CHEREDNIK ALGEBRAS, W-ALGEBRAS AND THE EQUIVARIANT COHOMOLOGY OF THE MODULI SPACE OF INSTANTONS ON $\mathbf{A}^2$

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ABSTRACT

We construct a representation of the affine W-algebra of $\mathfrak{gl}_r$ on the equivariant homology space of the moduli space of U_r-instantons, and we identify the corresponding module. As a corollary, we give a proof of a version of the AGT conjecture concerning pure $N=2$ gauge theory for the group $SU(r)$.

Our approach uses a deformation of the universal enveloping algebra of $W_{1+\infty}$, which acts on the above homology space and which specializes to $W(\mathfrak{gl}_r)$ for all r. This deformation is constructed from a limit, as n tends to ∞, of the spherical degenerate double affine Hecke algebra of GL_n.

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This research was partially supported by the ANR grant number ANR-10-BLAN-0110.

DOI 10.1007/s10240-013-0052-3

0. Introduction

In their recent study of $\mathrm{N}=2$ super-symmetric gauge theory in dimension four, the authors of [1] observed a striking relation with two-dimensional Conformal Field Theory. More precisely, they observed in some examples and conjectured in many other an equality between the conformal blocks of Liouville theory associated with a punctured Riemann surface and the group U_r on the one hand and the instanton part of the Nekrasov partition function for a suitable four-dimensional gauge theory associated with the group U_r on the other hand. Numerous partial results in this direction have been obtained in the physics literature, see e.g., [13] and the references therein. In mathematical terms, the AGT conjecture suggests in particular the existence of a representation of the affine W-algebra of G on the equivariant intersection cohomology of the moduli space of G^{L}-instantons on $\mathbf{R}^4$ satisfying some extra properties (relating the fundamental class and the Whittaker vector), see [7] and [18]. Here G, G^{L} are a pair of complex reductive groups which are dual to each other in the sense of Langlands. The purpose of this paper is to give for the gauge group $\mathrm{G}=\mathrm{G}^{\mathrm{L}}=\mathrm{GL}_r$ a construction of this action which is inspired by our previous work [33]. It is based on degenerate double affine Hecke algebras. For the same gauge group $\mathrm{G}=\mathrm{GL}_r$, a construction of this action has been given by Maulik and Okounkov using ideas from symplectic geometry, see [28].

Let us describe our main result more precisely. Let $\mathrm{M}_r=\bigsqcup_{n\geq 0}\mathrm{M}_{r,n}$ be the moduli space of rank r torsion free coherent sheaves on $\mathbf{P}^2$, equipped with a framing along $\mathbf{P}^1_\infty\subset\mathbf{P}^2$. For fixed n, $\mathrm{M}_{r,n}$ is a smooth symplectic variety of dimension $2rn$. It is acted upon by an $r+2$-dimensional torus $\widetilde{\mathrm{D}}=(\mathbf{C}^\times)^2\times\mathrm{D}$ where $(\mathbf{C}^\times)^2$ acts on $\mathbf{P}^2$ and $\mathrm{D}=(\mathbf{C}^\times)^r$ acts on the framing. When $r=1$, the moduli space $\mathrm{M}_{1,n}$ is isomorphic to the Hilbert scheme Hilb_n of n points on $\mathbf{C}^2$. In the mid 90s, Nakajima constructed a representation of the rank one Heisenberg algebra on the space

$$\widetilde{\mathbf{L}}^{(1)}=\bigoplus_{n\geq 0}\mathrm{H}_*(\mathrm{Hilb}_n)$$

by geometric methods, which identifies it with the standard level one Fock space, see [29] and [20]. The case of the equivariant Borel-Moore homology

$$\mathbf{L}^{(1)}=\bigoplus_{n\geq 0}\mathrm{H}_*^{\widetilde{\mathrm{D}}}(\mathrm{Hilb}_n)$$

was considered later in [40]. For $r\geq 1$ there is still a representation of a rank one Heisenberg algebra on the space

$$\mathbf{L}^{(r)}=\bigoplus_{n\geq 0}\mathrm{H}_*^{\widetilde{\mathrm{D}}}(\mathrm{M}_{r,n}),$$

but it is neither irreducible nor cyclic, see [3]. A construction of a representation of an r-dimensional Heisenberg algebra on $\mathbf{L}^{(r)}$ has also been given in [24]. Now, let

$$\mathrm{R}_r = \mathbf{C}[x, y, e_1, \ldots, e_r], \qquad \mathrm{K}_r = \mathbf{C}(x, y, e_1, \ldots, e_r)$$

be the cohomology ring of the classifying space of $\widetilde{\mathrm{D}}$ and its fraction field. The space $\mathbf{L}^{(r)}$ is an R_r-module. Set $\mathbf{L}^{(r)}_{\mathrm{K}} = \mathbf{L}^{(r)} \otimes_{\mathrm{R}_r} \mathrm{K}_r$. We'll abbreviate

$$\kappa = -y/x, \qquad \varepsilon_i = e_i/x, \qquad \xi = 1-\kappa, \quad i \in [1, r].$$

Let $\mathrm{W}_k(\mathfrak{gl}_r)$ be the level k affine W-algebra of $\mathfrak{gl}_r$. Recall that the cup product in equivariant cohomology yields a bilinear map

$$(\bullet, \bullet) : \mathbf{L}^{(r)}_{\mathrm{K}} \times \mathbf{L}^{(r)}_{\mathrm{K}} \to \mathrm{K}_r$$

called the *intersection pairing*. Set $\vec{e} = (e_1, e_2, \ldots, e_r)$, $\vec{\varepsilon} = \vec{e}/x$ and $\rho = (0, -1, -2, \ldots, 1-r)$. Here is the main result of this paper.

Theorem. — (a) *There is a representation of* $\mathrm{W}_k(\mathfrak{gl}_r)$ *of level* $k = \kappa - r$ *on* $\mathbf{L}^{(r)}_{\mathrm{K}}$, *identifying it with the Verma module* M_β *of highest weight* $\beta = -(\vec{\varepsilon} + \xi\rho)/\kappa$.

(b) *This action is quasi-unitary with respect to the intersection pairing on* $\mathbf{L}^{(r)}_{\mathrm{K}}$.

(c) *The* Gaiotto state $\mathrm{G} = \sum_{n\geq 0} \mathrm{G}_n$, $\mathrm{G}_n = [\mathrm{M}_{r,n}]$, *is a Whittaker vector of* M_β.

Parts (a) and (b) are proved in Theorem 8.33 and part (c) is proved in Proposition 9.4. Note that $\mathrm{W}_k(\mathfrak{gl}_1)$ is a Heisenberg algebra of rank one. So the above theorem may be seen as a generalization to higher ranks of the representation of the Heisenberg algebra on the equivariant cohomology of the Hilbert scheme. For instance, taking $r = 2$ we get an action of the Virasoro algebra on the cohomology of the moduli space of U_2-instantons on $\mathbf{R}^4$. The relation with the AGT conjecture for the pure $\mathrm{N} = 2$ supersymmetric gauge theory is the following. Recall that Nekrasov's partition function is the generating function of the integral of the equivariant cohomology class $1 \in \mathrm{H}^{\widetilde{\mathrm{D}}}_*(\mathrm{M}_{r,n})$, i.e., we have

$$\mathrm{Z}(x, y, \vec{e}; q) = \sum_{n\geq 0} q^n \big([\mathrm{M}_{r,n}], [\mathrm{M}_{r,n}]\big).$$

The element G belongs to the completed Verma module

$$\widehat{\mathrm{M}}_\beta = \prod_{n\geq 0} \mathrm{M}_{\beta,n}, \qquad \mathrm{M}_{\beta,n} = \mathrm{H}^{\widetilde{\mathrm{D}}}_*(\mathrm{M}_{r,n}) \otimes_{\mathrm{R}_r} \mathrm{K}_r.$$

Let $\{\mathrm{W}_{d,l};\ l \in \mathbf{Z},\ d \in [1, r]\}$ be the set of the Fourier modes of the generating fields of $\mathrm{W}_k(\mathfrak{gl}_r)$. Then M_β has a unique bilinear form $(\bullet, \bullet)$ such that the highest weight vector has norm 1 and the adjoint of $W_{d,-l}$ is $W_{d,l}$ for $l \geq 0$ (up to a sign). Then, the element G is uniquely determined by the Whittaker condition and we have

$$Z(x, y, \vec{e}; q) = \sum_{n \geq 0} q^n (G_n, G_n).$$

Let us now explain the main steps of the proof. Since W-algebras do not possess, beyond the case of $\mathfrak{gl}_3$, a presentation by generators and relations, we cannot hope to construct directly the action of $W_k(\mathfrak{gl}_r)$ on $\mathbf{L}_K^{(r)}$ by some correspondences. Our approach relies instead on an intermediate algebra $\mathbf{SH^c}$, defined over the field $F = \mathbf{C}(\kappa)$, which is interesting in its own right, and which does act on $\mathbf{L}_K^{(r)}$ by some correspondences. The actual definition of $\mathbf{SH^c}$ is rather involved. Its main properties are summarized below. Let $\mathbf{SH}_n$ denote the spherical degenerate double affine Hecke algebra of GL_n. Let $\Lambda = F[p_l; l \geq 1]$. Let

$$\mathscr{H} = \langle \mathbf{c}_0, b_l; l \in \mathbf{Z} \rangle$$

be the Heisenberg algebra of central charge $\mathbf{c}_0/\kappa$. The following is proved in Section 1 and Appendix F.

Proposition. — (a) *The algebra* $\mathbf{SH^c}$ *is* $\mathbf{Z}$*-graded,* $\mathbf{N}$*-filtered and has a triangular decomposition*

$$\mathbf{SH^c} = \mathbf{SH}^{>} \otimes \mathbf{SH}^{\mathbf{c},0} \otimes \mathbf{SH}^{<}, \qquad \mathbf{SH}^{\mathbf{c},0} = F[\mathbf{c}_l; l \geq 1] \otimes F[D_{0,l}; l \geq 1].$$

Here $F[\mathbf{c}_l; l \geq 1]$ *is a central subalgebra. The Poincaré polynomials of* $\mathbf{SH}^{<}$ *and* $\mathbf{SH}^{>}$ *are*

$$P_{\mathbf{SH}^{>}}(t, q) = \prod_{r>0} \prod_{l \geq 0} \frac{1}{1 - t^r q^l}, \qquad P_{\mathbf{SH}^{<}}(t, q) = \prod_{r<0} \prod_{l \geq 0} \frac{1}{1 - t^r q^l}.$$

(b) *Let* $\mathbf{SH}$ *be the specialization of* $\mathbf{SH^c}$ *at* $\mathbf{c}_0 = 0$ *and* $\mathbf{c}_l = -\kappa^l \omega^l$ *for* $l \geq 1$*. For* $n \geq 1$ *there is a surjective algebra homomorphism* $\Psi_n : \mathbf{SH} \to \mathbf{SH}_n$ *with* $\bigcap_n \operatorname{Ker} \Psi_n = \{0\}$.

(c) *The part of order* ≤ 0 *for the* $\mathbf{N}$*-filtration is* $\mathbf{SH^c}[\leq 0] = \mathscr{H} \otimes F[\mathbf{c}_l; l \geq 2]$*. The algebra* $\mathbf{SH^c}$ *is generated by* $\mathbf{SH^c}[\leq 0]$ *and* $D_{0,2}$.

(d) *Let* $\mathbf{SH}^{(1,0,\ldots)}$ *be the specialization of* $\mathbf{SH^c}$ *at* $\mathbf{c}_0 = 1$ *and* $\mathbf{c}_l = 0$ *for* $l \geq 1$*. It has a faithful representation in* Λ *such that* $\mathscr{H}$ *acts in the standard way and* $D_{0,2}$ *acts as the Laplace-Beltrami (or Calogero-Sutherland) operator*

$$D_{0,2} = \kappa \,\square = \frac{1}{2}\kappa(1-\kappa) \sum_{l \geq 1} (l-1) b_{-l} b_l + \frac{1}{2}\kappa^2 \sum_{l,k \geq 1} (b_{-l-k} b_l b_k + b_{-l} b_{-k} b_{l+k}).$$

(e) *The specialization of* $\mathbf{SH}^{(1,0,\ldots)}$ *at* $\kappa = 1$ *is isomorphic to the universal enveloping algebra of the Witt algebra* $W_{1+\infty}$.

We do not give a presentation of $\mathbf{SH^c}$ by generators and relations. We do not need it. However, the subalgebras $\mathbf{SH}^>$ and $\mathbf{SH}^<$ have realizations as shuffle algebras, see Theorem 4.7 and Corollary 6.4. The central subalgebra $\mathrm{F}[\mathbf{c}_l; l \geq 0]$ is not finitely generated, but only two of the generators are essential, i.e., the rest may be split off.

A construction of a similar limit $\mathbf{SH^c}$ of the spherical double affine Hecke algebras of GL_n as n tends to infinity appears in [32]. The algebra $\mathbf{SH^c}$ depends on two parameters t, q, and $\mathbf{SH^c}$ may be obtained by degeneration of $\mathbf{SH^c}$ as $t \mapsto 1$ and $q \mapsto 1$ with $t = q^{-\kappa}$, in much the same way as the trigonometric Cherednik algebra is obtained by degeneration of the elliptic Cherednik algebra, see Section 7. In [33] it was shown that $\mathbf{SH^c}$ acts on the space $\bigoplus_{n \geq 0} \mathrm{K}^{\widetilde{\mathrm{D}}}(\mathrm{M}_{r,n})$, where $\mathrm{K}^{\widetilde{\mathrm{D}}}$ is the equivariant algebraic K-theory. Adapting the arguments of *loc. cit.* to the equivariant cohomology setup, we prove the following in Theorem 3.2, Corollary 3.3 and Lemma 8.34. Let $\mathbf{SH}^{(r)}_{\mathrm{K}}$ be the specialization of $\mathbf{SH^c} \otimes \mathrm{K}_r$ to $\mathbf{c}_0 = r$ and $\mathbf{c}_i = p_i(\varepsilon_1, \ldots, \varepsilon_r)$.

Theorem **A.** — *There is a faithful representation $\rho^{(r)}$ of $\mathbf{SH}^{(r)}_{\mathrm{K}}$ on $\mathbf{L}^{(r)}_{\mathrm{K}}$ such that $\mathbf{L}^{(r)}_{\mathrm{K}}$ is generated by the fundamental class of $\mathrm{M}_{r,0}$.*

This representation is given by convolution with correspondences supported on the nested instanton spaces. In the proof of Theorem A an important role is played by the commuting varieties

$$\mathrm{C}_n = \left\{(u, v) \in (\mathfrak{gl}_n)^2; [u, v] = 0\right\}$$

and the *cohomological Hall algebra*, which is an associative algebra structure on

$$\mathbf{C}' = \bigoplus_{n \geq 0} \mathrm{H}^{\mathrm{T}}(\mathrm{C}_n), \quad \mathrm{T} = \left(\mathbf{C}^\times\right)^2.$$

Let $\mathfrak{U}(\mathrm{W}_k(\mathfrak{gl}_r))$ be the current algebra of $\mathrm{W}_k(\mathfrak{gl}_r)$. We'll use a quotient $\mathscr{U}(\mathrm{W}_k(\mathfrak{gl}_r))$ of $\mathfrak{U}(\mathrm{W}_k(\mathfrak{gl}_r))$ whose definition is given in Section 8.5. It is a $\mathbf{Z}$-graded, $\mathbf{N}$-filtered, degreewise topological associative algebra with 1 which is degreewise complete. Let $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})$ be the degreewise completion of $\mathbf{SH}^{(r)}_{\mathrm{K}}$, which is defined in Definition 8.8. Our main theorem is a consequence of the following results, proved in Theorem 8.22 and Corollary 8.29, and in Theorem 8.33 and Proposition 9.4. Put $k = \kappa - r$. First, we have

Theorem **B.** — *There is an embedding of graded and filtered algebras*

$$\Theta^{(r)} : \mathbf{SH}^{(r)}_{\mathrm{K}} \longrightarrow \mathscr{U}\left(\mathrm{W}_k(\mathfrak{gl}_r)\right)$$

which extends to a surjective morphism $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}}) \longrightarrow \mathscr{U}(\mathrm{W}_k(\mathfrak{gl}_r))$. The map $\Theta^{(r)}$ induces an equivalence between the categories of admissible $\mathbf{SH}^{(r)}_{\mathrm{K}}$ and $\mathscr{U}(\mathrm{W}_k(\mathfrak{gl}_r))$ modules.

This allows us to regard $\mathbf{L}^{(r)}_{\mathrm{K}}$ as a $\mathrm{W}_k(\mathfrak{gl}_r)$-module. Then, we have the following.

Theorem **C.** — *The representation* $\mathbf{L}_{\mathrm{K}}^{(r)}$ *of* $\mathrm{W}_k(\mathfrak{gl}_r)$ *is a Verma module. It is quasi-unitary with respect to the intersection pairing. The element* G *is a Whittaker vector for* $\mathrm{W}_k(\mathfrak{gl}_r)$.

Theorem C is proved by some simple explicit calculation. Let us briefly indicate how we prove Theorem B. Our approach rests upon the following crucial fact proved in Theorem 7.9.

Proposition. — *The algebra* $\mathbf{SH^c}$ *is equipped with a topological Hopf algebra structure. The comultiplication is uniquely determined by the following formulas*

$$\boldsymbol{\Delta}(\mathbf{c}_l) = \mathbf{c}_l \otimes 1 + 1 \otimes \mathbf{c}_l, \quad l \geq 0,$$
$$\boldsymbol{\Delta}(b_0) = b_0 \otimes 1 + 1 \otimes b_0 + \xi \mathbf{c}_0 \otimes \mathbf{c}_0, \qquad \boldsymbol{\Delta}(b_l) = b_l \otimes 1 + 1 \otimes b_l, \quad l \neq 0,$$
$$\boldsymbol{\Delta}(\mathrm{D}_{0,2}) = \mathrm{D}_{0,2} \otimes 1 + 1 \otimes \mathrm{D}_{0,2} + \kappa\xi \sum_{l \geq 1} l b_l \otimes b_{-l}.$$

Using this coproduct, we equip the category of admissible $\mathbf{SH^c}$-modules with a monoidal structure. In particular $(\mathbf{L}_{\mathrm{K}}^{(1)})^{\otimes r}$ is equipped with a faithful representation of $\mathbf{SH}_{\mathrm{K}}^{(r)}$. We call it the *free field realization* representation. We then compare this free field representation of $\mathbf{SH}_{\mathrm{K}}^{(r)}$ with the free field representation of $\mathrm{W}_k(\mathfrak{gl}_r)$ using some explicit computations in the cases $r = 1, 2$, the coassociativity of $\boldsymbol{\Delta}$ and the fundamental result of Feigin and Frenkel [14, 15] which characterizes $\mathrm{W}_k(\mathfrak{gl}_r)$ as the intersection of some *screening operators*.

One remark about the Hopf algebra structure on $\mathbf{SH^c}$ is in order. It was observed in [38] that, under Nakajima's realization of affine quantum groups in terms of equivariant K-theory of quiver varieties, the coproduct of the quantum groups could be constructed geometrically using some fixed subsets of the quiver varieties. In later works, a geometric construction of tensor products of representations in terms of both cohomology and K-theory of some quiver varieties was given in [26, 30]. In this paper, we do not give a geometric interpretation of our map $\boldsymbol{\Delta}$. In fact, we obtain it by degenerating a similar coproduct on the algebra $\mathbf{SH^c}$. The existence of a Hopf algebra structure on $\mathbf{SH^c}$, in equivariant K-theory, is not more natural than on $\mathbf{SH^c}$, in equivariant cohomology. However, since $\mathbf{SH^c}$ is identified with a central extension of the Drinfeld double of the spherical Hall algebra $\widehat{\boldsymbol{\mathcal{E}}}$ of an elliptic curve over a finite field, see [33], and since this Hall algebra has a coproduct,[1] the algebra $\mathbf{SH^c}$ is also equipped with a comultiplication. We do not know, however, of a similar isomorphism involving $\mathbf{SH^c}$ which would give directly the comultiplication.

Some of the methods and results of this paper generalize to the case of the moduli spaces of instantons on resolutions of simple Kleinian singularities, equivalently, the

[1] The correct choice of coproduct on $\widehat{\boldsymbol{\mathcal{E}}}$ here is not the standard one, but rather the standard one twisted by a *Fourier transform*, see (7.50).

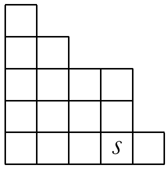


FIG. 1. — The partition $(5, 4^2, 2, 1)$ and a box in it

Nakajima quiver varieties attached to affine Dynkin diagrams. We'll come back to this question elsewhere.

To finish, let us say a few words concerning the organization of this paper. The construction and properties of the algebra $\mathbf{SH^c}$ are given in Section 1. In Sections 2 and 3 we define some convolution algebra acting on the space $\mathbf{L}_K^{(r)}$ and state our first main result, Theorem 3.2, which claims that this algebra is isomorphic to $\mathbf{SH}_K^{(r)}$. In Section 4 we introduce the commuting variety and its convolution algebra, the so-called *cohomological Hall algebra*. The proof of Theorem 3.2 is given in Sections 5 and 6. Section 7 is devoted to the construction of the Hopf algebra structure on $\mathbf{SH^c}$. Section 8 discusses the free field realizations of $\mathbf{SH}_K^{(r)}$ and $W_k(\mathfrak{gl}_r)$, and compares them (first for $r = 1$ then $r = 2$ and then for arbitrary r). Theorem 8.22 is proved in Section 8.9, and part (a) of our main Theorem is proved in Section 8.11, see Theorem 8.33. Finally, Section 9 is devoted to the Whittaker property of the Gaiotto state, with respect to both $\mathbf{SH}_K^{(r)}$ and $W_k(\mathfrak{gl}_r)$. Several technical lemmas are postponed to the appendices. In particular, the relation with $W_{1+\infty}$ is explained in Appendix F.

0.1. *Notation.* — We'll use the continental way of drawing a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, with rows going from the bottom up of successive length λ_1, λ_2, etc. If s is a box in the diagram of a partition λ, we denote by $x(s), y(s), l(s), a(s)$ the number of boxes lying *strictly* to the west, resp. south, resp. north, resp. east, of the box s.

Example **0.1.** — For the box s in the partition $(5, 4^2, 2, 1)$ depicted below we have $x(s) = 3, y(s) = 0, l(s) = 2$ and $a(s) = 1$.

When we need to stress the dependance on the partition λ we will write $a_\lambda(s)$ and $l_\lambda(s)$. This notation extends in an obvious way to boxes s which might lie outside of λ (in which case, $a_\lambda(s)$ or $l_\lambda(s)$ could be negative). For instance, if $\lambda = (5, 4^2, 2, 1)$ as in Figure 1 above and $x(s) = 4, y(s) = 2$ then $a_\lambda(s) = -1$ and $l_\lambda(s) = -2$. We will occasionaly refer to a box through its coordinates $s = (x(s) + 1, y(s) + 1)$. As usual, the length of a partition λ is denoted $l(\lambda)$, and the conjugate partition is denoted λ'. Finally, if s is a box of a partition λ then we denote by R_s and C_s, the set of all boxes of λ in the same row and same column respectively, as s, with s excepted. We call *r-partition of n* an r-tuple of partitions with total weight n. Given two r-partitions $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ and $\mu = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)})$ we write

$$\lambda \subset \mu \iff \lambda^{(a)} \subset \mu^{(a)}, \quad \forall a.$$

For any commutative ring A we set

$$(\mathbf{0.1}) \qquad \Lambda_{n,\mathrm{A}} = \mathrm{A}[\mathrm{X}_1, \dots, \mathrm{X}_n]^{\mathfrak{S}_n}, \qquad \Lambda_{\mathrm{A}} = \mathrm{A}[\mathrm{X}_1, \mathrm{X}_2, \dots]^{\mathfrak{S}_\infty}.$$

Note that Λ_{A} is the Macdonald algebra of symmetric functions. Let π_n be the obvious projection

$$(\mathbf{0.2}) \qquad \pi_n : \Lambda_{\mathrm{A}} \to \Lambda_{n,\mathrm{A}}.$$

For any ring A let δ be the map $\mathrm{A} \to \mathrm{A} \otimes \mathrm{A}$ given by

$$(\mathbf{0.3}) \qquad \delta(a) = a \otimes 1 + 1 \otimes a.$$

For $r \geq 1$ let $\delta^{r-1} : \mathrm{A} \to \mathrm{A}^{\otimes r}$ be the map obtained by iterating $r-1$ times the map δ. Let

$$(\mathbf{0.4}) \qquad \begin{aligned} e_l &= e_l(\mathrm{X}) = e_l(\mathrm{X}_1, \mathrm{X}_2, \dots), \\ p_l &= p_l(\mathrm{X}) = p_l(\mathrm{X}_1, \mathrm{X}_2, \dots), \\ m_\lambda &= m_\lambda(\mathrm{X}) = m_\lambda(\mathrm{X}_1, \mathrm{X}_2, \dots) \end{aligned}$$

be the lth elementary symmetric function, the lth power sum polynomial and the monomial symmetric function in Λ_{A}, see e.g., [25, Chap. I]. Let

$$(\mathbf{0.5}) \qquad \begin{aligned} e_l^{(n)} &= e_l^{(n)}(\mathrm{X}) = e_l(\mathrm{X}_1, \dots, \mathrm{X}_n), \\ p_l^{(n)} &= p_l^{(n)}(\mathrm{X}) = p_l(\mathrm{X}_1, \dots, \mathrm{X}_n), \\ m_\lambda^{(n)} &= m_\lambda^{(n)}(\mathrm{X}) = m_\lambda(\mathrm{X}_1, \mathrm{X}_2, \dots, \mathrm{X}_n) \end{aligned}$$

be the corresponding functions in $\Lambda_{\mathrm{A},n}$. If no confusion is possible we abbreviate

$$(\mathbf{0.6}) \qquad e_l = e_l^{(n)}, \qquad p_l = p_l^{(n)}, \qquad m_\lambda = m_\lambda^{(n)}.$$

We write also

$$(\mathbf{0.7}) \qquad \begin{aligned} \mathbf{Z}_0^2 &= \mathbf{Z}^2 \setminus (0,0), \\ \mathbf{N}_0^2 &= \mathbf{N}^2 \setminus (0,0), \\ \mathscr{E} &= \big\{(\epsilon, l); \epsilon = -1, 0, 1,\ l \in \mathbf{Z}_{\geq 0}\big\} \setminus (0,0), \\ \mathscr{E}^+ &= \big\{(\epsilon, l) \in \mathscr{E}; \epsilon \geq 0\big\}, \\ \mathscr{E}^- &= \big\{(\epsilon, l) \in \mathscr{E}; \epsilon \leq 0\big\}, \end{aligned}$$

1. The algebra $\mathbf{SH}^{\mathbf{c}}$

1.1. *The DDAHA.* — We define

$$\begin{gathered} \mathrm{G}=\mathrm{GL}_n, \qquad \mathrm{H}=\left(\mathbf{C}^{\times}\right)^n, \qquad \mathrm{H}^+=\mathbf{C}^n, \qquad \mathfrak{h}=\mathrm{L}ie(\mathrm{H}), \\ \mathbf{C}[\mathrm{H}]=\mathbf{C}\left[\mathrm{X}_1^{\pm 1},\ldots,\mathrm{X}_n^{\pm 1}\right], \qquad \mathbf{C}\left[\mathrm{H}^+\right]=\mathbf{C}[\mathrm{X}_1,\ldots,\mathrm{X}_n], \\ \mathbf{C}[\mathfrak{h}]=\mathbf{C}[x_1,\ldots,x_n], \qquad \mathbf{C}\left[\mathfrak{h}^*\right]=\mathbf{C}[y_1,\ldots,y_n]. \end{gathered} \tag{1.1}$$

Here $(y_1,\ldots,y_n)$ is the basis dual to $(x_1,\ldots,x_n)$. The symmetric group $\mathfrak{S}_n$ acts on H, $\mathfrak{h}$ and $\mathfrak{h}^*$. Let $s_1,\ldots,s_{n-1}$ be the standard generators of $\mathfrak{S}_n$. For $i\neq j$ let s_{ij} be the transposition (ij). Finally, set $\mathrm{F}=\mathbf{C}(\kappa)$ and $\mathrm{A}=\mathbf{C}[\kappa]$. The *degenerate double affine Hecke algebra* (=DDAHA) of G is the associative F-algebra $\mathbf{H}_n$ generated by F[H], F[$\mathfrak{h}^*$] and F[$\mathfrak{S}_n$] subject to the following set of relations

$$s\mathrm{X}_i^{\pm 1}=\mathrm{X}_{s(i)}^{\pm 1}s, \quad s\in\mathfrak{S}_n, \tag{1.2}$$

$$s_iy=s_i(y)s_i-\kappa\langle x_i-x_{i+1},y\rangle, \quad y\in\mathfrak{h}^*, \tag{1.3}$$

$$[y_i,\mathrm{X}_j]=\begin{cases}-\kappa\mathrm{X}_is_{ij} & \text{if } i<j,\\ \mathrm{X}_i+\kappa\left(\sum_{k<i}\mathrm{X}_ks_{ik}+\sum_{k>i}\mathrm{X}_is_{ik}\right) & \text{if } i=j,\\ -\kappa\mathrm{X}_js_{ij} & \text{if } i>j.\end{cases} \tag{1.4}$$

Let $\mathbf{S}=\frac{1}{n!}\sum_{s\in\mathfrak{S}_n}s$ be the complete idempotent in $\mathbf{C}[\mathfrak{S}_n]$. The *spherical DDAHA* of G is

$$\mathbf{SH}_n=\mathbf{S}\cdot\mathbf{H}_n\cdot\mathbf{S}. \tag{1.5}$$

Let $\mathbf{H}_n^+\subset\mathbf{H}_n$ be the F-subalgebra generated by $\mathfrak{S}_n$ and $\{y_i,\mathrm{X}_i;\, i\in[1,n]\}$. This is a deformation of the algebra of polynomial differential operators on H. Similarly, let $\mathbf{H}_n^-\subset\mathbf{H}_n$ be the subalgebra generated by $\mathfrak{S}_n$ and $\{y_i,\mathrm{X}_i^{-1};\, i\in[1,n]\}$. Write

$$\mathbf{SH}_n^{\pm}=\mathbf{S}\cdot\mathbf{H}_n^{\pm}\cdot\mathbf{S}, \qquad \mathbf{SH}_n^0=\mathbf{S}\cdot\mathrm{F}\left[\mathfrak{h}^*\right]\cdot\mathbf{S}. \tag{1.6}$$

Remark **1.1.** — Formally setting $\kappa=0$ in the relations of $\mathbf{H}_n$ yields a presentation of the crossed product $\mathrm{Diff}(\mathrm{H})\rtimes\mathfrak{S}_n$, with y_i degenerating to $\mathrm{X}_i\partial_{\mathrm{X}_i}$. The spherical DAHA is a deformation of the ring $\mathrm{Diff}(\mathrm{H})^{\mathfrak{S}_n}$ of symmetric differential operators on the torus H.

1.2. *Filtrations on* $\mathbf{H}_n$ *and* $\mathbf{SH}_n$. — We define the *order filtration* on $\mathbf{H}_n$ by letting y_i be of order 1 and s, $\mathrm{X}_i^{\pm 1}$ be of order 0. We define the *rank grading* on $\mathbf{H}_n$ by giving to s, y_i the degree 0 and to $\mathrm{X}_i^{\pm 1}$ the degree ± 1. Let $\mathbf{H}_n[r,\leq l]$ be the piece of $\mathbf{H}_n$ of degree r and of order $\leq l$. The piece of degree r and of order $\leq l$ in $\mathbf{SH}_n$ is

$$\mathbf{SH}_n[r,\leq l]=\mathbf{S}\cdot\mathbf{H}_n[r,\leq l]\cdot\mathbf{S}=\mathbf{SH}_n\cap\mathbf{H}_n[r,\leq l]. \tag{1.7}$$

Similarly, we set

$$\mathbf{SH}_n^+[r,\leq l]=\mathbf{S}\cdot\mathbf{H}_n^+[r,\leq l]\cdot\mathbf{S}=\mathbf{SH}_n^+\cap\mathbf{H}_n[r,\leq l]. \tag{1.8}$$

All the constructions given above make sense over the ring A. For instance, let $\mathbf{H}_{n,\mathrm{A}}\subset\mathbf{H}_n$ be the A-subalgebra generated by $\mathfrak{S}_n$, $\mathrm{A}[\mathfrak{h}^*]$ and $\mathrm{A}[\mathrm{H}]$, and put

$$\begin{gathered}\mathbf{SH}_{n,\mathrm{A}}=\mathbf{S}\cdot\mathbf{H}_{n,\mathrm{A}}\cdot\mathbf{S}=\mathbf{SH}_n\cap\mathbf{H}_{n,\mathrm{A}},\\ \mathbf{H}_{n,\mathrm{A}}[r,\leq l]=\mathbf{H}_{n,\mathrm{A}}\cap\mathbf{H}_n[r,\leq l],\\ \mathbf{SH}_{n,\mathrm{A}}^+[r,\leq l]=\mathbf{S}\cdot\mathbf{H}_{n,\mathrm{A}}^+[r,\leq l]\cdot\mathbf{S}=\mathbf{SH}_{n,\mathrm{A}}^+\cap\mathbf{H}_n[r,\leq l].\end{gathered} \tag{1.9}$$

The PBW theorem for $\mathbf{H}_{n,\mathrm{A}}$ implies that any element u of $\mathbf{H}_{n,\mathrm{A}}$ has a unique decomposition of the form

$$u=\sum_{s\in\mathfrak{S}_n}h_s(\mathrm{X})g_s(y)s,\quad g_s(y)\in\mathrm{A}\big[\mathfrak{h}^*\big],\quad h_s(\mathrm{X})\in\mathrm{A}[\mathrm{H}]. \tag{1.10}$$

Therefore, we have $\mathbf{H}_{n,\mathrm{A}}\otimes_{\mathrm{A}}\mathrm{F}=\mathbf{H}_n$. Since $\mathbf{SH}_{n,\mathrm{A}}$ is a direct summand of the A-module $\mathbf{H}_{n,\mathrm{A}}$, we have also $\mathbf{SH}_{n,\mathrm{A}}\otimes_{\mathrm{A}}\mathrm{F}=\mathbf{SH}_n$. A similar argument yields

$$\mathbf{H}_{n,\mathrm{A}}[r,\leq l]\otimes_{\mathrm{A}}\mathrm{F}=\mathbf{H}_n[r,\leq l],\qquad \mathbf{SH}_{n,\mathrm{A}}[r,\leq l]\otimes_{\mathrm{A}}\mathrm{F}=\mathbf{SH}_n[r,\leq l]. \tag{1.11}$$

Let $\overline{\mathbf{H}}_{n,\mathrm{A}}$ and $\overline{\mathbf{SH}}_{n,\mathrm{A}}$ be the graded A-algebras associated with the order filtrations on $\mathbf{H}_{n,\mathrm{A}}$ and $\mathbf{SH}_{n,\mathrm{A}}$ respectively. Let us state some useful consequences of the PBW theorem. Whenever this makes sense we may abbreviate $\mathrm{ad}(z)$ for the commutator with z.

Proposition **1.2.** — (a) *An element $u\in\mathbf{SH}_{n,\mathrm{A}}$ is of order $\leq k$ if and only if*

$$\mathrm{ad}(z_1)\circ\cdots\circ\mathrm{ad}(z_k)(u)\in\mathbf{S}\cdot\mathrm{A}[\mathrm{H}]\cdot\mathbf{S},\quad \forall z_1,\dots,z_k\in\mathbf{S}\cdot\mathrm{A}[\mathrm{H}]\cdot\mathbf{S}. \tag{1.12}$$

(b) *The obvious maps yield* A*-algebra isomorphisms*

$$\mathrm{A}\big[\mathrm{H}\times\mathfrak{h}^*\big]\rtimes\mathfrak{S}_n=\overline{\mathbf{H}}_{n,\mathrm{A}},\qquad \mathrm{A}\big[\mathrm{H}\times\mathfrak{h}^*\big]^{\mathfrak{S}_n}\cdot\mathbf{S}=\mathbf{S}\cdot\mathrm{A}\big[\mathrm{H}\times\mathfrak{h}^*\big]\cdot\mathbf{S}=\overline{\mathbf{SH}}_{n,\mathrm{A}}.$$

Proof. — Let $\mathbf{SH}_{n,\mathrm{A}}[\leq k]$ be the space of the elements of order $\leq k$ in $\mathbf{SH}_{n,\mathrm{A}}$. We have

$$\mathbf{H}_{n,\mathrm{A}}[\leq k]=\Big\{\sum_s h_s(\mathrm{X})g_s(y)s;\ \deg(g_s)\leq k,\ \forall s\Big\}. \tag{1.13}$$

Let U_k be the set of elements of $\mathbf{SH}_{n,\mathrm{A}}$ satisfying (1.12). The inclusion $\mathbf{SH}_{n,\mathrm{A}}[\leq k]\subset\mathrm{U}_k$ follows from (1.4). We prove the reverse inclusion by induction. For $k=0$ there is nothing to prove, so let us assume that $\mathrm{U}_l\subset\mathbf{SH}_{n,\mathrm{A}}[\leq l]$ for all $l<k$. We have $\mathrm{ad}(p_1(\mathrm{X}))(y_i)=\mathrm{X}_i$ for all i. From this and (1.13) we deduce that

$$\big\{u\in\mathbf{H}_{n,\mathrm{A}};\ \mathrm{ad}\big(p_1(\mathrm{X})\big)(u)\in\mathbf{H}_{n,\mathrm{A}}[<j]\big\}\subset\mathbf{H}_{n,\mathrm{A}}[\leq j]. \tag{1.14}$$

In particular, we have $\mathrm{U}_k\subset\mathbf{H}_{n,\mathrm{A}}[\leq k]$. We are done. □

Lemma **1.3.** — *The* F*-algebra* $\mathbf{SH}_n$ *is generated by* $\mathbf{S}\cdot \mathrm{F}[\mathfrak{h}^*]\cdot \mathbf{S}$ *and* $\mathbf{S}\cdot \mathrm{F}[\mathrm{H}]\cdot \mathbf{S}$. *The* F*-algebra* $\mathbf{SH}_n^+$ *is generated by* $\mathbf{S}\cdot \mathrm{F}[\mathfrak{h}^*]\cdot \mathbf{S}$ *and* $\mathbf{S}\cdot \mathrm{F}[\mathrm{H}^+]\cdot \mathbf{S}$.

Proof. — First, we have an isomorphism

$$\mathbf{SH}_{n,\mathrm{A}}/(\kappa) \simeq \mathbf{C}\big[\mathrm{X}_1^{\pm 1}, \mathrm{X}_1\partial_{\mathrm{X}_1}, \dots, \mathrm{X}_n^{\pm 1}, \mathrm{X}_n\partial_{\mathrm{X}_n}\big]^{\mathfrak{S}_n}. \tag{1.15}$$

A similar result holds for $\mathbf{SH}_{n,\mathrm{A}}^+ = \mathbf{S}\cdot \mathbf{H}_{n,\mathrm{A}}^+\cdot \mathbf{S}$. Next, the following is well-known.

Claim. — *The algebra* $\mathbf{C}[\mathrm{X}_1^{\pm 1}, \mathrm{X}_1\partial_{\mathrm{X}_1}, \dots, \mathrm{X}_n^{\pm 1}, \mathrm{X}_n\partial_{\mathrm{X}_n}]^{\mathfrak{S}_n}$ *is generated by* $\mathbf{C}[\mathrm{X}_1^{\pm 1}, \dots, \mathrm{X}_n^{\pm 1}]^{\mathfrak{S}_n}$ *and* $\mathbf{C}[\mathrm{X}_1\partial_{\mathrm{X}_1}, \dots, \mathrm{X}_n\partial_{\mathrm{X}_n}]^{\mathfrak{S}_n}$. *A similar result holds for* $\mathbf{C}[\mathrm{X}_1, \mathrm{X}_1\partial_{\mathrm{X}_1}, \dots, \mathrm{X}_n, \mathrm{X}_n\partial_{\mathrm{X}_n}]^{\mathfrak{S}_n}$.

We now prove the second statement of Lemma 1.3. We have

$$\mathbf{SH}_{n,\mathrm{A}}^+ = \bigoplus_{r\geq 0}\bigcup_{l\geq 0}\mathbf{SH}_{n,\mathrm{A}}^+[r, \leq l] \tag{1.16}$$

and $\mathbf{SH}_{n,\mathrm{A}}^+[r, \leq l]$ is a free A-module of finite rank such that

$$\mathbf{SH}_{n,\mathrm{A}}^+[r, \leq l]\otimes_{\mathrm{A}} \mathrm{F} = \mathbf{SH}_n^+[r, \leq l], \tag{1.17}$$

because $\mathbf{SH}_{n,\mathrm{A}}^+[r, \leq l]$ is a direct summand in the A-module $\mathbf{H}_{n,\mathrm{A}}^+$ and $\mathbf{H}_{n,\mathrm{A}}^+\otimes_{\mathrm{A}}\mathrm{F} = \mathbf{H}_n^+$. The claim above implies that $\mathbf{SH}_{n,\mathrm{A}}^+[r, \leq l]/(\kappa)$ is linearly spanned by a suitable set of monomials in the elements $\mathbf{S}p_l(\mathrm{X})\mathbf{S}$ and $\mathbf{S}p_l(y)\mathbf{S}$ for $l\geq 0$. Thus, by Nakayama's lemma and (1.17), we have that $\mathbf{SH}_n^+[r, \leq l]$ is linearly spanned over F by the same set of monomials. This proves the second statement in the lemma. The first one is now for instance a consequence of the fact that any element of $\mathbf{SH}_n$ belongs to $(\mathrm{X}_1\cdots\mathrm{X}_n)^{-l}\cdot \mathbf{SH}_n^+$ for l big enough. □

The assignment $\mathrm{X}_i\mapsto \mathrm{X}_i^{-1}$, $y_i\mapsto y_i$, $s\mapsto s^{-1}$ extends to an algebra antiautomorphism π of $\mathbf{H}_{n,\mathrm{A}}$, as may be directly seen from the defining relations. It restricts to an algebra antiautomorphism of $\mathbf{SH}_{n,\mathrm{A}}$ taking $\mathbf{SH}_{n,\mathrm{A}}^+$ to $\mathbf{SH}_{n,\mathrm{A}}^-$. Thus $\mathbf{SH}_{n,\mathrm{A}}^-$ may be identified with $(\mathbf{SH}_{n,\mathrm{A}}^+)^{\mathrm{op}}$.

Remark **1.4.** — Let A_1 be the localization of A at the ideal $(\kappa-1)$. We define $\mathbf{H}_{n,\mathrm{A}_1}$ and $\mathbf{SH}_{n,\mathrm{A}_1}$ in the obvious way. Lemma 1.3 holds true with F replaced by A_1. The proof is similar to the proof of [4, Thm. 4.6]. It suffices to observe that the specialization of $\mathbf{SH}_{n,\mathrm{A}_1}$ at $\kappa=1$ is a simple algebra, because it is a (Ore) localization of a simple spherical rational DAHA by [36, Prop. 4.1].

1.3. *The polynomial representation.* — The tautological representation of $\mathrm{Diff}(\mathrm{H}) \ltimes \mathfrak{S}_n$ on $\mathbf{C}[\mathrm{H}]$ can be deformed to a representation of $\mathbf{H}_{n,\mathrm{A}}$ on $\mathrm{A}[\mathrm{H}]$, see [10]. This representation is defined by the following explicit formulas

$$
\begin{aligned}
\rho_n(s) &= s,\\
\rho_n(\mathrm{X}_i^{\pm 1}) &= \mathrm{X}_i^{\pm 1},\\
\rho_n(y_i) &= \mathrm{X}_i \partial_{\mathrm{X}_i} + \kappa \sum_{k \neq i} \frac{1 - s_{ik}}{1 - \mathrm{X}_k/\mathrm{X}_i} + \kappa \sum_{k<i} s_{ik}.
\end{aligned}
\tag{1.18}
$$

From now on we'll write

$$
\mathbf{W}_{n,\mathrm{A}} = \mathrm{A}[\mathrm{H}], \qquad \mathbf{V}_{n,\mathrm{A}} = (\mathbf{W}_{n,\mathrm{A}})^{\mathfrak{S}_n}.
\tag{1.19}
$$

We'll abbreviate

$$
\Lambda = \Lambda_{\mathrm{F}}, \qquad \Lambda_n = \Lambda_{n,\mathrm{F}}, \qquad \mathbf{W}_n = \mathbf{W}_{n,\mathrm{F}}, \qquad \mathbf{V}_n = \mathbf{V}_{n,\mathrm{F}}.
\tag{1.20}
$$

Theorem **1.5** *(Cherednik).* — *The assignment ρ_n defines an embedding* $\rho_n : \mathbf{H}_{n,\mathrm{A}} \to \mathrm{End}(\mathbf{W}_{n,\mathrm{A}})$ *which takes* $\mathbf{SH}_{n,\mathrm{A}}$ *into* $\mathrm{End}(\mathbf{V}_{n,\mathrm{A}})$.

The representation ρ_n is called the *polynomial representation*. The space $\Lambda_{n,\mathrm{A}}$ is preserved by the action of the subalgebra $\mathbf{SH}^+_{n,\mathrm{A}}$. Let ρ_n^+ denote the corresponding faithful representation of $\mathbf{SH}^+_{n,\mathrm{A}}$ on $\Lambda_{n,\mathrm{A}}$. We set

$$
\tilde{\mathrm{D}}^{(n)}_{0,l} = \mathbf{S}\, p_l(y)\, \mathbf{S}/l, \qquad l \geq 1.
\tag{1.21}
$$

The elements $\tilde{\mathrm{D}}^{(n)}_{0,l}$ generate a commutative subalgebra called the algebra of *Sekiguchi operators*, see [34]. The joint spectrum in $\Lambda_{n,\mathrm{A}}$ of the operators $\tilde{\mathrm{D}}^{(n)}_{0,l}$ consists of the *Jack polynomials* $\mathrm{J}^{(n)}_\lambda$, for λ a partition with at most n parts, and their eigenvalues are described by Lemma 1.6 below. We refer to Section 1.6 for more details on the notation for Jack polynomials. Consider the generating function

$$
\Delta_n(u) = \mathbf{S} \prod_{i=1}^{n} (u + y_i) \mathbf{S} = \sum_{i=1}^{n} \mathbf{S}\, e_i \mathbf{S}\, u^{n-i}.
\tag{1.22}
$$

Lemma **1.6** *(Macdonald).* — *For $l(\lambda) \leq n$ we have*

$$
\Delta_n(u) \cdot \mathrm{J}^{(n)}_\lambda = \prod_{i=1}^{n} \big(u + \lambda_i + \kappa(n-i)\big) \mathrm{J}^{(n)}_\lambda.
$$

The above lemma only gives the eigenvalues of the elements $\mathbf{S}\, e_i \mathbf{S}$ for $i \in [1,n]$, but this is enough to determine the eigenvalues of all the operators $\tilde{\mathrm{D}}^{(n)}_{0,l}$. In fact, Lemma 1.6 has the following immediate corollary.

Corollary **1.7.** — *For* $f \in \mathbf{C}[y_1, \ldots, y_n]^{\mathfrak{S}_n}$ *and* $l(\lambda) \leq n$ *we have*

$$\mathbf{S} f \mathbf{S} \cdot \mathrm{J}_\lambda^{(n)} = f\big(\lambda_1 + \kappa(n-1), \lambda_2 + \kappa(n-2), \ldots, \lambda_n\big) \mathrm{J}_\lambda^{(n)}.$$

Since the joint spectrum of the $\tilde{\mathrm{D}}_{0,l}^{(n)}$ is simple, the Jack polynomials $\{\mathrm{J}_\lambda^{(n)}\}$ are completely determined by Lemma 1.6, up to a scalar. Following Stanley [35] we normalize this scalar by requiring that.

$$\mathrm{J}_\lambda^{(n)} \in \bigoplus_{(1^n) < \mu \leq \lambda} \mathrm{F} m_\mu + |\lambda|!\, \mathrm{X}_1 \cdots \mathrm{X}_n. \tag{1.23}$$

For future use, we state here the Pieri rules for Jack polynomials [35, thm 6.1]. For a pair of partitions $\mu \subset \lambda$ with $|\lambda| - |\mu| = 1$ we write

$$\psi_{\lambda\backslash\mu} = \prod_{s \in \mathrm{C}_{\lambda\backslash\mu}} \frac{h_\mu(s)}{h_\lambda(s)} \prod_{s \in \mathrm{R}_{\lambda\backslash\mu}} \frac{h^\mu(s)}{h^\lambda(s)} \tag{1.24}$$

where $\mathrm{C}_{\lambda\backslash\mu}$ and $\mathrm{R}_{\lambda\backslash\mu}$ are as in Section 0.1, and where for any box s of a partition λ, we have set

$$h_\lambda(s) = \kappa\, l_\lambda(s) + \big(a_\lambda(s) + 1\big), \qquad h^\lambda(s) = \kappa\big(l_\lambda(s) + 1\big) + a_\lambda(s). \tag{1.25}$$

Theorem **1.8** *(Stanley).* — *For* $l(\mu) \leq n$ *we have*

$$e_1 \mathrm{J}_\mu^{(n)} = \sum_\lambda \psi_{\lambda\backslash\mu} \mathrm{J}_\lambda^{(n)}, \tag{1.26}$$

where the sum ranges over all partitions λ *of length at most* n *with* $\mu \subset \lambda$ *and* $|\lambda| = |\mu| + 1$.

1.4. *The normalized Sekiguchi operators.* — The eigenvalues of the operators $\tilde{\mathrm{D}}_{0,l}^{(n)}$ on the Jack polynomial $\mathrm{J}_\lambda^{(n)}$ for $l(\lambda) \leq n$ do depend on n. In order to correct this, we will introduce a new set of diagonalisable operators $\mathrm{D}_{0,l}^{(n)}$, whose eigenvalues on the $\mathrm{J}_\lambda^{(n)}$'s are independent of n. We may think of these new operators as normalized Sekiguchi operators. We'll use the following simple combinatorial lemma. Given a box s in the diagram of a partition λ we'll write

$$c(s) = x(s) - \kappa y(s). \tag{1.27}$$

Lemma **1.9.** — *For* $l \in \mathbf{N}$ *there exists a unique element* $\mathrm{B}_l^{(n)} \in \mathrm{A}[y_1, \ldots, y_n]^{\mathfrak{S}_n}$ *such that*

$$\mathrm{B}_l^{(n)}(\lambda_1 - \kappa, \lambda_2 - 2\kappa, \ldots, \lambda_n - n\kappa) = \sum_{s \in \lambda} c(s)^l, \quad \forall\, \lambda,\ l(\lambda) \leq n.$$

Proof. — There exist polynomials $\mathrm{T}_{r,i}(z) \in \mathrm{A}[z]$ such that for all $l \geq 0$ we have

$$\mathrm{T}_{r,i}(l) = \sum_{j=1}^{l} \big((j-1) - \kappa(i-1)\big)^r.$$

Then, for $l(\lambda) \leq n$, we have

$$\sum_{s\in\lambda} c(s)^r = \sum_{i=1}^{n} \mathrm{T}_{r,i}(\lambda_i) = \sum_{i=1}^{n} \tilde{\mathrm{T}}_{r,i}(\lambda_i - \kappa i), \qquad \tilde{\mathrm{T}}_{r,i}(z) = \mathrm{T}_{r,i}(z+\kappa i).$$

The existence of $\mathrm{B}_r^{(n)}$ will be proved if we can show that $\tilde{\mathrm{T}}_{r,i}(z) - \tilde{\mathrm{T}}_{r,j}(z) \in \mathrm{A}$ for any i, j (as polynomials in z). For this, it is enough to show that for all i, j

$$(\mathbf{1.28}) \qquad \tilde{\mathrm{T}}_{r,i}(z) - \tilde{\mathrm{T}}_{r,i}(z-1) = \tilde{\mathrm{T}}_{r,j}(z) - \tilde{\mathrm{T}}_{r,j}(z-1).$$

We have $\mathrm{T}_{r,i}(z) - \mathrm{T}_{r,i}(z-1) = (z-1-\kappa(i-1))^r$, since this holds for any $z \in \mathbf{N}$. Therefore $\tilde{\mathrm{T}}_{r,i}(z) - \tilde{\mathrm{T}}_{r,i}(z-1) = (z-1+\kappa)^r$ for any i, from which (1.28) is immediate. The unicity statement is clear. □

Now, we define the operators

$$(\mathbf{1.29}) \qquad \mathrm{D}_{0,l}^{(n)} = \mathbf{S}\mathrm{B}_{l-1}^{(n)}(y_1 - n\kappa, y_2 - n\kappa, \ldots, y_n - n\kappa)\mathbf{S}, \qquad l \geq 1.$$

By Corollary 1.7 and Lemma 1.9 we have,

$$(\mathbf{1.30}) \qquad \mathrm{D}_{0,l}^{(n)} \cdot \mathrm{J}_\lambda^{(n)} = \sum_{s\in\lambda} c(s)^{l-1} \mathrm{J}_\lambda^{(n)}, \qquad \forall \lambda,\ l(\lambda) \leq n.$$

In particular, we have $\mathrm{D}_{0,l}^{(n)}(1) = 0$ and the eigenvalues of $\mathrm{D}_{0,l}^{(n)}$ are independent of n. It is easy to see from the proof of Lemma 1.9 that

$$(\mathbf{1.31}) \qquad \mathrm{B}_{l-1}^{(n)} = p_l/l + q_l,$$

with q_l a symmetric function of degree $< l$. Thus $\{\mathrm{B}_0^{(n)}, \ldots, \mathrm{B}_{n-1}^{(n)}\}$ is a system of generators of the A-algebra $\mathrm{A}[y_1, \ldots, y_n]^{\mathfrak{S}_n}$. Hence, we have the following.

Lemma **1.10.** — *The* A*-algebra* $\mathbf{SH}_{n,\mathrm{A}}^0$ *is generated by* $\{\mathrm{D}_{0,l}^{(n)};\ l \geq 1\}$.

Remark **1.11.** — For each partition λ set $n(\lambda) = \sum_i \lambda'_i(\lambda'_i - 1)/2$, where as usual λ' is the conjugate partition. The formula (1.30) yields

$$\mathrm{D}_{0,2}^{(n)} \cdot \mathrm{J}_\lambda^{(n)} = \big(n(\lambda') - \kappa n(\lambda)\big)\mathrm{J}_\lambda^{(n)}.$$

Thus, we have $\mathrm{D}_{0,2}^{(n)} = \kappa \,\square_n$, where $\square_n$ is the *Laplace–Beltrami operator*. See e.g., [25, Chap. VI, Sect. 4, Ex. 3], where $\square_n$ is denoted $\square_n^{\kappa^{-1}}$.

1.5. *The algebras* $\mathbf{SH}_n^+$ *and* $\mathbf{SH}_n^-$. — Our aim is to construct some limit of the algebra $\mathbf{SH}_n$ and of the representation ρ_n as n tends to infinity. The algebras $\mathbf{SH}_n$ do not seem to form a nice projective system. Instead, our method is as follows

- first we define limits $\mathbf{SH}^{\pm}$ for the subalgebras $\mathbf{SH}_n^{\pm}$,
- then we define $\mathbf{SH}$ as some amalgamated product of $\mathbf{SH}^+$ with $\mathbf{SH}^-$.

To implement the second point, we first need to understand some relations between $\mathbf{SH}_n^+$ and $\mathbf{SH}_n^-$ inside $\mathbf{SH}_n$. This is what we do in the present paragraph. For $l \geq 1$ we set

$$
\begin{aligned}
\mathrm{D}_{0,0}^{(n)} &= n\mathbf{S}, \\
\mathrm{D}_{\pm l,0}^{(n)} &= \mathbf{S} p_l^{(n)}(\mathrm{X}_1^{\pm 1}, \dots, \mathrm{X}_n^{\pm 1})\mathbf{S}, \\
\mathrm{D}_{1,l}^{(n)} &= [\mathrm{D}_{0,l+1}^{(n)}, \mathrm{D}_{1,0}^{(n)}], \\
\mathrm{D}_{-1,l}^{(n)} &= [\mathrm{D}_{-1,0}^{(n)}, \mathrm{D}_{0,l+1}^{(n)}].
\end{aligned}
\tag{\textbf{1.32}}
$$

By Lemma 1.3, the F-algebra $\mathbf{SH}_n^{\pm}$ is generated by $\{\mathrm{D}_{0,l}^{(n)}, \mathrm{D}_{\pm l,0}^{(n)};\, l \geq 1\}$.

Definition **1.12.** — *Let* $\mathbf{SH}_n^{>}$ *be the* F*-subalgebra of* $\mathbf{SH}_n^+$ *generated by* $\{\mathrm{D}_{1,l}^{(n)};\, l \geq 0\}$. *We define the* F*-subalgebra* $\mathbf{SH}_n^{<}$ *of* $\mathbf{SH}_n^-$ *in a similar way.*

Example **1.13.** — The following identities hold

$$
\begin{aligned}
&\mathrm{D}_{1,1}^{(n)} = \mathbf{S}\Big(\sum_i \mathrm{X}_i y_i\Big)\mathbf{S} - \kappa(n-1)\mathrm{D}_{1,0}^{(n)}/2, \\
&\mathrm{D}_{-1,1}^{(n)} = \mathbf{S}\Big(\sum_i y_i \mathrm{X}_i^{-1}\Big)\mathbf{S} - \kappa(n-1)\mathrm{D}_{-1,0}^{(n)}/2, \\
&[\mathrm{D}_{1,1}^{(n)}, \mathrm{D}_{l,0}^{(n)}] = l\mathrm{D}_{l+1,0}^{(n)}, \qquad [\mathrm{D}_{-l,0}^{(n)}, \mathrm{D}_{-1,1}^{(n)}] = l\mathrm{D}_{-l-1,0}^{(n)}, \qquad l \geq 0.
\end{aligned}
\tag{\textbf{1.33}}
$$

The following is immediate.

Proposition **1.14.** — *For* $l \geq 0$ *the following hold*
(a) $\mathrm{D}_{l,0}^{(n)} \in \mathbf{SH}_n^{>}$ *and* $\mathrm{D}_{-l,0}^{(n)} \in \mathbf{SH}_n^{<}$ *for* $l \neq 0$,
(b) *for* $l(\mu) \leq n$ *we have*

$$
\mathrm{D}_{1,l}^{(n)} \cdot \mathrm{J}_\mu^{(n)} = \sum_\lambda c(\lambda\backslash\mu)^l\, \psi_{\lambda\backslash\mu} \mathrm{J}_\lambda^{(n)}
\tag{\textbf{1.34}}
$$

where the sum ranges over all partitions λ *with* $l(\lambda) \leq n$, $\mu \subset \lambda$ *and* $|\lambda| = |\mu| + 1$.

Note that (1.34) and (1.30) imply that (1.32) holds also for $l = 0$. The next result describes some of the relations between the three algebras $\mathbf{SH}_n^{>}$, $\mathbf{SH}_n^{0}$ and $\mathbf{SH}_n^{<}$. As we will see in Proposition 1.26 below, these relations (which, thanks to the introduction of $\mathrm{D}^{(n)}_{0,0}$, do not depend on n) are the only ones which survive in the limit $n \to \infty$. For $l \geq 0$ we write

$$
\begin{aligned}
&\xi = 1-\kappa,\\
&\mathrm{G}_0(s) = -\log(s),\\
&\mathrm{G}_l(s) = \big(s^{-l}-1\big)/l, \qquad l \neq 0,\\
&\varphi_l(s) = \sum_{q=1,-\xi,-\kappa} s^l\big(\mathrm{G}_l(1-qs)-\mathrm{G}_l(1+qs)\big).
\end{aligned}
\tag{1.35}
$$

Proposition **1.15.** — *The following relations hold in* $\mathbf{SH}_n$

$$\big[\mathrm{D}^{(n)}_{0,l}, \mathrm{D}^{(n)}_{1,k}\big] = \mathrm{D}^{(n)}_{1,l+k-1}, \tag{1.36}$$

$$\big[\mathrm{D}^{(n)}_{0,l}, \mathrm{D}^{(n)}_{-1,k}\big] = -\mathrm{D}^{(n)}_{-1,l+k-1}, \tag{1.37}$$

$$\big[\mathrm{D}^{(n)}_{-1,k}, \mathrm{D}^{(n)}_{1,l}\big] = \mathrm{E}^{(n)}_{k+l} \tag{1.38}$$

where the elements $\mathrm{E}^{(n)}_{k+l}$ *are determined through the formula*

$$
\begin{aligned}
&1+\xi\sum_{l\geq 0}\mathrm{E}^{(n)}_l s^{l+1} = \mathrm{K}\big(\kappa, \mathrm{D}^{(n)}_{0,0}, s\big)\exp\Big(\sum_{l\geq 0}\mathrm{D}^{(n)}_{0,l+1}\varphi_l(s)\Big),\\
&\mathrm{K}(\kappa,\omega,s) = \frac{(1+\xi s)(1+\kappa\omega s)}{1+\xi s+\kappa\omega s}.
\end{aligned}
\tag{1.39}
$$

Proof. — The first two relations are easily deduced from (1.30) and (1.34), and from the faithfulness of the polynomial representation Λ_n. The third relation is the result of a direct computation, see Appendix B. □

From now on we'll abbreviate $\otimes = \otimes_{\mathrm{F}}$ (the tensor product of F-vector spaces).

