanifold* of N^{m+n}. As the mean curvature vector satisfies the relation $\vec{H} = \Delta f$, M^m is a biharmonic submanifold if and only if the mean curvature vector is harmonic.

3. A local-global integral estimate

Throughout the remainder of the paper we shall use the abbreviations $M := M^m$ and $N := N^{m+n}$. The ambient space N will always be assumed to be complete. We first prove the following lemma.

Lemma 2. *Suppose that $f : M \to N$ is an ε-superbiharmonic submanifold for some $\varepsilon > 0$ and*

$$\liminf_{\rho\to\infty} \frac{1}{\rho^2} \int_{f^{-1}(B_\rho)} |\vec{H}|^2 d\mu = 0\,. \tag{5}$$

Then $\nabla\vec{H} \equiv 0$.

Proof. Suppose $\tilde{\eta} : N \to \mathbb{R}$ is a smooth cutoff function on a geodesic ball B_ρ (centred anywhere) in N. Clearly we can choose $\tilde{\eta} \in C^1_c(N)$ such that $\tilde{\eta}(q) = 1$ for $q \in B_\rho$, $\tilde{\eta}(q) = 0$ for $q \notin B_{2\rho}$, $\tilde{\eta}(q) \in [0,1]$ for all q, and $|D\tilde{\eta}| \le \frac{c}{\rho}$ for some $c < \infty$. Define $\eta(p) = (\tilde{\eta} \circ f)(p)$ for $p \in M$.

Let us use ∇^* to denote the divergence operator on M with respect to ∇. Integrating $\nabla^*\Big(\Big\langle \vec{H}, \nabla\vec{H}\Big\rangle \eta^2\Big)$ and using the divergence theorem we have

$$\int_M \Big\langle \vec{H}, \Delta\vec{H}\Big\rangle \eta^2\,d\mu + \int_M |\nabla\vec{H}|^2\eta^2\,d\mu + s\int_M \Big\langle\Big\langle \nabla\vec{H}, \vec{H}\Big\rangle, \nabla\eta\Big\rangle \eta\,d\mu = 0\,,$$

which implies

$$\begin{aligned}(\varepsilon - 1)\int_M |\nabla\vec{H}|^2\eta^2 d\mu &\le \int_M \Big\langle \Delta\vec{H}, \vec{H}\Big\rangle \eta^2\,d\mu \\ &= -\int_M |\nabla\vec{H}|^2\eta^2 d\mu - 2\int_M \Big\langle \nabla\vec{H}, \nabla\eta\,\vec{H}\Big\rangle \eta\,d\mu\,,\end{aligned}$$

so

$$\int_{f^{-1}(B_\rho)} |\nabla\vec{H}|^2 d\mu \le \frac{c}{\varepsilon^2\rho^2}\int_{f^{-1}(B_{2\rho})} |\vec{H}|^2 d\mu\,.$$

Recalling the assumption (5) and that N is complete, the claim follows by taking $\rho \to \infty$. □

It is important to note that the statement of the previous lemma is stronger than $\vec{H}$ being parallel in the normal bundle; indeed, it is enough to guarantee that f is minimal.

Lemma 3. *Suppose the mean curvature $\vec{H}$ of $f : M \to N$ satisfies $\nabla\vec{H} \equiv 0$. Then f is minimal.*

Proof. Let $p \in M$ and choose an orthonormal basis $\{\tau_i\}_{i=1}^m$ for T_pM. We also choose an orthonormal basis $\{\nu^\alpha\}_{\alpha=1}^n$ of $(T_pM)^\perp$. For any i, j, we have at p that

$$0 = \Big\langle \nabla_{\tau_i}\vec{H}, \tau_j\Big\rangle = H_\alpha A^\alpha(\tau_i, \tau_j)\,.$$

Tracing over the metric g, we conclude

$$0 = \sum_{\alpha=1}^n (H_\alpha)^2$$

and we are done.

□

Combining Lemmas 2 and 3 concludes the proof of Theorem 1.

Acknowledgements

This work was completed and announced during the 2011 SIAM PDE conference at Mission Valley. The author is grateful for the hospitality and atmosphere provided by SIAM at this event. The author would like to thank in particular the participants of the mini-symposium “Topics in Higher Order Geometric Partial Differential Equations” for lively and enlightening mathematical discussions related to this work.

The author is supported by an Alexander von Humboldt research fellowship at the Otto-von-Guericke Universität Magdeburg.

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