Proposition **1.16.** — *The multiplication map induces isomorphisms*

$$\mathbf{SH}^{>}_n \otimes \mathbf{SH}^{0}_n \xrightarrow{\sim} \mathbf{SH}^{+}_n, \qquad \mathbf{SH}^{0}_n \otimes \mathbf{SH}^{<}_n \xrightarrow{\sim} \mathbf{SH}^{-}_n.$$

Proof. — By Lemma 1.3, the algebra $\mathbf{SH}^{+}_n$ is generated by the pair of subalgebras $\mathbf{SH}^{>}_n$, $\mathbf{SH}^{0}_n$. Next, (1.36) implies that $[\mathrm{D}^{(n)}_{0,l}, \mathbf{SH}^{>}_n] \subset \mathbf{SH}^{>}_n$ for $l \geq 0$. Thus we have $\mathbf{SH}^{0}_n \cdot \mathbf{SH}^{>}_n = \mathbf{SH}^{>}_n \cdot \mathbf{SH}^{0}_n$. The surjectivity of the multiplication map

$$m : \mathbf{SH}^{>}_n \otimes \mathbf{SH}^{0}_n \to \mathbf{SH}^{+}_n \tag{1.40}$$

follows. To show that m is injective, we may use a degeneration argument similar to the one in Lemma 1.3. We leave the details to the reader. □

Corollary **1.17.** — *The multiplication map induces a surjective map* $\mathbf{SH}_n^{>} \otimes \mathbf{SH}_n^{0} \otimes \mathbf{SH}_n^{<} \to \mathbf{SH}_n$.

Proof. — By Proposition 1.16 and Lemma 1.3, the F-algebra $\mathbf{SH}_n$ is generated by the triplet of subalgebras $\mathbf{SH}_n^{>}$, $\mathbf{SH}_n^{0}$, $\mathbf{SH}_n^{>}$, hence by the collection of generators $\{D_{1,l}^{(n)}, D_{0,l}^{(n)}, D_{-1,l}^{(n)}\}$. We must check that any monomial in these generators may be 'straightened' into a linear combination of monomials in which the generators $\{D_{1,l}^{(n)}, D_{0,l}^{(n)}, D_{-1,l}^{(n)}\}$ appear in that fixed order. It is not difficult to see that relations (1.36)–(1.38) enable one to do this. □

1.6. *The algebra* $\mathbf{SH}^+$. — Let us now address the problem of constructing a limit $\mathbf{SH}^+$ of $\mathbf{SH}_n^+$. The following result is well-known, see e.g., [35, Prop. 2.5].

Lemma **1.18.** — *For* $l(\lambda) \leq n$ *and for any positive integer* $m < n$ *we have*

$$J_\lambda^{(n)}(X_1, \dots, X_m, 0, \dots, 0) = \begin{cases} J_\lambda^{(m)}(X_1, \dots, X_m) & \text{if } l(\lambda) \leq m, \\ 0 & \text{if } l(\lambda) > m. \end{cases}$$

This lemma allows one to define the limit of the symmetric polynomials $J_\lambda^{(n)}$ as n tends to infinity. We will write $J_\lambda = J_\lambda(X)$ for this limit. It is called the *integral form of Jack's symmetric function associated with the parameter* $\alpha = 1/\kappa$. It is denoted by the symbol $J_\lambda^{(1/\kappa)}$ in [25, Chap. VI, (10.22-3)]. The family $\{J_\lambda; \lambda \in \Pi\}$ forms an F-basis of Λ, see [25, Chap. VI]. The map $\pi_n : \Lambda \to \Lambda_n$ is given by $\pi_n(J_\lambda) = J_\lambda^{(n)}$ if $l(\lambda) \leq n$ and $\pi_n(J_\lambda) = 0$ otherwise. The operators $D_{l,0}^{(n)}$, for $l \in \mathbf{N}$, being the multiplication in Λ_n by symmetric functions, obviously stabilize in the limit Λ, since Λ is a ring. For instance $D_{1,0}^{(n)}$ is given by the Pieri formula (1.26), whose coefficients are independent of n. In other words, we have

$$\pi_{n+1,n} \circ D_{l,0}^{(n+1)} = D_{l,0}^{(n)} \circ \pi_{n+1,n} \tag{1.41}$$

where we have denoted by

$$\pi_{n+1,n} : \Lambda_{n+1} \to \Lambda_n \tag{1.42}$$

the projection maps. The kernels of the maps $\pi_{n+1,n}$ are linearly spanned by Jack polynomials, and the operators $D_{0,l}^{(n)}$ are diagonalisable on the basis of Jack polynomials with eigenvalues independent of n. This implies that for all $n, l \geq 1$ we have

$$\pi_{n+1,n} \circ D_{0,l}^{(n+1)} = D_{0,l}^{(n)} \circ \pi_{n+1,n}. \tag{1.43}$$

Since the polynomial representation is faithful and since the F-algebra $\mathbf{SH}_n^+$ is generated by

$$\left\{D_{0,l}^{(n)}, D_{l,0}^{(n)}; l \geq 1\right\}, \tag{1.44}$$

we deduce that the assignement

$$\mathrm{D}_{0,l}^{(n+1)} \mapsto \mathrm{D}_{0,l}^{(n)}, \qquad \mathrm{D}_{l,0}^{(n+1)} \mapsto \mathrm{D}_{l,0}^{(n)} \tag{1.45}$$

extends to a well-defined and surjective F-algebra homomorphism

$$\Phi_{n+1,n} : \mathbf{SH}_{n+1}^{+} \to \mathbf{SH}_{n}^{+}. \tag{1.46}$$

This allows us to consider the following algebra.

Definition **1.19.** — *We define* $\mathbf{SH}^+$ *to be the* F*-subalgebra of* $\prod_{n\geq 1} \mathbf{SH}_n^+$ *generated by the families* $\mathrm{D}_{0,l} = (\mathrm{D}_{0,l}^{(n)})$ *and* $\mathrm{D}_{l,0} = (\mathrm{D}_{l,0}^{(n)})$ *with* $l \geq 1$.

By construction, there are surjective maps

$$\Phi_n\ : \mathbf{SH}^+ \to \mathbf{SH}_n^+, \quad \mathrm{D}_{0,l} \mapsto \mathrm{D}_{0,l}^{(n)}, \quad \mathrm{D}_{l,0} \mapsto \mathrm{D}_{l,0}^{(n)}, \quad l \geq 1, \tag{1.47}$$

such that $\bigcap_n \mathrm{Ker}(\Phi_n) = \{0\}$. Further, we have the following.

Proposition **1.20.** — *There is a faithful representation* ρ^+ *of* $\mathbf{SH}^+$ *on* Λ *such that, for* $l \geq 1$,

$$\rho^+(\mathrm{D}_{0,l})(\mathrm{J}_\lambda) = \sum_{s\in\lambda} c(s)^{l-1}\mathrm{J}_\lambda, \qquad \rho^+(\mathrm{D}_{l,0}) = \textit{multiplication by } p_l.$$

The map π_n *intertwines the representation* ρ^+ *with the representation* ρ_n^+ *of* $\mathbf{SH}_n^+$ *on* Λ_n.

Observe that $\{\mathrm{D}_{0,l};\, l \geq 1\}$ generate a free commutative algebra which is isomorphic to Λ. The same holds for $\{\mathrm{D}_{l,0};\, l \geq 1\}$. We define an $\mathbf{N}$-grading on $\mathbf{SH}^+$, called the *rank grading*, by putting $\mathrm{D}_{l,0}$ in degree l and $\mathrm{D}_{0,l}$ in degree 0. We define a $\mathbf{N}$-filtration on $\mathbf{SH}^+$, called the *order filtration*, such that an element u is of order $\leq k$ if

$$\mathrm{ad}(z_1) \circ \cdots \circ \mathrm{ad}(z_k)(u) \in \mathrm{F}[\mathrm{D}_{l,0};\, l \in \mathbf{N}], \quad \forall z_1, \ldots, z_k \in \mathrm{F}[\mathrm{D}_{l,0};\, l \in \mathbf{N}]. \tag{1.48}$$

Let $\mathbf{SH}^+[r, \leq l]$ the piece of degree r and order $\leq l$. Note that any element of $\mathbf{SH}^+$ has indeed a finite order. Consider the Poincaré polynomial

$$\mathrm{P}_{\mathbf{SH}^+}(t,q) = \sum_{r,l\geq 0} \dim\big(\overline{\mathbf{SH}}^+[r,l]\big)\, t^r q^l, \qquad \overline{\mathbf{SH}}^+[r,l] = \mathbf{SH}^+[r,\leq l]/\mathbf{SH}^+[r,<l]. \tag{1.49}$$

Lemma **1.21.** — *The Poincaré polynomial of* $\mathbf{SH}^+$ *is given by*

$$\mathrm{P}_{\mathbf{SH}^+}(t,q) = \prod_{r,l} \frac{1}{1 - t^r q^l}, \quad (r,l) \in \mathbf{N}_0^2.$$

Proof. — By Proposition 1.2(b), the F-vector space

$$\overline{\mathbf{SH}}_n^+[r,l] = \mathbf{SH}_n^+[r,\leq l]/\mathbf{SH}_n^+[r,<l] \tag{1.50}$$

is isomorphic to the subspace of polynomials in

$$\mathrm{F}[\mathrm{X}_1,\ldots,\mathrm{X}_n,y_1,\ldots,y_n]^{\mathfrak{S}_n} \tag{1.51}$$

of degree r in the X_i's and of degree l in the y_i's. By Proposition 1.2(a) we have

$$\Phi_n\big(\mathbf{SH}^+[r,\leq l]\big) = \mathbf{SH}_n^+[r,\leq l]. \tag{1.52}$$

Thus Φ_n induces a surjective map

$$\begin{aligned} &\overline{\Phi}_n : \overline{\mathbf{SH}}^+ \to \overline{\mathbf{SH}}_n^+, \qquad \overline{\mathrm{D}}_{0,l} \mapsto p_l^{(n)}(y_1,\ldots,y_n)/l, \\ &\overline{\mathrm{D}}_{l,0} \mapsto p_l^{(n)}(\mathrm{X}_1,\ldots,\mathrm{X}_n), \quad l \geq 1. \end{aligned} \tag{1.53}$$

Thus $\overline{\mathbf{SH}}^+[r,l]$ is identified with the space of symmetric polynomials in infinitely many variables

$$\mathrm{F}[\mathrm{X}_1,\mathrm{X}_2,\ldots,y_1,y_2,\ldots]^{\mathfrak{S}_\infty} \tag{1.54}$$

of degree r in the X_i's and degree l in the y_i's. By Weyl's theorem [41] the F-algebra (1.54) is freely generated by the invariants $\sum_{k\geq 1} \mathrm{X}_k^r y_k^l$ for $r,l \geq 0$ and $(r,l) \neq (0,0)$. The result easily follows. □

Remark **1.22.** — The order filtration on $\mathbf{SH}^+$ is *not* the same as the filtration given by putting $\mathrm{D}_{l,0}$ of order 0 and $\mathrm{D}_{0,l}$ of order $\leq l$ (see however Proposition 1.39).

Remark **1.23.** — We have $\rho^+(\mathrm{D}_{0,2}) = \kappa\,\square$, where $\square$ is the Laplace-Beltrami operator in infinitely many variables, i.e., $\square = \square^{\kappa^{-1}} = \varprojlim \square_n^{\kappa^{-1}}$ in Macdonald's notations, see Remark 1.11.

Remark **1.24.** — There is a unique F-algebra homomorphism $\varepsilon^+ : \mathbf{SH}^+ \to \mathrm{F}$ such that $\varepsilon^+(\mathrm{D}_{0,l}) = \varepsilon^+(\mathrm{D}_{l,0}) = 0$ for $l \geq 1$. Indeed, the sum of $\bigoplus_{r\geq 1} \mathbf{SH}^+[r]$ and of the augmentation ideal of $\mathrm{F}[\mathrm{D}_{0,l};\, l\geq 1]$ is a two-sided ideal of $\mathbf{SH}^+$.

1.7. *The algebra* **SH**. — Our next objective is the construction of the limit of the whole algebra $\mathbf{SH}_n$. We construct $\mathbf{SH}$ by 'gluing' together two copies of $\mathbf{SH}^+$, denoted $\mathbf{SH}^+$ and $\mathbf{SH}^-$, with $\mathbf{SH}^- = (\mathbf{SH}^+)^{\mathrm{op}}$, along the subalgebra

$$\mathbf{SH}^0 = \mathrm{F}[\mathrm{D}_{0,l};\, l \geq 0]. \tag{1.55}$$

The extra generator $\mathrm{D}_{0,0}$, which accounts for the limit of the $\mathrm{D}_{0,0}^{(n)}$'s, may be considered as a formal parameter. We'll write $\omega = \mathrm{D}_{0,0}$. For $l \geq 1$ let $\mathrm{D}_{-l,0} \in \mathbf{SH}^{-}$ be the element mapping to $\mathrm{D}_{-l,0}^{(n)}$ for any n. Consider the elements

$$(\mathbf{1.56}) \qquad \mathrm{D}_{1,l} = [\mathrm{D}_{0,l+1}, \mathrm{D}_{1,0}], \qquad \mathrm{D}_{-1,l} = [\mathrm{D}_{-1,0}, \mathrm{D}_{0,l+1}], \quad l \geq 0.$$

Let $\mathbf{SH}^{>}$ be the F-subalgebra of $\mathbf{SH}^{+}$ generated by $\{\mathrm{D}_{1,l};\, l \geq 0\}$. This is the limit of $\mathbf{SH}_n^{>}$ as n tends to infinity. Now put $\mathbf{SH}^{<} = (\mathbf{SH}^{>})^{\mathrm{op}}$. We may view $\mathbf{SH}^{-}$ and $\mathbf{SH}^{<}$ as the limits of $\mathbf{SH}_n^{-}$ and $\mathbf{SH}_n^{<}$ respectively. Note that $\mathbf{SH}^{<}$ is the F-subalgebra of $\mathbf{SH}^{-}$ generated by $\{\mathrm{D}_{-1,l};\, l \geq 0\}$. We define

$$(\mathbf{1.57}) \qquad \mathbf{SH}^{>}[r, \leq l] = \mathbf{SH}^{>} \cap \mathbf{SH}^{+}[r, \leq l], \qquad \mathbf{SH}^{>}[r, \leq l] = \mathbf{SH}^{>} \cap \mathbf{SH}^{+}[r, \leq l].$$

Definition **1.25.** — *Let* $\mathbf{SH}$ *be the* F*-algebra generated by* $\mathbf{SH}^{>}$, $\mathbf{SH}^{0}$ *and* $\mathbf{SH}^{<}$ *modulo the following set of relations*

$$(\mathbf{1.58}) \qquad \omega = \mathrm{D}_{0,0} \quad \textit{is central},$$

$$(\mathbf{1.59}) \qquad [\mathrm{D}_{0,l}, \mathrm{D}_{1,k}] = \mathrm{D}_{1,l+k-1}, \quad l \geq 1,$$

$$(\mathbf{1.60}) \qquad [\mathrm{D}_{0,l}, \mathrm{D}_{-1,k}] = -\mathrm{D}_{-1,l+k-1}, \quad l \geq 1,$$

$$(\mathbf{1.61}) \qquad [\mathrm{D}_{-1,k}, \mathrm{D}_{1,l}] = \mathrm{E}_{k+l}, \quad l, k \geq 0,$$

where the elements E_{k+l} *are determined through the formula*

$$(\mathbf{1.62}) \qquad 1 + \xi \sum_{l \geq 0} \mathrm{E}_l s^{l+1} = \mathrm{K}(\kappa, \omega, s) \exp\Big(\sum_{l \geq 0} \mathrm{D}_{0,l+1} \varphi_l(s)\Big).$$

By Proposition 1.15, there are surjective maps

$$(\mathbf{1.63}) \qquad \Phi_n : \mathbf{SH} \to \mathbf{SH}_n, \quad \mathrm{D}_{0,l} \mapsto \mathrm{D}_{0,l}^{(n)}, \quad \mathrm{D}_{\pm l,0} \mapsto \mathrm{D}_{\pm l,0}^{(n)}, \quad \omega \mapsto n\mathrm{S}, \quad l \geq 1.$$

As above, for each $l \geq 0$ we write $\mathrm{D}_{0,l}$ and $\mathrm{D}_{\pm l,0}$ for the families $(\mathrm{D}_{0,l}^{(n)})$ and $(\mathrm{D}_{\pm l,0}^{(n)})$ in $\prod_{n \geq 1} \mathbf{SH}_n$. The definition of $\mathbf{SH}$ is justified by the following result.

Proposition **1.26.** — (a) *The multiplication map induces isomorphisms*

$$\mathbf{SH}^{>} \otimes \mathbf{SH}^{0} \simeq \mathbf{SH}^{+} \otimes \mathrm{F}[\omega], \qquad \mathbf{SH}^{0} \otimes \mathbf{SH}^{<} \simeq \mathbf{SH}^{-} \otimes \mathrm{F}[\omega],$$

$$\mathbf{SH}^{>} \otimes \mathbf{SH}^{0} \otimes \mathbf{SH}^{<} \simeq \mathbf{SH}.$$

(b) *The map* $\prod_{n \geq 1} \Phi_n$ *identifies* $\mathbf{SH}$ *with the* F*-subalgebra of* $\prod_{n \geq 1} \mathbf{SH}_n$ *generated by* $\mathrm{D}_{0,l}$ *and* $\mathrm{D}_{\pm l,0}$ *with* $l \geq 0$.

Proof. — The surjectivity statements in (a) are proved as in Proposition 1.16 and Corollary 1.17. To prove (a) it thus remains to show that the multiplication map

$$m : \mathbf{SH}^{>} \otimes \mathbf{SH}^{0} \otimes \mathbf{SH}^{<} \to \mathbf{SH}$$

is injective. Consider the following commutative diagram

$$\begin{tikzcd} \mathbf{SH}^{>} \otimes \mathbf{SH}^{0} \otimes \mathbf{SH}^{<} \arrow[r, "m"] \arrow[d, "\Phi_n^{\otimes 3}"'] & \mathbf{SH} \arrow[d, "\Phi_n"] \\ \mathbf{SH}_n^{>} \otimes \mathbf{SH}_n^{0} \otimes \mathbf{SH}_n^{<} \arrow[r, "m"] & \mathbf{SH}_n. \end{tikzcd} \tag{1.64}$$

Let $u \in \mathrm{Ker}(m)$ and assume that $u \neq 0$. There exist positive integers r_1, r_3, l_1, l_2, l_3 such that

$$u \in \mathbf{SH}^{>}[\leq r_1, \leq l_1] \otimes \mathbf{SH}^{0}[\leq l_2] \otimes \mathbf{SH}^{<}[\leq r_3, \leq l_3].$$

By Definition 1.19 we have

$$\mathbf{SH}^{>} \subset \prod_{n\geq 1} \mathbf{SH}_n^{>}, \qquad \mathbf{SH}^{<} \subset \prod_{n\geq 1} \mathbf{SH}_n^{<}, \qquad \mathbf{SH}^{0} \subset \prod_{n\geq 1} \mathbf{SH}_n^{0}. \tag{1.65}$$

Since we have

$$\mathbf{SH}_n^{0} = \mathrm{F}\big[\mathrm{D}_{0,l}^{(n)};\, l \geq 1\big], \qquad \mathbf{SH}^{0} = \mathrm{F}[\mathrm{D}_{0,l};\, l \geq 0],$$

we have also an inclusion $\mathbf{SH}^{0} \subset \prod_{n\geq 1} \mathbf{SH}_n^{0}$ which identifies the element $\omega = \mathrm{D}_{0,0}$ with the family $(n\mathrm{S})$. Thus, for $n \gg 0$ we have $\Phi_n^{\otimes 3}(u) \neq 0$. By passing to the associated graded and using the PBW theorem, we see that the restriction to

$$\mathbf{SH}_n^{>}[\leq r_1, \leq l_1] \otimes \mathbf{SH}_n^{0}[\leq l_2] \otimes \mathbf{SH}_n^{<}[\leq r_3, \leq l_3]$$

of the map m is injective for $n \gg 0$. But then $\Phi_n \circ m(u) \neq 0$, a contradiction. This shows that $\mathrm{Ker}(m) = \{0\}$. Our argument also implies that $\bigcap_n \mathrm{Ker}(\Phi_n \circ m) = \{0\}$. Hence $\bigcap_n \mathrm{Ker}(\Phi_n) = \{0\}$ because m is surjective. This implies the part (b). □

As a direct consequence of Lemma 1.21 and Proposition 1.26(a) we have the following.

Corollary **1.27.** — *The Poincaré polynomials of* $\mathbf{SH}^{>}$ *and* $\mathbf{SH}^{<}$ *are respectively given by*

$$\mathrm{P}_{\mathbf{SH}^{>}}(t,q) = \prod_{r>0}\prod_{l\geq 0} \frac{1}{1-t^r q^l}, \qquad \mathrm{P}_{\mathbf{SH}^{<}}(t,q) = \prod_{r<0}\prod_{l\geq 0} \frac{1}{1-t^r q^l}.$$

For a future use, let us mention also the following basic facts.

Proposition **1.28.** — (a) *The* F*-algebra* $\mathbf{SH}$ *is generated by the elements* ω, $\mathrm{D}_{1,0}$, $\mathrm{D}_{-1,0}$, $\mathrm{D}_{0,2}$.
(b) *There is a unique anti-involution* π *of* $\mathbf{SH}$ *such that* $\pi(\mathrm{D}_{\pm 1,l}) = \mathrm{D}_{\mp 1,l}$, $\pi(\mathrm{D}_{0,l}) = \mathrm{D}_{0,l}$.

Proof. — From (1.59)–(1.60) we see that $\mathrm{D}_{\pm 1,l}$ is an iterated commutator of $\mathrm{D}_{\pm 1,0}$ and $\mathrm{D}_{0,2}$. From (1.61) we see that $\mathbf{SH}^0$ is generated by the commutators $[\mathrm{D}_{-1,k}, \mathrm{D}_{1,l}]$ for $k, l \geq 0$. This proves (a). Part (b) is obvious. □

Remark **1.29.** — Note that $\{\mathrm{D}_{l,0};\, l \in \mathbf{Z}\}$ generates a commutative subalgebra of $\mathbf{SH}$ (use Proposition 1.26(b) and the commutativity of the elements $\mathrm{D}^{(n)}_{l,0}$, $l \in \mathbf{Z}$, in $\mathbf{SH}_n$).

Remark **1.30.** — In Corollary 6.4 we'll give an explicit description of the subalgebra $\mathbf{SH}^{>}$ of $\mathbf{SH}^{+}$, as a certain shuffle algebra.

1.8. *The algebra* $\mathbf{SH}^{\mathbf{c}}$. — Now, we define a central extension $\mathbf{SH}^{\mathbf{c}}$ of $\mathbf{SH}$. To do this, we introduce a new family $\mathbf{c} = (\mathbf{c}_0, \mathbf{c}_1, \ldots)$ of formal parameters, and for $l \geq 0$ we set

$$\phi_l(s) = s^l \mathrm{G}_l(1+\xi s), \qquad \mathbf{SH}^{\mathbf{c},0} = \mathrm{F}^{\mathbf{c}}[\mathrm{D}_{0,l};\, l \geq 0], \qquad \mathrm{F}^{\mathbf{c}} = \mathrm{F}[\mathbf{c}_l;\, l \geq 0]. \tag{1.66}$$

Definition **1.31.** — *Let* $\mathbf{SH}^{\mathbf{c}}$ *be the* $\mathrm{F}^{\mathbf{c}}$*-algebra generated by* $\mathbf{SH}^{>}$, $\mathbf{SH}^{\mathbf{c},0}$, $\mathbf{SH}^{<}$ *modulo the following set of relations*

$$[\mathrm{D}_{0,l}, \mathrm{D}_{1,k}] = \mathrm{D}_{1,l+k-1}, \qquad l \geq 1, \tag{1.67}$$

$$[\mathrm{D}_{0,l}, \mathrm{D}_{-1,k}] = -\mathrm{D}_{-1,l+k-1}, \qquad l \geq 1, \tag{1.68}$$

$$[\mathrm{D}_{-1,k}, \mathrm{D}_{1,l}] = \mathrm{E}_{k+l}, \qquad l, k \geq 0, \tag{1.69}$$

where $\mathrm{D}_{0,0} = 0$ *and the elements* E_{k+l} *are determined through the formula*

$$1 + \xi \sum_{l \geq 0} \mathrm{E}_l\, s^{l+1} = \exp\Big(\sum_{l \geq 0} (-1)^{l+1} \mathbf{c}_l \phi_l(s)\Big) \exp\Big(\sum_{l \geq 0} \mathrm{D}_{0,l+1}\, \varphi_l(s)\Big). \tag{1.70}$$

Example **1.32.** — A direct computation yields, see Section A,

$$\mathrm{E}_0 = \mathbf{c}_0, \qquad \mathrm{E}_1 = -\mathbf{c}_1 + \mathbf{c}_0(\mathbf{c}_0 - 1)\xi/2,$$
$$\mathrm{E}_2 = \mathbf{c}_2 + \mathbf{c}_1(1-\mathbf{c}_0)\xi + \mathbf{c}_0(\mathbf{c}_0-1)(\mathbf{c}_0-2)\xi^2/6 + 2\kappa \mathrm{D}_{0,1}.$$

For $l \geq 2$ we have also

$$\mathrm{E}_l = l(l-1)\kappa \mathrm{D}_{0,l-1} \mod \mathbf{SH}^{\mathbf{c},0}[\leq l-2] \tag{1.71}$$

where $\mathbf{SH}^{\mathbf{c},0}[\leq l-2]$ is the space of elements of $\mathbf{SH}^{\mathbf{c},0}$ of order at most $l-2$.

Remark **1.33.** — Given a family $c = (c_0, c_1, \dots)$ of elements in an extension of the field F, let $\mathbf{SH}^c$ be the specialization of $\mathbf{SH}^{\mathbf{c}}$ at $\mathbf{c} = c$. The specialization at $\mathbf{c} = 0$ is canonically isomorphic to the specialization of $\mathbf{SH}$ at $\omega = 0$. Next, a direct computation shows that

$$\mathrm{K}(\kappa, \omega, s) = \exp\Big(\sum_{l \geq 0} (-1)^{l+1} \big(\delta_{l,0} - \kappa^l \omega^l\big) \phi_l(s)\Big).$$

Therefore, taking $c_0 = 0$ and $c_l = -\kappa^l \omega^l$ in $\mathrm{F}(\omega)$ for $l \geq 1$, we get an $\mathrm{F}(\omega)$-algebra isomorphism $\mathbf{SH}^c \to \mathbf{SH}$ such that $\mathrm{D}_{1,l} \mapsto \mathrm{D}_{1,l}$ and $\mathrm{D}_{-1,l} \mapsto \mathrm{D}_{-1,l}$ for each $l \geq 0$.

Remark **1.34.** — We abbreviate $\mathbf{SH}^{\mathbf{c}_0,\mathbf{c}_1}$ for the algebra associated with the family of parameters $(\mathbf{c}_0, \mathbf{c}_1, 0, \dots)$. By Appendix A, Remark A.1 there is an algebra isomorphism $\mathbf{SH}^{\mathbf{c}} \to \mathbf{SH}^{\mathbf{c}_0,\mathbf{c}_1} \otimes \mathrm{F}[\mathbf{c}_l; l \geq 2]$ such that $\mathrm{D}_{1,l} \mapsto \mathrm{D}_{1,l}$ and $\mathrm{D}_{-1,l} \mapsto \mathrm{D}_{-1,l}$ for each $l \geq 0$. In other words, the algebra $\mathbf{SH}^{\mathbf{c}}$ depends only on the parameters $\mathbf{c}_0$, $\mathbf{c}_1$ up to isomorphisms.

Proposition **1.35.** — (a) *The* $\mathrm{F}^{\mathbf{c}}$*-algebra* $\mathbf{SH}^{\mathbf{c}}$ *is generated by* $\mathbf{c}_l$, $\mathrm{D}_{1,0}$, $\mathrm{D}_{-1,0}$, $\mathrm{D}_{0,2}$.
(b) *There is a unique anti-involution* π *of* $\mathbf{SH}^{\mathbf{c}}$ *such that* $\pi(\mathbf{c}_l) = \mathbf{c}_l$, $\pi(\mathrm{D}_{\pm 1,l}) = \mathrm{D}_{\mp 1,l}$ *and* $\pi(\mathrm{D}_{0,l}) = \mathrm{D}_{0,l}$.

Proof. — Parts (a), (b) are proved as Proposition 1.28. □

The following specialization of the algebra $\mathbf{SH}^{\mathbf{c}}$ will be important for us.

Definition **1.36.** — *For a field extension* $\mathrm{F} \subset \mathrm{K}$ *and an integer* $r > 0$ *let* $\mathrm{K}_r = \mathrm{K}(\varepsilon_1, \dots, \varepsilon_r)$, *where* $\varepsilon_1, \dots, \varepsilon_r$ *are new formal variables. Consider the algebra homomorphism* $\mathrm{F}^{\mathbf{c}} \to \mathrm{K}_r$, $\mathbf{c}_l \mapsto c_l = p_l(\varepsilon_1, \dots, \varepsilon_r)$. *We define the* K_r*-algebra* $\mathbf{SH}^{(r)}_{\mathrm{K}} = \mathbf{SH}^{\mathbf{c}} \otimes_{\mathrm{F}^{\mathbf{c}}} \mathrm{K}_r$. *We write also* $\mathbf{SH}^{(r),>}_{\mathrm{K}} = \mathbf{SH}^{>} \otimes \mathrm{K}_r$ *and* $\mathbf{SH}^{(r),<}_{\mathrm{K}} = \mathbf{SH}^{<} \otimes \mathrm{K}_r$.

Like $\mathbf{SH}$, the algebra $\mathbf{SH}^{\mathbf{c}}$ has a triangular decomposition. More precisely,

Proposition **1.37.** — *The multiplication map* $\mathbf{SH}^{>} \otimes \mathbf{SH}^{\mathbf{c},0} \otimes \mathbf{SH}^{<} \to \mathbf{SH}^{\mathbf{c}}$ *is an isomorphism.*

Proof. — The injectivity follows from Corollary D.2 and the commutativity of the diagram

$$(\mathbf{1.72})\qquad \begin{array}{ccc} \mathbf{SH}^{(r),>}_{\mathrm{K}} \otimes \mathbf{SH}^{(r),0}_{\mathrm{K}} \otimes \mathbf{SH}^{(r),<}_{\mathrm{K}} & \longrightarrow & \mathbf{SH}^{(r)}_{\mathrm{K}} \\ \uparrow & & \uparrow \\ (\mathbf{SH}^{>} \otimes \mathbf{SH}^{\mathbf{c},0} \otimes \mathbf{SH}^{<}) \otimes \mathrm{K}_r & \longrightarrow & \mathbf{SH}^{\mathbf{c}} \otimes \mathrm{K}_r \end{array}$$

for all r. The vertical maps are given by the specialization and the horizontal ones by the multiplication. The proof of the surjectivity is similar to the proof of Corollary 1.17. Since $\mathbf{SH^c}$ is generated by $\mathbf{SH}^>$, $\mathbf{SH}^{\mathbf{c},0}$ and $\mathbf{SH}^<$, and since these subalgebras are respectively generated by $\{D_{1,l};\, l \geq 1\}$, $\{D_{0,l};\, l \geq 0\}$ and $\{D_{-1,l};\, l \geq 0\}$, it suffices to prove that any monomial $D_{\mathbf{x}_1} \cdots D_{\mathbf{x}_s}$ may be expressed as a linear combination of monomials in which the generators $D_{1,l}$, $D_{0,l}$, $D_{-1,l}$ appear in that fixed order. The relations (1.67)–(1.69) allow one to do that. $\square$

Remark **1.38.** — There is a unique F-algebra homomorphism $\varepsilon : \mathbf{SH^c} \to \mathrm{F}$ such that $\varepsilon|_{\mathbf{SH}^+} = \varepsilon^+$, $\varepsilon|_{\mathbf{SH}^-} = \varepsilon^+ \circ \pi$ and $\varepsilon(\mathbf{c}_l) = 0$ for each $l \in \mathbf{N}$. Use Remark 1.24 and Definition 1.31.

1.9. *The order filtration on* $\mathbf{SH^c}$. — In this section we extend the order filtration on $\mathbf{SH}^+$ to $\mathbf{SH^c}$. Let $\mathbf{SH^c}[s, \leq l]$ be the image by the multiplication map of the F-vector space

$$\sum_{s_1, s_3, l_1, l_2, l_3} \mathbf{SH}^>[s_1, \leq l_1] \otimes \mathbf{SH}^{\mathbf{c},0}[\leq l_2] \otimes \mathbf{SH}^<[s_3, \leq l_3]. \tag{1.73}$$

The sum is over all tuples such that $s_1 - s_3 = s$ and $l_1 + l_2 + l_3 = l$. The F-subspaces $\mathbf{SH}^>[s_1, \leq l_1]$ and $\mathbf{SH}^<[s_3, \leq l_3]$ are as in (1.57), and $\mathbf{SH}^{\mathbf{c},0}[\leq l_2]$ is the $\mathrm{F}^{\mathbf{c}}$-subalgebra of $\mathbf{SH}^{\mathbf{c},0}$ spanned by the polynomials in the elements $D_{0,l}$ of order $\leq l_2$. By Proposition 1.37, the F-algebra $\mathbf{SH^c}$ carries a $\mathbf{Z}$-grading and an $\mathbf{N}$-filtration

$$\begin{aligned} \mathbf{SH^c} &= \bigoplus_{s \in \mathbf{Z}} \mathbf{SH^c}[s], & \mathbf{SH^c} &= \bigcup_{l \in \mathbf{N}} \mathbf{SH^c}[\leq l], \\ \mathbf{SH^c}[s] &= \bigcup_{l \in \mathbf{N}} \mathbf{SH^c}[s, \leq l], & \mathbf{SH^c}[\leq l] &= \bigoplus_{s \in \mathbf{Z}} \mathbf{SH^c}[s, \leq l]. \end{aligned} \tag{1.74}$$

We will prove that $\mathbf{SH^c}$, with this filtration, is a filtered algebra. This will imply that the associated graded $\overline{\mathbf{SH}}^{\mathbf{c}}$ is an algebra. Following (1.33), we define inductively the element $D_{l,0} \in \mathbf{SH^c}$ so that $D_{-1,0}$, $D_{1,0}$ are as above and

$$[D_{1,1}, D_{l,0}] = l D_{l+1,0}, \qquad [D_{-l,0}, D_{-1,1}] = l D_{-l-1,0}, \qquad l \geq 0. \tag{1.75}$$

In addition, for $l, r \geq 1$, we set

$$D_{r,l} = [D_{0,l+1}, D_{r,0}], \qquad D_{-r,l} = [D_{-r,0}, D_{0,l+1}]. \tag{1.76}$$

This notation is compatible with the previous definition of $D_{\pm 1,l}$. The elements $D_{r,l}$ satisfy the following properties, see Lemma E.3,

$$D_{r,l} \in \mathbf{SH}^>, \qquad D_{-r,l} \in \mathbf{SH}^<, \qquad \pi(D_{l,0}) = D_{-l,0}, \qquad [D_{0,1}, D_{l,0}] = l D_{l,0}. \tag{1.77}$$

Proposition **1.39.** — (a) *The element* $\mathrm{D}_{r,d}$ *is of order* d*, i.e., we have*

$$\mathrm{D}_{r,d} \in \mathbf{SH}^{>}[\leq d] \setminus \mathbf{SH}^{>}[<d], \qquad \mathrm{D}_{-r,d} \in \mathbf{SH}^{<}[\leq d] \setminus \mathbf{SH}^{<}[<d], \quad r \geq 1.$$

The symbols of the elements $\mathrm{D}_{\pm r,d}$ *with* $(r,d) \in \mathbf{N}_0^2$ *freely generate* $\overline{\mathbf{SH}}^{\pm}$.

(b) *The order filtration on* $\mathbf{SH}^{\mathbf{c}}$ *is determined by assigning to* $\mathrm{D}_{r,d}$*,* $\mathbf{c}_l$ *the orders* d *and zero.*

(c) *For* $l_1, l_2 \geq 0$ *we have* $\mathbf{SH}^{\mathbf{c}}[\leq l_1] \cdot \mathbf{SH}^{\mathbf{c}}[\leq l_2] \subseteq \mathbf{SH}^{\mathbf{c}}[\leq l_1 + l_2]$.

Proof. — It is known that $\mathrm{D}_{0,d}^{(n)}$ is of order d in $\mathbf{SH}_n^+$ for any n hence $\mathrm{D}_{0,d}$ is of order d in $\mathbf{SH}^+$. Similarly, $\mathrm{D}_{r,0}$ is of order zero. It follows that $\mathrm{D}_{r,d}$ is of order at most d. Let $\overline{\mathrm{D}}_{\pm r,d}$, (resp. $\overline{\mathrm{D}}_{\pm r,d}^{(n)}$) be the symbol of the element $\mathrm{D}_{\pm r,d}$ (resp. $\mathrm{D}_{\pm r,d}^{(n)}$) in the associated graded $\overline{\mathbf{SH}}^{\pm}$ (resp. $\overline{\mathbf{SH}}_n^{\pm}$). A direct computation shows that

$$\overline{\mathrm{D}}_{r,d}^{(n)} = c_{r,d} \sum_i \mathbf{S}\, \overline{\mathrm{X}}_i^r \overline{y}_i^d\, \mathbf{S} \in \overline{\mathbf{SH}}_n^+, \qquad c_{r,d} \in \mathrm{F}^\times. \tag{1.78}$$

This means that $\mathrm{D}_{r,d}^{(n)}$, hence also $\mathrm{D}_{r,d}$, is of order d. Equation (1.78) also shows that the set $\{\overline{\mathrm{D}}_{\mathbf{x}}^{(n)}; \mathbf{x} \in \mathbf{N}_0^2\}$ generates $\overline{\mathbf{SH}}_n^+$, and therefore that $\{\overline{\mathrm{D}}_{\mathbf{x}}; \mathbf{x} \in \mathbf{N}_0^2\}$ likewise generates $\overline{\mathbf{SH}}^+$. Comparing graded dimensions we get that these same generators freely generate $\overline{\mathbf{SH}}^+$. This proves (a) for $\mathbf{SH}^+$. The same proof works for $\mathbf{SH}^-$.

We now turn to part (b). Let $\mathbf{SH}^{\mathbf{c}}[\preccurlyeq l]$ temporarily denote the degree at most l piece of $\mathbf{SH}^{\mathbf{c}}$ with respect to the filtration defined by (b). By (a) we have $\mathbf{SH}^{\pm}[\preccurlyeq l] = \mathbf{SH}^{\pm}[\leq l]$ for any l, and the same holds for $\mathbf{SH}^{>}$, $\mathbf{SH}^{\mathbf{c},0}$ and $\mathbf{SH}^{<}$. From the definition (1.73) we immediately have $\mathbf{SH}^{\mathbf{c}}[\leq l] \subseteq \mathbf{SH}^{\mathbf{c}}[\preccurlyeq l]$. By construction, we have

$$\mathbf{SH}^{\mathbf{c}}[\preccurlyeq l] = \big\{u_1 u_2 \cdots u_s;\ u_i \in \mathbf{SH}^{\epsilon_i}[\leq l_i],\ \epsilon_i \in \{>, 0, <\},\ l_1 + \cdots + l_s = l\big\}. \tag{1.79}$$

Thus, in order to show the inclusion $\mathbf{SH}^{\mathbf{c}}[\preccurlyeq l] \subseteq \mathbf{SH}^{\mathbf{c}}[\leq l]$, it is enough to prove that

$$\mathbf{SH}^{\mathbf{c}}[\leq l_1] \cdot \mathbf{SH}^{\mathbf{c}}[\leq l_2] \subseteq \mathbf{SH}^{\mathbf{c}}[\leq l_1 + l_2], \tag{1.80}$$

which reduces to

$$\mathrm{ad}(\mathrm{D}_{r,d})\big(\mathbf{SH}^{\mathbf{c}}[\leq l]\big) \subset \mathbf{SH}^{\mathbf{c}}[\leq l + d]. \tag{1.81}$$

Rather than using the elements $\mathrm{D}_{r,d}$ we introduce a more convenient set of elements. For $d \geq 0$ and $r = -1, 0, 1$ we set $\mathrm{Y}_{r,d} = \mathrm{D}_{r,d}$ and we define inductively, for $r \geq 2$,

$$\begin{aligned} \mathrm{Y}_{r,d} &= \begin{cases} [\mathrm{D}_{1,1}, \mathrm{Y}_{r-1,d}] & \text{if } r-1 \neq d \\ [\mathrm{D}_{1,0}, \mathrm{Y}_{r-1,d+1}] & \text{if } r-1 = d, \end{cases} \\ \mathrm{Y}_{-r,d} &= \begin{cases} [\mathrm{D}_{-1,1}, \mathrm{Y}_{1-r,d}] & \text{if } r-1 \neq d \\ [\mathrm{D}_{-1,0}, \mathrm{Y}_{1-r,d+1}] & \text{if } r-1 = d. \end{cases} \end{aligned} \tag{1.82}$$

We have $Y_{r,d} \in \mathbf{SH}^{>}$ and $Y_{-r,d} \in \mathbf{SH}^{<}$. One shows by arguments similar to those used in (a) above that $Y_{r,d}$ is of order exactly d and that the symbols $\overline{Y}_{r,d}$ freely generate $\overline{\mathbf{SH}}^{\pm}$. We will now prove that

$$(\mathbf{1.83}) \qquad \mathrm{ad}(Y_{r,d})\big(\mathbf{SH}^{\mathbf{c}}[\leq l]\big) \subset \mathbf{SH}^{\mathbf{c}}[\leq l+d].$$

Since $\mathrm{ad}(Y_{r,d})$ is an iterated commutator of operators $\mathrm{ad}(D_{0,l})$, $\mathrm{ad}(D_{\pm 1,1})$, $\mathrm{ad}(D_{\pm 1,0})$, it is enough to prove (1.83) for each of those. For $D_{0,l}$ this comes from the fact that $\mathbf{SH}^{\pm}$ are filtered algebras. For the others it is enough to show that

$$(\mathbf{1.84}) \qquad \mathrm{ad}(D_{\pm 1,1})(Y_{\pm r,l}) \in \mathbf{SH}^{\pm}[\leq l], \qquad \mathrm{ad}(D_{\pm 1,0})(Y_{\pm r,l}) \in \mathbf{SH}^{\pm}[< l] \quad r \geq 0,\ l \geq 0,$$

$$(\mathbf{1.85}) \qquad \mathrm{ad}(D_{\pm 1,1})(Y_{\mp r,l}) \in \mathbf{SH}^{\mp}[\leq l], \qquad \mathrm{ad}(D_{\pm 1,0})(Y_{\mp r,l}) \in \mathbf{SH}^{\mp}[< l] \quad r > 0,\ l \geq 0.$$

Both (1.84) and (1.85) easily follow from the inductive definition of $Y_{r,d}$ and from the relations

$$[D_{-1,1}, D_{1,1}] = E_2, \qquad [D_{-1,1}, D_{1,0}] = [D_{-1,0}, D_{1,1}] = E_1.$$

Statement (c) was proved on the way. $\square$

1.10. *Wilson operators on* $\mathbf{SH}^{>}$. — Recall that $\mathbf{SH}^{0} = F[D_{0,l};\, l \geq 1]$. By (1.67) the commutator with $D_{0,l}$ preserves $\mathbf{SH}^{>}$ and the operators $\mathrm{ad}(D_{0,l})$ commute with each other. This extends uniquely to an action of the algebra $\mathbf{SH}^{0}$ on $\mathbf{SH}^{>}$ satisfying

$$(\mathbf{1.86}) \qquad D_{0,l} \bullet u = [D_{0,l}, u], \quad u \in \mathbf{SH}^{>},\ l \geq 0.$$

Recall that Λ carries a comultiplication given by

$$(\mathbf{1.87}) \qquad \Delta(p_l) = p_l \otimes 1 + 1 \otimes p_l, \quad l \geq 1.$$

We'll use Sweedler's notation $\Delta(x) = \sum x_1 \otimes x_2$. We identify $\mathbf{SH}^{0}$ and Λ via $D_{0,l} \mapsto p_l$. We hence have an action

$$(\mathbf{1.88}) \qquad \bullet : \Lambda \otimes \mathbf{SH}^{>} \to \mathbf{SH}^{>},$$

which we call the action by *Wilson operators*. For a field extension $F \subset K$ let $\bullet$ denote again the corresponding action of Λ_K on $\mathbf{SH}^{>}_K$. The following lemma is left to the reader.

Lemma **1.40.** — (a) *The action of* Λ *on* $\mathbf{SH}^{>}$ *preserves each graded piece of* $\mathbf{SH}^{>}$,
(b) *the action of* Λ *on the degree* n *part of* $\mathbf{SH}^{>}$ *factors through* Λ_n,
(c) *the Wilson operators are compatible with the coproduct, namely*

$$x \bullet (uv) = \sum (x_1 \bullet u)(x_2 \bullet v), \quad x \in \Lambda,\ u, v \in \mathbf{SH}^{>}.$$

1.11. *The Heisenberg and Virasoro subalgebras.* — For $l \geq 1$ we define the following elements

$$
\begin{aligned}
&b_l = \kappa^{-l} \mathrm{D}_{-l,0}, \qquad b_{-l} = \mathrm{D}_{l,0}, \qquad b_0 = \mathrm{E}_1/\kappa, \\
&\mathrm{H}_l = \kappa^{-l} \mathrm{D}_{-l,1}/l + (1-l)\mathbf{c}_0 \xi\, b_l/2, \qquad \mathrm{H}_{-l} = \mathrm{D}_{l,1}/l + (1-l)\mathbf{c}_0 \xi\, b_{-l}/2, \\
&\mathrm{H}_0 = [\mathrm{H}_1, \mathrm{H}_{-1}]/2.
\end{aligned}
\tag{1.89}
$$

These elements will be important to define a Virasoro subalgebra in a completion of $\mathbf{SH}^{(r)}_{\mathrm{K}}$ in Section 8.11. In Appendix E we prove the following.

Proposition **1.41.** — *For $k, l \in \mathbf{Z}$ we have*

$$[b_l, b_{-k}] = l\, \delta_{l,k}\, \mathbf{c}_0/\kappa, \tag{1.90}$$

$$[\mathrm{H}_{-1}, b_l] = -l\, b_{l-1}, \qquad [\mathrm{H}_1, b_l] = -l b_{l+1}. \tag{1.91}$$

Let $\mathscr{H}$ be the Heisenberg subalgebra of $\mathbf{SH}^{\mathbf{c}}$ generated by $\{b_l;\, l \in \mathbf{Z}\}$ and $\mathbf{c}_0$.

2. Equivariant cohomology of the Hilbert scheme

In this section, we recall briefly the structure of the Hilbert scheme of points on the complex plane $\mathbf{C}^2$, and we define a convolution algebra acting on its (equivariant, Borel-Moore) homology groups. This is essentially a homology version of the K-theoretic construction given in [33], to which we refer the reader for a more detailed treatment. All the geometric properties of the Hilbert scheme which we use below may be found in [12, 40].

2.1. *Equivariant cohomology and Borel-Moore homology.* — Let G be a complex, connected, linear algebraic group and let X be a G-variety, that is an algebraic variety equipped with a rational G action. By a variety we always mean a complex quasi-projective variety. Let $\mathrm{H}^i_{\mathrm{G}}(\mathrm{X})$ and $\mathrm{H}^{\mathrm{G}}_i(\mathrm{X})$ be the equivariant cohomology group and the equivariant Borel-Moore homology group of X, with $\mathbf{C}$ coefficients. We write

$$\mathrm{H}_{\mathrm{G}}(\mathrm{X}) = \bigoplus_i \mathrm{H}^i_{\mathrm{G}}(\mathrm{X}), \qquad \mathrm{H}^{\mathrm{G}}(\mathrm{X}) = \bigoplus_i \mathrm{H}^{\mathrm{G}}_i(\mathrm{X}). \tag{2.1}$$

Both of these spaces are graded modules over the graded ring $\mathrm{R}_{\mathrm{G}} = \mathrm{H}_{\mathrm{G}}(\bullet)$, where $\bullet$ is a point with a trivial G-action. Recall that $\mathrm{H}^{\mathrm{G}}_i(\mathrm{X}) = \mathrm{H}^{-i}_{\mathrm{G}}(\mathrm{X}, \mathcal{D})$ where $\mathcal{D}$ is the G-equivariant dualizing complex, see [5, Def. 3.5.1] or [19, Sect. 5.8]. Recall that

$$\mathrm{H}^{\mathrm{G}}_i(\mathrm{X}) = \mathrm{H}_{i+2\dim \mathrm{E}-2\dim \mathrm{G}}(\mathrm{X} \times_{\mathrm{G}} \mathrm{E}), \tag{2.2}$$

where $\mathrm{E} \to \mathrm{E}/\mathrm{G}$ is a principal G-bundle such that $\mathrm{H}^j(\mathrm{E}) = 0$ for $j = 1, 2, \ldots, i$. The cup product endows $\mathrm{H}_{\mathrm{G}}(\mathrm{X})$ with the structure of a graded commutative R_{G}-algebra. We denote by $[Y] \in H^G(X)$ the fundamental class of a G-stable subvariety $Y \subset X$. If Y is pure of dimension d then the class [Y] has the degree $2d$. Let us now assume that X is smooth and connected. Then the map $\alpha \mapsto \alpha \cdot [X]$, where $\cdot$ denotes the cap product, defines a Poincaré duality isomorphism

$$H^i_G(X) \to H^G_{2\dim X - i}(X). \tag{2.3}$$

This allows us to define a product on $H^G(X)$, dual to the cup product on $H_G(X)$. If E is a G-equivariant vector bundle over X then we write

$$c^i(E) \in H^{2i}_G(X), \qquad c_i(E) = c^i(E) \cdot [X] \in H^G_{2\dim X - 2i}(X)$$

for the equivariant Chern classes of E. We write $eu(E) = c_r(E)$ where r is the rank of E. We call eu(E) the *Euler class* of E. We have

$$c_1\big(E \oplus E'\big) = c_1(E) + c_1\big(E'\big), \qquad eu\big(E \oplus E'\big) = eu(E)\ eu\big(E'\big).$$

Fix a morphism $f : X \to Y$ of complex G-varieties. If f is a proper map there is a direct image homomorphism

$$f_* : H^G_i(X) \to H^G_i(Y). \tag{2.4}$$

If f is a fibration or if X, Y are smooth complex G-varieties there is an inverse image homomorphism (given, in the second case, by the Poincaré duality isomorphism and the pull-back in equivariant cohomology)

$$f^* : H^G_i(Y) \to H^G_{i+2d}(X), \quad d = \dim X - \dim Y. \tag{2.5}$$

Note that if Y is smooth and $Z \subset Y$ is closed then $H^G(Z) = H_G(Y, Y \setminus Z)$. So, if X, Y are both smooth and $Z \subset Y$ is closed then the pull-back in equivariant cohomology gives a map

$$H^G(Z) = H_G(Y, Y \setminus Z) \to H_G\big(X, X \setminus f^{-1}(Z)\big) = H^G\big(f^{-1}(Z)\big). \tag{2.6}$$

These maps fit into the commutative square

$$\begin{tikzcd} H^G(Z) \arrow[r] \arrow[d] & H^G(f^{-1}(Z)) \arrow[d] \\ H^G(Y) \arrow[r] & H^G(X). \end{tikzcd} \tag{2.7}$$

Further, given a Cartesian square of smooth complex G-varieties

$$\begin{tikzcd} Y' \arrow[d, "i'"'] & X' \arrow[l, "g"'] \arrow[d, "i"] \\ Y & X, \arrow[l, "f"'] \end{tikzcd} \tag{2.8}$$

where i, i' are proper, we have the base change identity $f^*i'_* = i_*g^*$. If Y', X' are no longer smooth but i, i' are closed embeddings then the inverse morphism g^* is still well-defined and the base change identity above holds.

Example **2.1.** — Assume that T is a torus and that X is a point. Then $R_T = S(\mathfrak{t}^*)$, where $\mathfrak{t}$ is the Lie algebra of T. Let E be a finite dimensional representation of T. Write it as a sum of characters $E = \chi_1 \oplus \cdots \oplus \chi_r$ with $\chi_a : T \to \mathbf{C}^\times$. Then we have $c_l(E) = e_l(d\chi_1, \ldots, d\chi_r)$ where $d\chi_a \in \mathfrak{t}^*$ is the differential of χ_a. In particular,

$$c_1(E) = \sum_a d\chi_a \in R_T^2, \qquad eu(E) = \prod_a d\chi_a \in R_T^{2r}. \tag{2.9}$$

Note that, since the Euler class is multiplicative, we may consider the element eu(E) in K_T, the fraction field of R_T, for an arbitrary virtual T-module E. For any characters χ, χ' of T, we may abbreviate $\chi^* = \chi^{-1}$ and $\chi \otimes \chi' = \chi\,\chi'$. If $T = (\mathbf{C}^\times)^2$ we get

$$R_T = \mathbf{C}[x, y], \tag{2.10}$$

where $x = c_1(q) = dq$, $y = c_1(t) = dt$ and q, t are the characters of T given by

$$q(z_1, z_2) = z_1^{-1}, \qquad t(z_1, z_2) = z_2^{-1}. \tag{2.11}$$

2.2. *Correspondences.* — We can now define the *convolution product* in equivariant homology. Let X_1, X_2, X_3 be smooth connected algebraic G-varieties. Let us denote by $\pi_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ the projection along the factor not named. If X_1, X_2, X_3 are proper, there is a map

$$\begin{aligned} &\star : H^G(X_1 \times X_2) \otimes H^G(X_2 \times X_3) \to H^G(X_1 \times X_3), \\ &\alpha \otimes \beta \mapsto \pi_{13!}\big(\pi_{12}^*(\alpha) \cdot \pi_{23}^*(\beta)\big). \end{aligned} \tag{2.12}$$

If $X_1 = X_2 = X_3 = X$ then the map $\star$ equips $H^G(X \times X)$ with the structure of an associative R_G-algebra. If $X_1 = X_2 = X$ and $X_3 = \bullet$ then we obtain an action of the R_G-algebra $H^G(X \times X)$ on the R_G-module $H^G(X)$. If X_1, X_2, X_3 are not proper but there is a smooth closed G-subvariety $Z \subset X_1 \times X_2$ such that the projection $Z \to X_1$ is proper then any element $z \in H^G(Z)$ defines an R_G-linear operator

$$H^G(X_2) \to H^G(X_1), \qquad \alpha \mapsto z \star \alpha = \pi_{1!}\big(z \cdot \pi_2^*(\alpha)\big). \tag{2.13}$$

If the projection $Z \to X_1$ is not proper but $i : Z^c \to Z$ is the inclusion of a smooth closed G-subvariety such that $\pi_1 \circ i$ is proper, then any element $z^c \in H^G(Z^c)$ defines an R_G-linear operator

$$H^G(X_2) \to H^G(X_1), \qquad \alpha \mapsto z^c \star \alpha = \pi_{1!}^c\big(z^c \cdot \pi_2^{c,*}(\alpha)\big), \qquad \pi_a^c = \pi_a \circ i, \tag{2.14}$$

and the projection formula implies that $z^c \star \alpha = i_*(z^c) \star \alpha$.

Remark **2.2.** — The R_G-modules $H^G(X_1)$, $H^G(X_2)$ are graded by the *homological* and *cohomological* degrees, for which $H_i^G(X_a)$ has the degree i and $2\dim X_a - i$ respectively. Let deg denote the homological degree and cdeg the cohomological degree. Then, if $z \in H_i^G(Z)$ then the convolution by z is a homogeneous operator for the (co-)homological degrees, and we have

$$\deg(z \star \bullet) = i - 2\dim X_2, \qquad \operatorname{cdeg}(z \star \bullet) = 2\dim X_1 - i. \tag{2.15}$$

2.3. *The Hilbert scheme.* — Let Hilb_n denote the Hilbert scheme parametrizing length n subschemes of $\mathbf{C}^2$. By Fogarty's theorem it is a smooth irreducible variety of dimension $2n$. By associating to a closed point of Hilb_n its ideal sheaf we obtain a bijection (at the level of points)

$$\mathrm{Hilb}_n(\mathbf{C}) = \big\{I \subset \mathbf{C}[X, Y];\ I \text{ is an ideal of codimension } n\big\}.$$

Let us denote by $S = \mathbf{C}[X, Y]$ the ring of regular functions on $\mathbf{C}^2$. The tangent space $T_I\mathrm{Hilb}_n$ at a closed point $I \in \mathrm{Hilb}_n(\mathbf{C})$ is canonically isomorphic to the vector space $\mathrm{Hom}_S(I, S/I)$.

2.4. *The torus action on* Hilb_n. — Consider the torus $T = (\mathbf{C}^\times)^2$. The torus T acts on $\mathbf{A}^2$ via $(z_1, z_2) \cdot (u, v) = (z_1 u, z_2 v)$. There is an induced action on S given by $(z_1, z_2) \cdot P(X, Y) = P(z_1^{-1}X, z_2^{-1}Y)$ and one on Hilb_n such that

$$(z_1, z_2) \cdot I = \big\{P\big(z_1^{-1}X, z_2^{-1}Y\big);\ P(X, Y) \in I\big\}, \quad \forall I \in \mathrm{Hilb}_n(\mathbf{C}). \tag{2.16}$$

This action has a finite number of isolated fixed points, indexed by the set of partitions of the integer n. To such a partition $\lambda \vdash n$ corresponds the fixed point I_λ where

$$I_\lambda = \bigoplus_{s \notin \lambda} \mathbf{C} X^{x(s)} Y^{y(s)}. \tag{2.17}$$

When $I = I_\lambda$ is a T-fixed point, there is an induced T-action on $T_I\mathrm{Hilb}_n$. In order to describe this action, we fix a few notations concerning T. Consider the characters q, t as in Example 2.1. For V a T-module let [V] be its class in the Grothendieck group of T. We abbreviate $T_\lambda = [T_{I_\lambda}\mathrm{Hilb}_n]$. It is given by

$$T_\lambda = \sum_{s \in \lambda} \big(t^{l(s)} q^{-a(s)-1} + t^{-l(s)-1} q^{a(s)}\big). \tag{2.18}$$

We set $\mathrm{eu}_\lambda = \mathrm{eu}(T_\lambda^*)$.

2.5. *The Hecke correspondence* $\mathrm{Hilb}_{n,n+1}$. — Let $k \geq 0$. The nested Hilbert scheme $\mathrm{Hilb}_{n,n+k}$ is the reduced closed subscheme of $\mathrm{Hilb}_n \times \mathrm{Hilb}_{n+k}$ parametrizing pairs of ideals (I,J) where $\mathrm{J} \subset \mathrm{I}$. One defines the nested Hilbert scheme $\mathrm{Hilb}_{n+k,n}$ in a similar fashion. Of course $\mathrm{Hilb}_{n,n}$ is simply the diagonal of $\mathrm{Hilb}_n \times \mathrm{Hilb}_n$. The schemes $\mathrm{Hilb}_{n,n+k}$ are smooth if $k=0$ or $k=1$, see [9]. The tangent space at a point $(\mathrm{I},\mathrm{J}) \in \mathrm{Hilb}_{n,n+k}$ is the kernel of the obvious map

$$\psi : \mathrm{Hom}_{\mathrm{S}}(\mathrm{I},\mathrm{S}/\mathrm{I}) \oplus \mathrm{Hom}_{\mathrm{S}}(\mathrm{J},\mathrm{S}/\mathrm{J}) \to \mathrm{Hom}_{\mathrm{S}}(\mathrm{J},\mathrm{S}/\mathrm{I}). \tag{2.19}$$

When $k=1$ the map ψ is surjective. The diagonal T-action on $\mathrm{Hilb}_n \times \mathrm{Hilb}_{n+k}$ preserves $\mathrm{Hilb}_{n,n+k}$. The fixed points contained in $\mathrm{Hilb}_{n,n+k}$ are those pairs $\mathrm{I}_{\mu,\lambda} = (\mathrm{I}_\mu, \mathrm{I}_\lambda)$ for which $\mu \subset \lambda$. The character of the fiber at $\mathrm{I}_{\mu,\lambda}$ of the normal bundle to $\mathrm{Hilb}_{n,n+1}$ in $\mathrm{Hilb}_n \times \mathrm{Hilb}_{n+1}$ is

$$\mathrm{N}_{\mu,\lambda} = \sum_{s\in\mu}\big(t^{l_\mu(s)}q^{-a_\lambda(s)-1} + t^{-l_\lambda(s)-1}q^{a_\mu(s)}\big). \tag{2.20}$$

Of course, similar formulas hold for the nested Hilbert scheme $\mathrm{Hilb}_{n+1,n}$.

2.6. *The tautological bundles.* — Let $\Theta_n \subset \mathrm{Hilb}_n \times \mathbf{A}^2$ be the universal family and let $p : \mathrm{Hilb}_n \times \mathbf{A}^2 \to \mathrm{Hilb}_n$ be the projection. The *tautological bundle* of Hilb_n is the locally free sheaf $\tau_n = p_*(\mathcal{O}_{\Theta_n})$. The fiber of τ_n at a point $\mathrm{I} \in \mathrm{Hilb}_n(\mathbf{C})$ is S/I. The character of the T-action on its fiber at the fixed point I_λ is

$$\tau_\lambda = \sum_{s\in\lambda} t^{y(s)}q^{x(s)}. \tag{2.21}$$

Next, let π_1, π_2 be the projections of $\mathrm{Hilb}_n \times \mathrm{Hilb}_{n+1}$ to Hilb_n and Hilb_{n+1} respectively. Over $\mathrm{Hilb}_{n,n+1}$ there is a surjective map $\pi_2^*(\tau_{n+1}) \to \pi_1^*(\tau_n)$. Over the point (I,J) it specializes to the map $\mathrm{S}/\mathrm{J} \to \mathrm{S}/\mathrm{I}$. The kernel sheaf is a line bundle, which we call the *tautological bundle* of $\mathrm{Hilb}_{n,n+1}$ and which we denote by $\tau_{n,n+1}$. Over a T-fixed point $\mathrm{I}_{\mu,\lambda}$ its character is

$$\tau_{\mu,\lambda} = t^{y(s)}q^{x(s)} \tag{2.22}$$

where $s = \lambda\backslash\mu$ is the unique box of λ not contained in μ. Finally, let π_1, π_2 be the projections of $\mathrm{Hilb}_n \times \mathrm{Hilb}_n$ to Hilb_n. Over $\mathrm{Hilb}_{n,n}$ we have the vector bundle $\tau_{n,n} = \pi_2^*(\tau_n) = \pi_1^*(\tau_n)$. We call it the *tautological bundle* of $\mathrm{Hilb}_{n,n}$. Over a T-fixed point $\mathrm{I}_{\lambda,\lambda}$ its character is $\tau_{\lambda,\lambda} = \tau_\lambda$.

2.7. *The algebra* $\widetilde{\mathbf{E}}^{(1)}_{\mathrm{K}}$ *and the* $\widetilde{\mathbf{E}}^{(1)}_{\mathrm{K}}$*-module* $\widetilde{\mathbf{L}}^{(1)}_{\mathrm{K}}$. — Recall that

$$\mathrm{R}_{\mathrm{T}} = \mathbf{C}[x,y], \quad x = dq,\ y = dt. \tag{2.23}$$

Consider the fraction field

(**2.24**) $$\mathrm{K}_\mathrm{T} = \mathrm{Frac}(\mathrm{R}_\mathrm{T}) = \mathbf{C}(x,y).$$

If no confusion is possible we abbreviate

(**2.25**) $$\mathrm{R} = \mathrm{R}_\mathrm{T}, \qquad \mathrm{K} = \mathrm{K}_\mathrm{T}.$$

By the Atiyah-Bott localization theorem, the direct image by the inclusion $\mathrm{Hilb}_n^\mathrm{T} \subset \mathrm{Hilb}_n$ yields a canonical isomorphism

(**2.26**) $$\bigoplus_{\lambda \vdash n} \mathrm{K}[\mathrm{I}_\lambda] = \mathrm{H}^\mathrm{T}(\mathrm{Hilb}_n) \otimes_\mathrm{R} \mathrm{K}.$$

Similarly, there is an isomorphism

(**2.27**) $$\bigoplus_{\substack{\lambda \vdash n \\ \mu \vdash m}} \mathrm{K}[\mathrm{I}_{\lambda,\mu}] = \mathrm{H}^\mathrm{T}(\mathrm{Hilb}_n \times \mathrm{Hilb}_m) \otimes_\mathrm{R} \mathrm{K}, \quad \mathrm{I}_{\lambda,\mu} = (\mathrm{I}_\lambda, \mathrm{I}_\mu).$$

So, we may define a K-algebra structure on

(**2.28**) $$\widetilde{\mathbf{E}}_\mathrm{K}^{(1)} = \bigoplus_{k \in \mathbf{Z}} \prod_n \mathrm{H}^\mathrm{T}(\mathrm{Hilb}_{n+k} \times \mathrm{Hilb}_n) \otimes_\mathrm{R} \mathrm{K},$$

together with an action on the K-vector space

(**2.29**) $$\widetilde{\mathbf{L}}_\mathrm{K}^{(1)} = \bigoplus_n \widetilde{\mathbf{L}}_{n,\mathrm{K}}^{(1)} = \bigoplus_n \mathrm{H}^\mathrm{T}(\mathrm{Hilb}_n) \otimes_\mathrm{R} \mathrm{K}.$$

In (2.28), the product ranges over all values of $n \geq 0$ such that $n+k \geq 0$. The integer k provides a $\mathbf{Z}$-grading on $\widetilde{\mathbf{E}}_\mathrm{K}^{(1)}$, and the $\mathbf{N}$-grading on $\widetilde{\mathbf{L}}_\mathrm{K}^{(1)}$ turns it into a faithful graded $\widetilde{\mathbf{E}}_\mathrm{K}^{(1)}$-module.

2.8. *The algebra* $\widetilde{\mathbf{U}}_\mathrm{K}^{(1)}$. — Let $i : \mathrm{Hilb}_{n+1,n} \to \mathrm{Hilb}_{n+1} \times \mathrm{Hilb}_n$ be the closed embedding. For notational convenience, the pushforward $i_* \,\mathrm{c}_1(\tau_{n+1,n})$ of the Chern class of the line bundle $\tau_{n+1,n}$ on $\mathrm{Hilb}_{n+1,n}$ will simply be denoted by $\mathrm{c}_1(\tau_{n+1,n})$. We will use similar notation for the tautological bundles $\tau_{n,n+1}$ on $\mathrm{Hilb}_{n,n+1}$ and $\tau_{n,n}$ on $\mathrm{Hilb}_{n,n}$. For $l \geq 0$ we consider the following elements in $\widetilde{\mathbf{E}}_\mathrm{K}^{(1)}$

(**2.30**) $$f_{1,l} = \prod_{n \geq 0} \mathrm{c}_1(\tau_{n+1,n})^l, \qquad f_{-1,l} = \prod_{n \geq 0} \mathrm{c}_1(\tau_{n,n+1})^l, \qquad e_{0,l} = \prod_{n \geq 0} \mathrm{c}_l(\tau_{n,n}).$$

We used the convention that $\mathrm{c}_0(\tau_{n,n}) = n[\mathrm{Hilb}_{n,n}]$. Let $\widetilde{\mathbf{U}}_\mathrm{K}^{(1)}$ be the K-subalgebra of $\widetilde{\mathbf{E}}_\mathrm{K}^{(1)}$ generated by

$$\{f_{-1,l},\, e_{0,l},\, f_{1,l};\ l \geq 0\}.$$

Observe that since the $e_{0,l}$'s are supported on the diagonal of the Hilbert scheme, their convolution product is given by the cup product in the equivariant cohomology groups of the Hilbert scheme. Therefore, the subalgebra of $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1)}$ generated by $\{e_{0,l};\, l \geq 0\}$ is commutative. Let us introduce another set of elements $\{f_{0,l};\, l \geq 0\}$ defined through the following formula

$$(\mathbf{2.31}) \qquad \sum_{l\geq 1} f_{0,l}s^{l-1} = -\partial_s \log\big(e(s)\big), \quad e(s) = 1 + \sum_{k\geq 1}(-1)^k e_{0,k}s^k, \quad f_{0,0} = e_{0,0}.$$

Under restriction, the canonical representation of $\widetilde{\mathbf{E}}_{\mathrm{K}}^{(1)}$ on $\widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}$ yields a faithful representation of $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1)}$ on $\widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}$. We call it the *canonical representation* of $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1)}$ on $\widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}$.

Remark **2.3.** — Given a splitting into a sum of line bundles $\tau_{n,n} = \phi_1 \oplus \cdots \oplus \phi_n$, we get

$$f_{0,l} = \prod_{n\geq 0} p_l(\alpha_1, \ldots, \alpha_n), \quad \alpha_i = \mathrm{c}_1(\phi_i), \; l \geq 0.$$

2.9. *From* $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1)}$ *to* $\widetilde{\mathbf{SH}}_{\mathrm{K}}^{(1)}$. — Consider the inclusion

$$(\mathbf{2.32}) \qquad \mathrm{F} \to \mathrm{K}, \qquad \kappa \mapsto -y/x.$$

Let $\widetilde{\mathbf{SH}}_{\mathrm{K}}^{(1)}$ be the specialization of $\mathbf{SH}^{\mathbf{c}} \otimes \mathrm{K}$ at $\mathbf{c} = (1, 0, \ldots)$. It can be viewed as a specialization of the K_1-algebra $\mathbf{SH}_{\mathrm{K}}^{(1)}$ at $\varepsilon_1 = 0$. We set

$$(\mathbf{2.33}) \qquad h_{0,l+1} = x^{-l}f_{0,l}, \qquad h_{1,l} = x^{1-l}yf_{1,l}, \qquad h_{-1,l} = x^{-l}f_{-1,l}, \quad l \geq 0.$$

We can now state our first result, compare [33, Thm. 3.1]. The proof is given in Section 5. Recall the definition of $\mathscr{E}$ in (0.7).

Theorem **2.4.** — *There is a* K*-algebra isomorphism* $\Psi : \widetilde{\mathbf{SH}}_{\mathrm{K}}^{(1)} \to \widetilde{\mathbf{U}}_{\mathrm{K}}^{(1)}$ *such that* $\mathrm{D}_{\mathbf{x}} \mapsto h_{\mathbf{x}}$ *for* $\mathbf{x} \in \mathscr{E}$.

We identify $\widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}$ with Λ_{K} by the K-linear map

$$(\mathbf{2.34}) \qquad \Lambda_{\mathrm{K}} \to \widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}, \quad \mathrm{J}_\lambda \mapsto [\mathrm{I}_\lambda].$$

Corollary **2.5.** — *Under the map* Ψ, *the representation of* $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1)}$ *on* $\widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}$ *gives a faithful representation* $\widetilde{\rho}^{(1)}$ *of* $\widetilde{\mathbf{SH}}_{\mathrm{K}}^{(1)}$ *on* Λ_{K}.

Proposition **2.6.** — *We have*
(a) $\widetilde{\rho}^{(1)}(b_{-l}) =$ *multiplication by* p_l *and* $\widetilde{\rho}^{(1)}(b_l) = l\kappa^{-1}\partial_{p_l}$ *for* $l \geq 1$,
(b) $\widetilde{\rho}^{(1)}(\mathrm{D}_{0,1}) = \sum_i \mathrm{X}_i\partial_{\mathrm{X}_i}$ *and* $\widetilde{\rho}^{(1)}(\mathrm{D}_{0,2}) = \kappa\square$.

Proof. — The representation $\widetilde{\rho}^{(1)}$ extends the representation ρ^+ in Proposition 1.20, see the proof of Proposition 5.1 for details. Thus, for $l \geq 1$, the operators $\widetilde{\rho}^{(1)}(b_{-l})$, $\widetilde{\rho}^{(1)}(\mathrm{D}_{0,1})$ and $\widetilde{\rho}^{(1)}(\mathrm{D}_{0,2})$ are as in the proposition above by Remark 1.23. Next, we have $\widetilde{\rho}^{(1)}(b_l) \cdot 1 = 0$ and the map

$$\mathrm{K}[b_{-l}; l \geq 1] \to \Lambda_{\mathrm{K}}, \quad u \mapsto \widetilde{\rho}^{(1)}(u) \cdot 1 \tag{2.35}$$

is an isomorphism. Further, by Proposition 1.41 the elements b_l, $l \in \mathbf{Z}$, generate a Heisenberg algebra of central charge κ^{-1}. This forces $\widetilde{\rho}^{(1)}(b_l)$ to be given by the formula above. □

The representation $\widetilde{\rho}^{(1)}$ extends both the representation ρ^+ of $\mathbf{SH}^+$ in Proposition 1.20 and the standard Fock space of the Heisenberg algebra. We'll call it the Fock space of $\widetilde{\mathbf{SH}}^{(1)}_{\mathrm{K}}$.

3. Equivariant cohomology of the moduli space of torsion free sheaves

The Hilbert scheme of $\mathbf{C}^2$ is isomorphic to the moduli space of framed torsion free rank one coherent sheaves on $\mathbf{P}^2$. We now generalize the considerations above to higher ranks.

3.1. *The moduli space of torsion free sheaves.* — Fix integers $r > 0$, $n \geq 0$. Let $\mathrm{M}_{r,n}$ be the moduli space of framed torsion-free sheaves on $\mathbf{P}^2$ with rank r and second Chern class n. More precisely, $\mathbf{C}$-points of $\mathrm{M}_{r,n}$ are isomorphism classes of pairs $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a torsion-free sheaf which is locally free in a neighborhood of ℓ_∞ and $\Phi : \mathcal{E}|_{\ell_\infty} \to \mathcal{O}^r_{\ell_\infty}$ is a framing at infinity. Here $\ell_\infty = \{[x : y : 0] \in \mathbf{P}^2\}$ is the line at infinity. Recall that $\mathrm{M}_{r,n}$ is a smooth variety of dimension $2rn$ which admits the following alternative description. Let E be an n-dimensional vector space. We have $\mathrm{M}_{r,n} = \mathrm{M}^s_{r,\mathrm{E}}/\mathrm{GL}_{\mathrm{E}}$ where $\mathrm{M}^s_{r,\mathrm{E}} = \mathrm{N}^s_{r,\mathrm{E}} \cap \mathrm{M}_{r,\mathrm{E}}$, with

$$\begin{aligned}
\mathrm{N}^s_{r,\mathrm{E}} &= \big\{(a, b, \varphi, v) \in \mathrm{N}_{r,\mathrm{E}};\, (a, b, \varphi, v) \text{ is stable}\big\},\\
\mathrm{M}_{r,\mathrm{E}} &= \big\{(a, b, \varphi, v) \in \mathrm{N}_{r,\mathrm{E}};\, [a, b] + v \circ \varphi = 0\big\},\\
\mathrm{N}_{r,\mathrm{E}} &= \mathfrak{g}^2_{\mathrm{E}} \times \mathrm{Hom}\big(\mathrm{E}, \mathbf{C}^r\big) \times \mathrm{Hom}\big(\mathbf{C}^r, \mathrm{E}\big).
\end{aligned} \tag{3.1}$$

The GL_{E}-action is given by $g(a, b, \varphi, v) = (gag^{-1}, gbg^{-1}, \varphi g^{-1}, gv)$. The tuple (a, b, φ, v) is *stable* iff there is no proper subspace $\mathrm{E}_1 \subsetneq \mathrm{E}$ which is preserved by a, b and contains $v(\mathbf{C}^r)$. From now on we may abbreviate $\mathrm{G} = \mathrm{GL}_{\mathrm{E}}$ and $\mathfrak{g} = \mathfrak{g}_{\mathrm{E}}$.

3.2. *The torus action on* $\mathrm{M}_{r,n}$. — Put $\mathrm{D} = (\mathbf{C}^\times)^r$ and $\mathrm{T} = (\mathbf{C}^\times)^2$. We abbreviate $\widetilde{\mathrm{D}} = \mathrm{D} \times \mathrm{T}$. Set also $x = c_1(q)$, $y = c_1(t)$ and $e_a = c_1(\chi_a)$ for $a \in [1, r]$. We have

$$\mathrm{R}_r = \mathrm{R}_{\widetilde{\mathrm{D}}} = \mathbf{C}[x, y, e_1, \ldots, e_r], \qquad \mathrm{K}_r = \mathrm{K}_{\widetilde{\mathrm{D}}} = \mathbf{C}(x, y, e_1, \ldots, e_r). \tag{3.2}$$

The characters $q=\chi_x$, $t=\chi_y$ and $\chi_a=\chi_{e_a}$ of $\widetilde{\mathrm{D}}$ are given by

$$(\mathbf{3.3})\quad q(h,z_1,z_2)=z_1^{-1},\quad t(h,z_1,z_2)=z_2^{-1},\quad \chi_a(h,z_1,z_2)=h_a^{-1},\quad h=(h_1,h_2,\dots,h_r).$$

We equip the variety $\mathrm{N}_{r,\mathrm{E}}$ with the $\widetilde{\mathrm{D}}$-action given by

$$(\mathbf{3.4})\qquad (h,z_1,z_2)\cdot(a,b,\varphi,v)=\big(z_1a,z_2b,z_1z_2h\varphi,vh^{-1}\big).$$

This action preserves $\mathrm{M}^s_{r,\mathrm{E}}$, descends to $\mathrm{M}_{r,n}$, on which it has a finite number of isolated fixed points which are indexed by the set of r-partitions of n. To the r-partition λ corresponds a fixed point I_λ such that the character T_λ of the $\widetilde{\mathrm{D}}$-module $\mathrm{T}_{\mathrm{I}_\lambda}\mathrm{M}_{r,n}$ is given by [31, Thm. 2.11]

$$(\mathbf{3.5})\qquad \mathrm{T}_\lambda=\sum_{a,b=1}^r\sum_{s\in\lambda^{(a)}}\chi_a\chi_b^{-1}t^{l_{\lambda^{(b)}}(s)}q^{-a_{\lambda^{(a)}}(s)-1}+\sum_{a,b=1}^r\sum_{s\in\lambda^{(b)}}\chi_a\chi_b^{-1}t^{-l_{\lambda^{(a)}}(s)-1}q^{a_{\lambda^{(b)}}(s)}.$$

3.3. *The Hecke correspondence* $\mathrm{M}_{r,n,n+1}$. — Now, we assume that $\dim(\mathrm{E})=n+1$. The *Hecke correspondence* is the geometric quotient $\mathrm{M}_{r,n,n+1}=\mathrm{Z}^s_{r,\mathrm{E}}/\mathrm{G}$, where $\mathrm{Z}^s_{r,\mathrm{E}}$ is the variety of all tuples $(a,b,\varphi,v,\mathrm{E}_1)$ where $(a,b,\varphi,v)\in\mathrm{M}^s_{r,\mathrm{E}}$ and $\mathrm{E}_1\subset\operatorname{Ker}\varphi$ is a line preserved by a,b. We define also

$$(\mathbf{3.6})\qquad \mathrm{M}^c_{r,n,n+1}=\big\{(a,b,\varphi,v,\mathrm{E}_1)\in\mathrm{Z}^s_{r,\mathrm{E}};\ a|_{\mathrm{E}_1}=b|_{\mathrm{E}_1}=0\big\}/\mathrm{G}.$$

Write $\mathrm{E}_2=\mathrm{E}/\mathrm{E}_1$ and consider the induced linear maps

$$(\mathbf{3.7})\qquad \bar v=\pi\circ v,\qquad \bar a,\bar b\in\mathfrak{g}_{\mathrm{E}_2},\qquad \bar\varphi\in\operatorname{Hom}\big(\mathrm{E}_2,\mathbf{C}^r\big).$$

Let π_1,π_2 be the projections of $\mathrm{M}_{r,n}\times\mathrm{M}_{r,n+1}$ to $\mathrm{M}_{r,n}$, $\mathrm{M}_{r,n+1}$. The following is well-known.

Proposition **3.1.** — (a) *The variety* $\mathrm{Z}^s_{r,\mathrm{E}}$ *is a* G*-torsor over* $\mathrm{M}_{r,n,n+1}$.
(b) *The variety* $\mathrm{M}_{r,n,n+1}$ *is a smooth variety of dimension* $2rn+r+1$.
(c) *The closed subvariety* $\mathrm{M}^c_{r,n,n+1}$ *is also smooth.*
(d) *The map* $(a,b,\varphi,v)\mapsto(\bar a,\bar b,\bar\varphi,\bar v)$, (a,b,φ,v) *is a closed immersion* $\mathrm{M}_{r,n,n+1}\subset\mathrm{M}_{r,n}\times\mathrm{M}_{r,n+1}$. *The restriction of* π_2 *to* $\mathrm{M}_{r,n,n+1}$ *is proper. The restriction of* π_1 *to* $\mathrm{M}^c_{r,n,n+1}$ *is proper.*

The pair $\mathrm{I}_{\mu,\lambda}=(\mathrm{I}_\mu,\mathrm{I}_\lambda)$ belongs to $\mathrm{M}_{r,n,n+1}$ if and only if $\mu\subset\lambda$ and the r-partitions μ,λ have weight n, $n+1$ respectively. Let $\mathrm{N}_{\mu,\lambda}$ be the character of the fiber at $\mathrm{I}_{\mu,\lambda}$ of the normal bundle (in $\mathrm{M}_{r,n}\times\mathrm{M}_{r,n+1}$) of $\mathrm{M}_{r,n,n+1}$. We set also $\mathrm{N}_{\lambda,\mu}=\mathrm{N}_{\mu,\lambda}$. Finally, we define

$$(\mathbf{3.8})\qquad \mathrm{eu}_\lambda=\mathrm{eu}\big(\mathrm{T}^*_\lambda\big),\qquad \mathrm{eu}_{\lambda,\mu}=\mathrm{eu}_\lambda\,\mathrm{eu}_\mu\,.$$

3.4. *The tautological bundles.* — The *tautological bundle* of $M_{r,n}$ is the $\widetilde{D}$-equivariant bundle $\tau_n = M^s_{r,E} \times_G E$. The character of the $\widetilde{D}$-module $\tau_n|_{I_\lambda}$ is given by [31, Thm. 2.11], [37, Lemma 6]

$$\tau_\lambda = \sum_a \sum_{s\in\lambda^{(a)}} \chi_a^{-1} t^{y(s)} q^{x(s)}. \tag{3.9}$$

Set $v = (qt)^{-1}$. The characters T_λ and τ_λ are related by the following equation

$$T_\lambda = -\big(1-q^{-1}\big)\big(1-t^{-1}\big)\tau_\lambda \otimes \tau_\lambda^* + \tau_\lambda \otimes W^* + v\tau_\lambda^* \otimes W, \tag{3.10}$$

where $W = \chi_1^{-1} + \cdots + \chi_r^{-1}$ is the tautological representation of D. For $\mu \subset \lambda$ we have also

$$\begin{aligned} N_{\mu,\lambda} &= -\big(1-q^{-1}\big)\big(1-t^{-1}\big)\tau_\mu \otimes \tau_\lambda^* + \tau_\mu \otimes W^* + v\tau_\lambda^* \otimes W - v,\\ N_{\mu,\lambda} &= \sum_{a,b}\sum_{s\in\mu^{(a)}} \chi_a\chi_b^{-1} t^{l_{\mu^{(b)}}(s)} q^{-a_{\lambda^{(a)}}(s)-1} + \sum_{a,b}\sum_{s\in\lambda^{(b)}} \chi_a\chi_b^{-1} t^{-l_{\lambda^{(a)}}(s)-1} q^{a_{\mu^{(b)}}(s)} - v. \end{aligned} \tag{3.11}$$

Over $M_{r,n,n+1}$ there is a surjective map $\pi_2^*(\tau_{n+1}) \to \pi_1^*(\tau_n)$. The kernel sheaf is a line bundle called the *tautological bundle* of $M_{r,n,n+1}$ which we denote by $\tau_{n,n+1}$. Over $I_{\mu,\lambda}$ its character is

$$\tau_{\mu,\lambda} = \chi_a^{-1} t^{y(s)} q^{x(s)}, \quad \mu\subset\lambda. \tag{3.12}$$

Here $s = \lambda^{(a)} \backslash \mu^{(a)}$ is the unique box of λ not contained in μ. We define the Hecke correspondence $M_{r,n+1,n}$ and the tautological bundle $\tau_{n+1,n}$ over it in the obvious way, so that we get $\tau_{\lambda,\mu} = \tau_{\mu,\lambda}$.

3.5. *The algebra* $\mathbf{E}_K^{(r)}$ *and the* $\mathbf{E}_K^{(r)}$*-module* $\mathbf{L}_K^{(r)}$. — Consider the graded R_r-modules

$$\begin{aligned} &\mathbf{L}_n^{(r)} = H^{\widetilde{D}}(M_{r,n}), && \mathbf{L}^{(r)} = \bigoplus_{n\ge 0} \mathbf{L}_n^{(r)},\\ &\mathbf{E}_n^{(r)} = \prod_k H^{\widetilde{D}}(M_{r,n+k}\times M_{r,k}), && \mathbf{E}^{(r)} = \bigoplus_{n\in\mathbf{Z}} \mathbf{E}_n^{(r)}, \end{aligned} \tag{3.13}$$

where the product ranges over all integers $k\ge 0$ with $n+k\ge 0$. They are known to be torsion free. We abbreviate

$$\begin{aligned} &H^{\widetilde{D}}(M_{r,n}\times M_{r,m})_K = H^{\widetilde{D}}(M_{r,n}\times M_{r,m}) \otimes_{R_r} K_r,\\ &\mathbf{E}_{n,K}^{(r)} = \prod_k H^{\widetilde{D}}(M_{r,n+k}\times M_{r,k})_K, && \mathbf{E}_K^{(r)} = \bigoplus_{n\in\mathbf{Z}} \mathbf{E}_{n,K}^{(r)}\\ &\mathbf{L}_{n,K}^{(r)} = \mathbf{L}_n^{(r)} \otimes_{R_r} K_r, && \mathbf{L}_K^{(r)} = \bigoplus_{n\ge 0} \mathbf{L}_{n,K}^{(r)}. \end{aligned} \tag{3.14}$$

The variety $\mathrm{M}_{r,n}$ is not proper but it has a finite number of fixed points by the $\widetilde{\mathrm{D}}$-action. By the Atiyah-Bott localization theorem, the direct image by the obvious inclusion $\mathrm{M}_{r,n}^{\widetilde{\mathrm{D}}} \to \mathrm{M}_{r,n}$ provides us with canonical isomorphisms

$$(\mathbf{3.15}) \qquad \mathbf{L}^{(r)}_{n,\mathrm{K}} = \bigoplus_{\lambda} \mathrm{K}_r[\mathrm{I}_\lambda], \qquad \mathbf{E}^{(r)}_{n,\mathrm{K}} = \prod_k \bigoplus_{\lambda,\mu} \mathrm{K}_r[\mathrm{I}_{\lambda,\mu}],$$

where λ, μ run over the set of r-partitions of $n+k$, k respectively. This allows us to define, by convolution, an associative multiplication on $\mathbf{E}^{(r)}_{\mathrm{K}}$ and an action of $\mathbf{E}^{(r)}_{\mathrm{K}}$ on $\mathbf{L}^{(r)}_{\mathrm{K}}$.

3.6. *The algebras* $\mathbf{U}^{(r)}_{\mathrm{K}}$ *and* $\mathbf{SH}^{(r)}_{\mathrm{K}}$. — Consider the inclusion $\mathrm{F} \subset \mathrm{K}$ in (2.32). For $l \geq 0$ we define the following elements in $\mathbf{E}^{(r)}_{\mathrm{K}}$

$$(\mathbf{3.16}) \qquad f_{1,l} = \prod_{n\geq 0} \mathrm{c}_1(\tau_{n+1,n})^l, \qquad f_{-1,l} = \prod_{n\geq 0} \mathrm{c}_1(\tau_{n,n+1})^l, \qquad e_{0,l} = \prod_{n\geq 0} \mathrm{c}_l(\tau_{n,n}).$$

We define also the element $f_{0,l}$ through the relations (2.31). We abbreviate

$$(\mathbf{3.17}) \qquad h_{0,l+1} = x^{-l} f_{0,l}, \qquad h_{1,l} = x^{1-l} y f_{1,l}, \qquad h_{-1,l} = (-1)^{r-1} x^{-l} f_{-1,l}.$$

From (3.9) and the formulas above we get the following identity, compare (C.6),

$$(\mathbf{3.18}) \qquad f_{0,l}\big([\mathrm{I}_\lambda]\big) = \sum_a \sum_{s\in\lambda^{(a)}} c_a(s)^l\,[\mathrm{I}_\lambda], \qquad c_a(s) = x(s)\,x + y(s)\,y - e_a.$$

Recall the field $\mathrm{K}_r = \mathrm{K}(\varepsilon_1, \ldots, \varepsilon_r)$ from Definition 1.36. Write

$$(\mathbf{3.19}) \qquad \varepsilon_a = e_a/x, \qquad a \in [1, r].$$

We consider the K_r-subalgebras of $\mathbf{E}^{(r)}_{\mathrm{K}}$ given by

- $\mathbf{U}^{(r)}_{\mathrm{K}}$ is generated by $\{f_{-1,l}, e_{0,l}, f_{1,l};\ l \geq 0\}$,
- $\mathbf{U}^{(r),+}_{\mathrm{K}}$ by $\{e_{0,l}, f_{1,l};\ l \geq 0\}$ and $\mathbf{U}^{(r),-}_{\mathrm{K}}$ by $\{f_{-1,l}, e_{0,l};\ l \geq 0\}$,
- $\mathbf{U}^{(r),<}_{\mathrm{K}}$ by $\{f_{-1,l};\ l \geq 0\}$ and $\mathbf{U}^{(r),>}_{\mathrm{K}}$ by $\{f_{1,l};\ l \geq 0\}$,
- $\mathbf{U}^{(r),0}_{\mathrm{K}}$ by $\{f_{0,l};\ l \geq 0\}$,

Theorem **3.2.** — *The assignment* $\mathrm{D}_{\mathbf{x}} \mapsto h_{\mathbf{x}}$ *for* $\mathbf{x} \in \mathscr{E}$ *extends to a* K_r*-algebra isomorphism* $\Psi : \mathbf{SH}^{(r)}_{\mathrm{K}} \to \mathbf{U}^{(r)}_{\mathrm{K}}$ *which takes* $\mathbf{SH}^{(r),>}_{\mathrm{K}}$, $\mathbf{SH}^{(r),0}_{\mathrm{K}}$, $\mathbf{SH}^{(r),<}_{\mathrm{K}}$ *into* $\mathbf{U}^{(r),>}_{\mathrm{K}}$, $\mathbf{U}^{(r),0}_{\mathrm{K}}$, $\mathbf{U}^{(r),<}_{\mathrm{K}}$.

Corollary **3.3.** — *Under the map* Ψ, *the representation of* $\mathbf{U}^{(r)}_{\mathrm{K}}$ *on* $\mathbf{L}^{(r)}_{\mathrm{K}}$ *gives a faithful representation* $\rho^{(r)} : \mathbf{SH}^{(r)}_{\mathrm{K}} \longrightarrow \mathrm{End}(\mathbf{L}^{(r)}_{\mathrm{K}})$.

Proof. — The theorem is proved in Section 6. The faithfulness of $\rho^{(r)}$ is proved in Section D.2. ☐

Remark **3.4.** — We have $\rho^{(1)}(D_{0,2}) = x^{-1}f_{0,1}$. Therefore, comparing Proposition 1.20 with (3.18) we get the following formula $\rho^{(1)}(D_{0,2}) + \varepsilon_1\rho^{(1)}(D_{0,1}) = \kappa\,\square$.

Remark **3.5.** — Since $\mathbf{L}^{(r)}$ is torsion free as an R_r-module, we can view it as an R_r-submodule of $\mathbf{L}^{(r)}_K$. Since the projection $\pi_1 : M_{r,n+1,n} \to M_{r,n+1}$ is proper, we have $f_{1,l}(\mathbf{L}^{(r)}) \subset \mathbf{L}^{(r)}$. Since

$$xy\,c_1(\tau_{n,n+1}) \in \mathrm{Im}\big\{H^{\widetilde{D}}_*\big(M^c_{r,n,n+1}\big) \to H^{\widetilde{D}}_*(M_{r,n,n+1})\big\},$$

we have also $xyf_{-1,l}(\mathbf{L}^{(r)}) \subset \mathbf{L}^{(r)}$. Finally, we have $f_{0,l}(\mathbf{L}^{(r)}) \subset \mathbf{L}^{(r)}$. Therefore, the operators

$$\rho^{(r)}\big(x^{l-1}y^{-1}D_{1,l}\big), \qquad \rho^{(r)}\big(x^{l+1}yD_{-1,l}\big), \qquad \rho^{(r)}\big(x^{l}D_{0,l+1}\big), \quad l \geq 0, \tag{3.20}$$

preserve the lattice $\mathbf{L}^{(r)}$. More generally, using (1.76), we get that the operators

$$x^{d-1}y^{-l}\rho^{(r)}(D_{l,d}), \qquad x^{d-1+2l}y^{l}\rho^{(r)}(D_{-l,d}), \quad l \geq 0 \tag{3.21}$$

preserve also the lattice $\mathbf{L}^{(r)}$.

Remark **3.6.** — The R_r-module $\mathbf{L}^{(r)}$ is bi-graded: it is first graded by the c_2, for which the degree n piece is $\mathbf{L}^{(r)}_{n,K}$, and then by the (co-)homological degree, for which $H^{\widetilde{D}}_i(M_{r,n})$ has the degree $4rn - 2i$ or $2i$ respectively. The operator $\rho^{(r)}(D_{\mathbf{x}})$ is homogeneous for the (co-)homological degrees. For $\mathbf{x} \in \mathscr{E}$ with $\mathbf{x} = (\epsilon, d)$ we have

$$\mathrm{cdeg}\big(\rho^{(r)}(D_{\mathbf{x}})\big) = 2\epsilon(r+1). \tag{3.22}$$

More generally, using (1.76), the formula (3.22) is again true for any $\epsilon \in \mathbf{Z}$.

3.7. *The pairing on* $\mathbf{L}^{(r)}_K$. — The cup product equips the K_r-vector space $\mathbf{L}^{(r)}_K$ with a K_r-bilinear form $(\bullet, \bullet)$ such that for all r-partitions λ, μ we have

$$\big([I_\lambda], [I_\mu]\big) = \delta_{\lambda,\mu}\,\mathrm{eu}_\lambda\,. \tag{3.23}$$

Let f^* denote the adjoint of a K_r-linear operator f on $\mathbf{L}^{(r)}_K$ with respect to this pairing. Using this anti-involution, we can prove the following.

Proposition **3.7.** — *The assignment $h_{1,l} \mapsto h_{-1,l}$ and $h_{0,l} \mapsto h_{0,l}$ for $l \geq 0$ extends to an algebra anti-involution $\mathbf{U}^{(r),+}_K \to \mathbf{U}^{(r),-}_K$ which takes $\mathbf{U}^{(r),>}_K$ onto $\mathbf{U}^{(r),<}_K$.*

Proof. — By (7.85), for any r-partitions λ, π such that $\lambda \subset \pi$ and $|\lambda| = |\pi| - 1$, we have

$$\big([I_\pi], f_{1,l}[I_\lambda]\big) = c_1(\tau_{\lambda,\pi})^l\,\mathrm{eu}\big(N^*_{\lambda,\pi}\big) = \big(f_{-1,l}[I_\pi], [I_\lambda]\big). \tag{3.24}$$

Thus, we get the following

$$f_{1,l}^* = f_{-1,l}, \qquad h_{1,l}^* = (-1)^{r-1} xy\, h_{-1,l}. \tag{3.25}$$

Clearly, we have also

$$h_{0,l}^* = h_{0,l}. \tag{3.26}$$

The proposition follows. □

Proposition **3.8.** — *For* $(l,d) \in \mathbf{N}_0^2$ *we have* $\rho^{(r)}(\mathrm{D}_{l,d})^* = (-1)^{(r-1)l} x^l y^l \rho^{(r)}(\mathrm{D}_{-l,d})$.

Proof. — For $(l,d) \in \mathscr{E}$ this is (3.25), (3.26). The claim follows by applying (1.75), (1.76). □

Remark **3.9.** — The cup-product in cohomology gives a K-bilinear form $(\bullet, \bullet)$ on $\widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}$ such that

$$\big([\mathrm{I}_\lambda], [\mathrm{I}_\mu]\big) = \delta_{\lambda,\mu} (-1)^{|\lambda|} \prod_{s\in\lambda} \Big(xa(s) - y\big(l(s)+1\big)\Big)\Big(x\big(a(s)+1\big) - yl(s)\Big). \tag{3.27}$$

Under the map (2.34) this pairing is taken to $\bigoplus_{n\geq 0} (-y^2)^n (\bullet, \bullet)_{1/\kappa}$. Here $(\bullet, \bullet)_{1/\kappa}$ is the inner product which has Jack polynomials as an orthogonal basis of Λ_{K}. See [25, Chap. VI, Sect. 10] for details.

3.8. *Wilson operators on* $\mathbf{U}_{\mathrm{K}}^{(r),>}$ *and* $\mathbf{L}_{\mathrm{K}}^{(r)}$. — The product of the R_r-algebra homomorphisms

$$\mathrm{R}_{\widetilde{\mathrm{G}}} \to \mathrm{H}_{\widetilde{\mathrm{D}}}(\mathrm{M}_{r,n}), \qquad e_l^{(n)} \mapsto \mathrm{c}_l(\tau_n), \quad n \geq 0 \tag{3.28}$$

gives an R_r-algebra homomorphism

$$\Lambda_{\mathrm{R}_r} \to \prod_n \mathrm{H}_{\widetilde{\mathrm{D}}}(\mathrm{M}_{r,n}), \qquad p \mapsto p(\tau) = \big(p(\tau_n)\big). \tag{3.29}$$

Composing it with the cup product, we get a Λ_{K_r}-module structure $\bullet$ on $\mathbf{L}_{\mathrm{K}}^{(r)}$ which preserves the direct summand $\mathbf{L}_{n,\mathrm{K}}^{(r)}$ for each n. The Λ_{K_r}-action on $\mathbf{L}_{n,\mathrm{K}}^{(r)}$ factors through a Λ_{n,K_r}-action via the map π_n. We define an action of Λ_{K_r} on $\mathrm{End}(\mathbf{L}_{\mathrm{K}}^{(r)})$ by setting

$$p_l \bullet u = \big[p_l(\tau), u\big], \qquad u \in \mathrm{End}\big(\mathbf{L}_{\mathrm{K}}^{(r)}\big), \quad l \geq 1. \tag{3.30}$$

This action preserves each graded component of $\mathrm{End}(\mathbf{L}_{\mathrm{K}}^{(r)})$. Note that the K_r-subalgebra $\mathbf{U}_{\mathrm{K}}^{(r),>}$ of $\mathrm{End}(\mathbf{L}_{\mathrm{K}}^{(r)})$ carries an induced $\mathbf{N}$-grading with $f_{1,l}$ being of degree one for all l.

Example **3.10.** — The restriction of the Wilson operator p_l to $\mathbf{L}^{(r)}_{n,\mathrm{K}}$ is the cup product with

$$p_l(\tau_n) = p_l\big(\mathrm{c}_1(\rho_1), \dots, \mathrm{c}_1(\rho_n)\big), \tag{3.31}$$

if $\tau_n = \rho_1 + \dots + \rho_n$ is a sum of invertible $\widetilde{\mathrm{D}}$-equivariant bundles. Thus $p_l \bullet f_{1,k}$ is represented by the correspondence

$$\prod_{n\geq 0} \mathrm{c}_1(\tau_{n,n+1})^k\big(p_l(\tau_{n+1}) - p_l(\tau_n)\big) = \prod_{n\geq 0} \mathrm{c}_1(\tau_{n,n+1})^k p_l(\tau_{n,n+1}) = \prod_{n\geq 0} \mathrm{c}_1(\tau_{n,n+1})^{l+k}$$

from which we get

$$p_l \bullet f_{1,k} = f_{1,l+k}. \tag{3.32}$$

For r-partitions λ, μ with $\mu \subset \lambda$ and for any $p \in \Lambda_{\mathrm{K}_r}$ we write also

$$\tau_{\mu,\lambda} = \tau_\lambda - \tau_\mu, \qquad p(\tau_{\mu,\lambda}) = p\big(\mathrm{c}_1(\rho_1), \dots, \mathrm{c}_1(\rho_n)\big), \tag{3.33}$$

if $\tau_{\mu,\lambda} = \rho_1 + \dots + \rho_n$ is a sum of $\widetilde{\mathrm{D}}$-characters.

The following lemma is left to the reader. We'll use Sweedler's notation $\Delta(a) = \sum a_1 \otimes a_2$.

Lemma **3.11.** — (a) *The action of* Λ_{K_r} *on* $\mathrm{End}(\mathbf{L}^{(r)}_{\mathrm{K}})$ *preserves* $\mathbf{U}^{(r),>}_{\mathrm{K}}$,
(b) *the action of* Λ_{K_r} *on the degree n part of* $\mathbf{U}^{(r)}_{\mathrm{K}}$ *factors through* Λ_{n,K_r},
(c) *for* $a \in \Lambda_{\mathrm{K}_r}$, $u, u' \in \mathbf{U}^{(r),>}_{\mathrm{K}}$ *and* $v \in \mathbf{L}^{(r)}_{\mathrm{K}}$ *we have*

$$a \bullet u(v) = \sum (a_1 \bullet u)(a_2 \bullet v), \qquad a \bullet uu' = \sum (a_1 \bullet u)\big(a_2 \bullet u'\big),$$

(d) *the* K_r*-algebra isomorphism* $\Psi : \mathbf{SH}^{(r),>}_{\mathrm{K}} \to \mathbf{U}^{(r),>}_{\mathrm{K}}$ *intertwines the* Λ_{K_r}*-actions.*

For an element $u \in \mathbf{U}^{(r),>}_{\mathrm{K}}$ and for r-partitions λ, μ let $\langle\lambda; u; \mu\rangle$ be the coefficient of $[\mathrm{I}_\lambda]$ in $u([\mathrm{I}_\mu])$. This coefficient is zero unless $\mu \subset \lambda$. For $p \in \Lambda_{\mathrm{K}_r}$ we have

$$\langle\lambda; p \bullet u; \mu\rangle = p(\tau_{\mu,\lambda})\langle\lambda; u; \mu\rangle. \tag{3.34}$$

We will say an element $p \in \mathrm{Frac}(\Lambda_{n,\mathrm{K}_r})$ is *regular* if it is regular at $\tau_{\mu,\lambda}$ for any λ, μ with $|\lambda\backslash\mu| = n$. If p is regular then its action on $\mathbf{U}^{(r),>}_{\mathrm{K}}$ is well-defined. Indeed, it is well-defined on any operator $\gamma \in \mathrm{End}(\mathbf{L}^{(r)}_{\mathrm{K}})$ satisfying

$$\langle\lambda; \gamma; \mu\rangle \neq 0 \Rightarrow \mu \subset \lambda.$$

We now provide an explicit description of the action of some element of $\mathbf{U}^{(r),>}_{\mathrm{K}}$ on $\mathbf{L}^{(r)}_{\mathrm{K}}$ in terms of Wilson operators. For this we define a surjective K_r-linear map

$$\iota : \mathrm{K}_r[z_1, \dots, z_n] \to \mathbf{U}^{(r),>}_{\mathrm{K}}, \qquad z_1^{l_1} \cdots z_n^{l_n} \mapsto f_{1,l_1} \cdots f_{1,l_n} \tag{3.35}$$

and a twisted symmetrization map

$$\varpi_n\colon \quad \begin{cases} \mathrm{K}_r[z_1,\dots,z_n] \to \mathrm{K}_r(z_1,\dots,z_n)^{\mathfrak{S}_n} = \mathrm{Frac}(\Lambda_{n,\mathrm{K}_r}) \\ \mathrm{P}(z_1,\dots,z_n) \mapsto \mathrm{SYM}_n(g(z_1,\dots,z_n)\mathrm{P}(z_1,\dots,z_n)) \end{cases} \tag{3.36}$$

where SYM_n is the standard symmetrization map

$$\mathrm{SYM}_n: \quad \mathrm{K}_r(z_1,\dots,z_n) \to \mathrm{K}_r(z_1,\dots,z_n)^{\mathfrak{S}_n}, \qquad \mathrm{P} \mapsto \sum_{\sigma\in\mathfrak{S}_n} \sigma\cdot\mathrm{P}, \tag{3.37}$$

and where

$$g(z_1,\dots,z_n) = \prod_{i<j} g(z_i - z_j), \qquad g(z) = \frac{(z+x)(z+y)}{z(z+x+y)}. \tag{3.38}$$

For $n \geq 1$ consider the element γ_n in $\mathbf{E}^{(r)}_{n,\mathrm{K}}$ given by

$$\gamma_n = \prod_{\mu\subset\lambda} a_{\mu,\lambda}\,\mathrm{eu}_\mu^{-1}[\mathrm{I}_{\lambda,\mu}], \tag{3.39}$$

$$a_{\mu,\lambda} = \mathrm{eu}\big((1-q)(1-t)\big(\tau^*_{\mu,\lambda}\otimes\tau_\lambda\big) - \tau^*_{\mu,\lambda}\otimes \mathrm{W} - nv^{-1}\big), \tag{3.40}$$

where the product ranges over all r-partitions λ, μ such that $\mu\subset\lambda$, $|\lambda| = |\mu| + n$. This element gives rise to an operator of degree n in $\mathrm{End}(\mathbf{L}^{(r)}_{\mathrm{K}})$. Let γ_n denote also this operator. It does not belong to $\mathbf{U}^{(r),>}_{\mathrm{K}}$ unless $n=1$ (then it is the product of the fundamental classes of the correspondences $\mathrm{M}_{r,k,k+1}$ for $k\geq 0$).

Lemma **3.12.** — *For* $\mathrm{P}\in\mathrm{K}_r[z_1,\dots,z_n]$ *the element* $\varpi_n(\mathrm{P})$ *is regular and* $\iota(\mathrm{P}) = \varpi_n(\mathrm{P})\bullet\gamma_n$ *in* $\mathrm{End}(\mathbf{L}^{(r)}_{\mathrm{K}})$.

Proof. — See Appendix D.3. □

Proposition **3.13.** — *The action of* Λ_{n,K_r} *on the degree n part of* $\mathbf{U}^{(r),>}_{\mathrm{K}}$ *is torsion free.*

Proof. — By Lemma 3.12, it is enough to show that the map

$$\Lambda_{n,\mathrm{K}_r} \to \mathrm{End}\big(\mathbf{L}^{(r)}_{\mathrm{K}}\big), \qquad p\mapsto p\bullet\gamma_n \tag{3.41}$$

is injective. Now, an element $p\in\Lambda_{n,\mathrm{K}_r}$ annihilates γ_n if and only if

$$\mu\subset\lambda, \qquad |\lambda\backslash\mu| = n, \qquad a_{\mu,\lambda}\neq 0 \quad\Rightarrow\quad p(\tau_{\mu,\lambda}) = 0. \tag{3.42}$$

We claim that in fact $a_{\mu,\lambda}\neq 0$ for any pair satisfying $\mu\subset\lambda$ and $|\lambda\backslash\mu| = n$. This indeed implies that any p which annihilates γ_n must be zero because the collection of possible values of $(c_1(\rho_1), \dots, c_1(\rho_n))$ is Zariski dense in K_r^n. To prove the claim we must check that the trivial representation does not appear in

$$(1-q)(1-t)\big(\tau_{\mu,\lambda}^* \otimes \tau_\lambda\big) - \tau_{\mu,\lambda}^* \otimes W. \tag{3.43}$$

Recall that

$$\tau_{\mu,\lambda} = \sum_{a=1}^{r} \sum_{s \in \lambda^{(a)} \backslash \mu^{(a)}} \chi_a^{-1} t^{y(s)} q^{x(s)}. \tag{3.44}$$

The multiplicity of the trivial representation in (3.43) is a sum of contributions from each box $s \in \lambda \backslash \mu$. It is easy to check using (3.44) that this contribution is precisely zero for each box. We are done. □

4. Equivariant cohomology of the commuting variety

In this section we introduce an algebra **SCo** in the equivariant cohomology of the commuting variety. Then we provide a description of **SCo** in terms of shuffle algebras. In Section 6 we will construct an action of **SCo** on $\mathbf{L}^{(r)}$ and we'll compare **SCo** with $\mathbf{SH}^{>}$.

4.1. *Correspondences in equivariant Borel-Moore homology.* — Let G be a complex linear algebraic group. Let $P \subset G$ a parabolic subgroup and $M \subset P$ a Levi subgroup. Fix an M-variety Y. The group P acts on Y through the obvious group homomorphism $P \to M$. Let $X = G \times_P Y$ be the induced G-variety. Now assume that Y is smooth. For any smooth subvariety $O \subset Y$ let T_O^*Y be the conormal bundle to O. It is well-known that the induced M-action on T^*Y is Hamiltonian and that the zero set of the moment map is the closed M-subvariety

$$T_M^*Y = \bigsqcup_O T_O^*Y,$$

where O runs over the set of M-orbits. See e.g., [11, Prop. 1.4.8]. Further we have [33]

$$T^*X = T_P^*(G \times Y)/P, \qquad T_G^*X = G \times_P T_M^*Y. \tag{4.1}$$

So the induction yields a canonical isomorphism

$$H^M\big(T_M^*Y\big) = H^G\big(T_G^*X\big). \tag{4.2}$$

We'll call *fibration* a smooth morphism which is locally trivial in the analytic topology. Let X' be a smooth G-variety and V be a smooth M-variety. Fix M-equivariant homomorphisms

$$Y \xleftarrow{p} V \xrightarrow{q} X' \tag{4.3}$$

with p a fibration and q a closed embedding. Set $\mathrm{W} = \mathrm{G} \times_{\mathrm{P}} \mathrm{V}$ and consider the following maps

$$\begin{gathered} \mathrm{X} \xleftarrow{f} \mathrm{W} \xrightarrow{g} \mathrm{X}' \ , \\ f : (g, v) \bmod \mathrm{P} \mapsto \big(g, p(v)\big) \bmod \mathrm{P}, \\ g : (g, v) \bmod \mathrm{P} \mapsto gq(v). \end{gathered} \tag{4.4}$$

Note that V, W, X, X′ are smooth. Further, the map f is a G-equivariant fibration, the map g is a G-equivariant proper morphism, and the map $f \times g$ is a closed embedding $\mathrm{W} \subset \mathrm{X} \times \mathrm{X}'$. See [33] for details. We'll identify W with its image in $\mathrm{X} \times \mathrm{X}'$. The G-variety

$$\mathrm{Z} = \mathrm{T}^*_{\mathrm{W}}\big(\mathrm{X} \times \mathrm{X}'\big) \tag{4.5}$$

is again smooth and the obvious projections yield G-equivariant maps

$$\mathrm{T}^*\mathrm{X} \xleftarrow{\phi} \mathrm{Z} \xrightarrow{\psi} \mathrm{T}^*\mathrm{X}' \ . \tag{4.6}$$

We define the G-variety

$$\mathrm{Z}_{\mathrm{G}} = \mathrm{Z} \cap \big(\mathrm{T}^*_{\mathrm{G}}\mathrm{X} \times \mathrm{T}^*_{\mathrm{G}}\mathrm{X}'\big). \tag{4.7}$$

The following is immediate.

Lemma **4.1.** — (a) *The map* ψ *is proper, the varieties* $\mathrm{T}^*\mathrm{X}$, Z *and* $\mathrm{T}^*\mathrm{X}'$ *are smooth.*
(b) *We have* $\phi^{-1}(\mathrm{T}^*_{\mathrm{G}}\mathrm{X}) = \mathrm{Z}_{\mathrm{G}}$ *and* $\psi(\mathrm{Z}_{\mathrm{G}}) \subset \mathrm{T}^*_{\mathrm{G}}\mathrm{X}'$.

We'll abbreviate $\phi_{\mathrm{G}} = \phi|_{\mathrm{Z}_{\mathrm{G}}}$ and $\psi_{\mathrm{G}} = \psi|_{\mathrm{Z}_{\mathrm{G}}}$. We have the following diagram of singular varieties

$$\mathrm{T}^*_{\mathrm{G}}\mathrm{X} \xleftarrow{\phi_{\mathrm{G}}} \mathrm{Z}_{\mathrm{G}} \xrightarrow{\psi_{\mathrm{G}}} \mathrm{T}^*_{\mathrm{G}}\mathrm{X}' \ . \tag{4.8}$$

Since the map ψ_{G} is proper the direct image yields maps

$$\psi_{\mathrm{G},*} : \mathrm{H}^{\mathrm{G}}(\mathrm{Z}_{\mathrm{G}}) \to \mathrm{H}^{\mathrm{G}}\big(\mathrm{T}^*_{\mathrm{G}}\mathrm{X}'\big). \tag{4.9}$$

Since Z, $\mathrm{T}^*\mathrm{X}$ are smooth and $\phi^{-1}(\mathrm{T}^*_{\mathrm{G}}\mathrm{X}) = \mathrm{Z}_{\mathrm{G}}$, the pull-back by ϕ yields a map

$$\phi^*_{\mathrm{G}} : \mathrm{H}^{\mathrm{G}}\big(\mathrm{T}^*_{\mathrm{G}}\mathrm{X}\big) \to \mathrm{H}^{\mathrm{G}}(\mathrm{Z}_{\mathrm{G}}). \tag{4.10}$$

Composing $\psi_{\mathrm{G},*}$ and ϕ^*_{G} we get a map

$$\psi_{\mathrm{G},*} \circ \phi^*_{\mathrm{G}} : \mathrm{H}^{\mathrm{G}}\big(\mathrm{T}^*_{\mathrm{G}}\mathrm{X}\big) \to \mathrm{H}^{\mathrm{G}}\big(\mathrm{T}^*_{\mathrm{G}}\mathrm{X}'\big). \tag{4.11}$$

By (4.1) the induction yields also an isomorphism

$$\mathrm{H}^{\mathrm{M}}\big(\mathrm{T}^*_{\mathrm{M}}\mathrm{Y}\big) = \mathrm{H}^{\mathrm{G}}\big(\mathrm{T}^*_{\mathrm{G}}\mathrm{X}\big). \tag{4.12}$$

Composing it by $\psi_{\mathrm{G},*} \circ \phi^*_{\mathrm{G}}$ we obtain a map

$$\mathrm{H}^{\mathrm{M}}\big(\mathrm{T}^*_{\mathrm{M}}\mathrm{Y}\big) \to \mathrm{H}^{\mathrm{G}}\big(\mathrm{T}^*_{\mathrm{G}}\mathrm{X}'\big). \tag{4.13}$$

4.2. *The commuting variety.* — We'll apply the general construction above to the commuting variety. First, we fix some notation. Let E be a finite dimensional **C**-vector space. Write

$$\mathfrak{g}_{\mathrm{E}} = \mathrm{End}(\mathrm{E}), \qquad \mathrm{C}_{\mathrm{E}} = \big\{(a, b) \in \mathfrak{g}_{\mathrm{E}} \times \mathfrak{g}_{\mathrm{E}};\, [a, b] = 0\big\}. \tag{4.14}$$

We may abbreviate $\mathrm{G} = \mathrm{GL}_{\mathrm{E}}$, $\mathfrak{g} = \mathfrak{g}_{\mathrm{E}}$ and $\mathrm{C} = \mathrm{C}_{\mathfrak{g}} = \mathrm{C}_{\mathrm{E}}$. Put $\widetilde{\mathrm{G}} = \mathrm{T} \times \mathrm{G}$ with $\mathrm{T} = (\mathbf{C}^\times)^2$. The group $\widetilde{\mathrm{G}}$ acts on C: the subgroup G acts diagonally by the adjoint action on $\mathfrak{g}$, while T acts by $(e, f) \cdot (a, b) = (ea, fb)$. We set $\mathbf{Co}'_{\mathrm{E}} = \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{C})$. Let $\mathrm{K}_{\widetilde{\mathrm{G}}}$ be the fraction field of $\mathrm{R}_{\widetilde{\mathrm{G}}}$. Let $\mathscr{V}_n$ be the groupoid formed by all n-dimensional vector spaces with their isomorphisms and set $\mathscr{V} = \bigoplus_{n\geq 0} \mathscr{V}_n$. An isomorphism $\mathrm{E} \to \mathrm{E}'$ yields an R-module isomorphism $\mathbf{Co}'_{\mathrm{E}} \to \mathbf{Co}'_{\mathrm{E}'}$. Let $\mathbf{Co}'$ be the colimit of the system $(\mathbf{Co}'_{\mathrm{E}})$ where E varies in $\mathscr{V}$. It is an **N**-graded vector space. The piece $\mathbf{Co}'_n$ of degree n is the colimit over the groupoid $\mathscr{V}_n$.

4.3. *The cohomological Hall algebra.* — Fix a flag of finite dimensional vector spaces

$$0 \longrightarrow \mathrm{E}_1 \longrightarrow \mathrm{E} \longrightarrow \mathrm{E}_2 \longrightarrow 0. \tag{4.15}$$

Set $\mathrm{G} = \mathrm{GL}_{\mathrm{E}}$, $\mathrm{M} = \mathrm{GL}_{\mathrm{E}_1} \times \mathrm{GL}_{\mathrm{E}_2}$ and $\mathrm{P} = \{g \in \mathrm{G};\, g(\mathrm{E}_1) = \mathrm{E}_1\}$. Let $\mathfrak{g}$, $\mathfrak{m}$ and $\mathfrak{p}$ be the corresponding Lie algebras. Put $\mathrm{Y} = \mathfrak{m}$, $\mathrm{V} = \mathfrak{p}$, and $\mathrm{X}' = \mathfrak{g}$. The G-action on X' and the M-action on Y are the adjoint ones. Put

$$\begin{aligned} &\mathrm{C}_{\mathfrak{m}} = (\mathfrak{m} \times \mathfrak{m}) \cap \mathrm{C}, \qquad \mathrm{C}_{\mathfrak{p}} = (\mathfrak{p} \times \mathfrak{p}) \cap \mathrm{C}_{\mathfrak{g}}, \\ &\widetilde{\mathrm{C}}_{\mathfrak{m}} = \big\{(d, a, b) \in \mathfrak{p} \times \mathfrak{m} \times \mathfrak{m};\ d_{\mathfrak{m}} = [a, b]\big\}, \end{aligned} \tag{4.16}$$

where $p : \mathfrak{p} \to \mathfrak{m}$, $a \mapsto a_{\mathfrak{m}}$ is the canonical projection. We apply the general construction in Section 4.1 to the diagram (4.3) equal to

$$\mathfrak{m} \xleftarrow{\ p\ } \mathfrak{p} \xrightarrow{\ q\ } \mathfrak{g}, \tag{4.17}$$

where q is the obvious inclusion. The P-actions on $\mathfrak{p} \times \mathfrak{p}$ and on $\mathfrak{p} \times \mathfrak{m} \times \mathfrak{m}$ are the obvious ones. Further we identify $\mathfrak{g}^* = \mathfrak{g}$ and $\mathfrak{m}^* = \mathfrak{m}$ via the trace.

Lemma **4.2.** — (a) *There are isomorphisms of* G*-varieties*

$$\mathrm{T}^*\mathrm{X} = \mathrm{G} \times_{\mathrm{P}} \widetilde{\mathrm{C}}_{\mathfrak{m}}, \qquad \mathrm{Z} = \mathrm{G} \times_{\mathrm{P}} (\mathfrak{p} \times \mathfrak{p}), \qquad \mathrm{T}^*\mathrm{X}' = \mathfrak{g} \times \mathfrak{g}.$$

(b) *For* $a, b \in \mathfrak{p}$ *we have*

$$\phi\big((g, a, b) \mod \mathrm{P}\big) = \big(g, [a, b], a_{\mathfrak{m}}, b_{\mathfrak{m}}\big) \mod \mathrm{P},$$
$$\psi\big((g, a, b) \mod \mathrm{P}\big) = \big(gag^{-1}, gbg^{-1}\big).$$

(c) *There are isomorphisms of* G*-varieties*

$$\mathrm{T}^*_{\mathrm{G}}\mathrm{X} = \mathrm{G} \times_{\mathrm{P}} \mathrm{C}_{\mathfrak{m}}, \qquad \mathrm{Z}_{\mathrm{G}} = \mathrm{G} \times_{\mathrm{P}} \mathrm{C}_{\mathfrak{p}}, \qquad \mathrm{T}^*_{\mathrm{G}}\mathrm{X}' = \mathrm{C}_{\mathfrak{g}}.$$

(d) *The maps* ϕ, ψ, ϕ_{G}, ψ_{G} *in the following diagram are the obvious ones*

$$\mathrm{G} \times_{\mathrm{P}} \mathrm{C}_{\mathfrak{m}} \xleftarrow{\phi_{\mathrm{G}}} \mathrm{G} \times_{\mathrm{P}} \mathrm{C}_{\mathfrak{p}} \xrightarrow{\psi_{\mathrm{G}}} \mathrm{C}_{\mathfrak{g}},$$

$$\mathrm{G} \times_{\mathrm{P}} \widetilde{\mathrm{C}}_{\mathfrak{m}} \xleftarrow{\phi} \mathrm{G} \times_{\mathrm{P}} (\mathfrak{p} \times \mathfrak{p}) \xrightarrow{\psi} \mathfrak{g} \times \mathfrak{g}.$$

We define as in (4.13) an R-linear map

$$\mathrm{H}^{\widetilde{\mathrm{M}}}(\mathrm{C}_{\mathfrak{m}}) \to \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{C}_{\mathfrak{g}}). \tag{4.18}$$

By the Kunneth formula, it can be viewed as a map

$$\mathbf{Co}'_{\mathrm{E}_1} \otimes_{\mathrm{R}} \mathbf{Co}'_{\mathrm{E}_2} \to \mathbf{Co}'_{\mathrm{E}}. \tag{4.19}$$

The following is proved as in [33, Prop. 7.5].

Proposition **4.3.** — *The map* (4.19) *equips* $\mathbf{Co}'$ *with the structure of an* R*-algebra with 1.*

We call the **N**-graded R-algebra $\mathbf{Co}'$ the *cohomological Hall algebra*. Let $\mathbf{SCo}'$ be the R-subalgebra of $\mathbf{Co}'$ generated by $\mathbf{Co}'_1$. We'll abbreviate $\mathbf{SCo}'_n = \mathbf{Co}'_n \cap \mathbf{SCo}'$ and $\mathrm{G} = \mathrm{GL}_n$. The direct image by the obvious inclusion $\mathrm{C}_{\mathfrak{g}} \subset \mathfrak{g} \times \mathfrak{g}$, which is a proper map, yields an $\mathrm{R}_{\widetilde{\mathrm{G}}}$-module homomorphism

$$\mathbf{Co}'_n \to \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathfrak{g} \times \mathfrak{g}). \tag{4.20}$$

We conjecture that (4.20) is an injective map. Since the kernel of (4.20) is the torsion submodule $\mathbf{Co}^{\mathrm{tor}}_n$ of $\mathbf{Co}'_n$ by the localization theorem, this conjecture is equivalent to the following one.

Conjecture **4.4.** — *The* $\mathrm{R}_{\widetilde{\mathrm{G}}}$*-module* $\mathbf{Co}'_n$ *is torsion-free.*

Let $\mathbf{Co}_n$, $\mathbf{SCo}_n$ be the image of $\mathbf{Co}'_n$, $\mathbf{SCo}'_n$ by (4.20) and set

$$\mathbf{Co} = \bigoplus_{n\geq 0} \mathbf{Co}_n, \qquad \mathbf{SCo} = \bigoplus_{n\geq 0} \mathbf{SCo}_n. \tag{4.21}$$

We call $\mathbf{SCo}$ the *spherical subalgebra* of $\mathbf{Co}$.

Proposition **4.5.** — *The map* (4.20) *yields surjective* R*-algebra homomorphisms* $\mathbf{Co}' \to \mathbf{Co}$ *and* $\mathbf{SCo}' \to \mathbf{SCo}$.

Proof. — For $\mathrm{E} \in \mathscr{V}$ let $\mathbf{Co}_\mathrm{E}$ be the quotient of $\mathbf{Co}'_\mathrm{E}$ by its torsion $\mathrm{R}_{\widetilde{\mathrm{G}}\mathrm{L}_\mathrm{E}}$-submodule $\mathbf{Co}^{\mathrm{tor}}_\mathrm{E}$. Given $\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}$ as in (4.15), we must check that the map (4.19) fits into a commutative square

$$\begin{tikzcd} \mathbf{Co}'_{\mathrm{E}_1} \otimes_\mathrm{R} \mathbf{Co}'_{\mathrm{E}_2} \arrow[r] \arrow[d] & \mathbf{Co}'_\mathrm{E} \arrow[d] \\ \mathbf{Co}_{\mathrm{E}_1} \otimes_\mathrm{R} \mathbf{Co}_{\mathrm{E}_2} \arrow[r] & \mathbf{Co}_\mathrm{E}. \end{tikzcd} \tag{4.22}$$

Recall that $\mathbf{Co}_\mathrm{E}$ is identified with the image by the obvious map

$$\mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{C}_\mathfrak{g}) \to \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathfrak{g} \times \mathfrak{g}). \tag{4.23}$$

Similarly, since $\widetilde{\mathrm{C}}_\mathfrak{m}$ is isomorphic to $\mathfrak{u} \times \mathfrak{m} \times \mathfrak{m}$ as an $\widetilde{\mathrm{M}}$-module, where $\mathfrak{u}$ is the nilpotent radical of $\mathfrak{p}$, we can identify $\mathbf{Co}_{\mathrm{E}_1} \otimes_\mathrm{R} \mathbf{Co}_{\mathrm{E}_2}$ with the image of the direct image by the obvious inclusion

$$\mathrm{H}^{\widetilde{\mathrm{M}}}(\mathrm{C}_\mathfrak{m}) \to \mathrm{H}^{\widetilde{\mathrm{M}}}(\widetilde{\mathrm{C}}_\mathfrak{m}). \tag{4.24}$$

So the proposition follows from the commutativity of the diagram

$$\begin{tikzcd} \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{G} \times_\mathrm{P} \mathrm{C}_\mathfrak{m}) \arrow[r, "\phi^*_\mathrm{G}"] \arrow[d] & \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{G} \times_\mathrm{P} \mathrm{C}_\mathfrak{p}) \arrow[r, "\psi_{\mathrm{G},*}"] \arrow[d] & \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{C}_\mathfrak{g}) \arrow[d] \\ \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{G} \times_\mathrm{P} \widetilde{\mathrm{C}}_\mathfrak{m}) \arrow[r, "\phi^*"] & \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{G} \times_\mathrm{P} (\mathfrak{p} \times \mathfrak{p})) \arrow[r, "\psi_*"] & \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathfrak{g} \times \mathfrak{g}). \end{tikzcd} \tag{4.25}$$

□

For any commutative ring extension $\mathrm{R} \subset \mathrm{L}$ we abbreviate

$$\mathbf{Co}'_\mathrm{L} = \mathbf{Co}' \otimes_\mathrm{R} \mathrm{L}, \qquad \mathbf{SCo}'_\mathrm{L} = \mathbf{SCo}' \otimes_\mathrm{R} \mathrm{L}, \qquad \mathbf{SCo}_\mathrm{L} = \mathbf{SCo} \otimes_\mathrm{R} \mathrm{L}, \qquad \textit{etc.} \tag{4.26}$$

4.4. *The shuffle algebra.* — Fix $E \in \mathscr{V}_n$. Let $G = GL_n$ and let $D \subset G$ be a maximal torus. The Poincaré duality and the inverse image by the obvious inclusion $\{0\} \to \mathfrak{g} \times \mathfrak{g}$ yield an isomorphism $H^{\widetilde{G}}(\mathfrak{g} \times \mathfrak{g}) = R_{\widetilde{G}}$. Composing it with (4.20) we get an $R_{\widetilde{G}}$-linear map

$$\gamma_G : \mathbf{Co}'_n \to R_{\widetilde{G}}. \tag{4.27}$$

Taking the tensor power over R, we define an $R_{\widetilde{D}}$-linear map

$$\gamma_D = (\gamma_{\mathbf{C}^\times})^{\otimes n} : \big(\mathbf{Co}'_1\big)^{\otimes n} \to R_{\widetilde{D}}. \tag{4.28}$$

Recall that there are obvious isomorphisms

$$R_{\widetilde{D}} = R[z_1, z_2, \ldots, z_n], \qquad R_{\widetilde{G}} = R[z_1, z_2, \ldots, z_n]^{\mathfrak{S}_n}. \tag{4.29}$$

Let $K_{\widetilde{D}}$, $K_{\widetilde{G}}$ be the fraction fields and $\mathrm{SYM}_n : K_{\widetilde{D}} \to K_{\widetilde{G}}$ be the symmetrization operator.

Proposition **4.6.** — *We have the commutative diagram*

$$\begin{tikzcd} (\mathbf{Co}'_1)^{\otimes n} \arrow[r, "\mu_n"] \arrow[d, "\gamma_D"'] & \mathbf{Co}'_n \arrow[d, "\gamma_G"] \\ R_{\widetilde{D}} \arrow[r, "\nu_n"] & R_{\widetilde{G}}, \end{tikzcd}$$

where μ_n is the multiplication in $\mathbf{Co}'$ and ν_n is given by

$$\begin{aligned} &\nu_n\big(P(z_1, \ldots, z_n)\big) = \mathrm{SYM}_n\big(k(z_1, z_2, \ldots, z_n)P(z_1, z_2, \ldots, z_n)\big), \\ &k(z) = z^{-1}(x+y+z)(x-z)(y-z), \\ &k(z_1, z_2, \ldots, z_n) = \prod_{i<j} k(z_i - z_j). \end{aligned} \tag{4.30}$$

Proof. — Let $\mathfrak{d}$ be the Lie algebra of D. Since $C_\mathfrak{d}$ is a vector space, the $R_{\widetilde{D}}$-module $H^{\widetilde{D}}(C_\mathfrak{d})$ is spanned by the set

$$\big\{z^m \cdot [C_\mathfrak{d}];\, m \in \mathbf{N}^n\big\}, \qquad z^m = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}, \quad m = (m_1, m_2, \ldots m_n) \in \mathbf{N}^n. \tag{4.31}$$

Here $[C_\mathfrak{d}]$ is the fundamental class and $\cdot$ is the $R_{\widetilde{D}}$-module structure on $H^{\widetilde{D}}(C_\mathfrak{d})$. Note that

$$\gamma_D\big(z^m \cdot [C_\mathfrak{d}]\big) = z^m. \tag{4.32}$$

Let $\mathrm{B} \subset \mathrm{G}$ be a Borel subgroup containing T. Let $\mathfrak{b} = \mathrm{Lie}(\mathrm{B})$ and let $\mathfrak{n}$ be its nilpotent radical. We have

$$\begin{aligned} &\mathrm{T}^*_\mathrm{G}\mathrm{X} = \mathrm{G} \times_\mathrm{B} \mathrm{C}_\mathfrak{d}, \qquad \mathrm{T}^*\mathrm{X} = \mathrm{G} \times_\mathrm{B} (\mathfrak{n} \times \mathrm{C}_\mathfrak{d}), \\ &\mathrm{C}_\mathfrak{d} = \mathfrak{d} \times \mathfrak{d}, \qquad \mathrm{Z} = \mathrm{G} \times_\mathrm{B} (\mathfrak{b} \times \mathfrak{b}). \end{aligned} \tag{4.33}$$

Let Ind denote the induction

$$\mathrm{H}^{\widetilde{\mathrm{D}}}(\bullet) = \mathrm{H}^{\widetilde{\mathrm{B}}}(\bullet) \to \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{G} \times_\mathrm{B} \bullet). \tag{4.34}$$

Consider the elements in $\mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{T}^*_\mathrm{G}\mathrm{X})$ given by

$$\alpha_m = \mathrm{Ind}\big(z^m \cdot [\mathrm{C}_\mathfrak{d}]\big), \qquad m \geq 0. \tag{4.35}$$

For a future use, we consider also the following commutative diagram

$$\begin{tikzcd} \mathrm{T}^*_\mathrm{G}\mathrm{X} \arrow[d, "j"'] & & \mathrm{T}^*_\mathrm{G}\mathrm{X}' = \mathrm{C}_\mathfrak{g} \arrow[d] \\ \mathrm{T}^*\mathrm{X} & \mathrm{Z} \arrow[l, "\phi"'] \arrow[r, "\psi"] & \mathrm{T}^*\mathrm{X}' = \mathfrak{g} \times \mathfrak{g} \\ & \mathrm{G}/\mathrm{B} \arrow[u, "i"'] \arrow[r, "\pi"] & \{0\}. \arrow[u, "h"'] \end{tikzcd} \tag{4.36}$$

The vertical maps are the obvious inclusions. The multiplication (4.19) gives

$$\nu_n\big(z^m\big) = h^*\psi_*\phi^*j_*(\alpha_m). \tag{4.37}$$

Now, we compute the right hand side of (4.37). We have

$$j_*(\alpha_m) = \mathrm{Ind}\big(z^m\,\mathrm{eu}\big(v\mathfrak{n}^*\big) \cdot [\mathfrak{n} \times \mathrm{C}_\mathfrak{d}]\big). \tag{4.38}$$

Therefore we have also

$$\phi^*j_*(\alpha_m) = \mathrm{Ind}\big(z^m\,\mathrm{eu}\big(v\mathfrak{n}^*\big) \cdot [\mathfrak{b} \times \mathfrak{b}]\big). \tag{4.39}$$

Tensoring by $\mathrm{K}_{\widetilde{\mathrm{G}}}$, the maps i_*, i^* become invertible by the localization theorem. We have

$$\begin{aligned} \nu_n\big(z^m\big) &= h^*\psi_*i_*\,\mathrm{Ind}\big(z^m\,\mathrm{eu}\big(v\mathfrak{n}^*\big)\,\mathrm{eu}\big(q^{-1}\mathfrak{b}^* + t^{-1}\mathfrak{b}^*\big)^{-1} \cdot [\mathrm{G}/\mathrm{B}]\big), \\ &= h^*h_*\pi_*\,\mathrm{Ind}\big(z^m\,\mathrm{eu}\big(v\mathfrak{n}^*\big)\,\mathrm{eu}\big(q^{-1}\mathfrak{b}^* + t^{-1}\mathfrak{b}^*\big)^{-1} \cdot [\mathrm{G}/\mathrm{B}]\big), \\ &= \mathrm{eu}\big(q^{-1}\mathfrak{g}^* + t^{-1}\mathfrak{g}^*\big) \cdot \pi_*\,\mathrm{Ind}\big(z^m\,\mathrm{eu}\big(v\mathfrak{n}^*\big)\,\mathrm{eu}\big(q^{-1}\mathfrak{b}^* + t^{-1}\mathfrak{b}^*\big)^{-1} \cdot [\mathrm{G}/\mathrm{B}]\big), \\ &= \pi_*\,\mathrm{Ind}\big(z^m\,\mathrm{eu}\big(v\mathfrak{n}^* + q^{-1}\mathfrak{n} + t^{-1}\mathfrak{n}\big) \cdot [\mathrm{G}/\mathrm{B}]\big). \end{aligned} \tag{4.40}$$

Thus the integration over the set $(G/B)^{\widetilde{D}}$ yields the formula

$$\nu_n\big(z^m\big) = \mathrm{SYM}_n\big(k(z_1, z_2, \ldots, z_n)\, z^m\big). \tag{4.41}$$
□

Now, we equip the R-module

$$\mathbf{Sh} = \bigoplus_{n\geq 0} \mathbf{Sh}_n, \qquad \mathbf{Sh}_n = \mathrm{R}[z_1, \ldots, z_n]^{\mathfrak{S}_n} \tag{4.42}$$

with the shuffle multiplication given by

$$\begin{aligned}(\mathrm{P}\cdot \mathrm{Q})(z_1, \ldots, z_{m+n}) \\ = \frac{1}{n!m!}\,\mathrm{SYM}_{n+m}\bigg(\Big(\prod_{i,j} k(z_i - z_j)\Big)\mathrm{P}(z_1, \ldots, z_n)\mathrm{Q}(z_{n+1}, \ldots, z_{n+m})\bigg).\end{aligned} \tag{4.43}$$

The product runs over all i, j with $1 \leq i \leq n < j \leq n+m$. For $\dim \mathrm{E} = 1$ and $l \geq 0$ let

$$\theta_l = z^l \cdot [\mathrm{C_E}]. \tag{4.44}$$

The following direct consequence of Proposition 4.6 is the first main result of this chapter.

Theorem **4.7.** — *There is a unique* R*-algebra embedding* $\mathbf{SCo} \subset \mathbf{Sh}$ *such that* $\theta_l \mapsto (z_1)^l$*, where* z_1 *is viewed as an element in* $\mathbf{Sh}_1$.

We state one useful consequence.

Corollary **4.8.** — *For* $u \in \mathbf{C}$ *the assignment* $\theta_l \mapsto \sum_{i=0}^{l} \binom{l}{i} u^{l-i}\theta_i$ *extends to an algebra automorphism* τ_u *of* $\mathbf{SCo}$. *We have* $\tau_u \circ \tau_v = \tau_{u+v}$ *for* $u, v \in \mathbf{C}$.

Proof. — Under the embedding $\mathbf{SCo} \subset \mathbf{Sh}$ the map τ_u is the restriction of the automorphism induced by the substitution $z_i \mapsto z_i + u$. Observe that $k(z_1, \ldots, z_n)$ is invariant under this substitution. □

4.5. *Wilson operators on* **C** *and* **SCo**. — The canonical $\mathrm{R}_{\widetilde{\mathrm{GL}}_n}$-module structure on $\mathbf{Co}'_n$ gives a graded Λ-algebra structure on $\mathbf{Co}'$ and $\mathbf{Co}$, which we will denote by $\bullet$.

Lemma **4.9.** — (a) *The action of* Λ *on* $\mathbf{Co}$, $\mathbf{Co}'$ *preserves the spherical subalgebras* $\mathbf{SCo}'$, $\mathbf{SCo}$.

(b) *The* Λ*-action on* $\mathbf{Co}'_n$, $\mathbf{Co}_n$ *factors through* Λ_n.

(c) *For* $p \in \Lambda$ *and* $u, v \in \mathbf{Co}'$ *(or* $\mathbf{Co}$*) we have* $p \bullet (uv) = \sum (p_1 \bullet u)(p_2 \bullet v)$.

Proof. — Statement (b) is clear. Observe that $p_l \bullet \theta_k = \theta_{l+k}$, hence (c) implies (a). Finally (c) is a consequence of the commutativity of the following diagram

$$
\textbf{(4.45)} \qquad
\begin{tikzcd}
\Lambda_{\mathrm{K}} \arrow[r, "\pi_{n+m}"] \arrow[d, "\Delta"'] & \Lambda_{n+m,\mathrm{K}} \arrow[r, equal] & \mathrm{R}_{\widetilde{\mathrm{GL}}_{n+m}} \arrow[d] \\
\Lambda_{\mathrm{K}} \otimes_{\mathrm{K}} \Lambda_{\mathrm{K}} \arrow[r, "\pi_n \otimes \pi_m"] & \Lambda_{n,\mathrm{K}} \otimes_{\mathrm{K}} \Lambda_{m,\mathrm{K}} \arrow[r, equal] & \mathrm{R}_{\widetilde{\mathrm{M}}}
\end{tikzcd}
$$

where $\mathrm{M} \subset \mathrm{GL}_{n+m}$ is the standard parabolic with Levi $\mathrm{GL}_n \times \mathrm{GL}_m$, and where the rightmost arrow is the restriction map. □

5. Proof of Theorem 2.4

5.1. *Part 1: the positive and negative halves.* — We define subalgebras $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),\pm}$, $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),>}$, $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),<}$ and $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),0}$ of $\widetilde{\mathbf{E}}_{\mathrm{K}}^{(1)}$ in a way entirely similar to that used for the subalgebras $\mathbf{U}_{\mathrm{K}}^{(r),\pm}$, $\mathbf{U}_{\mathrm{K}}^{(r),>}$, $\mathbf{U}_{\mathrm{K}}^{(r),<}$ and $\mathbf{U}_{\mathrm{K}}^{(r),0}$ of $\mathbf{E}_{\mathrm{K}}^{(r)}$ in Section 3.6. Our first task is to construct an isomorphism $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),+} \to \mathbf{SH}_{\mathrm{K}}^{+}$. For this, we will use the canonical representation of $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),+}$ on $\widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}$. It is the restriction of the canonical representation of $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1)}$ on $\widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}$ considered in Section 2.8.

Proposition **5.1.** — (a) *The map* $\mathrm{D}_{\mathbf{x}} \mapsto h_{\mathbf{x}}$ *for* $\mathbf{x} \in \mathscr{E}^+$ *yields an algebra isomorphism* $\Psi^+ : \widetilde{\mathbf{SH}}_{\mathrm{K}}^{(1),+} \to \widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),+}$ *which takes* $\widetilde{\mathbf{SH}}_{\mathrm{K}}^{(1),>}$ *onto* $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),>}$.

(b) *The map* (2.34) *intertwines* ρ^+ *with the canonical representation of* $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),+}$ *on* $\widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}$.

Proof. — First, we compare the action of $\mathrm{D}_{\mathbf{x}}$ on Λ_{K} with the action of $h_{\mathbf{x}}$ on $\widetilde{\mathbf{L}}_{\mathrm{K}}^{(1)}$, under the isomorphism (2.34). These actions are described by the formulas (1.30) and (1.34) for $\mathrm{D}_{\mathbf{x}}$, and by the formulas (C.6) and (C.4) for $h_{\mathbf{x}}$. These formulas coincide, because $\psi_{\lambda\backslash\mu} = \mathrm{L}_{\mu,\lambda}$. Since ρ^+ is a faithful representation, see Proposition 1.20, this yields the isomorphism Ψ^+ above. □

Remark **5.2.** — By Propositions 1.35 and 3.7, the K-algebras $\widetilde{\mathbf{SH}}_{\mathrm{K}}^{(1),-}$ and $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),-}$ are isomorphic to the opposite K-algebras of $\widetilde{\mathbf{SH}}_{\mathrm{K}}^{(1),+}$ and $\widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),+}$ respectively. Thus, by Proposition 5.1, the assignment $\mathrm{D}_{\mathbf{x}} \mapsto h_{\mathbf{x}}$ for $\mathbf{x} \in \mathscr{E}^-$ extends to an algebra isomorphism $\Psi^- : \widetilde{\mathbf{SH}}_{\mathrm{K}}^{(1),-} \to \widetilde{\mathbf{U}}_{\mathrm{K}}^{(1),-}$.

5.2. *Part 2: glueing the positive and negative halves.* — We must prove that the two algebra isomorphisms Ψ^+, Ψ^- glue together to an algebra homomorphism

$$\textbf{(5.1)} \qquad \Psi : \widetilde{\mathbf{SH}}_{\mathrm{K}}^{(1)} \to \widetilde{\mathbf{U}}_{\mathrm{K}}^{(1)}.$$

It suffices to check (1.69). The proof of this relation follows from Appendix D by setting $r=1$ and $e_a=0$ there. To finish the proof of Theorem 2.4, it remains to show that the map Ψ is an isomorphism. Since it is clearly surjective, we only have to check that it is injective. Our argument is based on the existence of triangular decompositions for $\widetilde{\mathbf{SH}}^{(1)}_{\mathrm{K}}$ and $\widetilde{\mathbf{U}}^{(1)}_{\mathrm{K}}$. First, let us quote the following proposition whose proof is given in Appendix C.2.

Proposition **5.3.** — *The multiplication gives an isomorphism*

$$m:\widetilde{\mathbf{U}}^{(1),>}_{\mathrm{K}}\otimes_{\mathrm{K}}\widetilde{\mathbf{U}}^{(1),0}_{\mathrm{K}}\otimes_{\mathrm{K}}\widetilde{\mathbf{U}}^{(1),<}_{\mathrm{K}}\to\widetilde{\mathbf{U}}^{(1)}_{\mathrm{K}}.$$

Let $\Psi^>$, Ψ^0, $\Psi^<$ be the restrictions of Ψ^+, Ψ^- to $\widetilde{\mathbf{SH}}^{(1),>}_{\mathrm{K}}$, $\widetilde{\mathbf{SH}}^{(1),0}_{\mathrm{K}}$ and $\widetilde{\mathbf{SH}}^{(1),<}_{\mathrm{K}}$. We have the following commutative diagram

$$\begin{tikzcd}
\widetilde{\mathbf{SH}}^{(1),>}_{\mathrm{K}}\otimes_{\mathrm{K}}\widetilde{\mathbf{SH}}^{(1),0}_{\mathrm{K}}\otimes_{\mathrm{K}}\widetilde{\mathbf{SH}}^{(1),<}_{\mathrm{K}} \arrow[r, "\Psi^>\otimes\Psi^0\otimes\Psi^<"] \arrow[d, "m"'] & \widetilde{\mathbf{U}}^{(1),>}_{\mathrm{K}}\otimes_{\mathrm{K}}\widetilde{\mathbf{U}}^{(1),0}_{\mathrm{K}}\otimes_{\mathrm{K}}\widetilde{\mathbf{U}}^{(1),<}_{\mathrm{K}} \arrow[d, "m"] \\
\widetilde{\mathbf{SH}}^{(1)}_{\mathrm{K}} \arrow[r, "\Psi"] & \widetilde{\mathbf{U}}^{(1)}_{\mathrm{K}}.
\end{tikzcd} \tag{5.2}$$

Further, we have the following isomorphisms

$$\widetilde{\mathbf{SH}}^{(1),0}_{\mathrm{K}}=\mathrm{K}[\mathrm{D}_{0,l};\, l\geq 1],\qquad \widetilde{\mathbf{U}}^{(1),0}_{\mathrm{K}}=\mathrm{K}[h_{0,l};\, l\geq 1]. \tag{5.3}$$

Thus, by Proposition 5.1 and Proposition 5.3, the top arrow and the right one are isomorphisms. The left arrow is surjective by Proposition 1.37. Thus the left arrow and the bottom one are both isomorphisms. Theorem 2.4 is proved.

6. Proof of Theorem 3.2

6.1. *Part 1: the positive and negative halves.* — Given $\mathrm{E}_1,\mathrm{E}\in\mathscr{V}$ with $\mathrm{E}_1\subset\mathrm{E}$, we write

$$\begin{gathered}\mathrm{E}_2=\mathrm{E}/\mathrm{E}_1,\qquad \mathrm{M}=\mathrm{GL}_{\mathrm{E}_1}\times\mathrm{GL}_{\mathrm{E}_2},\qquad \mathrm{P}=\big\{g\in\mathrm{G};\, g(\mathrm{E}_1)=\mathrm{E}_1\big\},\\ \mathrm{X}'=\mathfrak{g}\times\mathrm{Hom}\big(\mathbf{C}^r,\mathrm{E}\big),\qquad \mathrm{Y}=\mathfrak{m}\times\mathrm{Hom}\big(\mathbf{C}^r,\mathrm{E}_2\big),\qquad \mathrm{V}=\mathfrak{p}\times\mathrm{Hom}\big(\mathbf{C}^r,\mathrm{E}\big).\end{gathered} \tag{6.1}$$

Here $\mathfrak{p}$, $\mathfrak{m}$ are the Lie algebras of P, M. Consider the obvious maps

$$\pi:\mathrm{E}\to\mathrm{E}_2,\qquad p:\mathrm{V}\to\mathrm{Y},\qquad q:\mathrm{V}\to\mathrm{X}'. \tag{6.2}$$

For $x\in\mathfrak{p}$ let $x_\mathfrak{m}$ be its projection in $\mathfrak{m}$, modulo the nilpotent radical $\mathfrak{u}$ of $\mathfrak{p}$. Let X, W, Z, $\mathrm{Z_G}$, ϕ, ψ be as in Section 4.1 and $\mathrm{M}_{r,\mathrm{E}}$, $\mathrm{N}_{r,\mathrm{E}}$ be as in (3.1). Define

$$\begin{aligned}
&\mathrm{N}_\mathfrak{m}=(\mathfrak{g}_{\mathrm{E}_1})^2\times\mathrm{N}_{r,\mathrm{E}_2}=\mathfrak{m}^2\times\mathrm{Hom}\big(\mathrm{E},\mathbf{C}^r\big)\times\mathrm{Hom}\big(\mathbf{C}^r,\mathrm{E}\big),\\
&\mathrm{M}_\mathfrak{m}=\mathrm{C}_{\mathrm{E}_1}\times\mathrm{M}_{r,\mathrm{E}_2}=\big\{(a,b,\varphi,v)\in\mathrm{N}_\mathfrak{m};\,0=[a,b]+v\circ\varphi\big\},\\
&\mathrm{N}_\mathfrak{p}=\mathfrak{p}^2\times\mathrm{Hom}\big(\mathrm{E}_2,\mathbf{C}^r\big)\times\mathrm{Hom}\big(\mathbf{C}^r,\mathrm{E}\big),\\
&\mathrm{M}_\mathfrak{p}=\mathrm{N}_\mathfrak{p}\cap\mathrm{M}_{r,\mathrm{E}}=\big\{(a,b,\varphi,v)\in\mathrm{N}_\mathfrak{p};\,0=[a,b]+v\circ\varphi\big\},\\
&\widetilde{\mathrm{N}}_\mathfrak{m}=\big\{(c,a,b,\varphi,v)\in\mathfrak{p}\times\mathrm{N}_\mathfrak{m};\,c_\mathfrak{m}=[a,b]+v\circ\varphi\big\}\simeq\mathfrak{u}\times\mathrm{N}_\mathfrak{m}.
\end{aligned}\tag{6.3}$$

We have the following technical lemma [33, Lem. 8.2].

Lemma **6.1.** — (a) *We have canonical isomorphisms of* G*-varieties*

$$\begin{aligned}
&\mathrm{T^*X}=\mathrm{G}\times_\mathrm{P}\widetilde{\mathrm{N}}_\mathfrak{m},\quad \mathrm{Z}=\mathrm{G}\times_\mathrm{P}\mathrm{N}_\mathfrak{p},\quad \mathrm{T^*X'}=\mathrm{N}_{r,\mathrm{E}},\\
&\mathrm{T^*_GX}=\mathrm{G}\times_\mathrm{P}\mathrm{M}_\mathfrak{m},\quad \mathrm{Z_G}=\mathrm{G}\times_\mathrm{P}\mathrm{M}_\mathfrak{p},\quad \mathrm{T^*_GX'}=\mathrm{M}_{r,\mathrm{E}}.
\end{aligned}$$

(b) *The maps* $\phi:\mathrm{Z}\to\mathrm{T^*X}$ *and* $\psi:\mathrm{Z}\to\mathrm{T^*X'}$ *in* (4.6) *are given, for* $(a,b,\varphi,v)\in\mathrm{N}_\mathfrak{p}$, *by*

$$\begin{aligned}
&\phi\big((g,a,b,\varphi,v)\bmod\mathrm{P}\big)=\big(g,[a,b]+v\circ\varphi,a_\mathfrak{m},b_\mathfrak{m},\varphi,\pi\circ v\big)\bmod\mathrm{P},\\
&\psi\big((g,a,b,\varphi,v)\bmod\mathrm{P}\big)=\big(gag^{-1},gbg^{-1},(\varphi\circ\pi)g^{-1},gv\big).
\end{aligned}$$

(c) *The inclusion* $\mathrm{T^*_GX}\subset\mathrm{T^*X}$ *is induced by the inclusion* $\mathrm{M}_\mathfrak{m}\to\widetilde{\mathrm{N}}_\mathfrak{m}$, $(a,b,\varphi,v)\mapsto(0,a,b,\varphi,v)$. *The inclusion* $\mathrm{Z_G}\subset\mathrm{Z}$ *is induced by the obvious inclusion* $\mathrm{M}_\mathfrak{p}\subset\mathrm{N}_\mathfrak{p}$. *The inclusion* $\mathrm{T^*_GX'}\subset\mathrm{T^*X'}$ *is the obvious one.*

Using this lemma, we can now prove the following.

Proposition **6.2.** — *There is a representation* η' *of* $\mathbf{Co}'$ *on* $\mathbf{L}^{(r)}$ *such that* $\eta'(\theta_l)=f_{1,l}$ *for* $l\in\mathbf{N}$.

Proof. — To define η' we consider the closed embeddings

$$\mathrm{N}_\mathfrak{p}\subset\mathrm{N}_{r,\mathrm{E}},\qquad \mathrm{M}_\mathfrak{p}\subset\mathrm{M}_{r,\mathrm{E}},\qquad (a,b,\varphi,v)\mapsto(a,b,\varphi\circ\pi,v).\tag{6.4}$$

Then, we set

$$\mathrm{N}^s_\mathfrak{p}=\mathrm{N}^s_{r,\mathrm{E}}\cap\mathrm{N}_\mathfrak{p},\qquad \mathrm{M}^s_\mathfrak{p}=\mathrm{M}^s_{r,\mathrm{E}}\cap\mathrm{M}_\mathfrak{p},\qquad \mathrm{Z}^s=\mathrm{G}\times_\mathrm{P}\mathrm{N}^s_\mathfrak{p},\qquad \mathrm{Z}^s_\mathrm{G}=\mathrm{G}\times_\mathrm{P}\mathrm{M}^s_\mathfrak{p}.\tag{6.5}$$

Note that $\mathrm{N}^s_\mathfrak{p}$, $\mathrm{M}^s_\mathfrak{p}$, Z^s, Z^s_G are open in $\mathrm{N}_\mathfrak{p}$, $\mathrm{M}_\mathfrak{p}$, Z and $\mathrm{Z_G}$. Next, the proper map $\psi:\mathrm{Z}\to\mathrm{T^*X'}$ restricts to a proper map $\psi_s:\mathrm{Z}^s\to\mathrm{N}^s_{r,\mathrm{E}}$, because $\mathrm{Z}^s=\mathrm{Z}\cap\psi^{-1}(\mathrm{N}^s_{r,\mathrm{E}})$. Finally, we have $\psi_s(\mathrm{Z}^s_\mathrm{G}) \subset \mathrm{M}^s_{r,\mathrm{E}}$, because $\mathrm{Z}^s_\mathrm{G} = \mathrm{Z}_\mathrm{G} \cap \mathrm{Z}^s$. Thus, taking the direct image by ψ_s, we get the commutative diagram

$$\begin{tikzcd} \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{Z}^s_\mathrm{G}) \arrow[r, "\psi_{s,*}"] \arrow[d] & \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{M}^s_{r,\mathrm{E}}) \arrow[d] \\ \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{Z}^s) \arrow[r, "\psi_{s,*}"] & \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{N}^s_{r,\mathrm{E}}). \end{tikzcd} \tag{6.6}$$

Now, set $\mathrm{N}^s_\mathfrak{m} = (\mathfrak{g}_{\mathrm{E}_1})^2 \times \mathrm{N}^s_{r,\mathrm{E}_2}$, $\mathrm{M}^s_\mathfrak{m} = \mathrm{C}_{\mathrm{E}_1} \times \mathrm{M}^s_{r,\mathrm{E}_2}$, $\widetilde{\mathrm{N}}^s_\mathfrak{m} = \widetilde{\mathrm{N}}_\mathfrak{m} \cap (\mathfrak{p} \times \mathrm{N}^s_\mathfrak{m})$ and

$$\mathrm{T}^*\mathrm{X}^s = \mathrm{G} \times_\mathrm{P} \widetilde{\mathrm{N}}^s_\mathfrak{m}, \qquad \mathrm{T}^*_\mathrm{G}\mathrm{X}^s = \mathrm{G} \times_\mathrm{P} \mathrm{M}^s_\mathfrak{m}.$$

For $(a, b, \varphi, v) \in \mathrm{N}^s_\mathfrak{p}$ we have $(a_\mathfrak{m}, b_\mathfrak{m}, \varphi, \pi \circ v) \in \mathrm{N}^s_\mathfrak{m}$. Thus the map $\phi : \mathrm{Z} \to \mathrm{T}^*\mathrm{X}$ restricts to a map $\phi_s : \mathrm{Z}^s \to \mathrm{T}^*\mathrm{X}^s$. The varieties Z^s and $\mathrm{T}^*\mathrm{X}^s$ are both smooth and we have $\phi_s^{-1}(\mathrm{T}^*_\mathrm{G}\mathrm{X}^s) = \mathrm{Z}^s \cap \mathrm{Z}_\mathrm{G} = \mathrm{Z}^s_\mathrm{G}$. Hence the pull-back by ϕ_s gives the commutative diagram

$$\begin{tikzcd} \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{T}^*_\mathrm{G}\mathrm{X}^s) \arrow[r, "\phi_s^*"] \arrow[d] & \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{Z}^s_\mathrm{G}) \arrow[d] \\ \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{T}^*\mathrm{X}^s) \arrow[r, "\phi_s^*"] & \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{Z}^s). \end{tikzcd} \tag{6.7}$$

Set $n_1 = \dim \mathrm{E}_1$, $n_2 = \dim \mathrm{E}_2$ and $n = n_1 + n_2$. Since $\mathrm{M}^s_{r,\mathrm{E}_2}$ is a $\mathrm{GL}_{\mathrm{E}_2}$-torsor over M_{r,n_2}, by descent we have an isomorphism

$$\mathbf{L}^{(r)}_{n_2} = \mathrm{H}^{\widetilde{\mathrm{GL}}_{\mathrm{E}_2}}\big(\mathrm{M}^s_{r,\mathrm{E}_2}\big). \tag{6.8}$$

Here we used the symbol $\widetilde{\mathrm{GL}}_{\mathrm{E}_2} = \mathrm{GL}_{\mathrm{E}_2} \times \mathrm{T}$ following the notation $\widetilde{\mathrm{G}}$ in Section 4.2. We have also $\mathbf{L}^{(r)}_n = \mathrm{H}^{\widetilde{\mathrm{G}}}(\mathrm{M}^s_{r,\mathrm{E}})$. Finally, the induction and the Kunneth formula yield an isomorphism

$$\mathrm{Ind} : \mathbf{Co}'_{n_1} \otimes_\mathrm{R} \mathbf{L}^{(r)}_{n_2} = \mathrm{H}^{\widetilde{\mathrm{M}}}\big(\mathrm{C}_{\mathrm{E}_1} \times \mathrm{M}^s_{r,\mathrm{E}_2}\big) = \mathrm{H}^{\widetilde{\mathrm{G}}}\big(\mathrm{T}^*_\mathrm{G}\mathrm{X}^s\big). \tag{6.9}$$

Thus, composing (6.9) with (6.6) and (6.7), we get a map

$$\psi_{s,*}\,\phi_s^*\,\mathrm{Ind} : \mathbf{Co}'_{n_1} \otimes_\mathrm{R} \mathbf{L}^{(r)}_{n_2} \to \mathbf{L}^{(r)}_n. \tag{6.10}$$

The same argument as in the proof of [33, Prop. 7.9] implies that (6.10) defines an R-linear representation of $\mathbf{Co}'$ on $\mathbf{L}^{(r)}$. Details are left to the reader. Let η' denote this representation.

Now, we compute the image of the element $\theta_l \in \mathbf{Co}'_1$ by the map η'. To do so, we change slightly the notation. Assume that $E_1 \in \mathscr{V}_1$ and $E \in \mathscr{V}_{n+1}$. Fix $x \in \mathbf{L}_n^{(r)}$ and let y be the image of $\theta_l \otimes x$ by the map

$$(\mathbf{6.11}) \qquad \mathbf{Co}'_1 \otimes_R \mathbf{L}_n^{(r)} \to \mathbf{L}_{n+1}^{(r)}$$

given in (6.10). We must check that y is equal to the element

$$(\mathbf{6.12}) \qquad c_1(\tau_{n+1,n})^l \cdot x = \pi_{1,*}\big(c_1(\tau_{n+1,n})^l \pi_2^*(x)\big).$$

By definition of (6.10) we have

$$(\mathbf{6.13}) \qquad y = \psi_{s,*}\phi_s^* \operatorname{Ind}(\theta_l \otimes x).$$

The variety Z_G^s is the set of all pairs $((a,b,\varphi,v),E_1)$ where $(a,b,\varphi,v) \in M_{r,E}^s$, $a,b \in \mathfrak{p}$ and $\varphi(E_1)=0$. It is a smooth G-torsor over $M_{r,n+1,n}$. Hence, by descent we have an isomorphism

$$(\mathbf{6.14}) \qquad H^{\widetilde{G}}\big(Z_G^s\big) \to H^T(M_{r,n+1,n}).$$

So we have the commutative diagram

$$(\mathbf{6.15}) \qquad \begin{array}{ccc} H^{\widetilde{G}}(Z_G^s) & \xrightarrow{\psi_{s,*}} & H^{\widetilde{G}}(M_{r,E}^s) \\ \downarrow & & \downarrow \\ H^T(M_{r,n+1,n}) & \xrightarrow{\pi_{1,*}} & H^T(M_{r,n+1}) \end{array}$$

where both vertical maps are given by descent. Comparing (6.12), (6.13) and (6.15), we see that it is enough to observe that the isomorphism (6.14) takes $\phi_s^* \operatorname{Ind}(\theta_l \otimes x)$ to $c_1(\tau_{n+1,n})^l \pi_2^*(x)$. □

We can now prove the following.

Theorem **6.3.** — *The map η' factors to a K_r-algebra isomorphism*

$$(\mathbf{6.16}) \qquad \eta : \mathbf{SCo}_{K_r} \to \mathbf{U}_K^{(r),>}, \qquad \theta_l \to f_{1,l}, \qquad l \in \mathbf{N}$$

which commutes with the action of Λ_{K_r}.

Proof. — Since the representation of $\mathbf{U}_K^{(r)}$ on $\mathbf{L}_K^{(r)}$ is faithful, the map η' gives a surjective K_r-algebra homomorphism

$$(\mathbf{6.17}) \qquad \eta' : \mathbf{SCo}'_{K_r} \to \mathbf{U}_K^{(r),>}, \qquad \theta_l \mapsto f_{1,l}, \qquad l \in \mathbf{N}.$$

First, we claim that η' commutes with Wilson operators. It is enough to check it on generators by Lemma 3.11(c) and Lemma 4.9(c). Example 3.10 gives

$$\eta'(p_l \bullet \theta_k) = \eta'(\theta_{l+k}) = f_{1,l+k} = p_l \bullet f_{1,k} = p_l \bullet \eta'(\theta_k), \quad l \geq 1, \; k \geq 0, \tag{6.18}$$

proving the claim. Next, by Proposition 3.13 the action of Λ_{n,K_r} on $\mathbf{U}_{\mathrm{K}}^{(r),>}[n]$ is torsion free. Hence the map η' factors to a surjective K_r-algebra homomorphism

$$\eta : \mathbf{SCo}_{\mathrm{K}_r} \to \mathbf{U}_{\mathrm{K}}^{(r),>} \tag{6.19}$$

taking θ_l to $f_{1,l}$. It remains to show that η is injective. Let $x \in \mathbf{SCo}_{n,\mathrm{K}_r}$ and assume that $\eta(x) = 0$. If $x \neq 0$ then, by the localization theorem, for any $y \in \mathbf{SCo}_{n,\mathrm{K}_r}$ there exists $p, p' \in \Lambda_{n,\mathrm{K}_r}$ such that $p \bullet x = p' \bullet y$. But, then, we have

$$p' \bullet \eta(y) = \eta\big(p' \bullet y\big) = \eta(p \bullet x) = p \bullet \eta(x) = 0. \tag{6.20}$$

It follows that $\eta(y)$ is torsion, hence $\eta(y) = 0$ by Proposition 3.13. This contradicts the surjectivity of η. We deduce that $x = 0$, i.e., that η is injective. □

Proposition 5.1 and Theorem 6.3 (for $r = 1$) yield the following.

Corollary **6.4.** — *There is a* K-*algebra isomorphism* $\mathbf{SCo}_{\mathrm{K}} \to \mathbf{SH}_{\mathrm{K}}^{>}$, $\theta_l \mapsto x^l \mathrm{D}_{1,l}$.

Remark **6.5.** — Proposition 3.7 and Theorem 6.3 give a K_r-algebra homomorphism

$$\eta^{\mathrm{op}} : (\mathbf{SCo}_{\mathrm{K}_r})^{\mathrm{op}} \to \mathbf{U}_{\mathrm{K}}^{(r),<}, \qquad \theta_l \mapsto f_{-1,l}. \tag{6.21}$$

Proposition 1.35 and Corollary 6.4 give a K-algebra isomorphism

$$(\mathbf{SCo}_{\mathrm{K}})^{\mathrm{op}} \to \mathbf{SH}_{\mathrm{K}}^{<}, \qquad \theta_l \mapsto x^l \mathrm{D}_{-1,l}. \tag{6.22}$$

We define $\mathbf{U}^{(r),>}$ and $\mathbf{U}^{(r),<}$ to be the images of $\mathbf{SCo}_{\mathrm{R}_r}$, $(\mathbf{SCo}_{\mathrm{R}_r})^{\mathrm{op}}$ by the maps η and η^{op}. We have R_r-algebra isomorphisms

$$\mathbf{SCo}_{\mathrm{R}_r} \to \mathbf{U}^{(r),>}, \qquad (\mathbf{SCo}_{\mathrm{R}_r})^{\mathrm{op}} \to \mathbf{U}^{(r),<}. \tag{6.23}$$

6.2. *Part 2: glueing the positive and negative halves.* — Theorem 6.3, Corollary 6.4 and Remark 6.5 give K_r-algebra isomorphisms

$$\Psi^{>} : \mathbf{SH}_{\mathrm{K}}^{(r),>} \to \mathbf{U}_{\mathrm{K}}^{(r),>}, \qquad \Psi^{<} : \mathbf{SH}_{\mathrm{K}}^{(r),<} \to \mathbf{U}_{\mathrm{K}}^{(r),<} \tag{6.24}$$

such that $\Psi^{>}(\mathrm{D}_{1,l}) = h_{1,l}$ and $\Psi^{<}(\mathrm{D}_{-1,l}) = h_{-1,l}$. Next, Appendix D gives the following.

Proposition **6.6.** — *The class* $[h_{-1,k}, h_{1,l}]$ *is supported on the diagonal of* $\mathrm{M}_{r,n} \times \mathrm{M}_{r,n}$ *and it coincides, as an element of* $\mathbf{U}_{\mathrm{K}}^{(r)}$*, with the operator* E_{k+l} *on* $\mathbf{L}_{\mathrm{K}}^{(r)}$ *given by*

$$1 + \xi \sum_{l \geq 0} \mathrm{E}_l \, s^{l+1} = \exp\Big(\sum_{l \geq 0} (-1)^{l+1} p_l(\varepsilon_a) \phi_l(s)\Big) \exp\Big(\sum_{l \geq 0} h_{0,l+1} \, \varphi_l(s)\Big).$$

We can now prove Theorem 3.2. First, note that we have a K_r-algebra homomorphism

$$\Psi \ : \mathbf{SH}_{\mathrm{K}}^{(r)} \to \mathbf{U}_{\mathrm{K}}^{(r)}, \qquad \mathrm{D}_{\mathbf{x}} \mapsto h_{\mathbf{x}}, \qquad \mathbf{x} \in \mathscr{E}. \tag{6.25}$$

Indeed, relation (1.69) follows from Proposition 6.6 and (1.67), (1.68) from Remark 2.3. Thus, we are reduced to check that the representation $\rho^{(r)}$ is faithful. A proof is given in Section D.2.

7. The comultiplication

So far, we have defined an algebra $\mathbf{SH}^{\mathbf{c}}$ and we have constructed a representation $\rho^{(r)}$ of $\mathbf{SH}^{\mathbf{c}}$ in $\mathbf{L}_{\mathrm{K}}^{(r)}$. In order to compare $\mathbf{SH}^{\mathbf{c}}$ with W-algebras, it is important to equip it with a Hopf algebra structure. We do not know how to construct the (topological) coproduct on $\mathbf{SH}^{\mathbf{c}}$ in an elementary algebraic way. Our argument uses our previous work [33]. First, we prove that $\mathbf{SH}^{\mathbf{c}}$ can be regarded as a degeneration of the elliptic Hall algebras which was studied there. This is Theorem 7.7. Next, using this result, we prove that the coproduct of the elliptic Hall algebra degenerates and induces a coproduct on $\mathbf{SH}^{\mathbf{c}}$. This is Theorem 7.9.

7.1. *The DAHA.* — We'll abbreviate $\mathbf{A} = \mathbf{C}[q^{\pm 1/2}, t^{\pm 1/2}]$, $\mathbf{K} = \mathbf{C}(q^{1/2}, t^{1/2})$ and $v^{1/2} = (qt)^{-1/2}$. Fix an integer $n > 1$. The *double affine Hecke algebra* (=DAHA) of GL_n is the associative $\mathbf{K}$-algebra $\widetilde{\mathbf{H}}_n$ generated by

$$\mathrm{X}_1^{\pm 1}, \dots, \mathrm{X}_n^{\pm 1}, \mathrm{Y}_1^{\pm 1}, \dots, \mathrm{Y}_n^{\pm 1}, \mathrm{T}_1, \dots, \mathrm{T}_{n-1} \tag{7.1}$$

subject to the following relations [10, Sect. 1.4.3]

$$\mathrm{T}_i \mathrm{X}_i \mathrm{T}_i = \mathrm{X}_{i+1}, \qquad \mathrm{T}_i^{-1} \mathrm{Y}_i \mathrm{T}_i^{-1} = \mathrm{Y}_{i+1}, \tag{7.2}$$

$$\mathrm{T}_i \mathrm{X}_j = \mathrm{X}_j \mathrm{T}_i, \qquad \mathrm{T}_i \mathrm{Y}_j = \mathrm{Y}_j \mathrm{T}_i, \quad j \neq i, i+1, \tag{7.3}$$

$$\big(\mathrm{T}_i + t^{1/2}\big)\big(\mathrm{T}_i - t^{-1/2}\big) = 0, \qquad \mathrm{T}_i \mathrm{T}_{i+1} \mathrm{T}_i = \mathrm{T}_{i+1} \mathrm{T}_i \mathrm{T}_{i+1}, \tag{7.4}$$

$$\mathrm{T}_i \mathrm{T}_j = \mathrm{T}_j \mathrm{T}_i, \quad j \neq i-1, i, i+1, \tag{7.5}$$

$$\mathrm{P}\mathrm{X}_i = \mathrm{X}_{i+1} \mathrm{P}, \qquad \mathrm{P}\mathrm{X}_n = q^{-1} \mathrm{X}_1 \mathrm{P}, \qquad \mathrm{P} = \mathrm{Y}_1^{-1} \mathrm{T}_1 \cdots \mathrm{T}_{n-1}, \quad i \neq n. \tag{7.6}$$

Let $\widetilde{\mathbf{H}}_n^+$ be the $\mathbf{K}$-subalgebra generated by

$$\mathrm{X}_1, \ldots, \mathrm{X}_n, \mathrm{Y}_1^{\pm 1}, \ldots, \mathrm{Y}_n^{\pm 1}, \mathrm{T}_1, \ldots, \mathrm{T}_{n-1}, \tag{7.7}$$

and let $\widetilde{\mathbf{S}}$ be the complete idempotent. We set

$$\widetilde{\mathbf{SH}}_n = \widetilde{\mathbf{S}}\,\widetilde{\mathbf{H}}_n\widetilde{\mathbf{S}}, \qquad \widetilde{\mathbf{SH}}_n^+ = \widetilde{\mathbf{S}}\,\widetilde{\mathbf{H}}_n^+\widetilde{\mathbf{S}}. \tag{7.8}$$

For $\mathbf{x} \in \mathbf{Z}_0^2$ we define an element $\mathrm{P}_{\mathbf{x}}^{(n)}$ of $\widetilde{\mathbf{SH}}_n$ as in [32, Sect. 2.2]. For $l \geq 1$ we have

$$\begin{aligned} \mathrm{P}_{l,0}^{(n)} &= q^l \widetilde{\mathbf{S}} p_l(\mathrm{X}_1, \ldots, \mathrm{X}_n)\widetilde{\mathbf{S}}, \qquad & \mathrm{P}_{-l,0}^{(n)} &= \widetilde{\mathbf{S}} p_l\big(\mathrm{X}_1^{-1}, \ldots, \mathrm{X}_n^{-1}\big)\widetilde{\mathbf{S}}, \\ \mathrm{P}_{0,l}^{(n)} &= \widetilde{\mathbf{S}} p_l(\mathrm{Y}_1, \ldots, \mathrm{Y}_n)\widetilde{\mathbf{S}}, \qquad & \mathrm{P}_{0,-l}^{(n)} &= q^l \widetilde{\mathbf{S}} p_l\big(\mathrm{Y}_1^{-1}, \ldots, \mathrm{Y}_n^{-1}\big)\widetilde{\mathbf{S}}. \end{aligned} \tag{7.9}$$

There is a unique $\mathbf{K}$-algebra automorphism [32, Sect. 3.1], [10, Sect. 3.2.2],

$$\sigma : \widetilde{\mathbf{SH}}_n \to \widetilde{\mathbf{SH}}_n, \qquad \mathrm{P}_{\mathbf{x}}^{(n)} \mapsto \mathrm{P}_{\sigma(\mathbf{x})}^{(n)}, \quad \sigma(i,j) = (j, -i). \tag{7.10}$$

Let $\widetilde{\mathbf{H}}_{n,\mathbf{A}}$ be the $\mathbf{A}$-subalgebra of $\widetilde{\mathbf{H}}_n$ generated by (7.1) and set $\widetilde{\mathbf{SH}}_{n,\mathbf{A}} = \widetilde{\mathbf{S}}\,\widetilde{\mathbf{H}}_{n,\mathbf{A}}\,\widetilde{\mathbf{S}}$. Note that

$$\widetilde{\mathbf{H}}_n = \widetilde{\mathbf{H}}_{n,\mathbf{A}} \otimes_{\mathbf{A}} \mathbf{K}, \qquad \widetilde{\mathbf{SH}}_n = \widetilde{\mathbf{SH}}_{n,\mathbf{A}} \otimes_{\mathbf{A}} \mathbf{K}. \tag{7.11}$$

We have an $\mathbf{A}$-basis of $\mathbf{H}_{n,\mathbf{A}}$ given by [10]

$$\big\{\mathrm{X}^\alpha \mathrm{Y}^\beta \mathrm{T}_w;\, \alpha \in \mathbf{Z}^n, \beta \in \mathbf{Z}^n, w \in \mathfrak{S}_n\big\}. \tag{7.12}$$

Consider the following $\mathbf{K}$-vector spaces, see (1.19),

$$\widetilde{\mathbf{W}}_n = \mathbf{W}_{n,\mathbf{K}}, \qquad \widetilde{\mathbf{V}}_n = \mathbf{V}_{n,\mathbf{K}}. \tag{7.13}$$

The $\mathbf{K}$-algebra $\widetilde{\mathbf{H}}_n$ is equipped with a faithful representation [32, Sect. 4.1]

$$\varphi_n : \widetilde{\mathbf{H}}_n \to \mathrm{End}(\widetilde{\mathbf{W}}_n) \tag{7.14}$$

called the *polynomial representation*. The subalgebras $\widetilde{\mathbf{SH}}_n$ and $\widetilde{\mathbf{SH}}_n^+$ act faithfully on the subspaces $\widetilde{\mathbf{W}}_n^{\mathfrak{S}_n}$ and $\Lambda_{n,\mathbf{K}}$. For a partition λ with at most n parts let $\mathrm{J}_\lambda(\mathrm{X}; q, t^{-1})$ be the integral form of the Macdonald polynomial $\mathrm{P}_\lambda(\mathrm{X}; q, t^{-1})$, see [25, Chap. VI, (8.3)]. We abbreviate

$$\mathrm{J}_\lambda^{(n)}\big(q, t^{-1}\big) = \mathrm{J}_\lambda\big(\mathrm{X}_1, \ldots, \mathrm{X}_n; q, t^{-1}\big). \tag{7.15}$$

This yields the following $\mathbf{K}$-basis of $\Lambda_{n,\mathbf{K}}$

$$\big\{\mathrm{J}_\lambda^{(n)}\big(q, t^{-1}\big);\, l(\lambda) \leq n\big\}. \tag{7.16}$$

Finally, for $\mathbf{x} \in \mathbf{Z}_0^2$ we define new elements $u_{\mathbf{x}}^{(n)}$, $\theta(u_{\mathbf{x}}^{(n)})$ of $\widetilde{\mathbf{SH}}_n$ as follows. First, we set

$$a_l = \left(q^l - 1\right)\left(t^l - 1\right), \quad l \geq 1. \tag{7.17}$$

Next, for $\mathbf{x} = (i,j)$ and $l = \gcd(\mathbf{x})$, we set

$$\mathrm{P}_{\mathbf{x}}^{(n)} = \left(q^l - 1\right) u_{\mathbf{x}}^{(n)}, \qquad \theta\left(u_{\mathbf{x}}^{(n)}\right) = t^{j(n-1)/2} u_{\mathbf{x}}^{(n)} - \delta_{i,0}\left(t^{jn} - 1\right)/a_j. \tag{7.18}$$

We have the following formula [33, Cor. 1.5]

$$\theta\left(u_{0,l}^{(n)}\right) \cdot \mathrm{J}_\lambda^{(n)}\left(q, t^{-1}\right) = \sum_{s \in \lambda} q^{lx(s)} t^{ly(s)} \mathrm{J}_\lambda^{(n)}\left(q, t^{-1}\right), \quad l \geq 1. \tag{7.19}$$

By [32], [33, Sect. 1.3] we have also

$$\left[u_{0,l}^{(n)}, u_{\pm 1,k}^{(n)}\right] = \pm \operatorname{sgn}(l)\, u_{\pm 1,k+l}^{(n)}, \tag{7.20}$$

where

$$\operatorname{sgn}(l) = 1, \qquad \operatorname{sgn}(-l-1) = -1, \quad l \geq 0. \tag{7.21}$$

To unburden the notation, let $\widetilde{\mathbf{SH}}_n$ denote also the smash product

$$\mathbf{K}\left[u_{0,0}^{(n)}\right] \otimes_{\mathbf{K}} \widetilde{\mathbf{SH}}_n, \tag{7.22}$$

where $u_{0,0}^{(n)}$ is a new formal variable and the commutator with $u_{0,0}^{(n)}$ is the $\mathbf{K}$-derivation

$$\left[u_{0,0}^{(n)}, u_{i,j}^{(n)}\right] = i u_{i,j}^{(n)}. \tag{7.23}$$

The element $u_{0,0}^{(n)}$ acts on $\mathbf{V}_n$ as the grading operator. We'll set $\theta(u_{0,0}^{(n)}) = u_{0,0}^{(n)}$.

7.2. *The degeneration of* $\widetilde{\mathbf{H}}_n$. — Our aim is to construct a degeneration from $\widetilde{\mathbf{H}}_n$ to $\mathbf{H}_n$. The degenerations of $\widetilde{\mathbf{H}}_n$ have been extensively studied, see e.g., [39]. Here we only need a very particular one introduced for the first time by Cherednik. We set

$$\mathscr{K} = \mathrm{F}\big((h)\big), \qquad \mathscr{A} = \mathrm{F}\big[[h]\big]. \tag{7.24}$$

We refer to [23] for a reminder on topological $\mathscr{A}$-modules (for the h-adic topology). Let $\widetilde{\otimes}$ denote the *topological tensor product* of $\mathscr{A}$-modules. An $\mathscr{A}$-module is *topologically free* if it is isomorphic to $\mathrm{V}[[h]]$ for an F-vector space V. Let $\mathrm{F}\langle \mathrm{X}\rangle$ be the free F-algebra on X. For a future use, recall that a complete separated $\mathscr{A}$-algebra $\mathbf{B}$ is *topologically generated* by a subset X if the obvious continuous map $\mathrm{F}\langle \mathrm{X}\rangle[[h]] \to \mathbf{B}$ is surjective.

First, consider the algebra embedding $\mathbf{A} \subset \mathscr{A}$ given by

$$q^{1/2} \mapsto \exp(h/2), \qquad t^{1/2} \mapsto \exp(-\kappa h/2). \tag{7.25}$$

Let I be the ideal of $\mathbf{A}$ given by $\mathrm{I} = (h) \cap \mathbf{A}$. Let $\mathbf{B}_n \subset \widetilde{\mathbf{H}}_n$ be the $\mathbf{A}$-subalgebra generated by $\widetilde{\mathbf{H}}_{n,\mathbf{A}}$ and the elements $(\mathrm{Y}_i - 1)/(q-1)$ with $i \in [1, n]$. We set

$$\mathcal{H}_{n,\mathscr{A}} = \varprojlim_k \big(\mathbf{B}_n / \mathrm{I}^k \mathbf{B}_n\big). \tag{7.26}$$

By (7.12) the $\mathbf{A}$-algebra $\mathbf{B}_n$ is topologically linearly spanned by the elements of the form $\mathrm{X}^\alpha f(\mathrm{Y})\mathrm{T}_w$ where $\alpha \in \mathbf{Z}^n$, $w \in \mathfrak{S}_n$, and $f(\mathrm{Y}) \in \mathbf{A}[\mathrm{Y}_i^{\pm 1}, (\mathrm{Y}_i - q^k t^l)/(q-1)]$ for $i \in [1, n]$ and $k, l \in \mathbf{Z}$. Consider the element y_i in $\mathcal{H}_{n,\mathscr{A}}$ given by

$$y_i = \sum_{l \geq 1} (-1)^{l-1} (\mathrm{Y}_i - 1)^l / lh. \tag{7.27}$$

We have faithful representations, see Section 1.3,

$$\varphi_n : \widetilde{\mathbf{H}}_n \to \mathrm{End}(\widetilde{\mathbf{W}}_n), \qquad \rho_n : \mathbf{H}_n \to \mathrm{End}(\mathbf{W}_n). \tag{7.28}$$

From (7.25) we get inclusions

$$\widetilde{\mathbf{W}}_n \subset \mathbf{W}_n\big((h)\big), \qquad \mathrm{End}(\widetilde{\mathbf{W}}_n) \subset \mathrm{End}(\mathbf{W}_n)\big((h)\big). \tag{7.29}$$

We abbreviate

$$\mathcal{O}\big(h^l\big) = h^l \mathcal{H}_{n,\mathscr{A}}, \qquad \mathcal{H}_n = \mathcal{H}_{n,\mathscr{A}} / h\mathcal{H}_{n,\mathscr{A}}, \quad l \in \mathbf{N}. \tag{7.30}$$

Lemma **7.1.** — *The $\mathscr{A}$-module $\mathcal{H}_{n,\mathscr{A}}$ is topologically free. As a topological $\mathscr{A}$-algebra it is generated by the set $\{\mathrm{T}_j, \mathrm{X}_i^{\pm 1}, y_i;\ i \in [1, n],\ j \in [1, n)\}$. We have*

$$\mathrm{X}_i^{\pm 1} \in \mathcal{O}(1), \qquad \mathrm{T}_j \in \mathcal{O}(1), \qquad \mathrm{Y}_i = 1 + \mathcal{O}(h), \qquad y_i = (\mathrm{Y}_i - 1)/h + \mathcal{O}(h). \tag{7.31}$$

The map φ_n yields a continuous embedding $\varphi_n : \mathcal{H}_{n,\mathscr{A}} \to \mathrm{End}(\mathbf{W}_n)[[h]]$.

Proof. — The $\mathscr{A}$-module $\mathcal{H}_{n,\mathscr{A}}$ is topologically free because it is separated, complete and torsion free. The other statements are easy and are left to the reader. □

Finally, we set

$$\mathcal{H}_{n,\mathscr{K}} = \mathcal{H}_{n,\mathscr{A}} \otimes_{\mathscr{A}} \mathscr{K}. \tag{7.32}$$

By base change, the map φ_n yields a continuous embedding

$$\varphi_n : \mathcal{H}_{n,\mathscr{K}} \to \mathrm{End}(\mathbf{W}_n)\big((h)\big). \tag{7.33}$$

The following is standard. The proof is left to the reader.

Proposition **7.2.** — (a) *We have* $\mathcal{H}_{n,\mathscr{A}} = \{x \in \mathcal{H}_{n,\mathscr{K}}\,;\, \varphi_n(x) \in \mathrm{End}(\mathbf{W}_n)[[h]]\}$.
(b) *The map* φ_n *factors to an injection* $\varphi'_n : \mathcal{H}_n \to \mathrm{End}(\mathbf{W}_n)$.
(c) *There is a unique* F*-algebra isomorphism* $\phi_n : \mathcal{H}_n \to \mathbf{H}_n$ *such that*

$$\phi_n(\mathrm{X}_j^{\pm1}) = \mathrm{X}_{n+1-j}^{\pm1}, \qquad \phi_n(y_j) = y_{n+1-j} - (n-1)\kappa/2, \qquad \phi_n(\mathrm{T}_i) = s_{n-i}.$$

(d) *We have* $\varphi'_n = \rho'_n \circ \phi_n$, *where* $\rho'_n = w_0\rho_n w_0$ *and* $w_0 \in \mathrm{Aut}(\mathbf{W}_n)$ *is given by* $\mathrm{X}_i \mapsto \mathrm{X}_{n+1-i}$.

7.3. *The degeneration of* $\widetilde{\mathbf{SH}}_n$. — We now turn our attention to the spherical subalgebras. Set

$$\mathcal{SH}_{n,\mathscr{A}} = \widetilde{\mathbf{S}} \cdot \mathcal{H}_{n,\mathscr{A}} \cdot \widetilde{\mathbf{S}}, \qquad \mathcal{SH}_n = \mathcal{SH}_{n,\mathscr{A}} / h\mathcal{SH}_{n,\mathscr{A}}. \tag{7.34}$$

The map φ'_n factors to an injective map

$$\varphi'_n : \mathcal{SH}_n \to \mathrm{End}(\mathbf{V}_n). \tag{7.35}$$

For $l \geq 1$ we consider the following elements

$$\begin{aligned} &\mathrm{Q}_{0,l}^{(n)} = h^{1-l} \sum_{k=0}^{l-1} \binom{l-1}{k} (-1)^k \theta\big(u_{0,l-1-k}^{(n)}\big), \\ &\mathrm{Q}_{l,0}^{(n)} = (-1)^l \kappa^l \mathrm{P}_{l,0}^{(n)}, \qquad \mathrm{Q}_{-l,0}^{(n)} = \mathrm{P}_{-l,0}^{(n)}, \qquad \mathrm{Q}_{1,l}^{(n)} = \big[\mathrm{Q}_{0,l+1}^{(n)}, \mathrm{Q}_{1,0}^{(n)}\big], \end{aligned} \tag{7.36}$$

Proposition **7.3.** — (a) *For* $l \geq 1$ *the elements* $\mathrm{Q}_{\pm l,0}^{(n)}$ *and* $\mathrm{Q}_{0,l}^{(n)}$ *belong to* $\mathcal{SH}_{n,\mathscr{A}}$.
(b) *The map* ϕ_n *restricts to an* F*-algebra isomorphism* $\mathcal{SH}_n \to \mathbf{SH}_n$ *such that we have* $\mathrm{P}_{\pm l,0}^{(n)} \mapsto \mathrm{D}_{\pm l,0}^{(n)}$ *and* $\mathrm{Q}_{0,l}^{(n)} \mapsto \mathrm{D}_{0,l}^{(n)}$ *for* $l \geq 1$.
(c) *The algebra* $\mathcal{SH}_{n,\mathscr{A}}$ *is topologically generated by* $\mathrm{P}_{\pm1,0}^{(n)}$ *and* $\mathrm{Q}_{0,2}^{(n)}$.

Proof. — We first prove (a). We have $\mathrm{P}_{\pm l,0}^{(n)} \in \mathcal{SH}_{n,\mathscr{A}}$. We consider the inclusions $\widetilde{\mathbf{V}}_n$, $\mathbf{V}_n \subset \mathbf{V}_{n,\mathscr{K}}$ associated with the obvious inclusion $\mathrm{F} \subset \mathscr{K}$ and with the embedding $\mathbf{K} \subset \mathscr{K}$ in (7.25). We have [25, Chap. VI, (10.23)]

$$\big(1 - t^{-1}\big)^{-|\lambda|} \mathrm{J}_\lambda^{(n)}\big(q, t^{-1}\big) = \mathrm{J}_\lambda^{(n)} \bmod h\mathbf{V}_{n,\mathscr{A}}. \tag{7.37}$$

By (7.19), for $l(\lambda) \leq n$, we have

$$\begin{aligned} \mathrm{Q}_{0,l}^{(n)} \cdot \mathrm{J}_\lambda^{(n)}\big(q, t^{-1}\big) &= h^{1-l} \sum_{s\in\lambda} \sum_{k=0}^{l-1} \binom{l-1}{k} (-1)^k q^{(l-1-k)c(s)} \mathrm{J}_\lambda^{(n)}\big(q, t^{-1}\big), \\ &= \sum_{s\in\lambda} \left(\frac{q^{c(s)} - 1}{h}\right)^{l-1} \mathrm{J}_\lambda^{(n)}\big(q, t^{-1}\big). \end{aligned} \tag{7.38}$$

By Proposition 7.2 the $\mathscr{A}$-algebra $\mathcal{SH}_{n,\mathscr{A}}$ is the subalgebra of

$$\mathcal{SH}_{n,\mathscr{K}} = \mathcal{SH}_{n,\mathscr{A}} \otimes_{\mathscr{A}} \mathscr{K} \tag{7.39}$$

which preserves the subspace $\mathbf{V}_{n,\mathscr{A}}$ of $\mathbf{V}_{n,\mathscr{K}}$. It entails that $Q^{(n)}_{0,l} \in \mathcal{SH}_{n,\mathscr{A}}$ as wanted. We now deal with (b) and (c). Note that

$$h\mathcal{H}_{n,\mathscr{A}} \cap \mathcal{SH}_{n,\mathscr{A}} = h\mathcal{SH}_{n,\mathscr{A}}.$$

Therefore the natural map gives an injection $\mathcal{SH}_n \to \mathcal{H}_n$. Since $\mathcal{SH}_{n,\mathscr{A}} = \widetilde{\mathbf{S}} \cdot \mathcal{H}_{n,\mathscr{A}} \cdot \widetilde{\mathbf{S}}$, the map ϕ_n in Proposition 7.2 restricts to an injection

$$\phi_n : \mathcal{SH}_n \to \mathbf{SH}_n. \tag{7.40}$$

The equality $\phi_n(Q^{(n)}_{0,l}) = D^{(n)}_{0,l}$ is a consequence of (1.30), (7.37) and (7.38). The equality $\phi_n(P^{(n)}_{\pm l,0}) = D^{(n)}_{\pm l,0}$ follows from (1.32) and (7.9). The map ϕ_n in (7.40) is surjective because, by Lemma 1.3, the F-algebra $\mathbf{SH}_n$ is generated by $\{D^{(n)}_{\pm l,0},\, D^{(n)}_{0,l};\, l \geq 1\}$. Claim (c) is a consequence of Nakayama's lemma together with the fact that $\mathbf{SH}_n$ is generated by $D^{(n)}_{0,2}$ and $D^{(n)}_{\pm 1,0}$. ☐

Let $\mathcal{SH}^{>}_{n,\mathscr{A}}$, $\mathcal{SH}^{<}_{n,\mathscr{A}}$ and $\mathcal{SH}^{0}_{n,\mathscr{A}}$ be the closed $\mathscr{A}$-subalgebras of $\mathcal{SH}_{n,\mathscr{A}}$ topologically generated respectively by the sets $\{Q^{(n)}_{1,l};\, l \geq 0\}$, $\{Q^{(n)}_{-1,l};\, l \geq 0\}$ and $\{Q^{(n)}_{0,l};\, l \geq 0\}$. We abbreviate

$$\begin{aligned} &\mathcal{SH}^{>}_{n} = \mathcal{SH}^{>}_{n,\mathscr{A}}/h\mathcal{SH}^{>}_{n,\mathscr{A}}, \qquad \mathcal{SH}^{<}_{n} = \mathcal{SH}^{<}_{n,\mathscr{A}}/h\mathcal{SH}^{<}_{n,\mathscr{A}}, \\ &\mathcal{SH}^{0}_{n} = \mathcal{SH}^{0}_{n,\mathscr{A}}/h\mathcal{SH}^{0}_{n,\mathscr{A}}. \end{aligned} \tag{7.41}$$

Using Proposition 7.3, we get the following.

Corollary **7.4.** — *The map ϕ_n gives* F*-algebra isomorphisms*

$$\mathcal{SH}^{>}_{n} \to \mathbf{SH}^{>}_{n}, \qquad \mathcal{SH}^{<}_{n} \to \mathbf{SH}^{<}_{n}, \qquad \mathcal{SH}^{0}_{n} \to \mathbf{SH}^{0}_{n}$$

such that $Q^{(n)}_{\pm l,0} \mapsto D^{(n)}_{\pm l,0}$, $Q_{\pm 1,l} \mapsto D_{\pm 1,l}$ *and* $Q^{(n)}_{0,l} \mapsto D^{(n)}_{0,l}$ *for* $l \geq 1$.

Proof. — Let $\mathcal{SH}^{>}_{n,\mathscr{K}}$ be the closed $\mathscr{K}$-subalgebra of $\mathcal{SH}_{n,\mathscr{K}}$ generated by $\{Q^{(n)}_{1,l};\, l \geq 0\}$. We have

$$\mathcal{SH}^{>}_{n,\mathscr{A}} \subset \mathcal{SH}^{>}_{n,\mathscr{K}} \cap \mathcal{SH}_{n,\mathscr{A}}, \tag{7.42}$$

and the map ϕ_n yields an isomorphism

$$\mathcal{SH}^{>}_{n,\mathscr{K}} \cap \mathcal{SH}_{n,\mathscr{A}}/h\big(\mathcal{SH}^{>}_{n,\mathscr{K}} \cap \mathcal{SH}_{n,\mathscr{A}}\big) \to \mathbf{SH}^{>}_{n}. \tag{7.43}$$

Since the induced map $\mathcal{SH}^>_{n,\mathscr{A}} \to \mathbf{SH}^>_n$ is surjective, we deduce that

$$\mathcal{SH}^>_{n,\mathscr{A}} = \mathcal{SH}^>_{n,\mathscr{K}} \cap \mathcal{SH}_{n,\mathscr{A}}. \tag{7.44}$$

In particular, we have also

$$\mathcal{SH}^>_{n,\mathscr{A}} \cap h\mathcal{SH}_{n,\mathscr{A}} = \mathcal{SH}^>_{n,\mathscr{K}} \cap h\mathcal{SH}_{n,\mathscr{A}} = h\mathcal{SH}^>_{n,\mathscr{A}}. \tag{7.45}$$

This shows the existence of an F-algebra isomorphism

$$\mathcal{SH}^>_n \to \mathbf{SH}^>_n, \qquad \mathrm{P}^{(n)}_{l,0} \mapsto \mathrm{D}^{(n)}_{l,0}, \qquad \big[\mathrm{Q}^{(n)}_{0,l+1}, \mathrm{P}^{(n)}_{1,0}\big] \mapsto \mathrm{D}^{(n)}_{1,l}. \tag{7.46}$$

Since $\mathcal{SH}^>_{n,\mathscr{A}}$ is $\mathbf{N}$-graded, there exists an automorphism of $\mathcal{SH}^>_{n,\mathscr{A}}$ sending $\mathrm{P}^{(n)}_{l,0}$ to $\mathrm{Q}^{(n)}_{l,0}$. This proves the corollary for $\mathcal{SH}^>_n$. The other cases are similar. □

7.4. *The algebra* $\widetilde{\mathbf{SH}}^{\mathbf{c}}$. — Consider the $\mathbf{K}$-algebra $\widehat{\boldsymbol{\mathcal{E}}}$ in [33, Sect. 1] associated with the parameters

$$\sigma^{1/2} = q^{-1/2}, \qquad \bar{\sigma}^{1/2} = t^{-1/2}. \tag{7.47}$$

It is generated by elements $u_{\mathbf{x}}$, $\kappa_{\mathbf{x}}$ with $\mathbf{x} \in \mathbf{Z}^2_0$, satisfying the relations in [33, Sect. 1.1]. For $\gcd(\mathbf{x}) = 1$ and $l \geq 1$, we set

$$\begin{gathered} \alpha_l = \big(1-q^l\big)\big(1-t^l\big)\big(1-v^{-l}\big)/l, \\ \mathrm{P}_{l\mathbf{x}} = \big(q^l - 1\big)\, u_{l\mathbf{x}}, \qquad \sum_{l\geq 0} \theta_{l\mathbf{x}}\, s^l = \exp\Big(\sum_{l\geq 1} \alpha_l\, u_{l\mathbf{x}}\, s^l\Big). \end{gathered} \tag{7.48}$$

Since $\widehat{\boldsymbol{\mathcal{E}}}$ is an *extended Hall algebra* in Ringel's sense, see e.g., [22, Sect. 1.6], it admits a *topological coproduct* Δ, which is given by the following formula, compare [8, Sect. 7],

$$\begin{aligned} &\Delta(\kappa_{\mathbf{x}}) = \kappa_{\mathbf{x}} \otimes \kappa_{\mathbf{x}}, \\ &\Delta(u_{0,l}) = u_{0,l} \otimes 1 + \kappa_{0,l} \otimes u_{0,l}, \qquad l \neq 0, \\ &\Delta(u_{1,l}) = u_{1,l} \otimes 1 + \sum_{k\geq 0} \kappa_{1,l-k}\theta_{0,k} \otimes u_{1,l-k}, \qquad l \in \mathbf{Z}. \end{aligned} \tag{7.49}$$

The expression "topological coproduct" means that Δ maps into some completion of the tensor square of $\widehat{\boldsymbol{\mathcal{E}}}$, see [8, Sect. 2] for details. By [8, Sect. 5], there is a unique $\mathbf{K}$-algebra automorphism

$$\sigma : \widehat{\boldsymbol{\mathcal{E}}} \to \widehat{\boldsymbol{\mathcal{E}}}, \quad \kappa_{\mathbf{x}} \mapsto \kappa_{\sigma(\mathbf{x})}, \quad u_{\mathbf{x}} \mapsto u_{\sigma(\mathbf{x})}. \tag{7.50}$$

Compare (7.10). We define a new topological coproduct on $\widehat{\boldsymbol{\mathcal{E}}}$ by the formula

$${}^{\sigma}\Delta = \big(\sigma^{-1} \otimes \sigma^{-1}\big) \circ \Delta \circ \sigma. \tag{7.51}$$

Now, we fix a family of formal parameters $\mathbf{c}_l$ with $l \in \mathbf{Z}$ and we set

$$\mathbf{K}^{\mathbf{c}} = \mathbf{K}[\tilde{\mathbf{c}}_l;\, l \in \mathbf{Z}]\big[\tilde{\mathbf{c}}_0^{-1}\big], \qquad \mathbf{A}^{\mathbf{c}} = \mathbf{A}[\tilde{\mathbf{c}}_l;\, l \in \mathbf{Z}]\big[\tilde{\mathbf{c}}_0^{-1}\big]. \tag{7.52}$$

Let $\mathcal{E}^{\mathbf{c}}$ be the specialization of $\widehat{\mathcal{E}} \otimes_{\mathbf{K}} \mathbf{K}^{\mathbf{c}}$ at $\kappa_{1,0} = \mathbf{c}_0$ and $\kappa_{0,1} = 1$. Let $u_{0,0}$ be a new formal variable, and consider the smash product

$$\widetilde{\mathbf{SH}}^{\mathbf{c}} = \mathbf{K}[u_{0,0}] \otimes_{\mathbf{K}} \mathcal{E}^{\mathbf{c}}, \tag{7.53}$$

where the commutator with $u_{0,0}$ is the $\mathbf{K}^{\mathbf{c}}$-derivation on $\mathcal{E}^{\mathbf{c}}$ such that

$$[u_{0,0}, u_{i,j}] = i u_{i,j}, \qquad (i,j) \in \mathbf{Z}_0^2. \tag{7.54}$$

The $\mathbf{K}^{\mathbf{c}}$-algebra $\mathbf{SH}^{\mathbf{c}}$ is $\mathbf{Z}^2$-graded with $\deg(u_{\mathbf{x}}) = \mathbf{x}$. It is equipped with the topological coproduct

$$\begin{aligned}
{}^{\sigma}\Delta(\tilde{\mathbf{c}}_0) &= \tilde{\mathbf{c}}_0 \otimes \tilde{\mathbf{c}}_0, \\
{}^{\sigma}\Delta(\tilde{\mathbf{c}}_l) &= \delta(\tilde{\mathbf{c}}_l) \qquad \text{if } l \neq 0, \\
{}^{\sigma}\Delta(u_{l,0}) &= u_{l,0} \otimes 1 + \tilde{\mathbf{c}}_0^l \otimes u_{l,0}, \\
{}^{\sigma}\Delta(u_{l,1}) &= u_{l,1} \otimes 1 + \tilde{\mathbf{c}}_0^l \otimes u_{l,1} + \sum_{k \geq 1} \tilde{\mathbf{c}}_0^{k+l} \theta_{-k,0} \otimes u_{k+l,1}.
\end{aligned} \tag{7.55}$$

Let $\widetilde{\mathbf{SH}}^{>}$, $\widetilde{\mathbf{SH}}^{\mathbf{c},0}$ and $\widetilde{\mathbf{SH}}^{<}$ be the $\mathbf{K}$-subalgebras generated respectively by

$$\{u_{1,l};\, l \in \mathbf{Z}\}, \qquad \mathbf{K}^{\mathbf{c}} \cup \{u_{0,l};\, l \in \mathbf{Z}\}, \qquad \{u_{-1,l};\, l \in \mathbf{Z}\}. \tag{7.56}$$

The following holds.

Lemma **7.5.** — (a) *The multiplication yields an isomorphism* $\widetilde{\mathbf{SH}}^{>} \otimes_{\mathbf{K}} \widetilde{\mathbf{SH}}^{\mathbf{c},0} \otimes_{\mathbf{K}} \widetilde{\mathbf{SH}}^{<} \to \widetilde{\mathbf{SH}}^{\mathbf{c}}$.

(b) *We have* $\widetilde{\mathbf{SH}}^{\mathbf{c},0} = \mathbf{K}^{\mathbf{c}}[u_{0,l};\, l \in \mathbf{Z}]$.

Proof. — Part (a) follows from [33, Sect. 1.1], which is proved using the formulas [33, Sect. 1.2]

$$\begin{aligned}
&[u_{0,l}, u_{\pm 1,k}] = \pm \mathrm{sgn}(l)\, u_{\pm 1,k+l}, \\
&[u_{-1,k}, u_{1,l}] = \begin{cases} \mathrm{sgn}(k+l)\, \tilde{\mathbf{c}}_0^{\mathrm{sgn}(k+l)}\, \theta_{0,k+l}/\alpha_1 & \text{if } k+l \neq 0, \\ (\tilde{\mathbf{c}}_0 - \tilde{\mathbf{c}}_0^{-1})/\alpha_1 & \text{else,} \end{cases}
\end{aligned} \tag{7.57}$$

where $\mathrm{sgn}(l)$ is as in (7.20). Part (b) is [8, Sect. 4]. □

Next, we consider the $\mathbf{A}^{\mathbf{c}}$-subalgebra $\widetilde{\mathbf{SH}}^{\mathbf{c}}_{\mathbf{A}}$ generated by the elements $u_{\mathbf{x}}$ with $\mathbf{x} \in \mathbf{Z}^2$. We have

$$(\mathbf{7.58}) \qquad \widetilde{\mathbf{SH}}^{\mathbf{c}} = \widetilde{\mathbf{SH}}^{\mathbf{c}}_{\mathbf{A}} \otimes_{\mathbf{A}^{\mathbf{c}}} \mathbf{K}^{\mathbf{c}}.$$

Finally, let $\widetilde{\mathbf{SH}}^{>}_n$ and $\widetilde{\mathbf{SH}}^{<}_n$ be the subalgebras of $\widetilde{\mathbf{SH}}_n$ generated by

$$(\mathbf{7.59}) \qquad \big\{u^{(n)}_{1,l};\, l \in \mathbf{Z}\big\}, \qquad \big\{u^{(n)}_{-1,l};\, l \in \mathbf{Z}\big\}.$$

By [32, Thm. 3.1], [33, Sect. 1.4] there is a unique surjective algebra homomorphism

$$(\mathbf{7.60}) \qquad \Psi_n : \widetilde{\mathbf{SH}}^{>} \to \widetilde{\mathbf{SH}}^{>}_n, \qquad u_{\mathbf{x}} \mapsto \theta\big(u^{(n)}_{\mathbf{x}}\big).$$

The map $\Psi = \prod_n \Psi_n$ is an embedding of $\widetilde{\mathbf{SH}}^{>}$ into $\prod_n \widetilde{\mathbf{SH}}^{>}_n$ by [32, Thm. 4.6].

7.5. *The degeneration of* $\widetilde{\mathbf{SH}}^{\mathbf{c}}$. — For $\mathbf{x} = (i,j)$ in $\mathbf{Z}^2_0$ and $l \geq 1$ we define

$$(\mathbf{7.61}) \qquad \begin{aligned}
&u^{\mathbf{c}}_{0,0} = u_{0,0}, \qquad u^{\mathbf{c}}_{\mathbf{x}} = u_{\mathbf{x}} + \delta_{i,0}\,\mathbf{c}_j/a_j,\\
&\mathrm{Q}_{0,l} = h^{1-l}\sum_{k=0}^{l-1}\binom{l-1}{k}(-1)^k u^{\mathbf{c}}_{0,l-1-k},\\
&\mathrm{Q}_{l,0} = (-1)^l\kappa^l \mathrm{P}_{l,0}, \qquad \mathrm{Q}_{-l,0} = \mathrm{P}_{-l,0},\\
&\mathrm{Q}_{1,l} = [\mathrm{Q}_{0,l+1}, \mathrm{Q}_{1,0}], \qquad \mathrm{Q}_{-1,l} = -[\mathrm{Q}_{0,l+1}, \mathrm{Q}_{-1,0}].
\end{aligned}$$

We have an inclusion of F-algebras $\mathbf{A}^{\mathbf{c}} \subset \mathscr{A}^{\mathbf{c}}$, where $\mathscr{A}^{\mathbf{c}} = \mathrm{F}^{\mathbf{c}}[[h]]$, which is given by

$$(\mathbf{7.62}) \qquad \begin{gathered}
q^{1/2} \mapsto \exp(h/2), \qquad t^{1/2} \mapsto \exp(-\kappa h/2), \qquad \mathbf{c}_0 \mapsto \exp(\xi h \mathbf{c}_0/2),\\
\mathbf{c}_l \mapsto \mathrm{sgn}(l)\sum_{k\geq 0}(-lh)^k \mathbf{c}_k/k!, \qquad l \neq 0.
\end{gathered}$$

Consider the ideal $\mathrm{I} = (h) \cap \mathbf{A}^{\mathbf{c}}$ in $\mathbf{A}^{\mathbf{c}}$. Let $\mathbf{B}^{\mathbf{c}} \subset \widetilde{\mathbf{SH}}^{\mathbf{c}}$ be the $\mathbf{A}^{\mathbf{c}}$-subalgebra generated by $\{\mathrm{Q}_{\pm l,0}, \mathrm{Q}_{0,l};\, l \geq 1\}$. We define an $\mathscr{A}^{\mathbf{c}}$-algebra by setting

$$(\mathbf{7.63}) \qquad \mathcal{SH}^{\mathbf{c}}_{\mathscr{A}} = \varprojlim_k \big(\mathbf{B}^{\mathbf{c}}/\mathrm{I}^k\,\mathbf{B}^{\mathbf{c}}\big).$$

Let $\mathcal{SH}^{>}_{\mathscr{A}}$ and $\mathcal{SH}^{<}_{\mathscr{A}}$ be the closed $\mathscr{A}^{\mathbf{c}}$-subalgebras of $\mathcal{SH}^{\mathbf{c}}_{\mathscr{A}}$ generated by the sets $\{\mathrm{Q}_{1,l};\, l \geq 0\}$ and $\{\mathrm{Q}_{-1,l};\, l \geq 0\}$. We write

$$(\mathbf{7.64}) \quad \mathcal{SH}^{>} = \mathcal{SH}^{>}_{\mathscr{A}}/h\mathcal{SH}^{>}_{\mathscr{A}}, \qquad \mathcal{SH}^{<} = \mathcal{SH}^{<}_{\mathscr{A}}/h\mathcal{SH}^{<}_{\mathscr{A}}, \qquad \mathcal{SH}^{\mathbf{c}} = \mathcal{SH}^{\mathbf{c}}_{\mathscr{A}}/h\mathcal{SH}^{\mathbf{c}}_{\mathscr{A}}.$$

Proposition **7.6.** — (a) *The* $\mathscr{A}$*-modules* $\mathcal{SH}^{>}_{\mathscr{A}}$ *and* $\mathcal{SH}^{<}_{\mathscr{A}}$ *are topologically free.*

(b) *There are* F*-algebra isomorphisms* $\phi : \mathcal{SH}^{>} \to \mathbf{SH}^{>}$ *and* $\phi : \mathcal{SH}^{<} \to \mathbf{SH}^{<}$ *such that we have* $\phi(\mathrm{Q}_{\pm l,0}) = \mathrm{D}_{\pm l,0}$ *and* $\phi(\mathrm{Q}_{\pm 1,l}) = \mathrm{D}_{\pm 1,l}$ *for* $l \geq 1$.

Proof. — Part (a) is obvious, because $\mathcal{SH}^{>}_{\mathscr{A}}$ and $\mathcal{SH}^{<}_{\mathscr{A}}$ are separated, complete and torsion free. Now we prove (b). First, consider the map Ψ_n. For $l \neq 0$ we have $\theta(u^{(n)}_{l,0}) = u^{(n)}_{l,0}$ by (7.18). Thus, by (7.18), (7.36), (7.48), (7.60) and (7.61) we have also

$$\Psi_n(\mathrm{Q}_{l,0}) = \mathrm{Q}^{(n)}_{l,0}. \tag{7.65}$$

Next, for $l \geq 1$, the formulas (7.48), (7.57), (7.60) and (7.61) give

$$\Psi_n(\mathrm{Q}_{1,l}) = \kappa(1-q)h^{-l}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\theta\big(u^{(n)}_{1,l-k}\big), \tag{7.66}$$

and by (7.18), (7.20) and (7.36) we have also

$$\mathrm{Q}^{(n)}_{1,l} = \kappa(1-q)h^{-l}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\theta\big(u^{(n)}_{1,l-k}\big). \tag{7.67}$$

Therefore, by (7.65), (7.67) the map Ψ_n gives rise to a continuous $\mathscr{A}$-algebra homomorphism

$$\Psi_n : \mathcal{SH}^{>}_{\mathscr{A}} \to \mathcal{SH}^{>}_{n,\mathscr{A}}, \qquad \Psi_n(\mathrm{Q}_{l,0}) = \mathrm{Q}^{(n)}_{l,0}, \qquad \Psi_n(\mathrm{Q}_{1,l}) = \mathrm{Q}^{(n)}_{1,l}, \quad l \geq 1. \tag{7.68}$$

The map Ψ is a closed embedding $\mathcal{SH}^{>}_{\mathscr{A}} \to \prod_n \mathcal{SH}^{>}_{n,\mathscr{A}}$. By Proposition 1.15 and Corollary 7.4, composing Ψ and $\prod_n \phi_n$ we get a map

$$\phi' : \mathcal{SH}^{>} \to \prod_n \mathbf{SH}^{>}_n, \qquad \phi'(\mathrm{Q}_{l,0}) = \big(\mathrm{D}^{(n)}_{l,0}\big), \qquad \phi'(\mathrm{Q}_{1,l}) = \big(\mathrm{D}^{(n)}_{1,l}\big). \tag{7.69}$$

By definition of $\mathbf{SH}^{>}$, there is an inclusion of F-algebras

$$i : \mathbf{SH}^{>} \to \prod_n \mathbf{SH}^{>}_n, \qquad i(\mathrm{D}_{l,0}) = \big(\mathrm{D}^{(n)}_{l,0}\big), \qquad i(\mathrm{D}_{0,l}) = \big(\mathrm{D}^{(n)}_{0,l}\big). \tag{7.70}$$

Thus, we have a surjective F-algebra homomorphism ϕ which is given by

$$\phi = i^{-1} \circ \phi' : \mathcal{SH}^{>} \to \mathbf{SH}^{>}. \tag{7.71}$$

We must prove that it is injective. We consider the partial order on $\mathbf{Z}^2$ given by

$$(r,d) \leq (r',d') \iff r \leq r' \text{ and } d \leq d'. \tag{7.72}$$

The $\mathbf{Z}^2$-grading on $\widetilde{\mathbf{SH}}^{\mathbf{c}}$ yields a filtration on $\mathcal{SH}^{>}_{\mathscr{A}}$ such that the piece $\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}]$ consists of the elements whose $\mathbf{Z}^2$-degree is $\leq \mathbf{x}$. The $\mathscr{A}$-module $\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}]$ has a finite rank and we have

$$\begin{gathered}\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}] \cap h\mathcal{SH}^{>}_{\mathscr{A}} = h\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}],\\ \mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}]/h\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}] \subset \mathcal{SH}^{>} = \bigcup_{\mathbf{x}} \mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}]/h\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}].\end{gathered} \tag{7.73}$$

We define $\mathcal{SH}^{>}_{n,\mathscr{A}}[\leq \mathbf{x}]$ in an identical fashion. From Corollary 7.4, we get

$$\mathcal{SH}^{>}_{n,\mathscr{A}}[\leq \mathbf{x}]/h\mathcal{SH}^{>}_{n,\mathscr{A}}[\leq \mathbf{x}] \subset \mathbf{SH}^{>}_{n}. \tag{7.74}$$

Next, given $\mathbf{x}$, for n large enough the map Ψ_n in (7.68) yields an isomorphism

$$\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}] \to \mathcal{SH}^{>}_{n,\mathscr{A}}[\leq \mathbf{x}]. \tag{7.75}$$

Thus it factors to an isomorphism

$$\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}]/h\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}] \to \mathcal{SH}^{>}_{n,\mathscr{A}}[\leq \mathbf{x}]/h\mathcal{SH}^{>}_{n,\mathscr{A}}[\leq \mathbf{x}]. \tag{7.76}$$

Composing (7.76) with (7.74) we obtain an inclusion, for n large enough,

$$\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}]/h\mathcal{SH}^{>}_{\mathscr{A}}[\leq \mathbf{x}] \subset \mathbf{SH}^{>}_{n}. \tag{7.77}$$

We conclude that ϕ' is injective. Hence ϕ is also injective. $\square$

Let $\mathcal{SH}^{0,\mathbf{c}}_{\mathscr{A}}$ be the closed $\mathscr{A}^{\mathbf{c}}$-subalgebra of $\mathcal{SH}^{\mathbf{c}}_{\mathscr{A}}$ topologically generated by $\{\mathrm{Q}_{0,l};\, l \geq 1\}$. By Lemma 7.6(b) we have an isomorphism

$$\mathcal{SH}^{0,\mathbf{c}}_{\mathscr{A}} = \mathrm{F}^{\mathbf{c}}[\mathrm{Q}_{0,l+1};\, l \geq 0][[h]]. \tag{7.78}$$

As above, we abbreviate $\mathcal{SH}^{0,\mathbf{c}} = \mathcal{SH}^{0,\mathbf{c}}_{\mathscr{A}}/h\mathcal{SH}^{0,\mathbf{c}}_{\mathscr{A}}$. We can now prove the following theorem.

Theorem **7.7.** — (a) *There is an* F*-algebra isomorphism* $\phi : \mathcal{SH}^{\mathbf{c}} \to \mathbf{SH}^{\mathbf{c}}$ *such that*

$$\phi(\mathrm{Q}_{0,l}) = \mathrm{D}_{0,l}, \qquad \phi(\mathrm{Q}_{\pm l,0}) = \mathrm{D}_{\pm l,0}, \qquad \phi(\mathrm{Q}_{\pm 1,l}) = \mathrm{D}_{\pm 1,l}, \quad l \geq 1.$$

(b) *The algebra* $\mathcal{SH}^{\mathbf{c}}_{\mathscr{A}}$ *is topologically generated by* $\mathrm{Q}_{-1,0}$, $\mathrm{Q}_{1,0}$ *and* $\mathrm{Q}_{0,2}$.

Proof. — By Proposition 1.35 the F-algebra $\mathbf{SH}^{\mathbf{c}}$ is generated by the elements $\mathbf{c}_l$, $\mathrm{D}_{\pm 1,0}$ and $\mathrm{D}_{0,2}$. Thus, part (b) follows from (a). Now, we prove (a). We'll identify the ring

$$\mathbf{A}_r = \mathbf{Z}\big[q^{\pm 1}, t^{\pm 1}, \chi_1^{\pm 1}, \dots, \chi_r^{\pm 1}\big] \tag{7.79}$$

with the Grothendieck ring of the group $\widetilde{\mathrm{D}}$ as in (3.3). Let $\mathbf{L}^{(r)}_{\mathrm{K}}$ be the *localized Grothendieck group* of the category of $\widetilde{\mathrm{D}}$-equivariant coherent sheaves on $\bigsqcup_{n\geq 0} \mathrm{M}_{r,n}$. The word localized means that the ring of scalars is extended from the ring $\mathbf{A}_r$ to the field

$$\mathbf{K}_r = \mathbf{K}(\chi_1, \dots, \chi_r). \tag{7.80}$$

The set of fixed points $\{\mathrm{I}_\lambda\}$ of $\mathrm{M}_{r,n}$ for the $\widetilde{\mathrm{D}}$-action gives bases in $\mathbf{L}^{(r)}_{\mathrm{K}}$ and $\mathbf{L}^{(r)}_{\mathrm{K}}$. Set

$$\mathscr{K}_r = \mathrm{K}_r\big((h)\big), \qquad \mathscr{A}_r = \mathrm{K}_r\big[[h]\big]. \tag{7.81}$$

We have an embedding $\mathbf{K}_r \subset \mathscr{K}_r$ given by the following formulas, compare (7.62),

$$
\begin{aligned}
&q = \exp(h), \qquad t = \exp(-\kappa h), \qquad \chi_a = \exp(\varepsilon_a h), \\
&\kappa = -y/x, \qquad \varepsilon_a = e_a/x.
\end{aligned} \tag{7.82}
$$

Identifying the bases above, we get inclusions of $\mathbf{L}_{\mathrm{K}}^{(r)} = \bigoplus_\lambda \mathrm{K}_r\,[\mathrm{I}_\lambda]$ and $\mathbf{L}_{\mathrm{K}}^{(r)} = \bigoplus_\lambda \mathbf{K}_r\,[\mathrm{I}_\lambda]$ into the $\mathscr{K}_r$-vector space

$$
\mathcal{L}^{(r)} = \bigoplus_\lambda \mathscr{K}_r\,[\mathrm{I}_\lambda]. \tag{7.83}
$$

Now, a representation of $\boldsymbol{\mathcal{E}} \otimes_{\mathbf{K}} \mathbf{K}_r$ in $\mathbf{L}_{\mathrm{K}}^{(r)}$ is constructed in [33, Sect. 8]. It can be upgraded to a representation of $\mathbf{SH^c} \otimes_{\mathbf{K}} \mathscr{K}_r$ on $\mathcal{L}^{(r)}$ in which $u_{0,0}$ acts as the grading operator. We have

$$
\begin{aligned}
&\tilde{\mathbf{c}}_0 = v^{-r/2}, \qquad \tilde{\mathbf{c}}_l = p_l\big(\chi_a^{-1}\big) \quad \text{for } l \neq 0, \qquad u_{0,l}^{\mathbf{c}} = \mathrm{sgn}(l)\,\mathbf{f}_{0,l}, \\
&u_{1,l} = v^{-1}(q-1)^r x^{1-r}\mathbf{f}_{1,l-r}, \qquad u_{-1,l} = (-1)^{r-1}\det(\mathrm{W})\,\tilde{\mathbf{c}}_0^{-1}(q-1)^{-r}x^{r-1}\mathbf{f}_{-1,l}, \\
&\mathbf{f}_{1,l}[\mathrm{I}_\lambda] = \sum_{\lambda\subset\pi} \tau_{\lambda,\pi}^l \Lambda\big(\mathrm{N}_{\lambda,\pi}^* - \mathrm{T}_\pi^*\big)\,[\mathrm{I}_\pi], \qquad \mathbf{f}_{-1,l}[\mathrm{I}_\lambda] = \sum_{\sigma\subset\lambda} \tau_{\sigma,\lambda}^l \Lambda\big(\mathrm{N}_{\sigma,\lambda}^* - \mathrm{T}_\sigma^*\big)\,[\mathrm{I}_\sigma], \\
&\mathbf{f}_{0,l}[\mathrm{I}_\lambda] = \sum_{a,s} \chi_a^{-l} q^{lx(s)} t^{ly(s)}[\mathrm{I}_\lambda].
\end{aligned} \tag{7.84}
$$

Here $l \geq 0$ and Λ is the Koszul complex and $\det(\mathrm{W}) = (\chi_1\chi_2\ldots\chi_r)^{-1}$. On the other hand, the representation $\rho^{(r)}$ is given by the following formulas, see Section 3.6 and Appendix D,

$$
\begin{aligned}
&\mathbf{c}_l = p_l(\varepsilon_a), \\
&\mathrm{D}_{1,l} = x^{1-l}yf_{1,l}, \qquad \mathrm{D}_{-1,l} = (-1)^{r-1}x^{-l}f_{-1,l}, \qquad \mathrm{D}_{0,l+1} = x^{-l}f_{0,l}, \\
&f_{1,l}[\mathrm{I}_\lambda] = \sum_{\lambda\subset\pi} \mathrm{c}_1(\tau_{\lambda,\pi})^l\,\mathrm{eu}\big(\mathrm{N}_{\lambda,\pi}^* - \mathrm{T}_\pi^*\big)\,[\mathrm{I}_\pi], \\
&f_{-1,l}[\mathrm{I}_\lambda] = \sum_{\sigma\subset\lambda} \mathrm{c}_1(\tau_{\sigma,\lambda})^l\,\mathrm{eu}\big(\mathrm{N}_{\sigma,\lambda}^* - \mathrm{T}_\sigma^*\big)\,[\mathrm{I}_\sigma], \\
&f_{0,l}[\mathrm{I}_\lambda] = \sum_{a,s} \mathrm{c}_1\big(\chi_a^{-1}q^{x(s)}t^{y(s)}\big)^l\,[\mathrm{I}_\lambda].
\end{aligned} \tag{7.85}
$$

The above formulas allow us to compare the action of $\mathrm{Q}_{\pm1,l}$, $\mathrm{Q}_{0,l}$ and of $\mathrm{D}_{\pm1,l}$, $\mathrm{D}_{0,l}$. Write

$$
\mathcal{O}\big(h^l\big) = \bigoplus_\lambda h^l \mathscr{A}_r\,[\mathrm{I}_\lambda], \quad l \in \mathbf{Z}. \tag{7.86}
$$

Using (7.61), (7.84) and (7.85), we get

$$\text{(7.87)}\qquad \mathrm{Q}_{0,l}[\mathrm{I}_\lambda] = \mathrm{D}_{0,l}[\mathrm{I}_\lambda] + \mathcal{O}(h), \quad l \geq 1.$$

Next, for $\sigma \subset \lambda \subset \pi$ such that $|\lambda| = |\sigma| + 1 = |\pi| - 1$ we have

$$\text{(7.88)}\qquad \dim(\mathrm{N}_{\lambda,\pi} - \mathrm{T}_\pi) = -r - 1, \qquad \dim(\mathrm{N}_{\sigma,\lambda} - \mathrm{T}_\sigma) = r - 1.$$

Therefore, we have the following estimates in $\mathscr{K}_r$

$$\text{(7.89)}\qquad \begin{aligned} &\Lambda\big(\mathrm{N}^*_{\lambda,\pi} - \mathrm{T}^*_\pi\big) \equiv (x/h)^{1+r}\,\mathrm{eu}\big(\mathrm{N}^*_{\lambda,\pi} - \mathrm{T}^*_\pi\big),\\ &\Lambda\big(\mathrm{N}^*_{\sigma,\lambda} - \mathrm{T}^*_\sigma\big) \equiv (x/h)^{1-r}\,\mathrm{eu}\big(\mathrm{N}^*_{\sigma,\lambda} - \mathrm{T}^*_\sigma\big)\end{aligned}$$

modulo lower terms for the h-adic topology. Finally, (7.61), (7.84), (7.85) and (7.89) give

$$\text{(7.90)}\qquad \mathrm{Q}_{1,0}[\mathrm{I}_\lambda] = \mathrm{D}_{1,0}[\mathrm{I}_\lambda] + \mathcal{O}(h), \qquad \mathrm{Q}_{-1,0}[\mathrm{I}_\lambda] = \mathrm{D}_{-1,0}[\mathrm{I}_\lambda] + \mathcal{O}(h).$$

By (1.67), (1.68) and (7.61) we have

$$\text{(7.91)}\qquad \begin{aligned} &\mathrm{D}_{1,l} = [\mathrm{D}_{0,l+1}, \mathrm{D}_{1,0}], \qquad \mathrm{D}_{-1,l} = -[\mathrm{D}_{0,l+1}, \mathrm{D}_{-1,0}],\\ &\mathrm{Q}_{1,l} = [\mathrm{Q}_{0,l+1}, \mathrm{Q}_{1,0}], \qquad \mathrm{Q}_{-1,l} = -[\mathrm{Q}_{0,l+1}, \mathrm{Q}_{-1,0}].\end{aligned}$$

Thus (7.87) and (7.90) imply that

$$\text{(7.92)}\qquad \mathrm{Q}_{1,l}[\mathrm{I}_\lambda] = \mathrm{D}_{1,l}[\mathrm{I}_\lambda] + \mathcal{O}(h), \qquad \mathrm{Q}_{-1,l}[\mathrm{I}_\lambda] = \mathrm{D}_{-1,l}[\mathrm{I}_\lambda] + \mathcal{O}(h).$$

Now, the algebra homomorphism $\mathrm{F}^{\mathbf{c}} \to \mathrm{K}_r$ in Definition 1.36 yields an algebra homomorphism $\mathscr{A}^{\mathbf{c}} \to \mathscr{A}_r$. Consider the algebras

$$\text{(7.93)}\qquad \mathcal{SH}^{(r)}_{\mathscr{A}} = \mathcal{SH}^{\mathbf{c}}_{\mathscr{A}} \otimes_{\mathscr{A}^{\mathbf{c}}} \mathscr{A}_r, \qquad \mathcal{SH}^{(r)} = \mathcal{SH}^{(r)}_{\mathscr{A}} / h\mathcal{SH}^{(r)}_{\mathscr{A}}.$$

Note that the composed map $\mathbf{A}^{\mathbf{c}} \to \mathscr{A}^{\mathbf{c}} \to \mathrm{K}_r$ is given by

$$\text{(7.94)}\qquad \tilde{\mathbf{c}}_0 = v^{-r/2}, \qquad \tilde{\mathbf{c}}_{\pm l} = \pm p_l\big(\chi_a^{\mp 1}\big).$$

We define $\mathcal{SH}^{(r),>}$, $\mathcal{SH}^{(r),<}$ and $\mathcal{SH}^{(r),0}$ in the same way, using $\mathcal{SH}^{>}_{\mathscr{A}}$, $\mathcal{SH}^{<}_{\mathscr{A}}$ and $\mathcal{SH}^{0,\mathbf{c}}_{\mathscr{A}}$. Formulas (7.87) and (7.92) imply that the $\mathbf{A}^{\mathbf{c}}$-subalgebra $\mathbf{B}^{\mathbf{c}} \subset \widetilde{\mathbf{SH}}^{\mathbf{c}}$ preserves the lattice $\mathcal{O}(1)$. This yields a representation of $\mathcal{SH}^{\mathbf{c}}_{\mathscr{A}}$ on $\mathcal{L}^{(r)}$ which preserves also $\mathcal{O}(1)$ and which factors to a representation of $\mathcal{SH}^{(r)}$ on $\mathcal{O}(1)/\mathcal{O}(h) = \mathbf{L}^{(r)}_{\mathrm{K}}$. Since $\rho^{(r)}$ is faithful, this yields also an algebra homomorphism

$$\text{(7.95)}\qquad \mathcal{SH}^{(r)} \to \mathbf{SH}^{(r)}_{\mathrm{K}}.$$

It is surjective, because $\mathbf{SH}^{(r)}_{\mathrm{K}}$ is generated by the elements $\mathrm{D}_{0,l+1}$, $\mathrm{D}_{-1,l}$ and $\mathrm{D}_{1,l}$ with $l \geq 0$.

Now, the $\mathscr{A}$-algebra embeddings of $\mathcal{SH}^{>}_{\mathscr{A}}$, $\mathcal{SH}^{>}_{\mathscr{A}}$ and $\mathcal{SH}^{0,\mathbf{c}}_{\mathscr{A}}$ into $\mathcal{SH}^{\mathbf{c}}_{\mathscr{A}}$ give obvious maps

$$\mathcal{SH}^{(r),>},\ \mathcal{SH}^{(r),0},\ \mathcal{SH}^{(r),<} \to \mathcal{SH}^{(r)}. \tag{7.96}$$

Composing them with (7.95) we get K_r-algebra homomorphisms

$$\mathcal{SH}^{(r),>} \to \mathbf{SH}^{(r),>}_{\mathrm{K}}, \qquad \mathcal{SH}^{(r),<} \to \mathbf{SH}^{(r),<}_{\mathrm{K}}, \qquad \mathcal{SH}^{(r),0} \to \mathbf{SH}^{(r),0}_{\mathrm{K}}, \tag{7.97}$$

which give the commutative square

$$\begin{tikzcd} \mathcal{SH}^{(r),>} \otimes_{\mathrm{K}_r} \mathcal{SH}^{(r),0} \otimes_{\mathrm{K}_r} \mathcal{SH}^{(r),<} \arrow[r, "m"] \arrow[d] & \mathcal{SH}^{(r)} \arrow[d] \\ \mathbf{SH}^{(r),>}_{\mathrm{K}} \otimes_{\mathrm{K}_r} \mathbf{SH}^{(r),0}_{\mathrm{K}} \otimes_{\mathrm{K}_r} \mathbf{SH}^{(r),<}_{\mathrm{K}} \arrow[r, "m"] & \mathbf{SH}^{(r)}_{\mathrm{K}}. \end{tikzcd} \tag{7.98}$$

Here m is the multiplication map.

Now, by Proposition 7.6 there are K_r-algebra isomorphisms

$$\begin{aligned} &\mathcal{SH}^{(r),>} \to \mathbf{SH}^{(r),>}_{\mathrm{K}}, \qquad \mathcal{SH}^{(r),<} \to \mathbf{SH}^{(r),<}_{\mathrm{K}}, \\ &\mathrm{Q}_{\pm l,0} \mapsto \mathrm{D}_{\pm l,0}, \qquad \mathrm{Q}_{\pm 1,l} \mapsto \mathrm{D}_{\pm 1,l}. \end{aligned} \tag{7.99}$$

Further, by (1.66) and (7.78), we have a K_r-algebra isomorphism

$$\mathcal{SH}^{(r),0} \to \mathbf{SH}^{(r),0}_{\mathrm{K}}, \qquad \mathrm{Q}_{0,l+1} \mapsto \mathrm{D}_{0,l+1}. \tag{7.100}$$

Thus the left vertical map in (7.98) is invertible. The bottom horizontal map is invertible by Proposition 1.37. Thus the upper map m is injective. Therefore, to prove that the right map is invertible it is enough to check the following.

Lemma **7.8.** — *The multiplication gives a surjective map*

$$m : \mathcal{SH}^{(r),>} \otimes_{\mathrm{K}_r} \mathcal{SH}^{(r),0} \otimes_{\mathrm{K}_r} \mathcal{SH}^{(r),<} \to \mathcal{SH}^{(r)}.$$

Proof. — It is enough to prove that

$$[\mathrm{Q}_{0,l}, \mathrm{Q}_{1,k}] \in \mathcal{SH}^{(r),>}, \qquad [\mathrm{Q}_{-1,l}, \mathrm{Q}_{0,k}] \in \mathcal{SH}^{(r),<}, \qquad [\mathrm{Q}_{-1,l}, \mathrm{Q}_{1,k}] \in \mathcal{SH}^{(r),0}. \tag{7.101}$$

The first two relations follow from a simple computation, since (7.57) implies that

$$[\mathrm{Q}_{0,l}, \mathrm{Q}_{1,k}] = \mathrm{Q}_{1,l+k-1}, \qquad [\mathrm{Q}_{-1,l}, \mathrm{Q}_{0,k}] = \mathrm{Q}_{-1,l+k-1}. \tag{7.102}$$

For the third one, we must check that $[\mathrm{Q}_{-1,l}, \mathrm{Q}_{1,k}]$ belongs to $\mathcal{SH}^{0,\mathbf{c}}_{\mathscr{A}}$. First, by (7.57) we have

$$[\mathrm{Q}_{-1,l}, \mathrm{Q}_{1,k}] \in \mathcal{SH}^{0,\mathbf{c}}_{\mathscr{A}} \otimes_{\mathscr{A}} \mathscr{K}. \tag{7.103}$$

Next, one can check that $[\mathrm{Q}_{-1,l}, \mathrm{Q}_{1,k}]$ lies indeed in $\mathcal{SH}^{0,\mathbf{c}}_{\mathscr{A}}$ by looking at its image by $\rho^{(r)}$. The details are left to the reader. □

We have proved that the assignment $\mathrm{Q}_{0,l} \mapsto \mathrm{D}_{0,l}$, $\mathrm{Q}_{\pm l,0} \mapsto \mathrm{D}_{\pm l,0}$ extends to an isomorphism of K_r-algebras $\mathcal{SH}^{(r)} = \mathbf{SH}^{(r)}_{\mathrm{K}}$ for any r. The theorem follows. □

7.6. *The coproduct of* $\mathbf{SH^c}$. — The F-algebra $\mathbf{SH^c}$ carries a $\mathbf{Z}$-grading $\mathbf{SH^c} = \bigoplus_{s\in\mathbf{Z}} \mathbf{SH^c}[s]$. We consider the topological tensor product $\mathbf{SH^c}\,\widehat{\otimes}\,\mathbf{SH^c}$ over F defined by

$$\begin{gathered}\mathbf{SH^c}\,\widehat{\otimes}\,\mathbf{SH^c} = \bigoplus_{s\in\mathbf{Z}} \varprojlim_{\mathrm{N}} \Big(\bigoplus_{t\in\mathbf{Z}} \big(\mathbf{SH^c}[s-t] \otimes \mathbf{SH^c}[t]\big)\Big) / \mathscr{I}_{\mathrm{N}}[s], \\ \mathscr{I}_{\mathrm{N}}[s] = \bigoplus_{t\geq \mathrm{N}} \big(\mathbf{SH^c}[s-t] \otimes \mathbf{SH^c}[t]\big).\end{gathered} \tag{7.104}$$

We can now prove the following.

Theorem **7.9.** — (a) *The map* ${}^{\sigma}\Delta$ *factors to an* F*-algebra homomorphism* $\boldsymbol{\Delta} : \mathbf{SH^c} \to \mathbf{SH^c}\,\widehat{\otimes}\,\mathbf{SH^c}$ *which is uniquely determined by the following formulas*

- $\boldsymbol{\Delta}(\mathbf{c}_l) = \delta(\mathbf{c}_l)$ *for* $l \geq 0$,
- $\boldsymbol{\Delta}(\mathrm{D}_{l,0}) = \delta(\mathrm{D}_{l,0})$ *for* $l \neq 0$,
- $\boldsymbol{\Delta}(\mathrm{D}_{0,1}) = \delta(\mathrm{D}_{0,1})$,
- $\boldsymbol{\Delta}(\mathrm{D}_{0,2}) = \delta(\mathrm{D}_{0,2}) + \xi \sum_{l\geq 1} l\kappa^{1-l} \mathrm{D}_{-l,0} \otimes \mathrm{D}_{l,0}$,
- $\boldsymbol{\Delta}(\mathrm{D}_{1,1}) = \delta(\mathrm{D}_{1,1}) + \xi \mathbf{c}_0 \otimes \mathrm{D}_{1,0}$ *and* $\boldsymbol{\Delta}(\mathrm{D}_{-1,1}) = \delta(\mathrm{D}_{-1,1}) + \xi \mathrm{D}_{-1,0} \otimes \mathbf{c}_0$.

(b) *The algebra homomorphism* $\varepsilon : \mathbf{SH^c} \to \mathrm{F}$ *in Remark 1.38 is a counit for* $\boldsymbol{\Delta}$.

For $l \in \mathbf{Z}$ we abbreviate $\mathcal{O}(h^l) = h^l \mathcal{SH}_{\mathscr{A}}$. First, let us quote the following formulas.

Lemma **7.10.** — *The following hold*
(a) $\alpha_l = \kappa\xi l^2 h^3 + \mathcal{O}(h^4)$,
(b) $\theta_{l,0} = \alpha_l u_{l,0} + \mathcal{O}(h^3) = \kappa\xi |l| h^2 \mathrm{P}_{l,0} + \mathcal{O}(h^3)$ *for* $l \neq 0$,
(c) $\mathrm{P}_{l,1} = \mathrm{P}_{l,0} + \mathcal{O}(h)$ *for* $l \neq 0$.

Proof. — Part (a) follows from (7.48). Note that $\mathrm{P}_{l,0} \in \mathcal{O}(1)$ by definition of $\mathcal{SH}^{\mathbf{c}}_{\mathscr{A}}$. Thus, (b) follows from (7.48), which gives the following formulas for $l \geq 1$

$$\mathrm{P}_{\pm l,0} = \big(q^l - 1\big)\, u_{\pm l,0}, \qquad \sum_{l\geq 0} \theta_{l,0}\, s^l = \exp\Big(\sum_{l\geq 1} \alpha_l\, u_{l,0}\, s^l\Big). \tag{7.105}$$

Finally, for $l \geq 1$, using (7.50), (7.57) we get

$$u_{\pm l,1} = \pm[u_{0,1}, u_{\pm l,0}]. \tag{7.106}$$

From (7.61) we get also

$$u_{0,1} = (q-1)\mathrm{Q}_{0,2} + u_{0,0} - \mathbf{c}_1/(q-1)(t-1). \tag{7.107}$$

Thus, part (c) follows from the following computation

$$\begin{aligned} \mathrm{P}_{\pm l,1} &= (q-1)u_{\pm l,1} \\ &= \pm(q-1)[u_{0,1}, \mathrm{P}_{\pm l,0}]/\big(q^l - 1\big) \\ &= \pm(q-1)^2[\mathrm{Q}_{0,2}, \mathrm{P}_{\pm l,0}]/\big(q^l-1\big) + l(q-1)\mathrm{P}_{\pm l,0}/\big(q^l-1\big) \\ &= \mathrm{P}_{\pm l,0} + \mathcal{O}(h). \end{aligned} \tag{7.108}$$

□

We can now turn to the proof of the theorem.

Proof. — We must prove that ${}^\sigma\Delta$ preserves the lattice $\mathcal{SH}_{\mathscr{A}}$ and we must compute the image of the elements $\mathrm{Q}_{l,0}$, $\mathrm{Q}_{0,1}$ and $\mathrm{Q}_{0,2}$. By (7.55) we have ${}^\sigma\Delta(\mathrm{P}_{l,0}) = \delta(\mathrm{P}_{l,0})$ for all $l \in \mathbf{Z}$. Thus, we have also $\boldsymbol{\Delta}(\mathrm{D}_{l,0}) = \delta(\mathrm{D}_{l,0})$. Next, using (7.55) and (7.61), we get

$$\mathrm{Q}_{0,1} = u_{0,0}, \qquad {}^\sigma\Delta(\mathrm{Q}_{0,1}) = \delta(\mathrm{Q}_{0,1}).$$

This implies that $\boldsymbol{\Delta}(\mathrm{D}_{0,1}) = \delta(\mathrm{D}_{0,1})$. Finally, using (7.55) and (7.61) again, we get

$$\begin{aligned} \mathrm{Q}_{0,2} &= (q-1)^{-1}\big(u^{\mathbf{c}}_{0,1} - u_{0,0}\big), \\ {}^\sigma\Delta(\mathrm{Q}_{0,2}) &= \delta(\mathrm{Q}_{0,2}) + (q-1)^{-1}\sum_{k\geq 1}\theta_{-k,0}\otimes u_{k,1}. \end{aligned} \tag{7.109}$$

Thus, by Lemma 7.10 we have

$${}^\sigma\Delta(\mathrm{Q}_{0,2}) = \delta(\mathrm{Q}_{0,2}) + \kappa h\xi\sum_{k\geq 1} k\mathrm{P}_{-k,0}\otimes \mathrm{P}_{k,0} + \mathcal{O}\big(h^2\big). \tag{7.110}$$

This implies that

$$\boldsymbol{\Delta}(\mathrm{D}_{0,2}) = \delta(\mathrm{D}_{0,2}) + \xi\sum_{l\geq 1} l\kappa^{1-l}\mathrm{D}_{-l,0}\otimes \mathrm{D}_{l,0}. \tag{7.111}$$

□

For future use, let us mention the following fact. For $l \geq 0$ we put

$$\mathbf{SH}^-[\leq -l] = \bigoplus_{s\geq l}\mathbf{SH}^-[-s], \qquad \mathbf{SH}^+[\geq l] = \bigoplus_{s\geq l}\mathbf{SH}^+[s] \tag{7.112}$$

where the grading is the rank mentioned above.

Lemma **7.11.** — *For* $l \geq 1$ *we have* $\boldsymbol{\Delta}(\mathrm{D}_{0,l}) = \delta(\mathrm{D}_{0,l})$ *modulo* $\mathbf{SH}^-[\leq -1]\widehat{\otimes}\,\mathbf{SH}^+[\geq 1]$.

Proof. — A simple computation shows that, modulo $\mathbf{SH}^-[\leq -1]\widehat{\otimes}\mathbf{SH}^+[\geq 2]$, we have

$$\begin{aligned}(7.113)\quad \boldsymbol{\Delta}(\mathrm{D}_{1,l}) &= \boldsymbol{\Delta}\big(\mathrm{ad}(\mathrm{D}_{0,2})^l(\mathrm{D}_{1,0})\big)\\ &= \mathrm{ad}\Big(\delta(\mathrm{D}_{0,2}) + \xi\sum_{l\geq 1} l\kappa^{1-l}\mathrm{D}_{-l,0}\otimes \mathrm{D}_{l,0}\Big)^l\big(\delta(\mathrm{D}_{0,1})\big)\\ &= \delta(\mathrm{D}_{1,l})\\ &\quad + \xi\sum_{k=1}^{l}\mathrm{ad}\big(\delta(\mathrm{D}_{0,2})\big)^{k-1}\circ \mathrm{ad}(\mathrm{D}_{-1,0}\otimes \mathrm{D}_{1,0})\circ \mathrm{ad}\big(\delta(\mathrm{D}_{0,2})\big)^{l-k}\big(\delta(\mathrm{D}_{1,0})\big)\\ &= \delta(\mathrm{D}_{1,l}) + \xi\sum_{k=1}^{l}\mathrm{E}_{l-k}\otimes \mathrm{D}_{1,k-1}.\end{aligned}$$

Applying the commutator with $\boldsymbol{\Delta}(\mathrm{D}_{-1,0})$, we get, modulo $\mathbf{SH}^-[\leq -1]\widehat{\otimes}\mathbf{SH}^+[\geq 1]$,

$$(\mathbf{7.114})\qquad \boldsymbol{\Delta}(\mathrm{E}_l) = \delta(\mathrm{E}_l) + \xi\sum_{k=1}^{l}\mathrm{E}_{l-k}\otimes \mathrm{E}_{k-1}.$$

It follows in particular that

$$(\mathbf{7.115})\qquad \boldsymbol{\Delta}(\mathrm{D}_{0,l}) \in \mathbf{SH}^{\mathbf{c},0}\otimes \mathbf{SH}^{\mathbf{c},0} + \mathbf{SH}^-[\leq -1]\widehat{\otimes}\mathbf{SH}^+[\geq 1].$$

Using (1.71), modulo the ideal $\mathbf{SH}^-[\leq -1]\widehat{\otimes}\mathbf{SH}^+[\geq 1]$, we deduce from (7.114) the desired estimate on $\boldsymbol{\Delta}(\mathrm{D}_{0,l})$. □

8. Relation to $\mathbf{W}_k(\mathfrak{gl}_r)$

8.1. *Vertex algebras.* — Fix a field $\mathbf{k}$ containing $\mathbf{C}$. By a *vertex algebra* we'll always mean a $\mathbf{Z}$-graded vertex $\mathbf{k}$-algebra, i.e., a $\mathbf{Z}$-graded $\mathbf{k}$-vector space V with a vacuum vector $|0\rangle$ and fields $\mathrm{Y}(v,z)=\sum_{n\in\mathbf{Z}} v_{(n)}z^{-n-1}$ in $(\mathrm{End}\,\mathrm{V})[[z,z^{-1}]]$ satisfying the usual axioms, see [2, 16]. We'll call the $v_{(n)}$'s the *Fourier coefficients* of the field $\mathrm{Y}(v,z)$ (or, equivalently, of v) and we call $v_{(0)}$ its *residue*. As usual, the symbol : : will denote the *normal ordering* (from right to left).

Let $\mathfrak{U}(\mathrm{V})$ be the *current algebra* of V, see [2, Sect. 3.11]. It is a *degreewise complete topological* $\mathbf{k}$*-algebra*. This means that it is a $\mathbf{Z}$-graded $\mathbf{k}$-algebra $\mathfrak{U}(\mathrm{V})=\bigoplus_{s\in\mathbf{Z}}\mathfrak{U}(\mathrm{V})[s]$ which is equipped with a *degreewise linear topology* such that the multiplication $\mathfrak{U}(\mathrm{V})[s]\times\mathfrak{U}(\mathrm{V})[s']\to\mathfrak{U}(\mathrm{V})[s+s']$ is continuous, and that each piece $\mathfrak{U}(\mathrm{V})[s]$ is complete. We call the degree with respect to this grading the *conformal degree*, and we call this degreewise linear topology the *standard degreewise topology*. See [27, Sect. 1] and [2, Sect. A.2] for the terminology. The $\mathbf{k}$-algebra $\mathfrak{U}(\mathrm{V})$ is equipped with a degreewise dense family of elements $\{v_{\{n\}};\, v \in \mathrm{V},\ n \in \mathbf{Z}\}$, see [2, Sect. 3.9, Prop. 3.11.1].

We define a V-module to be a $\mathfrak{U}(\mathrm{V})$-module. A V-module is *admissible* if it is a $\mathbf{Z}$-graded $\mathfrak{U}(\mathrm{V})$-module $\mathrm{M} = \bigoplus_{s\in\mathbf{Z}} \mathrm{M}[s]$ such that $\mathrm{M}[s] = 0$ for $s \gg 0$. If M is an admissible V-module the action $\mathfrak{U}(\mathrm{V})[s] \times \mathrm{M}[s'] \to \mathrm{M}[s+s']$ is continuous with respect to the topology on $\mathfrak{U}(\mathrm{V})[s]$ and the discrete topology on M.

8.2. *The vertex algebra* $\mathrm{W}_k(\mathfrak{gl}_r)$. — Fix an integer $r > 0$ and an element $k \in \mathbf{k}$. Let $\mathrm{W}_k(\mathfrak{sl}_r)_{\mathbf{k}}$ be the W-*algebra over* $\mathbf{k}$ *at level* k *associated with* $\mathfrak{sl}_r$. We may abbreviate $\mathrm{W}_k(\mathfrak{sl}_r) = \mathrm{W}_k(\mathfrak{sl}_r)_{\mathbf{k}}$. Recall that $\mathrm{W}_k(\mathfrak{sl}_r)$ is a $\mathbf{Z}$-graded vertex algebra with quasi-primary vectors $\widetilde{\mathrm{W}}_2, \widetilde{\mathrm{W}}_3, \dots, \widetilde{\mathrm{W}}_r$ of *conformal weight* $2, 3, \dots, r$. The corresponding fields are

$$(\mathbf{8.1})\qquad \widetilde{\mathrm{W}}_i(z) = \sum_{l\in\mathbf{Z}} \widetilde{\mathrm{W}}_{i,l}\, z^{-l-i}, \qquad \widetilde{\mathrm{W}}_{i,l} \in \mathrm{End}\big(\mathrm{W}_k(\mathfrak{sl}_r)\big).$$

The vacuum $|0\rangle$ of $\mathrm{W}_k(\mathfrak{sl}_r)$ has the degree zero, and $\widetilde{\mathrm{W}}_{i,l}$ is an operator of degree $-l$. We abbreviate $\widetilde{\mathrm{W}}_{i,(l)} = \widetilde{\mathrm{W}}_{i,l-i+1}$, so that we have $\widetilde{\mathrm{W}}_i = \widetilde{\mathrm{W}}_{i,(-1)}|0\rangle$. Then $\mathrm{W}_k(\mathfrak{gl}_r)$ is spanned, as a $\mathbf{k}$-vector space, by the elements

$$(\mathbf{8.2})\qquad \widetilde{\mathrm{W}}_{i_1,(-l_1)}\widetilde{\mathrm{W}}_{i_2,(-l_2)}\cdots\widetilde{\mathrm{W}}_{i_t,(-l_t)}|0\rangle, \quad l_i \geq 1,\ t \geq 0.$$

The vertex algebra $\mathrm{W}_k(\mathfrak{sl}_r)$ admits a *strict filtration*, in the sense of [2, Sects. 3.4, 3.8], such that the subspace $\mathrm{W}_k(\mathfrak{sl}_r)[\leq d]$ is spanned by the elements (8.2) with $i_1, i_2, \dots, i_t \geq 2$ and

$$(\mathbf{8.3})\qquad i_1 + i_2 + \cdots + i_t \leq d + t.$$

We'll call it the *order filtration*. This filtration differs from the *standard filtration* on any conformal vertex algebra [2, Sect. 3.5, Rem. 4.11.3]. The associated graded $\overline{\mathrm{W}}_k(\mathfrak{sl}_r)$ is a commutative vertex algebra. Let $\overline{\mathrm{W}}_{i,l}$ denote the symbol of $\widetilde{\mathrm{W}}_{i,l}$ in $\mathrm{End}(\overline{\mathrm{W}}_k(\mathfrak{sl}_r))$. The vectors $\widetilde{\mathrm{W}}_2, \dots, \widetilde{\mathrm{W}}_r$ *generate a PBW-basis* of $\mathrm{W}_k(\mathfrak{sl}_r)$, see [2, Sect. 3.6, Prop. 4.12.1]. This means that the map

$$(\mathbf{8.4})\qquad \mathbf{k}\big[w_{i,(-l)};\, i \in [2,r],\ l \geq 1\big] \to \overline{\mathrm{W}}_k(\mathfrak{sl}_r), \qquad f(w_{i,(-l)}) \mapsto f(\overline{\mathrm{W}}_{i,(-l)})|0\rangle$$

is invertible.

Let $\mathrm{W}_k(\mathfrak{gl}_r)$ be the W-*algebra over* $\mathbf{k}$ *at level* k *associated with* $\mathfrak{gl}_r$. It is the tensor product of $\mathrm{W}_k(\mathfrak{sl}_r)$ with the vertex algebra associated with a free bosonic field of conformal weight 1

$$(\mathbf{8.5})\qquad \widetilde{\mathrm{W}}_1(z) = \sum_{l\in\mathbf{Z}} \widetilde{\mathrm{W}}_{1,l}\, z^{-l-1}.$$

The results above generalize immediately to $\mathrm{W}_k(\mathfrak{gl}_r)$. In particular $\mathrm{W}_k(\mathfrak{gl}_r)$ admits a strict filtration such that the subspace $\mathrm{W}_k(\mathfrak{gl}_r)[\leq d]$ is spanned by the elements (8.2) with $i_1, i_2, \dots, i_t \geq 1$ as in (8.3). Finally, recall that $\widetilde{\mathrm{W}}_2$ is a conformal vector of central charge

$$(\mathbf{8.6})\qquad \mathrm{C}_k = (r-1) - r\big(r^2-1\big)(k+r-1)^2/(k+r).$$

In other words, the Fourier modes of the field $\widetilde{W}_2(z)$ satisfy the relations

$$[\widetilde{W}_{2,l},\widetilde{W}_{2,k}]=(l-k)\widetilde{W}_{2,l+k}+\big(l^3-l\big)\,\delta_{l,-k}\,C_k/12. \tag{8.7}$$

8.3. *The current algebra of* $W_k(\mathfrak{gl}_r)$. — Let $\mathfrak{U}(W_k(\mathfrak{gl}_r))$ be the *current algebra* of $W_k(\mathfrak{gl}_r)$. We'll abbreviate $\widetilde{W}_{i,l}=(\widetilde{W}_i)_{\{l+i-1\}}$. Thus $\widetilde{W}_{i,l}$ may be viewed both as a linear operator on $W_k(\mathfrak{gl}_r)$ and as an element of $\mathfrak{U}(W_k(\mathfrak{gl}_r))$ of conformal degree $-l$. We hope that this will not create any confusion. Note that the elements $\widetilde{W}_{i_1,l_1}\widetilde{W}_{i_2,l_2}\cdots\widetilde{W}_{i_t,l_t}$ with $i_1,i_2,\dots,i_t\geq 1$ and $l_1+l_2+\cdots+l_t=s$ span a dense subset of $\mathfrak{U}(W_k(\mathfrak{gl}_r))[s]$. Now, the order filtration on $W_k(\mathfrak{gl}_r)$ induces a filtration on $\mathfrak{U}(W_k(\mathfrak{gl}_r))$, called again the *order filtration*. The element $\widetilde{W}_{i,l}$ has order $i-1$. Let $\overline{W}_{i,l}$ denote its symbol in the piece [2, Thm. 3.13.3]

$$\overline{\mathfrak{U}}\big(W_k(\mathfrak{gl}_r)\big)[i]=\mathfrak{U}\big(W_k(\mathfrak{gl}_r)\big)[\leq i]/\mathfrak{U}\big(W_k(\mathfrak{gl}_r)\big)[<i]. \tag{8.8}$$

The conformal weight yields a $\mathbf{Z}$-grading on $\overline{\mathfrak{U}}(W_k(\mathfrak{gl}_r))[i]$ such that $\overline{W}_{i,l}$ has (conformal) degree $-l$. Note that $\overline{\mathfrak{U}}(W_k(\mathfrak{gl}_r))$ is also a degreewise complete topological $\mathbf{k}$-algebra. It is isomorphic to the *standard degreewise completion* of the algebra $\mathbf{k}[w_{i,l};\, i\in[1,r],\ l\in\mathbf{Z}]$ as a degreewise topological $\mathbf{k}$-vector space. Here $w_{i,l}$ is given the degree $-l$.

8.4. *The* $W_k(\mathfrak{gl}_r)$*-modules.* — Now, let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{gl}_r$. For $\beta\in\mathfrak{h}$, the *Verma module with the highest weight* β is an admissible module M_β with basis elements

$$\widetilde{W}_{i_1,-l_1}\widetilde{W}_{i_2,-l_2}\cdots\widetilde{W}_{i_t,-l_t}|\beta\rangle,\quad l_i\geq 1,\ t\geq 0. \tag{8.9}$$

Here $|\beta\rangle$ is the *highest weight vector*, see [2, Sect. 5.1]. We have the following relations

$$\widetilde{W}_{i,0}|\beta\rangle=e_i(\beta)|\beta\rangle,\qquad \widetilde{W}_{i,l}|\beta\rangle=0,\quad l\geq 1, \tag{8.10}$$

where $e_i(\beta)$ is the evaluation of the ith elementary symmetric function at β.

Remark **8.1.** — The order filtration on $W_k(\mathfrak{gl}_r)$ induces a filtration on M_β such that $M_\beta[\leq d]$ is spanned by the elements

$$\widetilde{W}_{i_1,-l_1}\widetilde{W}_{i_2,-l_2}\cdots\widetilde{W}_{i_t,-l_t}|\beta\rangle,\quad l_i\geq 1,\ t\geq 0, \tag{8.11}$$

with $i_1,i_2,\dots,i_t$ satisfying (8.3). By [2, Prop. 5.1.1], the associated graded is a $\overline{\mathfrak{U}}(W_k(\mathfrak{gl}_r))$-module $\overline{M}_\beta$. The conformal weight yields a $\mathbf{Z}$-grading on $\overline{M}_\beta$. As a graded vector space $\overline{M}_\beta$ is isomorphic to the polynomial ring $\mathbf{k}[w_{i,-l};\, i\in[1,r],\ l\geq 1]$, where $w_{i,-l}$ is given the degree l.

8.5. *The quantum Miura transform for* $\mathrm{W}_k(\mathfrak{gl}_r)$. — Let $b_1, b_2, \ldots, b_r$ be a basis of $\mathfrak{h}$ and let $b^{(1)}, b^{(2)}, \ldots, b^{(r)}$ be the dual basis. Let $\langle \bullet, \bullet \rangle$ denote both the canonical pairing $\mathfrak{h}^* \times \mathfrak{h} \to \mathbf{k}$ and the pairing $\mathfrak{h}^* \times \mathfrak{h}^* \to \mathbf{k}$ such that $(b^{(i)})$ is orthonormal. Fix $\kappa \in \mathbf{k}^\times$ and fix r commuting boson fields $b^{(1)}(z), b^{(2)}(z), \ldots, b^{(r)}(z)$ of level κ^{-1}. Thus, we have

$$(\mathbf{8.12}) \qquad \left[b_l^{(i)}, b_{-h}^{(j)}\right] = l\delta_{i,j}\delta_{l,h}/\kappa, \qquad b^{(i)}(z) = \sum_{l \in \mathbf{Z}} b_l^{(i)} z^{-l-1}.$$

Let $\mathscr{H}^{(r)}$ be the Heisenberg algebra generated by the elements $b_l^{(i)}$ with $i \in [1, r]$ and $l \in \mathbf{Z}$. For $\beta \in \mathfrak{h}$ let π_β be the $\mathscr{H}^{(r)}$-module generated by the vector $|\beta\rangle$ with the relations

$$(\mathbf{8.13}) \qquad b_l^{(i)}|\beta\rangle = \delta_{l,0}\big\langle b^{(i)}, \beta\big\rangle|\beta\rangle, \quad l \geq 0.$$

To avoid confusions we may write $\pi_\beta = \pi_{\beta,\mathbf{k}}$. Consider the fields

$$(\mathbf{8.14}) \qquad b(z) = \sum_i b^{(i)}(z)\, b_i, \qquad h(z) = \sum_i \langle h, b_i\rangle\, b^{(i)}(z), \qquad h \in \mathfrak{h}^*.$$

We call π_0 the *Fock space*. It has the structure of a conformal vertex algebra such that $\mathrm{Y}(b_{-1}^{(i)}|0\rangle, z) = b^{(i)}(z)$. As a vertex algebra π_0 is isomorphic to the rth tensor power $\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{\otimes r}$. The Virasoro field has central charge $r - 12\langle h, h\rangle/\kappa$ and is given by $\frac{\kappa}{2}\sum_i {:}b^{(i)}(z)^2{:} + \partial_z h(z)$. For each β the module π_β has the structure of a module over $\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{\otimes r}$.

Now, let $h^{(1)}, h^{(2)}, \ldots, h^{(r)}$ be the weights of the first fundamental representation of $\mathfrak{sl}_r$. Let also α_i, ω_i, with $i = 1, \ldots, r-1$, be the simple roots and the fundamental weights of $\mathfrak{sl}_r$, and ρ be the sum of the fundamental weights. Given $\mathrm{Q} \in \mathbf{k}$ we define the fields $\mathrm{W}_1(z), \mathrm{W}_2(z), \ldots, \mathrm{W}_r(z)$ in $\mathrm{End}(\pi_0)[[z^{-1}, z]]$ by the following formula

$$(\mathbf{8.15}) \qquad -\kappa{:}\prod_{i=1}^{r}\big(\mathrm{Q}\partial_z + h^{(i)}(z)\big){:} = \sum_{d=0}^{r} \mathrm{W}_d(z)\, (\mathrm{Q}\partial_z)^{r-d}.$$

Note that

$$\sum_{i=1}^{r} h^{(i)} = 0, \qquad -\sum_{i \neq j} h^{(i)} \otimes h^{(j)} = \sum_{i=1}^{r-1} \alpha_i \otimes \omega_i = \sum_{i=1}^{r} b^{(i)} \otimes b^{(i)} - \frac{1}{r}\mathrm{J} \otimes \mathrm{J},$$
$$\mathrm{J} = \sum_i b^{(i)}.$$

Therefore, we have

(**8.16**)
$$\begin{aligned}
W_0(z) &= 1,\\
W_1(z) &= 0,\\
W_2(z) &= -\kappa \sum_{i<j} {:}h^{(i)}(z)\, h^{(j)}(z){:} + \kappa Q\, \partial_z \rho(z)\\
&= \frac{\kappa}{2} \sum_{i=1}^{r-1} {:}\alpha_i(z)\omega_i(z){:} + \kappa Q\, \partial_z \rho(z)\\
&= \frac{\kappa}{2} \sum_{i=1}^{r} {:}b^{(i)}(z)^2{:} - \frac{\kappa}{2r} {:}J(z)^2{:} + \kappa Q\, \partial_z \rho(z).
\end{aligned}$$

For $r \geq 2$ the field $W_2(z)$ is a Virasoro field of central charge [21, Prop. 4.10]

(**8.17**)
$$C_Q = (r-1) - r\big(r^2 - 1\big)\kappa\, Q^2.$$

Although this notation is not compatible with (8.16), we'll write

(**8.18**)
$$W_1(z) = J(z) = \sum_{i=1}^{r} b^{(i)}(z).$$

Comparing (8.6) and (8.17) we get $C_k = C_Q$ if

(**8.19**)
$$Q = -\xi/\kappa, \quad \kappa = k + r.$$

Recall that we put $\xi = 1 - \kappa$, see (1.35). *We'll always assume that* (8.19) *holds*. Then, the fields $W_1(z), \ldots, W_r(z)$ generate a vertex subalgebra of $W_{\kappa-1}(\mathfrak{gl}_1)^{\otimes r}$ which is isomorphic to $W_{\kappa-r}(\mathfrak{gl}_r)$, see [16, Sect. 5.4.11]. In other words, there is a faithful representation of $W_{\kappa-r}(\mathfrak{gl}_r)$ in π_0 which is given by the fields $W_1(z), \ldots, W_r(z)$. An explicit expression of the field $W_d(z)$ yields complicated formulas. The following is enough for our purpose.

Proposition **8.2.** — *For $d \neq 1$, modulo lower terms in the order filtration of* $\mathfrak{U}(W_{\kappa-1}(\mathfrak{gl}_1))^{\widehat{\otimes} r}$,

$$W_d(z) = -\kappa \sum_{s=0}^{d} (-r)^{s-d} \binom{r-s}{r-d} \sum_{i_1<i_2<\cdots<i_s} {:}J(z)^{d-s} b^{(i_1)}(z) b^{(i_2)}(z) \cdots b^{(i_s)}(z){:}.$$

Proof. — Obvious because, modulo lower terms, we have

(**8.20**)
$$W_d(z) \equiv -\kappa \sum_{i_1<i_2<\cdots<i_d} {:}h^{(i_1)}(z)\, h^{(i_2)}(z) \cdots h^{(i_d)}(z){:}.$$

□

Since π_β is a module over the vertex algebra $\pi_0 = \mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{\otimes r}$, it is also a module over $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$. Let $\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ denote the image of $\mathfrak{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ in $\mathrm{End}(\pi_\beta)$. This image may depend on the choice of β. We hope this will not create any confusion. We have the following, see, e.g., [6].

Proposition **8.3.** — *The representation of* $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$ *on* π_β *is such that*

$$\mathrm{W}_{d,0}|\beta\rangle = w_d(\beta)|\beta\rangle, \qquad \mathrm{W}_{d,l}|\beta\rangle = 0, \quad l \geq 1,$$

$$w_1(\beta) = \sum_{i=1}^{r} \langle b^{(i)}, \beta\rangle,$$

$$w_d(\beta) = -\kappa \sum_{i_1 < i_2 < \cdots < i_d} \prod_{t=1}^{d} \big(\langle h^{(i_t)}, \beta\rangle + (d-t)\,\xi/\kappa\big), \quad d \geq 2.$$

8.6. *The free field representation of* $\mathbf{SH}_{\mathrm{K}}^{(r)}$. — A *composition* ν of r is a tuple $(\nu_1, \nu_2, \ldots, \nu_d)$ of positive integers summing to r. For each composition, we set

$$\begin{aligned} &\mathbf{SH}_{\mathrm{K}}^{\nu} = \widehat{\bigotimes_{1\leq i\leq d}} \mathbf{SH}_{\mathrm{K}_r}^{(\nu_i)}, \qquad \mathbf{L}_{\mathrm{K}}^{\nu} = \bigotimes_{1\leq i\leq d} \mathbf{L}_{\mathrm{K}_r}^{(\nu_i)}, \\ &\mathbf{SH}_{\mathrm{K}_r}^{(\nu_i)} = \mathbf{SH}_{\mathrm{K}}^{(\nu_i)} \otimes_{\mathrm{K}_{\nu_i}} \mathrm{K}_r, \qquad \mathbf{L}_{\mathrm{K}_r}^{(\nu_i)} = \mathbf{L}_{\mathrm{K}}^{(\nu_i)} \otimes_{\mathrm{K}_{\nu_i}} \mathrm{K}_r. \end{aligned} \tag{8.21}$$

Here, the symbol $\bigotimes$ denotes the tensor product over K_r and $\widehat{\bigotimes}$ is the topological tensor product over K_r as in Section 7.6. For instance, for $d = 2$, we have

$$\begin{aligned} &\mathbf{SH}_{\mathrm{K}}^{\nu} = \bigoplus_{s\in\mathbf{Z}} \mathbf{SH}_{\mathrm{K}}^{\nu}[s], \qquad \mathbf{SH}_{\mathrm{K}}^{\nu}[s] = \varprojlim_{\mathrm{N}} \Big(\bigoplus_{s_1+s_2=s} \bigotimes_{i=1,2} \mathbf{SH}_{\mathrm{K}_r}^{(\nu_i)}[s_i]\Big) \Big/ \mathscr{I}_{\mathrm{N}}[s], \\ &\mathscr{I}_{\mathrm{N}}[s] = \bigoplus_{s_2\geq \mathrm{N}} \bigotimes_{i=1,2} \mathbf{SH}_{\mathrm{K}_r}^{(\nu_i)}[s_i]. \end{aligned} \tag{8.22}$$

Taking only the terms in $\mathbf{SH}_{\mathrm{K}_r}^{(\nu_i)}[s_i]$ or $\mathbf{SH}_{\mathrm{K}_r}^{(\nu_i)}[\leq l_i]$ in the definition of $\mathbf{SH}_{\mathrm{K}}^{\nu}$, we get the subspaces

$$\mathbf{SH}_{\mathrm{K}}^{\nu}[s_1, \ldots, s_d], \qquad \mathbf{SH}_{\mathrm{K}}^{\nu}[\leq l_1, \ldots, \leq l_d]. \tag{8.23}$$

For future use, let us quote the following easy fact.

Proposition **8.4.** — *The map* $\boldsymbol{\Delta}^{d-1}$ *factors to an algebra embedding* $\boldsymbol{\Delta}^{\nu} : \mathbf{SH}_{\mathrm{K}}^{(r)} \to \mathbf{SH}_{\mathrm{K}}^{\nu}$ *and*

$$\boldsymbol{\Delta}^{\nu}\big(\mathbf{SH}_{\mathrm{K}}^{(r)}[s]\big) \subset \bigoplus_{s_1,\ldots,s_d} \mathbf{SH}_{\mathrm{K}}^{\nu}[s_1, \ldots, s_d],$$

$$\boldsymbol{\Delta}^{\nu}\big(\mathbf{SH}_{\mathrm{K}}^{(r)}[\leq l]\big) \subset \bigoplus_{l_1,\ldots,l_d} \mathbf{SH}_{\mathrm{K}}^{\nu}[\leq l_1, \ldots, \leq l_d].$$

Here the sums run over all tuples summing to s and l respectively.

Proof. — Since the coproduct $\boldsymbol{\Delta}$ admits a counit ε, the map $\boldsymbol{\Delta}^{d-1}$ is an injection

$$\mathbf{SH^c} \to (\mathbf{SH^c})^{\widehat{\otimes} d}, \tag{8.24}$$

because $\boldsymbol{\Delta}^{d-1}(x) = 0$ implies that $x = (\varepsilon^{\widehat{\otimes}(d-1)} \widehat{\otimes}\, \mathrm{id})(\boldsymbol{\Delta}^{d-1}(x)) = 0$. Here, the tensor product is taken over the field F. Composing this map with the base change $\bullet \otimes_{\mathrm{F^c}} \mathrm{K}_{\nu_i}$ and the obvious inclusion $\mathrm{K}_{\nu_i} \subset \mathrm{K}_r$ for $i \in [1, \nu_i]$, we get an F-linear map $\mathbf{SH^c} \to \mathbf{SH}^{\nu}_{\mathrm{K}}$ such that $\mathbf{c}_l \mapsto p_l(\varepsilon_1, \ldots, \varepsilon_r)$ for each $l \geq 0$, because $\boldsymbol{\Delta}(\mathbf{c}_l) = \delta(\mathbf{c}_l)$. Therefore, by construction of the map $\mathrm{F^c} \to \mathrm{K}_r$ in Definition 1.36, it factors to a K_r-algebra homomorphism $\boldsymbol{\Delta}^{\nu} : \mathbf{SH}^{(r)}_{\mathrm{K}} \to \mathbf{SH}^{\nu}_{\mathrm{K}}$. This map is again injective. □

Definition **8.5.** — *We define a representation ρ^{ν} of $\mathbf{SH}^{(r)}_{\mathrm{K}}$ on $\mathbf{L}^{\nu}_{\mathrm{K}}$ by composing $\boldsymbol{\Delta}^{\nu}$ with the representation of $\mathbf{SH}^{\nu}_{\mathrm{K}}$ on $\mathbf{L}^{\nu}_{\mathrm{K}}$ in Corollary 3.3.*

Corollary **8.6.** — *The representation ρ^{ν} is faithful.*

Proof. — Use Proposition 8.4 and Theorem 3.2. □

Remark **8.7.** — We will mostly be interested in the case $\nu = (1^r)$, where we abbreviate $(1^r) = (1, 1, \ldots, 1)$. In this case, we have

$$\mathbf{L}^{(1^r)}_{\mathrm{K}} = \big(\mathbf{L}^{(1)}_{\mathrm{K}_r}\big)^{\otimes r} = \mathbf{L}^{(1)}_{\mathrm{K}} \otimes_{\mathrm{K}} \mathbf{L}^{(1)}_{\mathrm{K}} \otimes_{\mathrm{K}} \cdots \otimes_{\mathrm{K}} \mathbf{L}^{(1)}_{\mathrm{K}}, \tag{8.25}$$

and the K_r-vector space structure is given by

$$\varepsilon_i = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \varepsilon_1 \otimes 1 \otimes \cdots \otimes 1, \qquad i \in [1, r], \tag{8.26}$$

where ε_1 is at the ith spot.

8.7. *The degreewise completion of* $\mathbf{SH}^{(r)}_{\mathrm{K}}$. — We refer to [27, Sects. 1.1–1.4] for the terminology concerning degreewise topological algebras. The K_r-algebra $\mathbf{SH}^{(r)}_{\mathrm{K}}$ carries a $\mathbf{Z}$-grading and an $\mathbf{N}$-filtration inherited from $\mathbf{SH}^{\mathbf{c}}_{\mathrm{K}}$, see Section 1.8.

Definition **8.8.** — *The* standard degreewise topology *of* $\mathbf{SH}^{(r)}_{\mathrm{K}}$ *is the* degreewise topology *defined by the sequence*

$$\mathscr{I}_{\mathrm{N}} = \bigoplus_{s \in \mathbf{Z}} \mathscr{I}_{\mathrm{N}}[s], \qquad \mathscr{I}_{\mathrm{N}}[s] = \sum_{t \geq \mathrm{N}} \mathbf{SH}^{(r)}_{\mathrm{K}}[t-s]\, \mathbf{SH}^{(r)}_{\mathrm{K}}[-t]. \tag{8.27}$$

The standard degreewise completion *of* $\mathbf{SH}^{(r)}_{\mathrm{K}}$ *is the $\mathbf{Z}$-graded algebra given by*

$$\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}}) = \bigoplus_{s \in \mathbf{Z}} \mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})[s], \qquad \mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})[s] = \varprojlim_{\mathrm{N}} \mathbf{SH}^{(r)}_{\mathrm{K}}[s] / \mathscr{I}_{\mathrm{N}}[s]. \tag{8.28}$$

The standard degreewise topology on $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})$ is the projective limit degreewise topology.

The standard degreewise topologies on $\mathbf{SH}^{(r)}_{\mathrm{K}}$ and $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})$ are linear. They equip $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})$ with the structure of a degreewise complete topological algebra and the canonical map $\mathbf{SH}^{(r)}_{\mathrm{K}} \to \mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})$ is a morphism of degreewise topological algebras with a *degreewise dense* image.

Definition **8.9.** — *A module* M *over* $\mathbf{SH}^{(r)}_{\mathrm{K}}$ *or* $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})$ *is* admissible *if* $\mathrm{M} = \bigoplus_{s\in\mathbf{Z}} \mathrm{M}[s]$ *is* $\mathbf{Z}$*-graded and* $\mathrm{M}[s] = 0$ *for* s *large enough.*

By an embedding of degreewise topological algebras we mean an injective morphism of degreewise topological algebras. The following is an immediate consequence of Corollary 8.6.

Proposition **8.10.** — (a) *The map* $\rho^{(1^r)}$ *is a faithful admissible representation of* $\mathbf{SH}^{(r)}_{\mathrm{K}}$ *on* $\mathbf{L}^{(1^r)}_{\mathrm{K}}$ *which extends to an admissible representation of* $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})$.

(b) *The canonical map* $\mathbf{SH}^{(r)}_{\mathrm{K}} \to \mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})$ *is an embedding of degreewise topological algebras.*

Remark **8.11.** — If M is admissible then the actions

$$\mathbf{SH}^{(r)}_{\mathrm{K}}[s] \times \mathrm{M}\big[s'\big] \to \mathrm{M}\big[s+s'\big], \qquad \mathfrak{U}\big(\mathbf{SH}^{(r)}_{\mathrm{K}}\big)[s] \times \mathrm{M}\big[s'\big] \to \mathrm{M}\big[s+s'\big] \tag{8.29}$$

are continuous with respect to the standard topology on $\mathbf{SH}^{(r)}_{\mathrm{K}}[s]$, $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})[s]$ and the discrete topology on $\mathrm{M}[s']$, $\mathrm{M}[s+s']$.

Remark **8.12.** — The order filtration on $\mathbf{SH}^{(r)}_{\mathrm{K}}$ induces a filtration on $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})$ called again the order filtration. By Proposition 1.39 it is determined by putting $\mathrm{D}_{r,d}$ in degree d for any r, d.

8.8. *From* $\mathbf{SH}^{(1)}_{\mathrm{K}}$ *to* $\mathrm{W}_k(\mathfrak{gl}_1)$. — In this section we set $\mathbf{k} = \mathrm{K}_1$ and $\kappa = k+1$. Recall that $\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1) = \pi_0$, the vertex algebra associated with the Heisenberg algebra $\mathscr{H}^{(1)}$. We abbreviate

$$\mathrm{W}_1(z) = b(z) = \sum_{l\in\mathbf{Z}} b_l\, z^{-l-1}. \tag{8.30}$$

Thus $\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)$ is spanned, as a vector space, by elements

$$b_{-l_1}\cdots b_{-l_t}|0\rangle, \quad l_i \geq 1,\ t\geq 0. \tag{8.31}$$

Definition **8.13.** — *The subspace* $\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)[\preccurlyeq d]$ *of* standard order *at most* d *is the span of the elements in (8.31) with* $t \leq d$.

The *standard filtration* on $\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)$ should not be confused with the order filtration. The associated graded of $\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)$ with respect to the standard filtration is a commutative vertex algebra. The current algebra $\mathfrak{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1))$ has a standard filtration as well, for which the elements b_l are of standard order 1. Now, we consider the $\mathscr{H}^{(1)}$-module

(**8.32**) $$\pi^{(1)} = \pi_\beta, \qquad \beta = -\varepsilon_1/\kappa.$$

Recall that $\mathscr{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1))$ is the image of $\mathfrak{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1))$ in $\mathrm{End}(\pi^{(1)})$. By Proposition 1.41 there is a unique isomorphism K_1-vector space $\mathbf{L}_{\mathrm{K}}^{(1)} \to \pi^{(1)}$ such that $[\mathrm{I}_\emptyset] \mapsto |\beta\rangle$ which intertwines the operator $\rho^{(1)}(b_{-l}) = \rho^{(1)}(y^l \mathrm{D}_{l,0})$ on $\mathbf{L}_{\mathrm{K}}^{(1)}$ with the operator b_{-l} on $\pi^{(1)}$. Following (2.34), we identify $\pi^{(1)}$ with Λ_{K_1} in the usual way. This yields an isomorphism

(**8.33**) $$\mathbf{L}_{\mathrm{K}}^{(1)} = \pi^{(1)} = \Lambda_{\mathrm{K}_1}.$$

Our next result describes the action of the element H_l introduced in (1.89).

Proposition **8.14.** — *We have the following relation in* $\mathrm{End}(\pi^{(1)})$

$$\rho^{(1)}(\mathrm{H}_l) = \frac{\kappa}{2} \sum_{h \in \mathbf{Z}} \rho^{(1)}(:b_{l-h} b_h:), \qquad l \in \mathbf{Z}.$$

Proof. — To unburden the notation we omit the symbol $\rho^{(1)}$ everywhere. We must prove that

(**8.34**) $$\mathrm{H}_0 = \kappa \sum_{l \geq 1} b_{-l} b_l + \kappa b_0^2/2, \qquad \mathrm{H}_k = \kappa \sum_{l \in \mathbf{Z}} b_{k-l} b_l/2, \quad k \neq 0.$$

Recall that b_{-l} acts on Λ_{K_1} by multiplication by p_l and that b_l acts by the operator $\kappa^{-1} l \partial_{p_l}$. Next, the computation in the proof of [35, Thm. 3.1] implies that

(**8.35**) $$\square(p_\lambda) = \kappa^{-1} \xi n(\lambda') p_\lambda + \frac{1}{2\kappa} \sum_{r \neq s} \lambda_r \lambda_s p_{\lambda_r}^{-1} p_{\lambda_s}^{-1} p_{\lambda_r + \lambda_s} p_\lambda + \frac{1}{2} \sum_r \sum_{j=1}^{\lambda_r - 1} \lambda_r p_{\lambda_r}^{-1} p_j p_{\lambda_r - j} p_\lambda.$$

So we have the following formula

(**8.36**) $$\square = \xi \sum_{l \geq 1} (l-1) b_{-l} b_l/2 + \kappa \sum_{l,k \geq 1} (b_{-l-k} b_l b_k + b_{-l} b_{-k} b_{l+k})/2.$$

Now, Remark 3.4 yields

(**8.37**) $$\mathrm{D}_{0,2} + \varepsilon_1 \mathrm{D}_{0,1} = \kappa \square.$$

Further, a direct computation (left to the reader) using (1.89), (8.36) and (8.37) gives

(**8.38**) $$[\mathrm{H}_k, b_l] = -l b_{l+k}, \qquad [\mathrm{H}_{-k}, b_l] = -l b_{l-k}, \quad l \in \mathbf{Z}, \ k \geq 1.$$

This implies the formula for H_k and $k \neq 0$. Next, a direct computation using (1.89) yields

(**8.39**) $$\mathrm{H}_0 = \mathrm{D}_{0,1} + \kappa b_0^2/2.$$

Further, by Lemma E.3 we have $[\mathrm{D}_{0,1}, \mathrm{D}_{l,0}] = l\mathrm{D}_{l,0}$ and by (3.18) we have $\mathrm{D}_{0,1}([\mathrm{I}_\emptyset]) = 0$. This yields the following formula for $\mathrm{D}_{0,1}$, which implies the formula for H_0,

$$\mathrm{D}_{0,1} = \kappa \sum_{l\geq 1} b_{-l} b_l. \tag{8.40}$$

□

Equations (8.36), (8.37) and (8.40) give the expression for the action of $\mathrm{D}_{0,1}$ and $\mathrm{D}_{0,2}$ on $\pi^{(1)}$. Since $\mathbf{SH}^{(1)}_{\mathrm{K}}$ is generated by $\{\mathrm{D}_{0,2},\ b_l;\ l \in \mathbf{Z}\}$, the proof above also gives the following.

Proposition **8.15.** — *There is an embedding* $\Theta^{(1)} : \mathbf{SH}^{(1)}_{\mathrm{K}} \to \mathscr{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)),\ b_l \mapsto b_l$ *which intertwines the representations of* $\mathbf{SH}^{(1)}_{\mathrm{K}}$ *and* $\mathscr{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1))$ *on* $\pi^{(1)}$.

Remark **8.16.** — From (8.36), (8.37), (8.40) we get $\rho^{(1)}(\mathrm{D}_{0,2}) = \mathrm{V}^{(1)} + \kappa\xi \sum_{l\geq 1} \frac{l\,b_{-l}b_l}{2}$, where $\mathrm{V}^{(1)} = u_{(2)}$ for some element $u \in \mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)$ of degree 3. Note that the infinite sum $\sum_{l\geq 1} \frac{l\,b_{-l}b_l}{2}$ belongs to $\mathscr{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1))$.

Thanks to Proposition 8.15, we may speak of the standard order of an element of $\mathbf{SH}^{(1)}_{\mathrm{K}}$.

Proposition **8.17.** — *For $d \geq 1$ we have*

$$\rho^{(1)}(\mathrm{D}_{0,d}) \equiv \frac{\kappa^d}{d(d+1)} \sum_{l_0,\ldots,l_d} \rho^{(1)}(:b_{l_0} b_{l_1} \cdots b_{l_d}:). \tag{8.41}$$

The sum runs over all tuples of integers with sum 0. *The symbol* $\equiv$ *means that the equality holds modulo the action of terms of standard order* $\preccurlyeq d-1$.

Proof. — To unburden the notation we omit the symbol $\rho^{(1)}$ everywhere. Further, for any integers $m_1, \ldots, m_d$ we abbreviate

$$b_{m_1,\ldots,m_d} = \mathrm{ad}(b_{m_1}) \circ \mathrm{ad}(b_{m_2}) \circ \cdots \circ \mathrm{ad}(b_{m_d}). \tag{8.42}$$

Recall that $:b_{m_1} \cdots b_{m_d}:$ is the monomial obtained from $b_{m_1} \cdots b_{m_d}$ by moving all b_{m_i}, $m_i < 0$, to the left of all b_{m_j} with $m_j \geq 0$. First, we prove that for any $m_1, \ldots, m_d$ we have

$$b_{m_1,\ldots,m_d}(\mathrm{D}_{0,d}) = (d-1)!\,(m_1 m_2 \cdots m_d)\, b_m, \qquad m = m_1 + \cdots + m_d. \tag{8.43}$$

We proceed by induction on d. Note that (8.36)–(8.40) imply that

$$\mathrm{D}_{0,1} \equiv \frac{\kappa}{2} \sum_{l_0,l_1} :b_{l_0} b_{l_1}:, \qquad \mathrm{D}_{0,2} \equiv \frac{\kappa^2}{6} \sum_{l_0,l_1,l_2} :b_{l_0} b_{l_1} b_{l_2}:, \tag{8.44}$$

where the l_i's are integers which sum to 0. This implies the claim for $d=1,2$. Assume that (8.43) is proved for d. Applying $\mathrm{ad}(\mathrm{D}_{1,1})$ to (8.43), the formula (1.91) gives

$$(d-1)!(m_1m_2\cdots m_d)mb_{m-1} = b_{m_1,\dots,m_d}(\mathrm{D}_{1,d}) + \sum_i m_i b_{m_1,\dots,m_i-1,\dots,m_d}(\mathrm{D}_{0,d})$$
$$= b_{m_1,\dots,m_d}(\mathrm{D}_{1,d}) + (d-1)!\sum_i (m_i-1)(m_1m_2\cdots m_d)b_{m-1}.$$

This implies the formula

(8.45) $$b_{m_1,\dots,m_d}(\mathrm{D}_{1,d}) = d!(m_1m_2\cdots m_d)b_{m-1}.$$

Similarly, we have

(8.46) $$b_{m_1,\dots,m_d}(\mathrm{D}_{-1,d}) = \kappa\, d!(m_1m_2\cdots m_d)b_{m+1}.$$

Next, we compute

(8.47) $$\begin{aligned} b_{m_1,\dots,m_{d+1}}(\mathrm{E}_{d+2}) &= b_{m_1,\dots,m_{d+1}}\big([\mathrm{D}_{-1,2},\mathrm{D}_{1,d}]\big) \\ &= \sum_{i<j}\big[b_{m_i,m_j}(\mathrm{D}_{-1,2}),\, b_{m_1,\dots,\widehat{m}_i,\dots,\widehat{m}_j,\dots,m_{d+1}}(\mathrm{D}_{1,d})\big] \\ &\quad + \sum_i \big[b_{m_i}(\mathrm{D}_{-1,2}),\, b_{m_1,\dots,\widehat{m}_i,\dots,m_{d+1}}(\mathrm{D}_{1,d})\big], \end{aligned}$$

where the symbol $\widehat{m}_i$ means that the index m_i is omitted. Write $m^+ = m + m_{d+1}$. The first sum on the right hand side of (8.47) is equal to

$$\begin{aligned} &\sum_{i<j} 2\kappa\, l_i l_j b_{m_i+m_j+1,m_1,\dots,\widehat{m}_i,\dots,\widehat{m}_j,\dots,m_{d+1}}(\mathrm{D}_{1,d}) \\ &\quad = 2\kappa\, d!(m_1\cdots m_{d+1})\sum_{i<j}(m_i+m_j+1)b_{m^+} \\ &\quad = 2\kappa\, d!(m_1\cdots m_{d+1})\big(dm^+ + d(d+1)/2\big)b_{m^+} \end{aligned}$$

while the second sum evaluates to

$$\begin{aligned} &-d!\sum_i (m_1\cdots\widehat{m}_i\cdots m_{d+1})b_{m^+-m_i-1,m_i}(\mathrm{D}_{-1,2}) \\ &\quad = -2\kappa\, d!(m_1\cdots m_{d+1})\sum_i \big(m^+ - m_i - 1\big)\, b_{m^+} \\ &\quad = 2\kappa\, d!(m_1\cdots m_{d+1})\big(-dm^+ + (d+1)\big)b_{m^+}. \end{aligned}$$

We obtain

$$b_{m_1,\dots,m_{d+1}}(\mathrm{E}_{d+2}) = \kappa(d+2)!(m_1\cdots m_{d+1})b_{m^+}. \tag{8.48}$$

By (1.71) we have $\mathrm{E}_{d+2} = \kappa(d+2)(d+1)\mathrm{D}_{0,d+1} + u$ where u is a polynomial in $\mathrm{D}_{0,1},\dots,\mathrm{D}_{0,d}$ of order $\leq d$. Thus, we have

$$b_{m_1,\dots,m_{d+1}}(\mathrm{E}_{d+2}) = \kappa(d+2)(d+1)b_{m_1,\dots,m_{d+1}}(\mathrm{D}_{0,d+1}). \tag{8.49}$$

From this we finally deduce relation (8.43) for $d+1$. We are done.

Now, relation (8.41) follows from (8.43). Indeed, given integers $l_0, l_1, \dots, l_d$ we have

$$\mathrm{ad}(b_m)(b_{l_0}b_{l_1}\cdots b_{l_d}) = \frac{m}{\kappa}\sum_{l_i=-m} b_{l_0}\cdots b_{l_{i-1}}b_{l_{i+1}}\cdots b_{l_d}, \tag{8.50}$$

where the sum is over all i's with $l_i = -m$. Thus, if $l_0, l_1, \dots, l_d$ sum to 0 we have

$$b_{m_1,\dots,m_d}(:b_{l_0}b_{l_1}\cdots b_{l_d}:) = c\, b_m, \tag{8.51}$$

for some constant c which is zero unless $l_0, l_1, \dots, l_d$ are equal to $m, -m_1, -m_2, \dots, -m_d$, up to a permutation, and which, in this case, is equal to $(m_1\cdots m_d)/\kappa^d$ times the number $c_{l_0,\dots,l_d}$ of permutations σ of $\{0,1,\dots,d\}$ such that $l_{\sigma(0)} = m$ and $l_{\sigma(s)} = -m_s$ for $s = 1, 2, \dots, d$. In other words, if $l_0, l_1, \dots, l_d$ are equal to $m, -m_1, -m_2, \dots, -m_d$ up to a permutation, then we have

$$b_{m_1,\dots,m_d}\left(\mathrm{D}_{0,d} - \frac{\kappa^d(d-1)!}{c_{l_0,\dots,l_d}}:b_{l_0}b_{l_1}\cdots b_{l_d}:\right) = 0. \tag{8.52}$$

Therefore, for any integers $m_1, m_2, \dots, m_d$ we have

$$b_{m_1,\dots,m_d}\left(\mathrm{D}_{0,d} - \frac{\kappa^d}{d(d+1)}\sum_{l_0,\dots,l_d} :b_{l_0}b_{l_1}\cdots b_{l_d}:\right) = 0. \tag{8.53}$$

The sum runs over all tuples of integers summing to 0. To conclude, we use the following lemma.

Lemma **8.18.** — *Let $u \in \mathfrak{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1))$ be annihilated by $b_{m_1,\dots,m_d}$ for any integers $m_1,\dots,m_d$. Then u is of standard order $\preccurlyeq d-1$.*

Proof. — We may express u as an infinite sum

$$u = \sum_{s\geq 0}\sum_{l_1,\dots,l_s} a_{l_1,\dots,l_s}:b_{l_1}\cdots b_{l_s}:. \tag{8.54}$$

Now observe that

$$s < t \Rightarrow b_{m_1,\dots,m_t}(:b_{l_1}\cdots b_{l_s}:) = 0, \qquad b_{m_1,\cdots,m_s}(:b_{l_1}\cdots b_{l_s}:) = c_{m_1,\dots,m_s}, \tag{8.55}$$

where $c_{m_1,\dots,m_s} \neq 0$ if and only if $l_1,\dots,l_s$ are equal, up to a permutation, to $-m_1,\dots,-m_s$. The lemma follows easily. ☐

8.9. *From* $\mathbf{SH}^{(2)}_{\mathrm{K}}$ *to* $\mathrm{W}_k(\mathfrak{gl}_2)$. — We are interested in higher rank analogues of the inclusion $\Theta^{(1)}$. In this section we deal with the case $r=2$. We set $\mathbf{k}=\mathrm{K}_2$ and $\kappa=k+2$. We write

$$\pi^{(1^2)}=\pi_\beta, \qquad \big\langle b^{(i)},\beta\big\rangle=-\varepsilon_i/\kappa+(i-1)\xi/\kappa, \quad i=1,2. \tag{8.56}$$

Recall that $\mathscr{U}(\mathrm{W}_{\kappa-2}(\mathfrak{gl}_2))$ is the image of $\mathfrak{U}(\mathrm{W}_{\kappa-2}(\mathfrak{gl}_2))$ in $\mathrm{End}(\pi^{(1^2)})$. The isomorphism (8.33) yields an isomorphism $\mathbf{L}^{(1^2)}_{\mathrm{K}}=\Lambda^{\otimes 2}_{\mathrm{K}_2}$. Composing it with the isomorphism $\Lambda^{\otimes 2}_{\mathrm{K}_2}=\pi^{(1^2)}$ such that $1\otimes 1\mapsto|\beta\rangle$ which intertwines the operators $b_{-l}\otimes 1$, $1\otimes b_{-l}$ on $\Lambda^{\otimes 2}_{\mathrm{K}_2}$ with the operators $b^{(1)}_{-l}$, $b^{(2)}_{-l}$ on $\pi^{(1^2)}$, we get an isomorphism

$$\mathbf{L}^{(1^2)}_{\mathrm{K}}=\pi^{(1^2)}=\Lambda^{\otimes 2}_{\mathrm{K}_2} \tag{8.57}$$

which identifies $[\mathrm{I}_\emptyset]^{\otimes 2}$, $|\beta\rangle$ and $1^{\otimes 2}$. Using (8.57) together with Propositions 8.3 and 8.10 we get inclusions of $\mathscr{U}(\mathrm{W}_{\kappa-2}(\mathfrak{gl}_2))$ and $\mathbf{SH}^{(2)}_{\mathrm{K}}$ into $\mathrm{End}(\pi^{(1^2)})$.

Proposition **8.19.** — *The representation* $\rho^{(1^2)}$ *yields an embedding of degreewise topological* K_2*-algebras* $\Theta^{(2)}:\mathbf{SH}^{(2)}_{\mathrm{K}}\to\mathscr{U}(\mathrm{W}_{\kappa-2}(\mathfrak{gl}_2))$.

Proof. — It is enough to check that $\rho^{(1^2)}(b_l)$ and $\rho^{(1^2)}(\mathrm{D}_{0,2})$ belong to $\mathscr{U}(\mathrm{W}_{\kappa-2}(\mathfrak{gl}_2))$. For b_l, this follows from the easily checked relation

$$\rho^{(1^2)}\big(b(z)\big)=\mathrm{J}(z), \qquad b(z)=\sum_{l\in\mathbf{Z}}b_lz^{-l-1}\in\mathbf{SH}^{(2)}_{\mathrm{K}}\big[\big[z,z^{-1}\big]\big]. \tag{8.58}$$

For $\mathrm{D}_{0,2}$, this is a consequence of the lemma below. □

Lemma **8.20.** — *There is a constant c such that*

$$\begin{aligned}\rho^{(1^2)}(\mathrm{D}_{0,2})=&\frac{\kappa}{2}\sum_{l\in\mathbf{Z}}{:}\mathrm{W}_{1,-l}\mathrm{W}_{2,l}{:}+\frac{\kappa^2}{24}\sum_{k,l\in\mathbf{Z}}{:}\mathrm{W}_{1,-k-l}\mathrm{W}_{1,k}\mathrm{W}_{1,l}{:}\\&+\frac{\kappa\xi}{4}\sum_{l\in\mathbf{Z}}\big(|l|-1\big){:}\mathrm{W}_{1,-l}\mathrm{W}_{1,l}{:}+\xi\mathrm{W}_{2,0}+c.\end{aligned}$$

Proof. — First, note that (8.36), (8.37), (8.40) and Theorem 7.9 imply that

$$\begin{aligned}\rho^{(1^2)}(\mathrm{D}_{0,2})=&\frac{\kappa^2}{2}\sum_{k,l\geq 1}\big(b^{(1)}_{-l-k}b^{(1)}_lb^{(1)}_k+b^{(1)}_{-l}b^{(1)}_{-k}b^{(1)}_{l+k}+b^{(2)}_{-l-k}b^{(2)}_lb^{(2)}_k+b^{(2)}_{-l}b^{(2)}_{-k}b^{(2)}_{l+k}\big)\\&+\frac{\kappa\xi}{2}\sum_{l\geq 1}(l-1)\big(b^{(1)}_{-l}b^{(1)}_l+b^{(2)}_{-l}b^{(2)}_l\big)-\kappa\sum_{l\geq 1}\big(\varepsilon_1b^{(1)}_{-l}b^{(1)}_l+\varepsilon_2b^{(2)}_{-l}b^{(2)}_l\big)\\&+\kappa\xi\sum_{l\geq 1}lb^{(2)}_{-l}b^{(1)}_l.\end{aligned} \tag{8.59}$$

Using $\varepsilon_1 = -\kappa b_0^{(1)}$ and $\varepsilon_2 = \xi - \kappa b_0^{(2)}$, we can rewrite (8.59) in the following way

$$\begin{aligned}\rho^{(1^2)}(\mathrm{D}_{0,2}) = \frac{\kappa^2}{6}\sum_{k,l\in\mathbf{Z}} &{:}b^{(1)}_{-l-k}b^{(1)}_l b^{(1)}_k + b^{(2)}_{-l-k}b^{(2)}_l b^{(2)}_k{:} \\ &+ \frac{\kappa\xi}{4}\sum_{l\in\mathbf{Z}}\big(|l|-2\big){:}b^{(1)}_{-l}b^{(1)}_l + b^{(2)}_{-l}b^{(2)}_l{:} + \frac{\kappa\xi}{4}\sum_{l\in\mathbf{Z}}{:}b^{(1)}_{-l}b^{(1)}_l - b^{(2)}_{-l}b^{(2)}_l{:} \\ &+ \kappa\xi\sum_{l\geq 1} l b^{(2)}_{-l}b^{(1)}_l + c_1\end{aligned} \tag{8.60}$$

for some constant c_1. Next, recall that

$$\begin{aligned}\mathrm{W}_1(z) &= b^{(1)}(z) + b^{(2)}(z), \\ \mathrm{W}_2(z) &= \frac{\kappa}{2}{:}b^{(1)}(z)^2 + b^{(2)}(z)^2{:} - \frac{\kappa}{4}{:}\mathrm{W}_1(z)^2{:} - \xi\,\partial_z\rho(z).\end{aligned} \tag{8.61}$$

This implies that

$$\mathrm{W}_{2,l} = -\frac{\kappa}{2}\sum_{k\in\mathbf{Z}}{:}b^{(1)}_{l-k}b^{(2)}_k{:} + \frac{\kappa}{4}\sum_{k\in\mathbf{Z}}{:}b^{(1)}_{l-k}b^{(1)}_k + b^{(2)}_{l-k}b^{(2)}_k{:} + \frac{\xi}{2}(l+1)\big(b^{(1)}_l - b^{(2)}_l\big). \tag{8.62}$$

Further, we have the following formulas

$$\begin{aligned}\sum_{k,l\in\mathbf{Z}}{:}\mathrm{W}_{1,-k-l}\mathrm{W}_{1,k}\mathrm{W}_{1,l}{:} = 3\sum_{k,l\in\mathbf{Z}}&{:}b^{(1)}_{-k-l}b^{(1)}_k b^{(2)}_l + b^{(2)}_{-k-l}b^{(2)}_k b^{(1)}_l{:} \\ &+ \sum_{k,l\in\mathbf{Z}}{:}b^{(1)}_{-k-l}b^{(1)}_k b^{(1)}_l + b^{(2)}_{-k-l}b^{(2)}_k b^{(2)}_l{:}\end{aligned} \tag{8.63}$$

$$\sum_{l\in\mathbf{Z}}|l|{:}\mathrm{W}_{1,-l}\mathrm{W}_{1,l}{:} = 2\sum_{l\in\mathbf{Z}}|l|{:}b^{(1)}_{-l}b^{(2)}_l{:} + \sum_{l\in\mathbf{Z}}|l|{:}b^{(1)}_{-l}b^{(1)}_l + b^{(2)}_{-l}b^{(2)}_l{:} \tag{8.64}$$

$$\begin{aligned}\sum_{l\in\mathbf{Z}}{:}\mathrm{W}_{1,-l}\mathrm{W}_{2,l}{:} = -\frac{\kappa}{4}\sum_{k,l\in\mathbf{Z}}&{:}b^{(1)}_{-k-l}b^{(1)}_k b^{(2)}_l + b^{(2)}_{-k-l}b^{(2)}_k b^{(1)}_l{:} \\ &+ \frac{\kappa}{4}\sum_{k,l\in\mathbf{Z}}{:}b^{(1)}_{-k-l}b^{(1)}_k b^{(1)}_l + b^{(2)}_{-k-l}b^{(2)}_k b^{(2)}_l{:} \\ &+ \frac{\xi}{2}\sum_{l\in\mathbf{Z}}{:}b^{(1)}_{-l}b^{(1)}_l - b^{(2)}_{-l}b^{(2)}_l{:} - \xi\sum_{l\in\mathbf{Z}} l{:}b^{(1)}_{-l}b^{(2)}_l{:}.\end{aligned} \tag{8.65}$$

Therefore, we get

$$\begin{aligned}(\mathbf{8.66})\qquad &\rho^{(1^2)}(\mathrm{D}_{0,2}) - \frac{\kappa}{2}\sum_{l\in\mathbf{Z}}{:}\mathrm{W}_{1,-l}\mathrm{W}_{2,l}{:} - \frac{\kappa^2}{24}\sum_{k,l\in\mathbf{Z}}{:}\mathrm{W}_{1,-k-l}\mathrm{W}_{1,k}\mathrm{W}_{1,l}{:}\\ &\quad - \frac{\kappa\xi}{4}\sum_{l\in\mathbf{Z}}|l|{:}\mathrm{W}_{1,-l}\mathrm{W}_{1,l}{:}\\ &= -\frac{\kappa\xi}{2}\sum_{l\in\mathbf{Z}}{:}b^{(1)}_{-l}b^{(1)}_{l} + b^{(2)}_{-l}b^{(2)}_{l}{:} + c_1.\end{aligned}$$

Next, observe that

$$(\mathbf{8.67})\qquad \begin{aligned}&\mathrm{W}_{2,0} = -\frac{\kappa}{2}\sum_{l\in\mathbf{Z}}{:}b^{(1)}_{-l}b^{(2)}_{l}{:} + \frac{\kappa}{4}\sum_{l\in\mathbf{Z}}{:}b^{(1)}_{-l}b^{(1)}_{l} + b^{(2)}_{-l}b^{(2)}_{l}{:} + \frac{\xi}{2}\big(b^{(1)}_{0} - b^{(2)}_{0}\big),\\ &\sum_{l\in\mathbf{Z}}{:}\mathrm{W}_{1,-l}\mathrm{W}_{1,l}{:} = 2\sum_{l\in\mathbf{Z}}{:}b^{(1)}_{-l}b^{(2)}_{l}{:} + \sum_{l\in\mathbf{Z}}{:}b^{(1)}_{-l}b^{(1)}_{l} + b^{(2)}_{-l}b^{(2)}_{l}{:}.\end{aligned}$$

Therefore, we have

$$(\mathbf{8.68})\qquad \frac{\kappa\xi}{2}\sum_{l\in\mathbf{Z}}{:}b^{(1)}_{-l}b^{(1)}_{l} + b^{(2)}_{-l}b^{(2)}_{l}{:} = \frac{\kappa\xi}{4}\sum_{l\in\mathbf{Z}}{:}\mathrm{W}_{1,-l}\mathrm{W}_{1,l}{:} + \xi\mathrm{W}_{2,0} - \frac{\xi^2}{2}\big(b^{(1)}_{0} - b^{(2)}_{0}\big).$$

The lemma follows, the constant c being given by

$$(\mathbf{8.69})\qquad c = p_3(\vec{\varepsilon})/6\kappa + p_2(\vec{\varepsilon})\xi/4\kappa - p_1(\vec{\varepsilon})\xi^2/2\kappa + \xi^3/12\kappa. \qquad \square$$

Remark **8.21.** — Lemma 8.20 yields $\rho^{(1^2)}(\mathrm{D}_{0,2}) = \mathrm{V}^{(1^2)} + \kappa\xi\sum_{l\geq 1} l\,\mathrm{W}_{1,-l}\mathrm{W}_{1,l}/2$ where $\mathrm{V}^{(1^2)}$ is a linear combination of Fourier coefficients of fields of the vertex algebra $\mathrm{W}_{\kappa-2}(\mathfrak{gl}_2)$. An easy computation using Theorem 7.9 yields

$$\mathrm{V}^{(1^2)} = \mathrm{V}^{(1)} + \mathrm{V}^{(2)} + \kappa\xi\sum_{l\in\mathbf{Z}} l\, b^{(2)}_{-l}b^{(1)}_{l}/2,$$

where $\mathrm{V}^{(i)}$ denotes the operator $\mathrm{V}^{(1)}$ in Remark 8.16 acting on the ith spot of $\pi^{(1^2)} = \pi^{(1)}\otimes\pi^{(1)}$. So, we have $\mathrm{V}^{(1^2)} = v_{(2)}$ for some element $v\in\mathrm{W}_{\kappa-2}(\mathfrak{gl}_2)$ of degree 3. Finally, note that the infinite sum $\sum_{l\geq 1} l\,\mathrm{W}_{1,-l}\mathrm{W}_{1,l}/2$ belongs to $\mathscr{U}(\mathrm{W}_{\kappa-2}(\mathfrak{gl}_2))$.

8.10. *From* $\mathbf{SH}^{(r)}_{\mathrm{K}}$ *to* $\mathrm{W}_k(\mathfrak{gl}_r)$. — Now r is arbitrary. We set $\mathbf{k} = \mathrm{K}_r$ and $\kappa = k + r$. We write

$$(\mathbf{8.70})\qquad \pi^{(1^r)} = \pi_\beta, \qquad \big\langle b^{(i)}, \beta\big\rangle = -\varepsilon_i/\kappa + (i-1)\xi/\kappa, \quad i\in[1,r].$$

Recall that $\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ is the image of $\mathfrak{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ in $\mathrm{End}(\pi^{(1^r)})$. We construct as in (8.57) a K_r-linear isomorphism

$$\mathbf{L}_{\mathrm{K}}^{(1^r)} = \pi^{(1^r)} = \Lambda_{\mathrm{K}_r}^{\otimes r} \tag{8.71}$$

which identifies $[\mathrm{I}_\emptyset]^{\otimes r}$, $|\beta\rangle$ and $1^{\otimes r}$ and which intertwines the operator $b_{-l}^{(i)}$ on $\pi^{(1^r)}$ with

$$1 \otimes \cdots \otimes 1 \otimes b_{-l} \otimes 1 \otimes \cdots \otimes 1 \quad (b_{-l} \text{ is at the } i\text{th spot}) \tag{8.72}$$

on $\Lambda_{\mathrm{K}_r}^{\otimes r}$ and with the operator on $\mathbf{L}_{\mathrm{K}}^{(1^r)}$ given by

$$1 \otimes \cdots \otimes 1 \otimes \rho^{(1)}(\mathrm{D}_{l,0}) \otimes 1 \otimes \cdots \otimes 1. \tag{8.73}$$

Propositions 8.3, 8.10 then provide inclusions of $\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ and $\mathbf{SH}_{\mathrm{K}}^{(r)}$ into $\mathrm{End}(\pi^{(1^r)})$. We equip $\mathbf{SH}_{\mathrm{K}}^{(r)}$, $\mathfrak{U}(\mathbf{SH}_{\mathrm{K}}^{(r)})$ and $\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ with the standard degreewise topologies.

Theorem **8.22.** — *The representation $\rho^{(1^r)}$ yields an embedding of degreewise topological K_r-algebras $\Theta^{(r)} : \mathbf{SH}_{\mathrm{K}}^{(r)} \to \mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ with a degreewise dense image. The morphism $\Theta^{(r)}$ is compatible with the order filtrations.*

The theorem is a direct consequence of Lemmas 8.23, 8.26 below. Note that the map $\Theta^{(r)}$ is homogeneous of degree zero relatively to the rank degree on $\mathbf{SH}_{\mathrm{K}}^{(r)}$ and the conformal degree on $\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$.

Lemma **8.23.** — (a) *We have $\rho^{(1^r)}(\mathrm{D}_{0,2}) = \mathrm{V}^{(1^r)} + \kappa\xi \sum_{l\geq 1} l\,\mathrm{W}_{1,-l}\mathrm{W}_{1,l}/2$, where $\mathrm{V}^{(1^r)}$ is a Fourier coefficient of a field of $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$.*

(b) *The representation $\rho^{(1^r)}$ yields an embedding of degreewise topological K_r-algebras $\Theta^{(1^r)} : \mathbf{SH}_{\mathrm{K}}^{(r)} \to \mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$.*

Proof. — Part (b) is a consequence of (a), because $\mathbf{SH}_{\mathrm{K}}^{(r)}$ is generated by the elements $\mathrm{D}_{0,2}$, b_l, $l \in \mathbf{Z}$, because the infinite sum $\sum_{l\geq 1} l\,\mathrm{W}_{1,-l}\mathrm{W}_{1,l}/2$ belongs to $\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ and because $\rho^{(1^r)}$ takes the formal series $b(z) \in \mathbf{SH}_{\mathrm{K}}^{(r)}[[z^{-1}, z]]$ to $\mathrm{W}_1(z)$, viewed as a field in $(\mathrm{End}\,\pi^{(1^r)})[[z^{-1}, z]]$.

Let us concentrate on part (a). The cases $r = 1, 2$ have been considered in Remarks 8.16, 8.21. So, we may assume that $r \geq 2$. Theorem 7.9 yields

$$\rho^{(1^r)}(\mathrm{D}_{0,2}) = \sum_{i=1}^{r} \mathrm{D}_{0,2}^{(i)} + \kappa\xi \sum_{i<j} \sum_{l\geq 1} l\, b_{-l}^{(j)} b_l^{(i)},$$

where $\mathrm{D}_{0,2}^{(i)}$ is the operator $\rho^{(1)}(\mathrm{D}_{0,2})$ acting on the ith spot of $\pi^{(1^r)} = (\pi^{(1)})^{\otimes r}$. Let $\mathrm{V}^{(i)} \in \mathrm{End}(\pi^{(1^r)})$ denote the operator $\mathrm{V}^{(1)}$ in Remark 8.16 acting on the ith spot of $\pi^{(1^r)} =$

$(\pi^{(1)})^{\otimes r}$. Then, a short computation yields

$$\rho^{(1^r)}(\mathrm{D}_{0,2}) = \mathrm{V}^{(1^r)} + \kappa\xi \sum_{l\geq 1} l\,\mathrm{W}_{1,-l}\mathrm{W}_{1,l}/2, \tag{8.74}$$

$$\mathrm{V}^{(1^r)} = \sum_{i=1}^{r} \mathrm{V}^{(i)} + \kappa\xi \sum_{i<j}\sum_{l\in\mathbf{Z}} l\, b_{-l}^{(j)} b_{l}^{(i)}/2.$$

Recall that under the quantum Miura transform we can view $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$ as a vertex subalgebra of $\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{\otimes r}$. Set $\mathrm{W}^{[i]} = \mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{\otimes(i-1)} \otimes \mathrm{W}_{\kappa-2}(\mathfrak{gl}_2) \otimes \mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{\otimes(r-i-1)}$. We have the following classical result due to Feigin and Frenkel.[2]

Theorem **8.24**. — *We have the equality* $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r) = \bigcap_{i=1}^{r-1} \mathrm{W}^{[i]}$ *in* $\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{\otimes r}$.

This is a direct corollary of the characterization of $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$ as the intersection of screening operators associated with the simple roots of $\mathfrak{gl}_r$, see [15, Thm. 4.6.9]. The above formulation appears in [16, Sect. 15.4.15].

Therefore, the part (a) is a consequence of the decomposition (8.74) and of the following.

Claim. — *There is an element* $w \in \bigcap_i \mathrm{W}^{[i]}$ *such that* $\mathrm{V}^{(1^r)} = w_{(2)}$.

Note that

$$\mathrm{V}^{(1^r)} = \sum_{i=1}^{r} \mathrm{V}^{(i)} + \kappa\xi \sum_{i<j}\sum_{l\in\mathbf{Z}} l{:}b_{-l}^{(j)} b_{l}^{(i)}{:}/2. \tag{8.75}$$

Using this expression and Remark 8.16 it is easy to see that there exists an element $w \in \mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{\otimes r}$ such that $\mathrm{V}^{(1^r)} = w_{(2)}$. This element is uniquely determined and admits a unique expression of the form $w = p|0\rangle$ where $p \in \mathbf{C}[b_{-k}^{(i)}; k > 0,\ i \in [1,r]]$. We must check that $w \in \bigcap_i \mathrm{W}^{[i]}$. For each i we can write $\mathrm{V}^{(1^r)} = \mathrm{A}_i + \mathrm{B}_i$ with

$$\mathrm{A}_i = \mathrm{V}^{(i,i+1)} + \sum_{j\neq i,i+1} x_j\kappa\xi \sum_{l\in\mathbf{Z}} l\Big(\sum_{i+1<j} b_{-l}^{(j)}\big(b_l^{(i)} + b_l^{(i+1)}\big) + \sum_{j<i}\big(b_{-l}^{(i)} + b_{-l}^{(i+1)}\big)\, b_l^{(j)}\Big)/2, \tag{8.76}$$

$$\mathrm{B}_i = \sum_{j\neq i,i+1} \mathrm{V}^{(j)} + \kappa\xi \sum_{l\in\mathbf{Z}} \sum_{\substack{j<k\\ j,k\neq i,i+1}} l\, b_{-l}^{(k)} b_l^{(j)}/2,$$

where $\mathrm{V}^{(i,i+1)}$ denotes the operator $\mathrm{V}^{(1^2)}$ in Remark 8.21 acting on the $(i,i+1)$th spot. Note that $b_{-1}^{(i)}|0\rangle + b_{-1}^{(i+1)}|0\rangle \in \mathrm{W}^{[i]}$. Thus, we have $\mathrm{A}_i = v_{(2)}^{(i,i+1)} + \sum_{j\neq i,i+1}(x_j)_{(2)}$ where

[2] Feigin-Frenkel's theorem is also used in the approach by A. Okounkov and D. Maulik, see [28, Sect. 19.2].

the element $v^{(i,i+1)}$ is the element v from Remark 8.21 at the $(i,i+1)$th spot and $x_j \in \mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{(j)} \otimes \mathrm{W}_{\kappa-2}(\mathfrak{gl}_2)^{(i,i+1)}$. Similarly, we have $\mathrm{B}_i = \sum_{j\neq i,i+1} u^{(j)}_{(2)} + \sum_{j<k,j,k\neq i,i+1}(x_{j,k})_{(2)}$ where the element $u^{(i)}$ is the element u from Remark 8.16 and $x_{j,k} \in \mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{(j)} \otimes \mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)^{(k)}$. Therefore we have $w \in \mathrm{W}^{[i]}$. □

Remark **8.25.** — Now, for each $i \in [1,r)$ we consider the composition $\omega_i = (1,\dots,1,2,1,\dots,1)$ of r where 2 is at the ith spot. Let $\Delta^{[i]} : \mathbf{SH}^{(r)}_{\mathrm{K}} \to \mathbf{SH}^{\omega_i}_{\mathrm{K}}$ be the K_r-algebra homomorphism given by the iterated coproduct. We can identify $\pi^{(1^r)}$ with the tensor product $\pi^{[i]} = \pi^{(1)} \otimes \cdots \otimes \pi^{(1)} \otimes \pi^{(1^2)} \otimes \pi^{(1)} \otimes \cdots \otimes \pi^{(1)}$ in the obvious way. Let $\rho^{[i]}$ be the representation of $\mathbf{SH}^{\omega_i}_{\mathrm{K}}$ on $\pi^{(1^r)}$ given by $\rho^{[i]} = \rho^{(1)} \otimes \cdots \otimes \rho^{(1)} \otimes \rho^{(1^2)} \otimes \rho^{(1)} \otimes \cdots \otimes \rho^{(1)}$. The coassociativity of the coproduct implies that $\rho^{(1^r)} = \rho^{[i]} \circ \Delta^{[i]}$. By Propositions 8.15, 8.19 the representations $\rho^{(1)}$ and $\rho^{(1^2)}$ give inclusions

$$\mathbf{SH}^{(1)}_{\mathrm{K}} \subset \mathscr{U}\big(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)\big), \qquad \mathbf{SH}^{(2)}_{\mathrm{K}} \subset \mathscr{U}\big(\mathrm{W}_{\kappa-2}(\mathfrak{gl}_2)\big). \tag{8.77}$$

Therefore, for each i, the decomposition $\rho^{(1^r)} = \rho^{[i]} \circ \Delta^{[i]}$ gives an inclusion $\mathbf{SH}^{(r)}_{\mathrm{K}} \subset \mathscr{U}^{[i]}$ where

$$\mathscr{U}^{[i]} = \mathscr{U}\big(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)\big)^{\widehat{\otimes}(i-1)} \,\widehat{\otimes}\, \mathscr{U}\big(\mathrm{W}_{\kappa-2}(\mathfrak{gl}_2)\big) \,\widehat{\otimes}\, \mathscr{U}\big(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)\big)^{\widehat{\otimes}(r-i-1)}. \tag{8.78}$$

Theorem 8.24 gives an inclusion $\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)) \subset \bigcap_{i=1}^{r-1} \mathscr{U}^{[i]}$. We do not know if this inclusion is an equality. So, we can not deduce that $\mathbf{SH}^{(r)}_{\mathrm{K}} \subset \mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ from this. This explains the need for the more precise computations in the proof above.

Lemma **8.26.** — *The inclusion $\Theta^{(r)}$ gives a surjective morphism of degreewise topological K_r-algebras $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}}) \to \mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ which is compatible with the order filtrations.*

Proof. — By the universal property of completions, the inclusion $\mathbf{SH}^{(r)}_{\mathrm{K}}[s] \to \mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))[s]$, which is a continuous map, extends uniquely to a continuous map $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})[s] \to \mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))[s]$, for each integer s. Taking the sum over all s we get a map

$$\Theta^{(r)} : \mathfrak{U}\big(\mathbf{SH}^{(r)}_{\mathrm{K}}\big) \to \mathscr{U}\big(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)\big). \tag{8.79}$$

It is a morphism of degreewise topological K_r-algebras. We must prove that it is surjective. We have already seen that $\Theta^{(r)}(b(z)) = \mathrm{W}_1(z)$. We now consider the fields $\mathrm{W}_d(z)$ with $d > 1$. The quantum Miura transform yields an embedding

$$\mathscr{U}\big(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)\big) \subset \mathscr{U}\big(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1)\big)^{\widehat{\otimes} r}. \tag{8.80}$$

By Propositions 8.10, 8.15, the representation $\rho^{(1^r)}$ yields a map $\mathfrak{U}(\mathbf{SH}^{(r)}_{\mathrm{K}}) \to \mathscr{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1))^{\widehat{\otimes} r}$. Let $\mathscr{U}(\mathbf{SH}^{(r)}_{\mathrm{K}})$ be its image. The standard filtration on $\mathscr{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1))$ introduced in Section 8.8 induces the standard filtrations

$$\mathscr{U}\big(\mathbf{SH}^{(r)}_{\mathrm{K}}\big) = \bigcup_d \mathscr{U}\big(\mathbf{SH}^{(r)}_{\mathrm{K}}\big)[\preccurlyeq d], \qquad \mathscr{U}\big(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)\big) = \bigcup_d \mathscr{U}\big(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)\big)[\preccurlyeq d]. \tag{8.81}$$

Proposition 8.2 yields the following.

Claim **8.27.** — *For $d \neq 1$, under the inclusion* (8.80) *we have*

$$\mathrm{W}_d(z) = -\kappa \sum_{s=0}^{d} (-r)^{s-d} \binom{r-s}{r-d} \sum_{i_1 < i_2 < \cdots < i_s} {:}\mathrm{J}(z)^{d-s} b^{(i_1)}(z) b^{(i_2)}(z) \cdots b^{(i_s)}(z){:}$$

modulo terms of standard order $\preccurlyeq d-1$ *in* $\mathfrak{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1))^{\widehat{\otimes} r}[[z, z^{-1}]]$.

Recall the elements $\mathrm{Y}_{r,d}$ defined in (1.82).

Claim **8.28.** — *For l, d with $d \geq 0$ there is a constant $c(l, d) \neq 0$ such that*

$$\rho^{(1^r)}(\mathrm{Y}_{l,d}) \equiv c(l, d) \sum_{i=1}^{r} \sum_{l_0, \ldots, l_d} {:}b^{(i)}_{l_0} \cdots b^{(i)}_{l_d}{:}. \tag{8.82}$$

The sum runs over all tuples of integers with sum $-l$. The symbol $\equiv$ means that the equality holds modulo terms of standard order $\preccurlyeq d$ in $\mathfrak{U}(\mathrm{W}_{\kappa-1}(\mathfrak{gl}_1))^{\widehat{\otimes} r}[[z, z^{-1}]]$.

Proof. — First, we prove the following estimate

$$\rho^{(1^r)}(\mathrm{Y}_{l,d}) \equiv \delta^{r-1}\big(\rho^{(1)}(\mathrm{Y}_{l,d})\big). \tag{8.83}$$

Equation (8.83) is clear from the definition of the coproduct on $\mathbf{SH}^{\mathbf{c}}$ for $d = 0, 1$ or for $l = 0$, $d = 2$. Next, the operator $\mathrm{ad}(\rho^{(1^r)}(\mathrm{D}_{0,2}))$ increases the standard order by at most one, see e.g., formula (8.59) in the case $r = 2$. Hence using relations

$$\mathrm{ad}(\mathrm{D}_{0,2})^d(\mathrm{D}_{1,0}) = \mathrm{D}_{1,d}, \qquad \mathrm{ad}(\mathrm{D}_{0,2})^d(\mathrm{D}_{-1,0}) = (-1)^d \mathrm{D}_{-1,d} \tag{8.84}$$

we deduce (8.83) for $l = \pm 1$. Likewise, the operator $\mathrm{ad}(\rho^{(1^r)}(\mathrm{D}_{\pm 1,1}))$ preserves the standard filtration and the operator $\mathrm{ad}(\rho^{(1^r)}(\mathrm{D}_{1,0}))$ decreases the standard filtration by one. This implies that (8.83) holds for $l \neq 0$. Thus we have also

$$\rho^{(1^r)}(\mathrm{E}_d) \equiv \delta^{r-1}\big(\rho^{(1)}(\mathrm{E}_d)\big). \tag{8.85}$$

This implies that (8.83) also holds for $\mathrm{D}_{0,d}$ for any d.

Next, combining (8.83) and Proposition 8.17 yields (8.82) for $l = 0$ with

$$c(0, d) = \frac{\kappa^d}{d(d+1)}. \tag{8.86}$$

Finally, acting by

$$\begin{aligned} &\mathrm{ad}\big(\rho^{(1^r)}(\mathrm{D}_{\pm 1,1})\big) \equiv \mathrm{ad}\big(\delta^{r-1}\big(\rho^{(1)}(\mathrm{D}_{\pm 1,1})\big)\big), \\ &\mathrm{ad}\big(\rho^{(1^r)}(\mathrm{D}_{\pm 1,0})\big) \equiv \mathrm{ad}\big(\delta^{r-1}\big(\rho^{(1)}(\mathrm{D}_{\pm 1,0})\big)\big) \end{aligned} \tag{8.87}$$

now yields (8.82) for all values of l, d. We are done. Note that, since (1.91) implies that

$$[b_l, D_{1,1}] = lb_{l-1}, \qquad [b_l, D_{-1,1}] = \kappa\, lb_{l+1}, \tag{8.88}$$

we get

$$c(1,d) = \kappa^d/(d+1), \qquad c(-1,d) = -\kappa^{d+1}/(d+1). \tag{8.89}$$ □

By Lemma 8.23, we have $\mathrm{Im}(\Theta^{(r)}) \subseteq \mathscr{U}(W_{\kappa-r}(\mathfrak{gl}_r))$. Using Claims 8.27 and 8.28 we see that the associated graded of $\mathrm{Im}(\Theta^{(r)})$ and $\mathscr{U}(W_{\kappa-r}(\mathfrak{gl}_r))$ with respect to the standard filtration are equal. This implies that

$$\mathrm{Im}\big(\Theta^{(r)}\big) = \mathscr{U}\big(W_{\kappa-r}(\mathfrak{gl}_r)\big). \tag{8.90}$$

To finish, we prove the compatibility of $\Theta^{(r)}$ with the order filtration $\mathscr{U}(W_{\kappa-r}(\mathfrak{gl}_r))[\leq d]$ defined in Section 8.3. Recall the filtration $\mathscr{U}(W_{\kappa-r}(\mathfrak{gl}_r))[\preccurlyeq d]$ defined in (8.81). By Claims 8.27 and 8.28, there exists for any l, d an explicit element

$$u_{l,d} \in \mathscr{U}\big(W_{\kappa-r}(\mathfrak{gl}_r)\big)[\leq d] \tag{8.91}$$

such that

$$\Theta^{(r)}(D_{l,d}) - u_{l,d} \in \mathscr{U}\big(W_{\kappa-r}(\mathfrak{gl}_r)\big)[\preccurlyeq d]. \tag{8.92}$$

But from the definition of the filtrations, we have

$$\mathscr{U}\big(W_{\kappa-r}(\mathfrak{gl}_r)\big)[\preccurlyeq d] \subseteq \mathscr{U}\big(W_{\kappa-r}(\mathfrak{gl}_r)\big)[< d]. \tag{8.93}$$

By Remark 8.12, the order filtration on $\mathfrak{U}(\mathbf{SH}_K^{(r)})$ is determined by putting $D_{r,d}$ (or equivalently $Y_{r,d}$) in degree d for any (r,d). Thus Lemma 8.26 is proved. □

Theorem 8.22 has the following consequence.

Corollary **8.29.** — *The pull-back by the morphism $\Theta^{(r)} : \mathbf{SH}_K^{(r)} \to \mathscr{U}(W_{\kappa-r}(\mathfrak{gl}_r))$ is an equivalence from the category of admissible $\mathscr{U}(W_{\kappa-r}(\mathfrak{gl}_r))$-modules to the category of admissible $\mathbf{SH}_K^{(r)}$-modules. This equivalence takes $\pi^{(1^r)}$ to $\rho^{(1^r)}$.*

Proof. — Since the image of $\mathbf{SH}_K^{(r)}$ in $\mathscr{U}(W_{\kappa-r}(\mathfrak{gl}_r))$ is degreewise dense, this functor is fully faithful. Thus, it is enough to check that it is essentially surjective. To do that, let M be an admissible $\mathbf{SH}_K^{(r)}$-module. View $\mathbf{SH}_K^{(r)}$ as a degreewise dense degreewise topological subalgebra of $\mathscr{U}(W_{\kappa-r}(\mathfrak{gl}_r))$. Then, for any s, s' the action map

$$\mathbf{SH}_K^{(r)}[s] \times M\big[s'\big] \to M\big[s+s'\big] \tag{8.94}$$

extends uniquely to a continuous map

$$\mathscr{U}\big(W_{\kappa-r}(\mathfrak{gl}_r)\big)[s] \times M\big[s'\big] \to M\big[s+s'\big]. \tag{8.95}$$

This yields an admissible $\mathscr{U}(W_{\kappa-r}(\mathfrak{gl}_r))$-module structure on M. The corollary follows. □

8.11. *The Virasoro field.* — We set $\mathbf{k} = \mathrm{K}_r$ and $\kappa = k + r$. Now, we describe the preimage under the map $\Theta^{(r)}$ in Theorem 8.22 of the Virasoro field $\mathrm{W}_2(z)$. We keep all the conventions of the previous section. We have introduced in (1.89) some elements b_l, H_l. Consider the fields in $\mathbf{SH}^{(r)}_{\mathrm{K}}[[z, z^{-1}]]$ given by

$$\mathrm{H}(z) = \sum_{l\in\mathbf{Z}} \mathrm{H}_l\, z^{-l-2}, \qquad b(z) = \sum_{l\in\mathbf{Z}} b_l z^{-l-1}. \tag{8.96}$$

Recall the field $\rho(z)$ in $\mathrm{End}(\pi^{(1^r)})[[z^{-1}, z]]$ given by

$$\rho(z) = \sum_{i=1}^{r} (r/2 - i + 1/2)\, b^{(i)}(z). \tag{8.97}$$

Proposition **8.30.** — *We have the following equalities*

$$\rho^{(1^r)}\big(b(z)\big) = \mathrm{J}(z), \qquad \rho^{(1^r)}\big(\mathrm{H}(z)\big) = \frac{\kappa}{2}\sum_i {:}b^{(i)}(z)^2{:} - \xi\,\partial_z\rho(z).$$

Proof. — The first claim is obvious. Note, indeed, that we have

$$\rho^{(1^r)}(b_0) = -p_1(\varepsilon_1, \dots, \varepsilon_r)/\kappa + r(r-1)\xi/2\kappa = \sum_{i=1}^{r}\big\langle b^{(i)}, \beta\big\rangle. \tag{8.98}$$

Let us concentrate on the second one. For $k \geq 1$ we set

$$\mathrm{H}'_k = \mathrm{H}_k + (r-1)(k-1)\xi\, b_k/2, \qquad \mathrm{H}'_{-k} = \mathrm{H}_{-k} + (r-1)(k-1)\xi\, b_{-k}/2.$$

We must prove the following formulas

$$\begin{aligned}
&\rho^{(1^r)}(\mathrm{H}_0) = \kappa\sum_i\sum_{l\geq 1} b^{(i)}_{-l} b^{(i)}_l + \kappa\sum_i \big(b^{(i)}_0\big)^2/2 + \xi\rho_0,\\
&\rho^{(1^r)}\big(\mathrm{H}'_{-k}\big) = \kappa\sum_i\sum_l b^{(i)}_{-k-l} b^{(i)}_l/2 - (k-1)\xi\rho_{-k} + (r-1)(k-1)\xi\mathrm{J}_{-k}/2,\\
&\rho^{(1^r)}\big(\mathrm{H}'_{k}\big) = \kappa\sum_i\sum_l b^{(i)}_{k-l} b^{(i)}_l/2 + (k+1)\xi\rho_{k} + (r-1)(k-1)\xi\mathrm{J}_{k}/2.
\end{aligned} \tag{8.99}$$

Write

$$\mathrm{H}^{(i)}_k = 1\otimes 1\otimes\cdots\otimes 1\otimes\rho^{(1)}(\mathrm{H}_k)\otimes 1\otimes\cdots\otimes 1, \qquad i\in[1, r], \tag{8.100}$$

where H_k is at the ith spot. We have

$$\mathrm{H}'_k = \kappa^{-k}\mathrm{D}_{-k,1}/k + (1-k)\xi\, b_k/2, \qquad \mathrm{H}'_{-k} = \mathrm{D}_{k,1}/k + (1-k)\xi\, b_{-k}/2. \tag{8.101}$$

Thus, Theorem 7.9 yields

$$\begin{aligned}\rho^{(1^r)}(\mathrm{H}'_{-k}) &= \sum_i \mathrm{H}^{(i)}_{-k} + k\xi\sum_i (i-1)b^{(i)}_{-k},\\ \rho^{(1^r)}(\mathrm{H}'_{k}) &= \sum_i \mathrm{H}^{(i)}_{k} + k\xi\sum_i (r-i)b^{(i)}_{k}.\end{aligned} \tag{8.102}$$

Proposition 8.14 now yields

$$\begin{aligned}\rho^{(1^r)}(\mathrm{H}'_{-k}) &= \kappa\sum_i\sum_{l\neq 0,-k} b^{(i)}_{-k-l}b^{(i)}_l/2 + \sum_i\big(-\varepsilon_i + k(i-1)\xi\big)b^{(i)}_{-k},\\ &= \kappa\sum_i\sum_l b^{(i)}_{-k-l}b^{(i)}_l/2 + (k-1)\xi\sum_i (i-1)b^{(i)}_{-k},\\ &= \kappa\sum_i\sum_l b^{(i)}_{-k-l}b^{(i)}_l/2 - (k-1)\xi\rho_{-k} + (r-1)(k-1)\xi \mathrm{J}_{-k}/2,\\ \rho^{(1^r)}(\mathrm{H}'_{k}) &= \kappa\sum_i\sum_{l\neq 0,k} b^{(i)}_{k-l}b^{(i)}_l/2 + \sum_i\big(-\varepsilon_i + k(r-i)\xi\big)b^{(i)}_{k},\\ &= \kappa\sum_i\sum_l b^{(i)}_{k-l}b^{(i)}_l/2 + \xi\sum_i\big((k-1)(r-1)\\ &\quad + (k+1)(r-2i+1)\big)b^{(i)}_k/2,\\ &= \kappa\sum_i\sum_l b^{(i)}_{k-l}b^{(i)}_l/2 + (k+1)\xi\rho_k + (r-1)(k-1)\xi \mathrm{J}_k/2.\end{aligned} \tag{8.103}$$

We have $[\mathrm{H}^{(i)}_k, b^{(i)}_l] = -lb^{(i)}_{k+l}$. Therefore, we get

$$\begin{aligned}\rho^{(1^r)}(\mathrm{H}_0) &= \sum_i \mathrm{H}^{(i)}_0 + \xi\sum_i (i-1)\big[\mathrm{H}^{(i)}_1, b^{(i)}_{-1}\big]/2 - \xi\sum_i (r-i)\big[\mathrm{H}^{(i)}_{-1}, b^{(i)}_1\big]/2\\ &\quad + \xi^2\sum_i (r-i)(i-1)\big[b^{(i)}_1, b^{(i)}_{-1}\big]/2\\ &= \sum_i \mathrm{H}^{(i)}_0 - (r-1)\xi\sum_i \varepsilon_i/2\kappa + \xi^2\sum_i (r-i)(i-1)/2\kappa\\ &= \kappa\sum_i\sum_{l\geq 1} b^{(i)}_{-l}b^{(i)}_l + \sum_i\big(\varepsilon_i-(i-1)\xi\big)\big(\varepsilon_i-(r-i)\xi\big)/2\kappa\\ &= \kappa\sum_i\sum_{l\geq 1} b^{(i)}_{-l}b^{(i)}_l - \sum_i b^{(i)}_0\big(\varepsilon_i-(r-i)\xi\big)/2\\ &= \kappa\sum_i\sum_{l\geq 1} b^{(i)}_{-l}b^{(i)}_l + \kappa\sum_i\big(b^{(i)}_0\big)^2/2 + \xi\sum_i b^{(i)}_0(r-2i+1)/2\\ &= \kappa\sum_i\sum_{l\geq 1} b^{(i)}_{-l}b^{(i)}_l + \kappa\sum_i\big(b^{(i)}_0\big)^2/2 + \xi\rho_0.\end{aligned} \tag{8.104}$$

$\square$

The fields $\mathrm{W}_1(z)$ and $\mathrm{W}_2(z)$ give two fields in $\mathrm{End}(\pi^{(1^r)})[[z^{-1},z]]$. Let us denote them again by $\mathrm{W}_1(z)$ and $\mathrm{W}_2(z)$. Consider the field $\mathrm{L}(z)$ in $\mathfrak{U}(\mathbf{SH}_{\mathrm{K}}^{(r)})[[z^{-1},z]]$ given by

$$\mathrm{L}(z) = \mathrm{H}(z) - \frac{\kappa}{2r}{:}b(z)^2{:}. \tag{8.105}$$

Proposition 8.30 and (8.16) imply that

$$\rho^{(1^r)}\big(b(z)\big) = \mathrm{W}_1(z), \qquad \rho^{(1^r)}\big(\mathrm{L}(z)\big) = \mathrm{W}_2(z). \tag{8.106}$$

Therefore, by definition of the map $\Theta^{(r)}$, we have the following.

Corollary **8.31.** — *We have* $\Theta^{(r)}(b(z)) = \mathrm{W}_1(z)$ *and* $\Theta^{(r)}(\mathrm{L}(z)) = \mathrm{W}_2(z)$.

8.12. *The representation* $\pi^{(r)}$ *of* $\mathrm{W}_k(\mathfrak{gl}_r)$ *on* $\mathbf{L}_{\mathrm{K}}^{(r)}$. — We set $\mathbf{k} = \mathrm{K}_r$ and $\kappa = k + r$. Let β be as in (8.70). The representation $\rho^{(r)}$ of $\mathbf{SH}_{\mathrm{K}}^{(r)}$ on $\mathbf{L}_{\mathrm{K}}^{(r)}$ is admissible.

Definition **8.32.** — *Let* $\pi^{(r)}$ *be the unique admissible representation of* $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$ *which is taken to* $\rho^{(r)}$ *by the equivalence of categories in Corollary 8.29.*

By Corollary 8.31 we have

$$\rho^{(r)}\big(b(z)\big) = \pi^{(r)}\big(\mathrm{W}_1(z)\big), \qquad \rho^{(r)}\big(\mathrm{L}(z)\big) = \pi^{(r)}\big(\mathrm{W}_2(z)\big). \tag{8.107}$$

Write $|0\rangle$ for the element $[\mathrm{I}_\emptyset]$ of $\mathbf{L}_{\mathrm{K}}^{(r)}$. Write $|\beta\rangle$ for the rth tensor power of the element $[\mathrm{I}_\emptyset]$ in $\mathbf{L}_{\mathrm{K}}^{(1)}$. We view $|\beta\rangle$ as an element of $\mathbf{L}_{\mathrm{K}}^{(1^r)}$. The following is one of the main results of this paper.

Theorem **8.33.** — *The representation* $\pi^{(r)}$ *of* $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$ *on* $\mathbf{L}_{\mathrm{K}}^{(r)}$ *is isomorphic to the Verma module whose highest weight is given by the following rules*

$$\pi^{(r)}(\mathrm{W}_{d,0})|0\rangle = w_d|0\rangle, \qquad \pi^{(r)}(\mathrm{W}_{d,l})|0\rangle = 0, \quad l \geq 1,$$

$$w_1 = \sum_{i=1}^{r}\langle b^{(i)},\beta\rangle, \qquad w_d = -\kappa \sum_{i_1<i_2<\cdots<i_d}\prod_{t=1}^{d}\big(\langle h^{(i_t)},\beta\rangle + (d-t)\xi/\kappa\big), \quad d \geq 2.$$

This Verma module is irreducible. Further, for $l \geq 0$ *and* $d \in [2,r]$ *we have*

$$\pi^{(r)}(\mathrm{W}_{1,-l})^* = (-1)^{rl}\pi^{(r)}(\mathrm{W}_{1,l}), \qquad \pi^{(r)}(\mathrm{W}_{d,-l})^* = (-1)^{rl+d}\pi^{(r)}(\mathrm{W}_{d,l}). \tag{8.108}$$

Proof. — The homomorphism $\Theta^{(r)} : \mathbf{SH}_{\mathrm{K}}^{(r)} \to \mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ is compatible with the $\mathbf{Z}$-gradings. Therefore $\mathbf{L}_{\mathrm{K}}^{(r)}$ is an $\mathbf{N}$-graded $\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$-module. Thus $|0\rangle$ is a highest weight vector of $\mathbf{L}_{\mathrm{K}}^{(r)}$, because it has the degree 0. Next, we must prove that $|0\rangle$ is a generator of $\mathbf{L}_{\mathrm{K}}^{(r)}$ over $\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$. Since $\mathbf{L}_{\mathrm{K}}^{(r)}$ is admissible and $\mathbf{SH}_{\mathrm{K}}^{(r)}$ is degreewise dense in $\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$, it is enough to prove the following.

Lemma **8.34.** — *We have* $\mathbf{L}_{\mathrm{K}}^{(r)} = \rho^{(r)}(\mathbf{SH}_{\mathrm{K}}^{(r)})|0\rangle$.

Proof. — We must check that $[\mathrm{I}_\lambda]$ belongs to the right hand side for each λ. We proceed by induction on the weight $|\lambda|$ of the r-partition λ. Assume that $|\lambda| = n$ and that $[\mathrm{I}_\mu]$ belongs to $\rho^{(r)}(\mathbf{SH}_{\mathrm{K}}^{(r)})|0\rangle$ whenever $|\mu| < n$. The formulas from Section D imply that there is an r-partition μ of $n-1$ such that the coefficient of $[\mathrm{I}_\lambda]$ in $\rho^{(r)}(\mathrm{D}_{1,l})([\mathrm{I}_\mu])$ is non zero (in K_r) for some $l \in \mathbf{N}$. Next, we have

$$\rho^{(r)}(\mathrm{D}_{0,l+1})\big([\mathrm{I}_\lambda]\big) = \sum_a \sum_{s\in\lambda^{(a)}} \big(c_a(s)/x\big)^l\,[\mathrm{I}_\lambda], \qquad l \geq 0. \tag{8.109}$$

We can regard $[\mathrm{I}_\lambda]$ as the set

$$\big\{c_a(s)/x;\ a = 1,\ldots,r,\ s\in\lambda^{(a)}\big\}. \tag{8.110}$$

Then, the action of $\mathrm{D}_{0,l+1}$ on $[\mathrm{I}_\lambda]$ is simply the evaluation of the lth power sum polynomial on the K_r-point $[\mathrm{I}_\lambda]$ of $(\mathrm{K}_r)^n/\mathfrak{S}_n$. Since all these points are distinct, by Hilbert's Nullstellensatz, for each λ there is a polynomial f in the $\mathrm{D}_{0,l+1}$'s such that $f([\mathrm{I}_\lambda]) = 1$ and $f([\mathrm{I}_\sigma]) = 0$ for any r-partition σ of n different from λ. This finishes the proof. □

Next, the graded dimension of $\mathbf{L}_{\mathrm{K}}^{(r)}$ is given by the number of r-partitions. Therefore, the previous arguments imply that $\mathbf{L}_{\mathrm{K}}^{(r)}$ is a Verma module with highest weight vector $|0\rangle$. Now, let us compute the weight of $|0\rangle$. We claim that it is the same as the weight of the element $|\beta\rangle$ in $\mathbf{L}_{\mathrm{K}}^{(1^r)}$. The later has been computed in Proposition 8.3 because $\mathbf{L}_{\mathrm{K}}^{(1^r)}$ is isomorphic to $\pi^{(1^r)}$ as a $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$-module by Corollary 8.29. So the claim implies the first part of the theorem. To prove the claim observe first that we have

Lemma **8.35.** — (a) *We have* $\rho^{(r)}(\mathrm{D}_{\mathbf{x}})|0\rangle = 0$ *for* $\mathbf{x}\in\mathscr{E}^-$.
(b) *We have* $\rho^{(1^r)}(\mathrm{D}_{\mathbf{x}})|0\rangle = 0$ *for* $\mathbf{x}\in\mathscr{E}^-$.

Proof. — Part (a) follows from (3.18), and (b) from (a) and Lemma 7.11. □

Now, for each $d \geq 1$, we fix an element $\mathrm{W}'_{d,0}$ in $\mathfrak{U}(\mathbf{SH}_{\mathrm{K}}^{(r)})$ which is taken to $\mathrm{W}_{d,0}$ by the map $\Theta^{(r)}$ in Lemma 8.26. We must prove that it acts in the same way on the vacua of $\mathbf{L}_{\mathrm{K}}^{(r)}$ and $\mathbf{L}_{\mathrm{K}}^{(1^r)}$. By Proposition 1.37 the element $\mathrm{W}'_{d,0}$ is an infinite sum of monomials

$$\mathrm{D}_{k_1,l_1}\mathrm{D}_{k_2,l_2}\cdots\mathrm{D}_{k_r,l_r}, \qquad (k_s,l_s)\in\mathscr{E}, \qquad k_1+k_2+\cdots+k_r = 0, \tag{8.111}$$

where the $\mathrm{D}_{0,l}$'s and the $\mathrm{D}_{-1,l}$'s are on the right. Thus the claim follows from Lemma 8.35.

Now, we must check that $\rho^{(r)}$ is irreducible. It is enough to check that $\mathbf{L}_{\mathrm{K}}^{(r)}$ is irreducible as an $\mathbf{SH}_{\mathrm{K}}^{(r)}$-module. The bilinear form $(\bullet,\bullet)$ on $\mathbf{L}_{\mathrm{K}}^{(r)}$ is nondegenerate, because the elements $[\mathrm{I}_\lambda]$ form an orthogonal basis. Further, by Lemmas 8.34 and 8.35, we have

$$\mathbf{L}_{\mathrm{K}}^{(r)} = \mathrm{K}_r|0\rangle \oplus \rho^{(r)}\big(\mathbf{SH}_{\mathrm{K}}^{(r),>}\big)|0\rangle. \tag{8.112}$$

Thus, by Proposition 3.8, any element in $\mathbf{L}_{\mathrm{K}}^{(r)}$ which is killed by $\rho^{(r)}(\mathbf{SH}_{\mathrm{K}}^{(r),<})$ is proportional to $|0\rangle$. This implies that $\mathbf{L}_{\mathrm{K}}^{(r)}$ does not contain any proper $\mathbf{SH}_{\mathrm{K}}^{(r)}$-submodule.

Finally, we must prove (8.108). By Proposition 1.35 there is a unique anti-involution

$$\textbf{(8.113)}\qquad \omega:\mathbf{SH}_{\mathrm{K}}^{(r)}\to\mathbf{SH}_{\mathrm{K}}^{(r)},\qquad \mathrm{D}_{l,d}\mapsto(-1)^{(r-1)l}x^ly^l\mathrm{D}_{-l,d},\quad d,l\geq 0.$$

Further, by Proposition 3.8 we have

$$\textbf{(8.114)}\qquad \rho^{(r)}(u)^*=\rho^{(r)}\big(\omega(u)\big),\quad u\in\mathbf{SH}_{\mathrm{K}}^{(r)}.$$

Next, recall that $\mathbf{L}_{\mathrm{K}}^{(1^r)}=(\mathbf{L}_{\mathrm{K}_r}^{(1)})^{\otimes r}$ and that $\mathbf{L}_{\mathrm{K}}^{(1)}$ is equipped with the pairing in (3.23). Thus we can equip $\mathbf{L}_{\mathrm{K}}^{(1^r)}$ with the unique K_r-bilinear form such that

$$\textbf{(8.115)}\qquad (u_1\otimes\cdots\otimes u_r,\, v_r\otimes\cdots\otimes v_1)=(u_1,v_1)\cdots(u_r,v_r),\quad u_i,v_i\in\mathbf{L}_{\mathrm{K}_r}^{(1)}.$$

Let f^* denote the adjoint of a K_r-linear operator f on $\mathbf{L}_{\mathrm{K}}^{(1^r)}$ with respect to this pairing. Note that we used the same symbol for the adjoint with respect to the pairing on $\mathbf{L}_{\mathrm{K}}^{(r)}$ in Section 3.7. We claim that

$$\textbf{(8.116)}\qquad \rho^{(1^r)}(u)^*=\rho^{(1^r)}\big(\varpi(u)\big),\quad u\in\mathbf{SH}_{\mathrm{K}}^{(r)},$$

where ϖ is the anti-involution

$$\textbf{(8.117)}\qquad \varpi:\mathbf{SH}_{\mathrm{K}}^{(r)}\to\mathbf{SH}_{\mathrm{K}}^{(r)},\qquad \mathrm{D}_{l,d}\mapsto x^ly^l\mathrm{D}_{-l,d},\quad d,l\geq 0.$$

Indeed, it is enough to prove (8.116) for $u=\mathrm{D}_{l,0},\ \mathrm{D}_{0,2}$. Then, it follows from the formulas

$$\textbf{(8.118)}\qquad \begin{aligned}\rho^{(1^r)}(\mathrm{D}_{l,0})&=\sum_{i=1}^r\rho^{(1)}(\mathrm{D}_{l,0})^{(i)},\\ \rho^{(1^r)}(\mathrm{D}_{0,2})&=\sum_{i=1}^r\rho^{(1)}(\mathrm{D}_{0,2})^{(i)}+\xi\sum_{l\geq 1}\sum_{i<j}l\kappa^{1-l}\rho^{(1)}(\mathrm{D}_{-l,0})^{(i)}\rho^{(1)}(\mathrm{D}_{l,0})^{(j)}\end{aligned}$$

which are proved in Theorem 7.9, and from the formulas

$$\textbf{(8.119)}\qquad \rho^{(1)}(\mathrm{D}_{l,0})^*=x^ly^l\rho^{(1)}(\mathrm{D}_{-l,0}),\qquad \rho^{(1)}(\mathrm{D}_{0,2})^*=\rho^{(1)}(\mathrm{D}_{0,2}),$$

which in turn follow from (8.114). On the other hand, there is a unique anti-involution

$$\textbf{(8.120)}\qquad \begin{gathered}\varpi:\mathfrak{U}\big(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)\big)\to\mathfrak{U}\big(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)\big),\\ \mathrm{W}_{d,-l}\mapsto(-1)^{l+d}\mathrm{W}_{d,l},\qquad \mathrm{W}_{1,-l}\mapsto(-1)^l\mathrm{W}_{1,l},\quad d\geq 2,\ l\geq 0.\end{gathered}$$

By (8.71) we have $\mathbf{L}_{\mathrm{K}}^{(1^r)}=\pi^{(1^r)}$. Let $\pi^{(1^r)}$ denote also the map $\mathfrak{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))\to\mathrm{End}(\pi^{(1^r)})$. An easy computation using (8.15) yields

$$\textbf{(8.121)}\qquad \pi^{(1^r)}(u)^*=\pi^{(1^r)}\big(\varpi(u)\big),\qquad u\in\mathfrak{U}\big(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)\big).$$

Finally, by Corollary 8.29 we have

$$\rho^{(1^r)} = \pi^{(1^r)} \circ \Theta^{(r)}, \qquad \rho^{(r)} = \pi^{(r)} \circ \Theta^{(r)}, \tag{8.122}$$

and $\Theta^{(r)}$ is compatible with the rank grading on $\mathbf{SH}^{(r)}_{\mathrm{K}}$ and the conformal grading on $\mathfrak{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$. Therefore, comparing (8.114), (8.116) and (8.121), we get (8.108). □

9. The Gaiotto state

9.1. *The definition of the element* G. — Let $[\mathrm{M}_{r,n}]$ denote the fundamental class of $\mathrm{M}_{r,n}$. It is characterized, up to a scalar, by the fact that it lies in $\mathbf{L}^{(r)}_n$ and has the cohomological degree zero. Further, we have the following formula, consequence of the Atiyah-Bott localization theorem

$$[\mathrm{M}_{r,n}] = \sum_{\lambda} \mathrm{eu}_{\lambda}^{-1}[\mathrm{I}_{\lambda}], \tag{9.1}$$

where the sum runs over all r-partitions of size n. We define an element in $\widehat{\mathbf{L}}^{(r)}_{\mathrm{K}} = \prod_{n \geq 0} \mathbf{L}^{(r)}_n$ by

$$\mathrm{G} = \sum_{n \geq 0} [\mathrm{M}_{r,n}]. \tag{9.2}$$

Proposition **9.1.** — *The element* G *satisfies the following properties*

$$\rho^{(r)}(\mathrm{D}_{-l,d})(\mathrm{G}) = 0, \quad l \geq 1,\, d \in [0, r-2], \tag{9.3}$$

$$\rho^{(r)}(\mathrm{D}_{-1,r-1})(\mathrm{G}) = x^{-r}y^{-1}\mathrm{G}, \qquad \rho^{(r)}(\mathrm{D}_{-l,r-1})(\mathrm{G}) = 0, \quad l \geq 2, \tag{9.4}$$

$$\rho^{(r)}(\mathrm{D}_{-1,r})(\mathrm{G}) = -x^{-r}y^{-1}\Big(\sum_i \varepsilon_i\Big)\mathrm{G}. \tag{9.5}$$

Proof. — See Appendix G. □

Remark **9.2.** — It is not true that G is an eigenvector for the operators $\rho^{(r)}(\mathrm{D}_{-1,l})$ with $l > r$.

9.2. *The Whittaker condition for* G. — Now, we give a characterization of G using only the representation $\pi^{(r)}$ of $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$. Let χ be a character of the subalgebra of $\mathfrak{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ generated by $\mathrm{W}_{d,l}$ for $l \geq 1$ and $d \in [1, r]$.

Definition **9.3.** — *An element* v *of* $\widehat{\mathbf{L}}_{\mathrm{K}}^{(r)}$ *is a Whittaker vector for* $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$ *associated with* χ *if*

$$(\mathbf{9.6})\qquad \pi^{(r)}(\mathrm{W}_{d,l})v = \chi(\mathrm{W}_{d,l})v, \quad d\in[1,r],\ l\geq 1.$$

Proposition **9.4.** — *The element* G *is a Whittaker vector for* $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$ *associated with the character* χ *given by*

$$(\mathbf{9.7})\qquad \chi(\mathrm{W}_{r,1}) = y^{1-r}x^{-1}, \qquad \chi(\mathrm{W}_{d,l}) = 0 \quad \text{if } d\neq r \text{ or } d=r,\ l\neq 1.$$

It is characterized, up to a scalar, by this property.

Proof. — We will work in the representation $\mathbf{L}_{\mathrm{K}}^{(r)}$ and omit to write the symbol $\rho^{(r)}$ to unburden the notations. Equation (9.3) implies that

$$(\mathbf{9.8})\qquad \mathfrak{U}\big(\mathbf{SH}_{\mathrm{K}}^{(r)}\big)[\leq r-2]\cdot \mathrm{G} = 0.$$

By Lemma 8.26 the map $\Theta^{(r)}$ gives a surjective morphism of degreewise topological K_r-algebras $\Theta^{(r)}:\mathfrak{U}(\mathbf{SH}_{\mathrm{K}}^{(r)})\to\mathscr{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r))$ which is compatible with the order filtrations. This implies that

$$(\mathbf{9.9})\qquad \mathfrak{U}\big(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)\big)[\leq r-2]\cdot \mathrm{G} = 0.$$

Since $\mathrm{W}_{d,l}$ has order $d-1$ by (8.3), this implies that

$$(\mathbf{9.10})\qquad \mathrm{W}_{d,l}\cdot \mathrm{G} = 0, \quad d<r.$$

Let us now assume that $d=r$. It will be convenient to use the elements $\mathrm{Y}_{l,n}$ from Section 1.9. We have

$$(\mathbf{9.11})\qquad \mathbf{SH}_{\mathrm{K}}^{-}[l,\leq n] = \mathbf{SH}_{\mathrm{K}}^{-}[l,<n]\oplus \mathrm{K}_r\mathrm{Y}_{-l,n}$$

with $\mathrm{Y}_{l,n}=\mathrm{D}_{l,n}$ for $l=-1,0,1$ and

$$(\mathbf{9.12})\qquad \mathrm{Y}_{-l,n} = \begin{cases} [\mathrm{D}_{-1,0},\mathrm{Y}_{1-l,n+1}] & \text{if } l-1=n,\\ [\mathrm{D}_{-1,1},\mathrm{Y}_{1-l,n}] & \text{if } l-1\neq n.\end{cases}$$

Assume first that $r=2$. Then

$$(\mathbf{9.13})\qquad \mathrm{Y}_{-2,1}\cdot \mathrm{G} = [\mathrm{D}_{-1,0},\mathrm{D}_{-1,2}]\cdot \mathrm{G} = 0.$$

More generally, we have

$$(\mathbf{9.14})\qquad \mathrm{Y}_{-l-1,1}\cdot \mathrm{G} = [\mathrm{D}_{-1,1},\mathrm{Y}_{-l,1}]\cdot \mathrm{G} = 0$$

for any $l \geq 2$. Next, let us assume that $r > 2$. Then

$$\begin{aligned} \mathrm{Y}_{-2,r} \cdot \mathrm{G} &= [\mathrm{D}_{-1,1}, \mathrm{D}_{-1,r}] \cdot \mathrm{G} = 0, \\ \mathrm{Y}_{-2,r-1} \cdot \mathrm{G} &= [\mathrm{D}_{-1,1}, \mathrm{D}_{-1,r-1}] \cdot \mathrm{G} = 0 \end{aligned} \tag{9.15}$$

and, acting by $\mathrm{ad}(\mathrm{D}_{-1,1})$,

$$\mathrm{Y}_{-l,r} \cdot \mathrm{G} = \mathrm{Y}_{-n,r-1} \cdot \mathrm{G} = 0, \quad 2 \leq l \leq r, 2 \leq n \leq r-1. \tag{9.16}$$

Therefore we have

$$\mathrm{Y}_{-r,r-1} \cdot \mathrm{G} = [\mathrm{D}_{-1,0}, \mathrm{Y}_{1-r,r}] \cdot \mathrm{G} = 0 \tag{9.17}$$

from which we deduce, by acting by $\mathrm{ad}(\mathrm{D}_{-1,1})$ again, that $\mathrm{Y}_{-l,r-1} \cdot \mathrm{G} = 0$ for $l > r$. We have thus proved that $\mathfrak{U}(\mathbf{SH}^{(r)})[-l, \leq r-1] \cdot \mathrm{G} = 0$ for $l > 1$, and hence that

$$\mathfrak{U}\big(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)\big)[l, \leq r-1] \cdot \mathrm{G} = 0, \quad l > 1. \tag{9.18}$$

In particular, we have $\mathrm{W}_{r,l} \cdot \mathrm{G} = 0$ for $l > 1$. To prove that G is a Whittaker vector, it now remains to compute $\mathrm{W}_{r,1} \cdot \mathrm{G}$. We will do this by expressing $\mathrm{W}_{r,1}$ in terms of the elements $\mathrm{D}_{l,n}$ up to terms of order $< r-1$. We will use the representation $\rho^{(1^r)}$ of $\mathbf{SH}^{(r)}$. Let us first introduce some notation. If $f = f(z_1, \ldots, z_r) = \sum_{\underline{i}} a_{\underline{i}} z_1^{i_1} \cdots z_r^{i_r}$ is a polynomial then we write

$$:f(\underline{z}): = \sum_{\underline{i}} a_{\underline{i}} :b^{(1)}(z)^{i_1} \cdots b^{(r)}(z)^{i_r}:. \tag{9.19}$$

Further, if $u(z) = \sum_i u_i z^{-i-d}$ is a field of conformal dimension d then we write $(u(z))_i = u_i$. By Claim 8.27 we have, up to terms of order $< r-1$ in the order filtration on $\mathfrak{U}(\mathrm{W}_{\kappa-r}(\mathfrak{gl}_1))^{\hat{\otimes} r}$

$$\mathrm{W}_r(z) = -\kappa \sum_{s=0}^{r} (-r)^{s-r} :p_1(\underline{z})^{r-s} e_s(\underline{z}): \tag{9.20}$$

while by Claim 8.28 and (8.89) we have, again up to order $< r-1$,

$$\rho^{(1^r)}(\mathrm{D}_{-1,d}) = -\frac{y^{-1}\kappa^{d+1}}{d+1}\big(:p_{d+1}(\underline{z}):\big)_{-1}, \qquad \rho^{(1^r)}(\mathrm{D}_{0,d}) = \frac{\kappa^{d+1}}{d(d+1)}\big(:p_{d+1}(\underline{z}):\big)_0. \tag{9.21}$$

Combining (9.20) and (9.21) and using the identity

$$\big(p_r, p_1^{r-s} e_s\big) = \delta_{r,s}(-1)^{r-1}/r \tag{9.22}$$

from the theory of symmetric functions we deduce that, up to terms of order $< r-1$,

$$\pi^{(1^r)}(\mathrm{W}_{r,1}) = (-1)^{r-1} y \kappa^{1-r} \rho^{(1^r)}(\mathrm{D}_{-1,r-1}) + u \tag{9.23}$$

where u is a linear combination of monomials $\rho^{(1^r)}(\mathrm{D}_{0,d_1}\cdots\mathrm{D}_{0,d_s}\mathrm{D}_{-1,d})$ with $d<r-1$. Acting on G and using Proposition 9.1 we obtain

$$\mathrm{W}_{r,1}\cdot\mathrm{G}=y^{1-r}x^{-1}\mathrm{G}. \tag{9.24}$$

To finish the proof of Proposition 9.4, we now show that there is, up to a scalar, at most one Whittaker vector of $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$ in $\mathbf{L}^{(r)}_{\mathrm{K}}$ associated with the character χ. So, assume that $v=\sum_{n\geq 0}v_n$ is a Whittaker vector, with $v_n\in\mathbf{L}^{(r)}_{n,\mathrm{K}}$ for all n. Assume also that we have proved that for some $n_0\geq 1$ we have $v_n=[\mathrm{M}_{r,n}]$ for all $n<n_0$. Then Equation (9.7) for G and v gives the following identities in $\mathbf{L}^{(r)}_{n_0-l,\mathrm{K}}$ for any $l\geq 1$

$$\pi^{(r)}(\mathrm{W}_{d,l})\big(v_{n_0}-[\mathrm{M}_{r,n_0}]\big)=0,\quad d\in[1,r]. \tag{9.25}$$

Since $\mathbf{L}^{(r)}_{\mathrm{K}}$ is irreducible as a $\mathrm{W}_{\kappa-r}(\mathfrak{gl}_r)$-module, this implies that $v_{n_0}=[\mathrm{M}_{r,n_0}]$. The proposition follows easily. $\square$

Acknowledgements

We would like to thank T. Arakawa, A. Braverman, D. Maulik, H. Nakajima, A. Okounkov, V. Pasquier and S. Yanagida for useful discussions and correspondences. O.S. would especially like to thank H. Nakajima for his suggestion to construct a coproduct on the algebra $\mathbf{SH^c}$, leading to the free field realization. Part of this work was done while the second author was visiting Columbia University. E.V. is grateful to A. Okounkov for his kind invitation.

Appendix A: Some useful formulas

In this section we gathered a few formulas concerning the functions G_l, φ_l and ϕ_l which are used throughout the paper. Recall that, for $l\geq 0$, we have

$$\mathrm{G}_0(s)=-\log(s),\qquad \mathrm{G}_l(s)=\big(s^{-l}-1\big)/l,\quad l\neq 0, \tag{A.1}$$

$$\begin{aligned}\varphi_l(s)=&\,s^l\mathrm{G}_l(1-s)+s^l\mathrm{G}_l(1+\kappa s)+s^l\mathrm{G}_l(1+\xi s)-s^l\mathrm{G}_l(1+s)\\&-s^l\mathrm{G}_l(1-\kappa s)-s^l\mathrm{G}_l(1-\xi s),\end{aligned} \tag{A.2}$$

$$\phi_l(s)=s^l\mathrm{G}_l(1+\xi s). \tag{A.3}$$

In particular, we have

$$\begin{gathered}\varphi_l(s)=(l+2)(l+1)\kappa\xi s^{l+3}+\mathcal{O}\big(s^{l+4}\big),\\ \phi_l(s)=-\xi s^{l+1}+(l+1)\xi^2s^{l+2}/2-(l+2)(l+1)\xi^3s^{l+3}/6+\mathcal{O}\big(s^{l+4}\big).\end{gathered} \tag{A.4}$$

Note also that for each a, b we have

$$\log\big(1+s(a+b)\big)=\sum_{l\geq 1}(-1)^{l+1}(a+b)^l s^l/l=\sum_{l\geq 0}(-1)^{l+1}a^l s^l\,\mathrm{G}_l(1+bs). \tag{A.5}$$

Remark **A.1.** — Note that for each $l\in\mathbf{N}$ there is a non-zero constant $a\in\mathrm{F}$ such that $\phi_{l+2}-a\varphi_l$ is a formal series in $\varphi_{l+1},\varphi_{l+2},\ldots$.

Appendix B: Proof of Proposition 1.15

B.1 *The reduction.* — We begin with the proof of the relation (1.38). We will use the polynomial representation ρ_n of $\mathbf{SH}_n$ in $\mathbf{V}_n$ in order to compute the expression (B.2) below. However, because the theory of Jack polynomials is only well-behaved for symmetric polynomials (as opposed to symmetric Laurent polynomials), we will need to somehow restrict ourselves to the subspace Λ_n. For this we will use the inner automorphism

$$\sigma=\mathrm{Ad}(e_n)\in\mathrm{Aut}(\mathbf{SH}_n),\qquad e_n=\mathrm{X}_1\mathrm{X}_2\cdots\mathrm{X}_n.$$

Note that $\mathbf{V}_n=\Lambda_n[(e_n)^{-1}]$

Lemma **B.1.** — *Let* $\mathrm{U}\subset\mathbf{SH}_n$ *be a finite dimensional subspace which is stable under* σ *and let* $u\in\mathrm{U}$. *If* $u((e_n)^k\Lambda_n)=\{0\}$ *for some integer* k *then* $u=0$.

Proof. — For $k\in\mathbf{Z}$ let $\mathrm{Z}_k\subset\mathbf{SH}_n$ be the annihilator of $(e_n)^k\Lambda_n$. We have $\mathrm{Z}_k\subset\mathrm{Z}_{k+1}$ and $\sigma(\mathrm{Z}_k)=\mathrm{Z}_{k+1}$. Further, since ρ_n is faithful we have also $\bigcap_k\mathrm{Z}_k=\{0\}$. Thus, since U is finite dimensional, there exists $l\in\mathbf{Z}$ such that $\mathrm{U}\cap\mathrm{Z}_l=\{0\}$. But $\sigma(\mathrm{U}\cap\mathrm{Z}_k)=\mathrm{U}\cap\mathrm{Z}_{k+1}$ for all k. Thus, we have $\mathrm{U}\cap\mathrm{Z}_k=\{0\}$ for all k. □

For $k\geq 0$ let $\mathrm{A}(k)$ be the subspace of elements of $\mathrm{F}[\mathrm{D}^{(n)}_{0,1},\ldots,\mathrm{D}^{(n)}_{0,k}]$ of degree k. Here $\mathrm{D}^{(n)}_{0,l}$ is in degree l. Consider the following finite-dimensional subspace of $\mathbf{SH}_n$

$$\mathrm{B}(k)=\big[\mathrm{D}^{(n)}_{-1,0},\big[\mathrm{A}(k),\mathrm{D}^{(n)}_{1,0}\big]\big]+\mathrm{A}(k).$$

We claim that

$$\big[\mathrm{D}^{(n)}_{-1,l},\mathrm{D}^{(n)}_{1,k}\big]=\big[\mathrm{D}^{(n)}_{-1,0},\big[\mathrm{D}^{(n)}_{0,l+k+1},\mathrm{D}^{(n)}_{1,0}\big]\big],\qquad\forall\, l,k\geq 0. \tag{B.1}$$

To see this, first observe that $[\mathrm{D}^{(n)}_{-1,0},\mathrm{D}^{(n)}_{1,0}]=0$. Applying $\mathrm{ad}(\mathrm{D}^{(n)}_{0,l+1})$ and using (1.36), (1.37) we get

$$\big[\mathrm{D}^{(n)}_{-1,l},\mathrm{D}^{(n)}_{1,0}\big]=\big[\mathrm{D}^{(n)}_{-1,0},\mathrm{D}^{(n)}_{1,l}\big]. \tag{B.2}$$

We now prove (B.1) by induction on $l+k$. Fix $r>0$ and assume that (B.1) holds for all pairs (l,k) with $l+k<r$. Applying $\mathrm{ad}(\mathrm{D}^{(n)}_{0,2})$ to the series of equalities

$$\textbf{(B.3)}\qquad \big[\mathrm{D}^{(n)}_{-1,0},\mathrm{D}^{(n)}_{1,r-1}\big]=\big[\mathrm{D}^{(n)}_{-1,1},\mathrm{D}^{(n)}_{1,r-2}\big]=\cdots=\big[\mathrm{D}^{(n)}_{-1,r-1},\mathrm{D}^{(n)}_{1,0}\big]$$

yields

$$\big[\mathrm{D}^{(n)}_{-1,0},\mathrm{D}^{(n)}_{1,r}\big]-\big[\mathrm{D}^{(n)}_{-1,1},\mathrm{D}^{(n)}_{1,r-1}\big]=\cdots=\big[\mathrm{D}^{(n)}_{-1,r-1},\mathrm{D}^{(n)}_{1,1}\big]-\big[\mathrm{D}^{(n)}_{-1,r},\mathrm{D}^{(n)}_{1,0}\big].$$

Denote by u this common value. Adding all the above equalities together and using (B.2) we get

$$ru=\big[\mathrm{D}^{(n)}_{-1,0},\mathrm{D}^{(n)}_{1,r}\big]-\big[\mathrm{D}^{(n)}_{-1,r},\mathrm{D}^{(n)}_{1,0}\big]=0$$

hence $u=0$. This implies that (B.3) holds with r in place of $r-1$. The induction step is completed and (B.1) is proved.

By (B.1) both sides of (1.38) belong to $\mathrm{B}(k+l)$. One checks that

$$\sigma(y_l)=y_l-1,\qquad \sigma\big(\mathrm{D}^{(n)}_{\pm1,0}\big)=\mathrm{D}^{(n)}_{\pm1,0},$$

from which we see that the subspace $\mathrm{B}(k+l)$ is stable under σ. By Lemma B.1 it is hence enough to check (1.38) in $(e_n)^k\Lambda_n$ for some $k\in\mathbf{Z}$. This is what we will do in the next paragraphs.

B.2 *The Pieri formula for e_{-1}.* — We state here a Pieri formula for the multiplication of Jack polynomials by the elementary symmetric Laurent polynomial

$$\textbf{(B.4)}\qquad e_{-1}=\mathrm{X}_1^{-1}+\cdots+\mathrm{X}_n^{-1}.$$

Since the product $e_{-1}\cdot\mathrm{J}^{(n)}_\lambda$ may not be a polynomial, we need to restrict the range of application. For $\lambda=(\lambda_1,\dots,\lambda_n)$ we write

$$\textbf{(B.5)}\qquad \lambda-\big(1^n\big)=(\lambda_1-1,\dots,\lambda_n-1),\qquad \big(1^n\big)=(1,1,\dots,1).$$

We'll use the following result [35, Sect. 5]. Recall the definitions of h^λ and h_λ from (1.25).

Lemma **B.2.** — *Let λ be a partition of length n. We have*

$$\mathrm{J}^{(n)}_\lambda=c_\lambda(\kappa)\,e_n\mathrm{J}^{(n)}_{\lambda-(1^n)},\qquad c_\lambda(\kappa)=\prod_{i=1}^n h^\lambda(i,1).$$

Thus, we have $e_n\Lambda_n=\bigoplus_\lambda \mathrm{F}\mathrm{J}^{(n)}_\lambda$, where the sum runs over the partitions of length n.

Proposition **B.3.** — *Let* λ *be a partition of length* n. *We have*

$$e_{-1}\mathrm{J}_\lambda^{(n)} = \sum_{\mu\subset\lambda} \phi_{\lambda\backslash\mu}\mathrm{J}_\mu^{(n)}$$

where the sum ranges over all $\mu\subset\lambda$ *with* $|\mu| = |\lambda| - 1$ *and where*

$$\phi_{\lambda\backslash\mu} = \frac{1}{\kappa}\frac{h^\lambda(1,j)}{h^\mu(1,j)}\prod_{s\in\mathrm{C}_{\lambda\backslash\mu}}\frac{h^\lambda(s)}{h^\mu(s)}\prod_{s\in\mathrm{R}_{\lambda\backslash\mu}}\frac{h_\lambda(s)}{h_\mu(s)}, \quad j = y(\lambda\backslash\mu)+1.$$

Proof. — We have

$$e_{-1}\mathrm{J}_\lambda^{(n)} = c_\lambda(\kappa)\, e_{-1}\, e_n\mathrm{J}_{\lambda-(1^n)}^{(n)} = c_\lambda(\kappa)\, e_{n-1}\mathrm{J}_{\lambda-(1^n)}^{(n)}. \tag{B.6}$$

This allows us to use the Pieri formulas for the multiplication by $e_{n-1}^{(n)}$ given in [35, Thm. 6.1], using the duality [35, Thm. 3.3]. Note that the inner product in [35] is given, in our notation, by the following formula

$$\langle \mathrm{J}_\lambda^{(n)}, \mathrm{J}_\mu^{(n)}\rangle = \delta_{\lambda,\mu}\kappa^{-2|\lambda|}\prod_{s\in\lambda} h_\lambda(s)h^\lambda(s).$$

We leave the details to the reader. □

B.3 *Proof of Proposition 1.15.* — For a linear operator f on Λ_n we define $\langle\mu; f;\lambda\rangle$ by

$$f(\mathrm{J}_\lambda^{(n)}) = \sum_\mu \langle \mu; f;\lambda\rangle \mathrm{J}_\mu^{(n)}.$$

Using the explicit expressions of the Pieri rules for $e_{\pm1}^{(n)}$ it is easy to check that

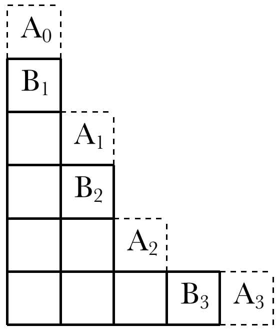
$$\langle\mu; [\mathrm{D}_{-1,0}^{(n)}, [\mathrm{D}_{0,l}^{(n)}, \mathrm{D}_{1,0}^{(n)}]];\lambda\rangle = 0 \tag{B.7}$$

for any $\mu\neq\lambda$ with $l(\lambda) = n$, compare [33, App. A]. In the remainder of this paragraph, we compute precisely the coefficient arising in (B.7) for $\mu=\lambda$. We will use the following notation introduced by Garsia and Tesler (Figure 2). Label the removable boxes of λ by $\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_r$ from left to right, and the addable boxes $\mathrm{A}_0, \ldots, \mathrm{A}_r$ also from left to right. Set $\mathrm{I} = \{0,\ldots,r\}, \mathrm{J} = \{1,\ldots,r\}$ and

$$a_i = c(\mathrm{A}_i), \qquad b_j = c(\mathrm{B}_j), \quad i\in\mathrm{I},\ j\in\mathrm{J} \tag{B.8}$$

where $c(s)$ is defined in (1.27). Observe that we have

$$x(\mathrm{A}_0) = y(\mathrm{A}_r) = 0, \qquad x(\mathrm{A}_j) = x(\mathrm{B}_j)+1, \qquad y(\mathrm{A}_{j-1}) = y(\mathrm{B}_j)+1, \quad j\in\mathrm{J}.$$



$$b_1 = -4\kappa$$
$$b_2 = 1 - 2\kappa$$
$$b_3 = 3$$
$$a_0 = -5\kappa$$
$$a_1 = 1 - 3\kappa$$
$$a_2 = 2 - \kappa$$
$$a_3 = 4$$

FIG. 2. — Garsia and Tesler's variables

Example **B.4.** — Here is an example with $\lambda = (4, 2^2, 1^2)$

Let us begin by rewriting the expressions appearing in the Pieri rules in terms of Garsia and Tesler's notation. Let λ be a fixed partition and let B_j, A_i, a_i, b_j be associated with λ as above. A direct computation yields

Lemma **B.5.** — *For $i \in I$, $j \in J$ we have*

$$\prod_{s \in C_{A_i}} \frac{h_\lambda(s)}{h_\sigma(s)} \prod_{s \in R_{A_i}} \frac{h^\lambda(s)}{h^\sigma(s)} = \prod_{j \in J} (a_i - \xi - b_j) \prod_{k \in I \setminus \{i\}} \frac{1}{a_i - a_k}, \qquad \sigma = \lambda + A_i,$$

$$\prod_{s \in C_{B_j}} \frac{h^\lambda(s)}{h^\nu(s)} \prod_{s \in R_{B_j}} \frac{h_\lambda(s)}{h_\nu(s)} = -\frac{1}{\kappa} \prod_{i \in I} (a_i - \xi - b_j) \prod_{k \in J \setminus \{j\}} \frac{1}{b_k - b_j}, \qquad \nu = \lambda - B_j.$$

Set $I^\times = I \setminus \{0\}$. The above lemma yields the following.

Corollary **B.6.** — *If λ has length n then, for $l \geq 0$, we have*

$$\langle \lambda ; [D^{(n)}_{-1,0}, [D^{(n)}_{0,l}, D^{(n)}_{1,0}]] ; \lambda \rangle = \sum_{i \in I^\times} a_i^l \prod_{k \in I^\times \setminus \{i\}} \frac{a_i - a_k + \xi}{a_i - a_k} \prod_{j \in J} \frac{a_i - b_j - \xi}{a_i - b_j} - \sum_{j \in J} b_j^l \prod_{i \in I^\times} \frac{b_j - a_i + \xi}{b_j - a_i} \prod_{k \in J \setminus \{j\}} \frac{b_j - b_k - \xi}{b_j - b_k}. \tag{B.9}$$

Note that in (B.9) the variable a_0 never appears. In fact, we have $a_0 = -n\kappa$ since $l(\lambda) = n$. Let us now form the generating series

$$X^{(n)}(t) = \sum_{l \geq 0} \langle \lambda ; [D^{(n)}_{-1,0}, [D^{(n)}_{0,l}, D^{(n)}_{1,0}]] ; \lambda \rangle t^l.$$

By (B.9) we have

$$\begin{aligned}\mathrm{X}^{(n)}(t) = &\sum_{i\in\mathrm{I}^\times}\frac{1}{1-a_it}\prod_{k\in\mathrm{I}^\times\setminus\{i\}}\frac{a_i-a_k+\xi}{a_i-a_k}\prod_{j\in\mathrm{J}}\frac{a_i-b_j-\xi}{a_i-b_j}\\ &-\sum_{j\in\mathrm{J}}\frac{1}{1-b_jt}\prod_{i\in\mathrm{I}^\times}\frac{b_j-a_i+\xi}{b_j-a_i}\prod_{k\in\mathrm{J}\setminus\{j\}}\frac{b_j-b_k-\xi}{b_j-b_k}.\end{aligned}$$

Lemma **B.7.** — *Given two disjoint sets of commutative formal variables* $\{a_i;\, i\in\mathrm{I}^\times\}$ *and* $\{b_j; j\in\mathrm{J}\}$ *we have*

$$\begin{aligned}&\sum_{i\in\mathrm{I}^\times}\frac{t\xi}{1-a_it}\prod_{k\in\mathrm{I}^\times\setminus\{i\}}\frac{a_i-a_k+\xi}{a_i-a_k}\prod_{j\in\mathrm{J}}\frac{a_i-b_j-\xi}{a_i-b_j}\\ &\quad-\sum_{j\in\mathrm{J}}\frac{t\xi}{1-b_jt}\prod_{i\in\mathrm{I}^\times}\frac{b_j-a_i+\xi}{b_j-a_i}\prod_{k\in\mathrm{J}\setminus\{j\}}\frac{b_j-b_k-\xi}{b_j-b_k}\\ &=\prod_{i\in\mathrm{I}^\times}\frac{1-t(a_i-\xi)}{1-ta_i}\prod_{j\in\mathrm{J}}\frac{1-t(b_j+\xi)}{1-tb_j}-1.\end{aligned}$$

Proof. — Both sides of the equality are rational functions in t of degree 0, with at most simple poles. One checks that the poles and residues are the same. This implies the equality, up to a possible constant. But both sides vanish at $t=0$. So this constant is zero. □

The above lemma implies the equality

$$\begin{aligned}1+t\xi\mathrm{X}^{(n)}(t) &= \prod_{i\in\mathrm{I}^\times}\frac{1-t(a_i-\xi)}{1-ta_i}\prod_{j\in\mathrm{J}}\frac{1-t(b_j+\xi)}{1-tb_j}\\ &=\exp\Big(\sum_{l\geq 1}\big(p_l(a_i)^\times-p_l(a_i-\xi)^\times+p_l(b_j)-p_l(b_j+\xi)\big)t^l/l\Big),\end{aligned}$$

where

$$p_l(a_i)^\times=\sum_{i\in\mathrm{I}^\times}a_i^l,\qquad p_l(a_i-\xi)^\times=\sum_{i\in\mathrm{I}^\times}(a_i-\xi)^l,\qquad p_l(b_j)=\sum_{j\in\mathrm{J}}b_j^l,\quad \text{etc.}$$

The last step is to identify the expression above with the eigenvalue of an element in $\mathbf{SH}_n^0$ on the Jack polynomial $\mathrm{J}_\lambda(\mathrm{X}_1,\ldots,\mathrm{X}_n)$. From (1.30) we get

$$\langle\mu;\mathrm{D}_{0,l}^{(n)};\mu\rangle=\sum_{s\in\mu}c(s)^{l-1}.$$

We'll use the following notation

$$p_l(a_i) = \sum_{i\in\mathrm{I}} a_i^l,$$
$$p_l(a_i-\xi) = \sum_{i\in\mathrm{I}} (a_i-\xi)^l,$$
$$\sigma_l(x) = (x+1)^l - (x-1)^l + (x+\xi-1)^l - (x-\xi+1)^l + (x-\xi)^l - (x+\xi)^l.$$

Lemma **B.8.** — *We have*

$$p_l(a_i) - p_l(a_i-\xi) + p_l(b_j) - p_l(b_j+\xi) = (-1)^{l+1}\xi^l + \sum_{s\in\lambda}\sigma_l\big(c(s)\big). \tag{B.10}$$

Proof. — The proof is by induction on $|\lambda|$. If $|\lambda|=0$ then $r=0$ and $a_0=0$. Assume that (B.10) holds for all partitions of size at most $m-1$ and let λ be a partition of size m. Let $\mu\subset\lambda$ be a subpartition of λ of size $m-1$, and set $s=\lambda\backslash\mu$. Let r, b_j, a_i and r', b'_j, a'_i be associated with λ and μ respectively. Note that we may have $r'=r$, $r'=r-1$ or $r'=r+1$. One checks that

$$\begin{aligned} &p_l(a_i) - p_l(a_i-\xi) + p_l(b_j) - p_l(b_j+\xi) \\ &\quad = p_l\big(a'_i\big) - p_l\big(a'_i-\xi\big) + p_l\big(b'_j\big) - p_l\big(b'_j+\xi\big) + \sigma_l\big(c(s)\big), \end{aligned}$$

which closes the induction step. We leave the details to the reader. □

Using Lemma B.8 and the fact that for $l(\lambda)=n$ we have

$$a_0 = -n\kappa = -\kappa\mathrm{D}^{(n)}_{0,0},$$

we get that the formal series $1+\xi t\mathrm{X}^{(n)}(t)$ is equal to

$$\begin{aligned} &\exp\Big(\sum_{l\geq 1}(-1)^{l+1}\xi^l t^l/l\Big)\exp\Big(\sum_{l\geq 1}(-1)^l\big(\big(\xi+\kappa\mathrm{D}^{(n)}_{0,0}\big)^l - \big(\kappa\mathrm{D}^{(n)}_{0,0}\big)^l\big)t^l/l\Big) \\ &\quad\times\exp\Big(\sum_{l\geq 1}\sum_{s\in\lambda}\sigma_l\big(c(s)\big)t^l/l\Big). \end{aligned}$$

Now, from (A.5) we get

$$\frac{1+at}{1+at+\xi t} = \exp\Big(\sum_{l\geq 0}(-1)^l a^l\phi_l(t)\Big). \tag{B.11}$$

Using this, we may finally write

$$1+\xi t\mathrm{X}^{(n)}(t)=\mathrm{K}\big(\kappa,\mathrm{D}^{(n)}_{0,0},t\big)\exp\Bigg(\sum_{l\geq 0}\sum_{s\in\lambda}c(s)^l\varphi_l(t)\Bigg),$$

$$\mathrm{K}(\kappa,\omega,t)=\frac{(1+\xi t)(1+\kappa\omega t)}{1+\xi t+\kappa\omega t}=(1+\xi t)\exp\Bigg(\sum_{l\geq 0}(-1)^l\kappa^l\omega^l\phi_l(t)\Bigg).$$

Therefore, by (1.30), we see that (1.39) holds when applied to J_λ. Since this is true for all λ of length n, the identity (1.39) holds when applied to any v in $e_n\mathbf{V}_n^+$ by Lemma B.2. But then, by Lemma B.1, (1.39) holds unconditionally. This concludes the proof of Proposition 1.15.

Appendix C: Complements on Section 5

C.1 *The canonical representation of* $\widetilde{\mathbf{U}}^{(1),+}_{\mathrm{K}}$ *on* $\widetilde{\mathbf{L}}^{(1)}_{\mathrm{K}}$. — In this section we describe the canonical representation of $\widetilde{\mathbf{U}}^{(1),+}_{\mathrm{K}}$ on $\widetilde{\mathbf{L}}^{(1)}_{\mathrm{K}}$ explicitly. The following lemma is well-known.

Lemma **C.1.** — (a) *The convolution product gives* $[\mathrm{I}_{\lambda\mu}]\dot{[}\mathrm{I}_\nu]=\delta_{\mu,\nu}\,\mathrm{eu}_\nu[\mathrm{I}_\lambda]$.
(b) *For a* T*-equivariant vector bundle* $\mathcal{V}$ *over* Hilb_n *of rank* r *we have*

$$\mathrm{c}_l(\mathcal{V})=\sum_{\lambda\vdash n}\mathrm{eu}_\lambda^{-1}\,\mathrm{c}_l(\mathcal{V}|_{\mathrm{I}_\lambda})[\mathrm{I}_\lambda],\qquad l\in[1,r].$$

From the above lemma we obtain the formulas

$$\mathrm{c}_1(\tau_{n,n+1})^l=\sum_{\mu\subset\lambda}\mathrm{c}_1(\tau_{\mu,\lambda})^l\,\mathrm{eu}\big(\mathrm{N}^*_{\mu,\lambda}\big)\,\mathrm{eu}^{-1}_{\mu,\lambda}[\mathrm{I}_{\mu,\lambda}] \tag{C.1}$$

$$\mathrm{c}_1(\tau_{n+1,n})^l=\sum_{\mu\subset\lambda}\mathrm{c}_1(\tau_{\lambda,\mu})^l\,\mathrm{eu}\big(\mathrm{N}^*_{\lambda,\mu}\big)\,\mathrm{eu}^{-1}_{\lambda,\mu}[\mathrm{I}_{\lambda,\mu}] \tag{C.2}$$

$$\mathrm{c}_l(\tau_{n,n})=\sum_{\mu\vdash n}\mathrm{c}_l(\tau_{\mu,\mu})\,\mathrm{eu}_\mu^{-1}[\mathrm{I}_{\mu,\mu}], \tag{C.3}$$

where the first two sums range over all pairs μ,λ with $\mu\subset\lambda$ and $\mu\vdash n$, $\lambda\vdash n+1$. Combining the above (C.1)–(C.3) with the explicit expressions deduced from (2.18) and (2.20)

$$\mathrm{eu}_\lambda=\prod_{s\in\lambda}\big(l(s)y-\big(a(s)+1\big)x\big)\big(-\big(l(s)+1\big)y+a(s)x\big)$$

$$\mathrm{eu}\big(\mathrm{N}^*_{\lambda,\mu}\big)=\mathrm{eu}\big(\mathrm{N}^*_{\mu,\lambda}\big)=\prod_{s\in\mu}\big(l_\mu(s)y-\big(a_\lambda(s)+1\big)x\big)\big(-\big(l_\lambda(s)+1\big)y+a_\mu(s)x\big)$$

we get the following formulas

$$f_{1,l}\big([\mathrm{I}_\mu]\big) = x^l \sum_{\lambda\supset\mu} c(\lambda\backslash\mu)^l \mathrm{L}_{\mu,\lambda}(x,y)\,[\mathrm{I}_\lambda], \tag{C.4}$$

$$f_{-1,l}\big([\mathrm{I}_\lambda]\big) = x^l \sum_{\mu\subset\lambda} c(\lambda\backslash\mu)^l \mathrm{L}_{\lambda,\mu}(x,y)\,[\mathrm{I}_\mu], \tag{C.5}$$

$$f_{0,l}\big([\mathrm{I}_\lambda]\big) = x^l \sum_{s\in\lambda} c(s)^l\,[\mathrm{I}_\lambda]. \tag{C.6}$$

Here $c(s)$ is defined in (1.27), we have

$$\mathrm{L}_{\mu,\lambda}(x,y) = \prod_{s\in\mathrm{C}_{\lambda\backslash\mu}} \frac{l_\mu(s)y-(a_\mu(s)+1)x}{(l_\mu(s)+1)y-(a_\mu(s)+1)x} \prod_{s\in\mathrm{R}_{\lambda\backslash\mu}} \frac{(l_\mu(s)+1)y-a_\mu(s)x}{(l_\mu(s)+1)y-(a_\mu(s)+1)x}$$

and the sum in (C.4) ranges over all λ containing μ satisfying $|\lambda|=|\mu|+1$. We set also

$$\mathrm{L}_{\lambda,\mu}(x,y) = \prod_{s\in\mathrm{C}_{\lambda\backslash\mu}} \frac{(l_\lambda(s)+1)y-a_\lambda(s)x}{l_\lambda(s)y-a_\lambda(s)x} \prod_{s\in\mathrm{R}_{\lambda\backslash\mu}} \frac{l_\lambda(s)y-(a_\lambda(s)+1)x}{l_\lambda(s)y-a_\lambda(s)x}$$

and the sum in (C.5) ranges over all μ which are contained in λ and satisfy $|\mu|=|\lambda|-1$.

C.2 *The triangular decomposition of* $\widetilde{\mathbf{U}}^{(1)}_{\mathrm{K}}$. — We begin with the following lemma.

Lemma **C.2.** — *There are one parameter subgroups* $\tau^\pm\colon \mathbf{C}\to\mathrm{Aut}(\widetilde{\mathbf{U}}^{(1),\pm}_{\mathrm{K}})$ *defined by*

$$\tau^\pm_u(f_{0,l}) = \tau_u(f_{0,l}) = \sum_{i=0}^{l}\binom{l}{i}u^{l-i}f_{0,i}, \qquad \tau^\pm_u(f_{\pm1,l}) = \sum_{i=0}^{l}\binom{l}{i}u^{l-i}f_{\pm1,i}, \quad l\geq 0.$$

Proof. — It is enough to deal with τ^+. By Theorem 6.3 there is an algebra isomorphism

$$\eta:\mathbf{SCo}_{\mathrm{K}}\to\widetilde{\mathbf{U}}^{(1),>}_{\mathrm{K}}, \qquad \theta_l\mapsto f_{1,l}, \quad l\geq 0.$$

By Corollary 4.8, the assignment $\theta_l\mapsto\sum_{i=0}^{l}\binom{l}{i}u^{l-i}\theta_i$ extends to an automorphism of $\mathbf{SCo}_{\mathrm{K}}$. This shows that τ^+_u is well-defined on $\widetilde{\mathbf{U}}^{(1),>}_{\mathrm{K}}$. Next, since $\widetilde{\mathbf{U}}^{(1),0}_{\mathrm{K}}=\mathrm{K}[f_{0,l};\,l\geq 1]$, the map τ^+_u is well-defined on $\widetilde{\mathbf{U}}^{(1),0}_{\mathrm{K}}$ as well. To finish the proof, it remains to observe that we have

$$\widetilde{\mathbf{U}}^{(1),+}_{\mathrm{K}} = \widetilde{\mathbf{U}}^{(1),0}_{\mathrm{K}} \ltimes \widetilde{\mathbf{U}}^{(1),>}_{\mathrm{K}} \tag{C.7}$$

with respect to the adjoint action $[f_{0,l}, f_{1,n}] = f_{1,l+n}$, and that

$$\begin{aligned}\left[\tau_u^+(f_{0,l}),\, \tau_u^+(f_{1,n})\right] &= \sum_{i=0}^{l}\sum_{j=0}^{n}\binom{l}{i}\binom{n}{j}u^{l+n-i-j}[f_{0,i}, f_{1,j}] \\ &= \sum_{k=0}^{l+n}\binom{l+n}{k}u^{l+n-k}f_{1,k} = \tau_u^+(f_{1,n+l}).\end{aligned}$$

□

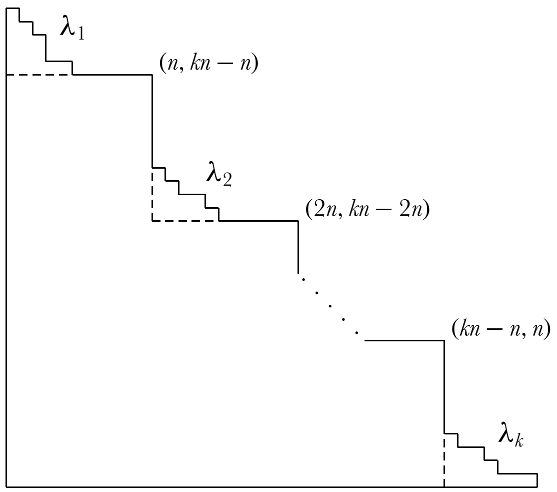
We now turn to the proof of Proposition 5.3. It is adapted from the proof of [33, Prop. 4.8]. The same argument as for **SH** implies that the multiplication map is surjective

$$m : \widetilde{\mathbf{U}}_K^{(1),>} \otimes_K \widetilde{\mathbf{U}}_K^{(1),0} \otimes_K \widetilde{\mathbf{U}}_K^{(1),<} \to \widetilde{\mathbf{U}}_K^{(1)}.$$

We only have to prove its injectivity. We argue by contradiction. Let $x = \sum_i \mathrm{P}_i \otimes \mathrm{R}_i \otimes \mathrm{Q}_i$ be a nonzero homogeneous element in $\mathrm{Ker}(m)$. We may assume that the elements R_i are linearly independent polynomials in the $f_{0,l}$'s and that $\mathrm{P}_i, \mathrm{Q}_i \neq 0$. Multiplying by an element of $\widetilde{\mathbf{U}}_K^{(1),>}$ or $\widetilde{\mathbf{U}}_K^{(1),<}$ if necessary, we may also assume that x is of degree zero. For all partition λ we have

$$\sum_i \mathrm{P}_i \circ \mathrm{R}_i \circ \mathrm{Q}_i\big([\mathrm{I}_\lambda]\big) = 0. \tag{C.8}$$

We'll apply (C.8) to certain partitions. Given partitions $\lambda_1, \lambda_2, \ldots, \lambda_k$ and given an integer $n \gg |\lambda_1|, \ldots, |\lambda_k|$, let the symbol $\lambda_1 \circledast \cdots \circledast \lambda_k$ denote the following partition



Note that $\lambda_1 \circledast \cdots \circledast \lambda_k$ is well-defined as soon as $n > \sup_i(l(\lambda_i), l(\lambda_i'))$. Put

$$t = \sup_i\big(\deg(\mathrm{P}_i)\big) = \sup_i\big(-\deg(\mathrm{Q}_i)\big). \tag{C.9}$$

For an operator f on $\widetilde{\mathbf{L}}^{(1)}_{\mathrm{K}}$ we denote by $\langle \mu; f; \lambda\rangle$ the coefficient of $[\mathrm{I}_\mu]$ in $f([\mathrm{I}_\lambda])$. For n large enough we consider the coefficients

$$\langle \bar\lambda_1 \circledast \lambda_2 \circledast \bar\lambda_3; \mathrm{P}_i\mathrm{R}_i\mathrm{Q}_i; \lambda_1 \circledast \lambda_2 \circledast \lambda_3\rangle, \quad \bar\lambda_1 \subset \lambda_1, \lambda_3 \subset \bar\lambda_3, |\lambda_1\backslash\bar\lambda_1| = |\bar\lambda_3\backslash\lambda_3| = t.$$

Since Q_i is an annihilation operator and P_i is a creation operator, by (C.9) the only way to obtain $\bar\lambda_1 \circledast \lambda_2 \circledast \bar\lambda_3$ from $\lambda_1 \circledast \lambda_2 \circledast \lambda_3$ is to use all of Q_i to reduce λ_1 to $\bar\lambda_1$ and to use all of P_i to increase λ_3 to $\bar\lambda_3$. Therefore we have

$$\begin{aligned}\langle \bar\lambda_1 \circledast \lambda_2 \circledast \bar\lambda_3; \mathrm{P}_i\mathrm{R}_i\mathrm{Q}_i; \lambda_1 \circledast \lambda_2 \circledast \lambda_3\rangle \\ = \langle \bar\lambda_1 \circledast \lambda_2 \circledast \bar\lambda_3; \mathrm{P}_i; \bar\lambda_1 \circledast \lambda_2 \circledast \lambda_3\rangle\, \langle \bar\lambda_1 \circledast \lambda_2 \circledast \lambda_3; \mathrm{R}_i; \bar\lambda_1 \circledast \lambda_2 \circledast \lambda_3\rangle \\ \times \langle \bar\lambda_1 \circledast \lambda_2 \circledast \lambda_3; \mathrm{Q}_i; \lambda_1 \circledast \lambda_2 \circledast \lambda_3\rangle.\end{aligned} \tag{C.10}$$

Note that (C.10) is zero unless $\deg(\mathrm{P}_i) = -\deg(\mathrm{Q}_i) = t$.

Lemma **C.3.** — *There are non-zero $c, d \in \mathrm{K}$ such that, for $\mathrm{P} \in \widetilde{\mathbf{U}}^{(1),>}_{\mathrm{K}}[t]$ and $\mathrm{Q} \in \widetilde{\mathbf{U}}^{(1),<}_{\mathrm{K}}[-t]$,*

$$\langle \bar\lambda_1 \circledast \lambda_2 \circledast \lambda_3; \mathrm{Q}; \lambda_1 \circledast \lambda_2 \circledast \lambda_3\rangle = c\big\langle \bar\lambda_1; \tau^-_{2ny}(\mathrm{Q}); \lambda_1\big\rangle, \tag{C.11}$$

$$\langle \bar\lambda_1 \circledast \lambda_2 \circledast \bar\lambda_3; \mathrm{P}; \bar\lambda_1 \circledast \lambda_2 \circledast \lambda_3\rangle = d\big\langle \bar\lambda_3; \tau^+_{2nx}(\mathrm{P}); \lambda_3\big\rangle. \tag{C.12}$$

Proof. — We prove (C.11). The proof of (C.12) is identical. If $\mathrm{Q} = f_{-1,k_t}\cdots f_{-1,k_1}$ then

$$\langle \bar\lambda_1 \circledast \lambda_2 \circledast \lambda_3; \mathrm{Q}; \lambda_1 \circledast \lambda_2 \circledast \lambda_3\rangle = \sum \prod_{i=1}^{t} c(s_i)^{k_i}\, \mathrm{L}_{\mu_i\circledast\lambda_2\circledast\lambda_3,\, \mu_{i+1}\circledast\lambda_2\circledast\lambda_3}(x, y). \tag{C.13}$$

In (C.13) the sum runs over all sequences

$$\lambda_1 = \mu_1 \supsetneq \mu_2 \cdots \supsetneq \mu_{t+1} = \bar\lambda_1$$

and we have set $s_i = \mu_i\backslash\mu_{i+1}$. For partitions $\alpha \supset \beta$ with $|\alpha| = |\beta| + 1$ we have

$$\mathrm{L}_{\alpha,\beta}(x, y) = \prod_{s\in \mathrm{C}_{\alpha\backslash\beta}} \frac{(l_\alpha(s)+1)y - a_\alpha(s)x}{l_\alpha(s)y - a_\alpha(s)x} \cdot \prod_{s\in \mathrm{R}_{\alpha\backslash\beta}} \frac{l_\alpha(s)y - (a_\alpha(s)+1)x}{l_\alpha(s)x - a_\alpha(s)y}.$$

Next, in (C.13) again, for a box s in μ_i we have $x(s) = x_{\lambda_1}(s)$ and $y(s) = y_{\lambda_1}(s) + 2n$, where x_{λ_1} and y_{λ_1} denote the coordinate values when we place the origin at the bottom left corner of λ_1, i.e., at the point $(0, 2n)$, as opposed to the point $(0, 0)$ which is the origin of $\lambda_1 \circledast \lambda_2 \circledast \lambda_3$. Similarly, we have

$$\mathrm{R}(s) = \mathrm{R}_{\lambda_1}(s), \qquad \mathrm{C}(s) = \mathrm{C}_{\lambda_1}(s) \sqcup \mathrm{C}'(s),$$

$$\mathrm{C}'(s)=\big\{\big(x(s),0\big),\dots,\big(x(s),2n-1\big)\big\}.$$

Finally, observe that the armlength $a(s)$ or the leglength $l(s)$ are the same whether we consider s as belonging to μ_i or to $\mu_i \circledast \lambda_2 \circledast \lambda_3$. Now, write $\sigma_i = \mu_i \circledast \lambda_2 \circledast \lambda_3$ and $\sigma = \lambda_1 \circledast \lambda_2 \circledast \lambda_3$. From the above formulae we deduce that

$$c(s_i) = c_{\lambda_1}(s_i) + 2ny,$$

$$\prod_{i=1}^{t} \mathrm{L}_{\mu_i\circledast\lambda_2\circledast\lambda_3,\,\mu_{i+1}\circledast\lambda_2\circledast\lambda_3}(x,y)$$
$$= \prod_{i=1}^{t} \mathrm{L}_{\mu_i,\mu_{i+1}}(x,y) \prod_{s\in \mathrm{C}'(s_i)} \frac{(l_{\sigma_i}(s)+1)y - a_{\sigma_i}(s)x}{l_{\sigma_i}(s)y - a_{\sigma_i}(s)x}.$$

Note also that the quantity

$$\prod_{i=1}^{t}\prod_{s\in\mathrm{C}'(s_i)} \frac{(l_{\mu_i}(s)+1)y - a_{\mu_i}(s)x}{l_{\mu_i}(s)y - a_{\mu_i}(s)x} = \prod_{s\in\lambda_1\setminus\bar\lambda_1}\prod_{u\in\mathrm{C}'(s)} \frac{(y(s)-y(u)+1)y - a_\sigma(u)x}{(y(s)-y(u))y - a_\sigma(u)x}$$

is independent of the choice of the chain of subdiagrams (μ_i), and that

$$\sum\prod_{i=1}^{t}\big(c_{\lambda_1}(s_i)+2ny\big)^{k_i}\,\mathrm{L}_{\mu_i,\mu_{i+1}}(x,y) = \big\langle\bar\lambda_1;\,\tau^-_{2ny}(\mathrm{Q});\,\lambda_1\big\rangle.$$

The lemma is proved. □

Using (C.10) and Lemma C.3, the linear relation (C.8) gives

$$\sum_i \big\langle\bar\lambda_3;\,\tau^+_{2nx}(\mathrm{P}_i);\,\lambda_3\big\rangle\,\langle\bar\lambda_1\circledast\lambda_2\circledast\lambda_3;\,\mathrm{R}_i;\,\bar\lambda_1\circledast\lambda_2\circledast\lambda_3\rangle\,\big\langle\bar\lambda_1;\,\tau^-_{2ny}(\mathrm{Q}_i);\,\lambda_1\big\rangle=0 \tag{C.14}$$

for all $\lambda_1,\bar\lambda_1,\lambda_2,\lambda_3,\bar\lambda_3$ as above and all large enough n. Since $\mathrm{P}_i,\mathrm{Q}_i\neq 0$, by Corollary 2.5 we may choose $\lambda_3,\bar\lambda_3$ and $\lambda_1,\bar\lambda_1$ such that

$$\big\langle\bar\lambda_3;\,\tau^+_{2nx}(\mathrm{P}_i);\,\lambda_3\big\rangle,\ \big\langle\bar\lambda_1;\,\tau^-_{2ny}(\mathrm{Q}_i);\,\lambda_1\big\rangle\neq 0$$

for some i. Fix the integer n and let us vary the partition λ_2. We abbreviate $\lambda=\emptyset\circledast\emptyset\circledast\emptyset$. By (C.6) the matrix coefficient $\langle\bar\lambda_1\circledast\lambda_2\circledast\lambda_3;f_{0,l};\,\bar\lambda_1\circledast\lambda_2\circledast\lambda_3\rangle$ is equal to

$$\big\langle\bar\lambda_1;\,\tau_{2ny}(f_{0,l});\,\bar\lambda_1\big\rangle+\big\langle\lambda_2;\,\tau_{nx+ny}(f_{0,l});\,\lambda_2\big\rangle+\big\langle\lambda_3;\,\tau_{2nx}(f_{0,l});\,\lambda_3\big\rangle+\langle\lambda;f_{0,l};\,\lambda\rangle.$$

Recall that $\mathrm{R}_i=\mathrm{R}_i(f_{0,1},f_{0,2},\dots)$ is a polynomial in the operators $f_{0,l}$. Set

$$\mathrm{R}'_i=\big\langle\bar\lambda_3;\,\tau^+_{2nx}(\mathrm{P}_i);\,\lambda_3\big\rangle\big\langle\bar\lambda_1;\,\tau^-_{2ny}(\mathrm{Q}_i);\,\lambda_1\big\rangle\mathrm{R}_i\big(\tau_{nx+ny}(f_{0,1})+\alpha_1,\,\tau_{nx+ny}(f_{0,2})+\alpha_2,\dots\big)$$

where

$$\alpha_l = \langle\bar\lambda_1;\, \tau_{2ny}(f_{0,l});\, \bar\lambda_1\rangle + \langle\lambda_3;\, \tau_{2nx}(f_{0,l});\, \lambda_3\rangle + \langle\lambda;f_{0,l};\,\lambda\rangle, \qquad l \geq 1.$$

We may rewrite (C.14) as

$$\textbf{(C.15)} \qquad \sum_i \langle\lambda_2;\, \mathrm{R}'_i;\, \lambda_2\rangle = 0.$$

Since this holds for all λ_2 with $l(\lambda_2)$, $l(\lambda'_2) < n$ and n is large enough, we deduce that $\sum_i \mathrm{R}'_i = 0$. Remember that the R_i's were chosen to be linearly independent. Then the R'_i's are also linearly independent and we arrive at a contradiction. This concludes the proof of Proposition 5.3. □

Appendix D: Complements on Sections 3 and 6

D.1 *Proof of Proposition 6.6.* — The proof is adapted from the computations in [37, Sect. 4.5]. First, we have the following formulas, compare (C.1), (C.2) and (C.3),

$$\textbf{(D.1)} \qquad \mathrm{c}_1(\tau_{n-1,n})^l = \sum_{\sigma\subset\lambda} \mathrm{c}_1(\tau_{\sigma,\lambda})^l\ \mathrm{eu}\big(\mathrm{N}^*_{\sigma,\lambda}\big)\ \mathrm{eu}^{-1}_{\sigma,\lambda}\,[\mathrm{I}_{\sigma,\lambda}],$$

$$\textbf{(D.2)} \qquad \mathrm{c}_1(\tau_{n+1,n})^l = \sum_{\lambda\subset\pi} \mathrm{c}_1(\tau_{\pi,\lambda})^l\ \mathrm{eu}\big(\mathrm{N}^*_{\pi,\lambda}\big)\ \mathrm{eu}^{-1}_{\pi,\lambda}\,[\mathrm{I}_{\pi,\lambda}],$$

$$\textbf{(D.3)} \qquad \mathrm{c}_l(\tau_{n,n}) = \sum_{\lambda} \mathrm{c}_l(\tau_\lambda)\ \mathrm{eu}^{-1}_{\lambda}\,[\mathrm{I}_{\lambda,\lambda}].$$

Here σ, λ and π are r-partitions of $n-1$, n and $n+1$ respectively. Now, assume that λ, μ are r-partitions of n and that σ, π are r-partitions of $n-1$, $n+1$ respectively, with $\sigma\subset\lambda$, $\mu\subset\pi$ and $\lambda\neq\mu$. Then, the r-partitions σ, π are completely determined by λ, μ and (3.9) gives the following identity

$$\tau_\lambda + \tau_\mu = \tau_\sigma + \tau_\pi.$$

Therefore, using the identities from Sections 3.4, 3.3, a short computation gives

$$\textbf{(D.4)} \qquad \mathrm{N}_{\lambda,\sigma} + \mathrm{N}_{\mu,\sigma} - \mathrm{T}_\sigma = \mathrm{N}_{\pi,\lambda} + \mathrm{N}_{\pi,\mu} - \mathrm{T}_\pi.$$

Therefore, using (D.1), (D.2) and (3.12), (D.4) we get

$$\textbf{(D.5)} \qquad [h_{-1,k}, h_{1,l}]\big([\mathrm{I}_\lambda]\big) = c_{\lambda,k+l}\,[\mathrm{I}_\lambda]$$

for some constant $c_{\lambda,k+l}$ which remains to be computed. To do so, observe first that

$$f_{-1,k}f_{1,l}\big([\mathrm{I}_\lambda]\big) = \sum_{\lambda\subset\pi} \mathrm{c}_1(\tau_{\lambda,\pi})^{k+l}\,\mathrm{eu}\big(\mathrm{N}^*_{\lambda,\pi}+\mathrm{N}^*_{\pi,\lambda}\big)\,\mathrm{eu}^{-1}_{\lambda,\pi}\,[\mathrm{I}_\lambda],$$

$$f_{1,l}f_{-1,k}\big([\mathrm{I}_\lambda]\big) = \sum_{\sigma\subset\lambda} \mathrm{c}_1(\tau_{\sigma,\lambda})^{k+l}\,\mathrm{eu}\big(\mathrm{N}^*_{\lambda,\sigma}+\mathrm{N}^*_{\sigma,\lambda}\big)\,\mathrm{eu}^{-1}_{\lambda,\sigma}\,[\mathrm{I}_\lambda],$$

modulo $[\mathrm{I}_\mu]$'s with $\mu\neq\lambda$. Next, set $\mathrm{H}_\lambda = (1-q)(1-t)\tau_\lambda - \mathrm{W}$. For $\lambda\subset\pi$ we have

$$\begin{aligned}
\mathrm{H}_\lambda &= \mathrm{H}_\pi - (1-q)(1-t)\tau_{\lambda,\pi},\\
\mathrm{N}_{\lambda,\pi} - \mathrm{T}_\lambda &= -v\tau^*_{\lambda,\pi}\mathrm{H}_\lambda - v\\
&= -v\tau^*_{\lambda,\pi}\mathrm{H}_\pi + 1 - q^{-1} - t^{-1},\\
\mathrm{N}_{\pi,\lambda} - \mathrm{T}_\pi &= \tau_{\lambda,\pi}\mathrm{H}^*_\pi - v\\
&= \tau_{\lambda,\pi}\mathrm{H}^*_\lambda + 1 - q^{-1} - t^{-1}.
\end{aligned} \tag{D.6}$$

Now, we consider the following sums

$$\mathrm{B}_\lambda = \sum_{\sigma\subset\lambda}\tau_{\sigma,\lambda}, \qquad \mathrm{A}_\lambda = \sum_{\lambda\subset\pi}\tau_{\lambda,\pi}. \tag{D.7}$$

The proof of the following lemma is left to the reader, compare [37, Lem. 7].

Lemma **D.1.** — *For each r-partition λ of n we have the equality of characters*

$$\mathrm{H}_\lambda = v^{-1}\mathrm{B}_\lambda - \mathrm{A}_\lambda.$$

Thus, we get

$$\mathrm{N}^*_{\lambda,\pi} + \mathrm{N}^*_{\pi,\lambda} - \mathrm{T}^*_\lambda - \mathrm{T}^*_\pi = v^{-1}\big(\tau_{\lambda,\pi}\mathrm{A}^*_\lambda + \tau^*_{\lambda,\pi}\mathrm{B}_\lambda - 1\big) - \big(\tau_{\lambda,\pi}\mathrm{B}^*_\lambda + \tau^*_{\lambda,\pi}\mathrm{A}_\lambda - 1\big) - q - t,$$

$$\mathrm{N}^*_{\lambda,\sigma} + \mathrm{N}^*_{\sigma,\lambda} - \mathrm{T}^*_\lambda - \mathrm{T}^*_\sigma = v^{-1}\big(\tau_{\sigma,\lambda}\mathrm{A}^*_\lambda + \tau^*_{\sigma,\lambda}\mathrm{B}_\lambda - 1\big) - \big(\tau_{\sigma,\lambda}\mathrm{B}^*_\lambda + \tau^*_{\sigma,\lambda}\mathrm{A}_\lambda - 1\big) - q - t.$$

We get

$$\begin{aligned}
(-1)^r c_{\lambda,k}\,x^k = &\sum_{\sigma\subset\lambda}\mathrm{c}_1(\tau_{\sigma,\lambda})^k\,\frac{\mathrm{eu}(v^{-1}\tau_{\sigma,\lambda}\mathrm{A}^*_\lambda + v^{-1}\tau^*_{\sigma,\lambda}\mathrm{B}_\lambda - v^{-1})}{\mathrm{eu}(\tau^*_{\sigma,\lambda}\mathrm{A}_\lambda + \tau_{\sigma,\lambda}\mathrm{B}^*_\lambda - 1)}\\
&- \sum_{\lambda\subset\pi}\mathrm{c}_1(\tau_{\lambda,\pi})^k\,\frac{\mathrm{eu}(v^{-1}\tau_{\lambda,\pi}\mathrm{A}^*_\lambda + v^{-1}\tau^*_{\lambda,\pi}\mathrm{B}_\lambda - v^{-1})}{\mathrm{eu}(\tau^*_{\lambda,\pi}\mathrm{A}_\lambda + \tau_{\lambda,\pi}\mathrm{B}^*_\lambda - 1)}.
\end{aligned}$$

Consider the formal series

$$\mathrm{C}(s) = \sum_{k\geq 0} c_{\lambda,k}\,x^k s^k.$$

For $u=x+y$, we have

$$\mathrm{C}(s)=-\sum_{j\in\mathrm{J}}\frac{1}{1-b_js}\prod_{i\in\mathrm{I}}\frac{b_j-a_i+u}{b_j-a_i}\prod_{k\in\mathrm{J}\setminus\{j\}}\frac{b_j-b_k-u}{b_j-b_k}$$
$$+\sum_{i\in\mathrm{I}}\frac{1}{1-a_is}\prod_{k\in\mathrm{I}\setminus\{i\}}\frac{a_i-a_k+u}{a_i-a_k}\prod_{j\in\mathrm{J}}\frac{a_i-b_j-u}{a_i-b_j},$$

with

$$\{b_j;j\in\mathrm{J}\}=\big\{\mathrm{c}_1(\tau_{\sigma,\lambda});\sigma\subset\lambda\big\},\qquad\{a_i;i\in\mathrm{I}\}=\big\{\mathrm{c}_1(\tau_{\lambda,\pi});\lambda\subset\pi\big\}.$$

Thus, by Lemma B.7, we get the following equality

$$1+us\mathrm{C}(s)=\prod_{i\in\mathrm{I}}\frac{1-s(a_i-u)}{1-sa_i}\prod_{j\in\mathrm{J}}\frac{1-s(b_j+u)}{1-sb_j}$$
$$=\prod_{\sigma\subset\lambda}\frac{1-s\,\mathrm{c}_1(v^{-1}\tau_{\sigma,\lambda})}{1-s\,\mathrm{c}_1(v^{-1}\tau_{\sigma,\lambda})+su}\Big/\prod_{\lambda\subset\pi}\frac{1-s\,\mathrm{c}_1(\tau_{\lambda,\pi})}{1-s\,\mathrm{c}_1(\tau_{\lambda,\pi})+su}.$$

Now, fix splitting sums of one-dimensional characters

$$\tau_\lambda^*=\phi_{\lambda,1}+\cdots+\phi_{\lambda,n},\qquad \mathrm{W}^*=\chi_1+\cdots+\chi_r.$$

Set $f_{\lambda,i}=\mathrm{eu}(\phi_{\lambda,i})$ and $e_a=\mathrm{eu}(\chi_a)$. Then, by Lemma D.1, we have

$$\mathrm{H}_\lambda=\sum_{\sigma\subset\lambda}v^{-1}\tau_{\sigma,\lambda}-\sum_{\lambda\subset\pi}\tau_{\lambda,\pi}=\sum_i(1-q)(1-t)\phi_{\lambda,i}^*-\sum_a\chi_a^*.$$

Therefore, we get

$$1+us\mathrm{C}(s)=\prod_{i=1}^n\frac{1+s(f_{\lambda,i}+x)}{1+s(f_{\lambda,i}-x)}\,\frac{1+s(f_{\lambda,i}+y)}{1+s(f_{\lambda,i}-y)}\,\frac{1+s(f_{\lambda,i}-u)}{1+s(f_{\lambda,i}+u)}\prod_{a=1}^r\frac{1+s(e_a+u)}{1+se_a}.$$

Recall that $u=x\xi$. Using (A.5) and (A.2), we finally get

$$1+\xi\sum_{k\geq 0}c_{\lambda,k}\,s^{k+1}=\prod_{a=1}^r\frac{1+s\varepsilon_a+s\xi}{1+s\varepsilon_a}\,\exp\Big(\sum_{l\geq 0}(-1)^lp_l(f_{\lambda,i})\,x^{-l}\varphi_l(s)\Big).$$

Now, Remark 2.3 gives

$$h_{0,l+1}\,[\mathrm{I}_\lambda]=(-1)^lx^{-l}p_l(f_{\lambda,i})\,[\mathrm{I}_\lambda].$$

Thus, we obtain

$$\Big(1+\xi\sum_{k\geq 0}c_{\lambda,k}\,s^{k+1}\Big)\,[\mathrm{I}_\lambda]=\prod_{a=1}^{r}\frac{1+s\varepsilon_a+s\xi}{1+s\varepsilon_a}\,\exp\Big(\sum_{l\geq 0}h_{0,l+1}\,\varphi_l(s)\Big)\,[\mathrm{I}_\lambda],$$
$$=\exp\Big(\sum_{l\geq 0}(-1)^{l+1}p_l(\vec{\varepsilon})\phi_l(s)\Big)\,\exp\Big(\sum_{l\geq 0}h_{0,l+1}\,\varphi_l(s)\Big)\,[\mathrm{I}_\lambda]$$

where $p_l(\vec{\varepsilon})=\sum_a\varepsilon_a^l$. Comparing this expression with (D.5), we get the proposition.

D.2 *Proof of Corollary 3.3.* — Now, we prove that the representation $\rho^{(r)}$ of $\mathbf{SH}_{\mathrm{K}}^{(r)}$ on $\mathbf{L}_{\mathrm{K}}^{(r)}$ is faithful. For an operator f on $\mathbf{L}_{\mathrm{K}}^{(r)}$ and r-partitions λ, μ we denote by $\langle\mu;f;\lambda\rangle$ the coefficient of $[\mathrm{I}_\mu]$ in $f([\mathrm{I}_\lambda])$. Given partitions $\lambda_1,\lambda_2,\dots,\lambda_k$ and given an integer $n\gg|\lambda_1|,\dots,|\lambda_k|$, let the symbol $\lambda_1\circledast\cdots\circledast\lambda_k$ denote the r-partition whose first part is the partition $\lambda_1\circledast\cdots\circledast\lambda_k$ from Section C.2 and the $r-1$ other partitions are empty. Given a finite family of elements

$$\mathrm{P}_i\in\mathbf{SH}_{\mathrm{K}}^{(r),>},\qquad \mathrm{R}_i\in\mathbf{SH}_{\mathrm{K}}^{(r),0},\qquad \mathrm{Q}_i\in\mathbf{SH}_{\mathrm{K}}^{(r),<},$$

we set $x=\sum_i\mathrm{P}_i\mathrm{R}_i\mathrm{Q}_i$. Assume that $\rho^{(r)}(x)=0$. We may also assume that $\mathrm{P}_i\mathrm{Q}_i\mathrm{R}_i$ is homogeneous of degree 0 (for the rank grading) for each i. Then

$$\sum_i\big\langle\mu;\,\rho^{(r)}(\mathrm{P}_i\mathrm{R}_i\mathrm{Q}_i);\,\lambda\big\rangle=0$$

for each r-partitions λ, μ. For n large enough we consider the coefficients

$$\begin{aligned}&\big\langle\bar\lambda_1\circledast\lambda_2\circledast\bar\lambda_3;\,\rho^{(r)}(\mathrm{P}_i\mathrm{R}_i\mathrm{Q}_i);\,\lambda_1\circledast\lambda_2\circledast\lambda_3\big\rangle,\quad \bar\lambda_1\subset\lambda_1,\ \lambda_3\subset\bar\lambda_3,\\ &|\lambda_1\backslash\bar\lambda_1|=|\bar\lambda_3\backslash\lambda_3|=t\end{aligned}\tag{D.8}$$

with $t=\sup_i(\deg(\mathrm{P}_i))=\sup_i(-\deg(\mathrm{Q}_i))$. Since Q_i is an annihilation operator and P_i is a creation operator, the coefficient

$$\big\langle\bar\lambda_1\circledast\lambda_2\circledast\bar\lambda_3;\,\rho^{(r)}(\mathrm{P}_i\mathrm{R}_i\mathrm{Q}_i);\,\lambda_1\circledast\lambda_2\circledast\lambda_3\big\rangle\tag{D.9}$$

factorizes as in (C.10), and it is zero unless $\deg(\mathrm{P}_i)=-\deg(\mathrm{Q}_i)=t$. We claim that (C.11), (C.12) hold again for some non-zero $c,d\in\mathrm{K}_r$. Then (C.14) hold again for all $\lambda_1,\bar\lambda_1,\lambda_2,\lambda_3,\bar\lambda_3$ as above and all large enough n. If $x\neq 0$ we may assume that the elements R_i are linearly independent polynomials in the $f_{0,l}$'s and that $\mathrm{P}_i,\mathrm{Q}_i\neq 0$. Then the same argument as in Section C.2 yields a contradiction. The proof of the claim is the same as the proof of Lemma C.3. It is left to the reader. □

The proof above has the following corollary.

Corollary **D.2.** — *The multiplication map* $\mathbf{SH}_{\mathrm{K}}^{(r),>}\otimes\mathbf{SH}_{\mathrm{K}}^{(r),0}\otimes\mathbf{SH}_{\mathrm{K}}^{(r),<}\to\mathbf{SH}_{\mathrm{K}}^{(r)}$ *is injective.*

D.3 *Proof of Lemma 3.12.* — Fix r-multipartitions μ, λ such that $\mu \subset \lambda$ and $|\lambda \backslash \mu| = n$. Let $l_1, \dots, l_n$ be integers ≥ 0. We need to prove that

$$\langle \lambda; f_{1,l_1} \cdots f_{1,l_n}; \mu \rangle = \varpi_n\big(z_1^{l_1} \cdots z_n^{l_n}\big)(\tau_{\mu,\lambda})\, a_{\mu,\lambda}\, \mathrm{eu}_\mu \,. \tag{D.10}$$

Let $s_1, \dots, s_n$ be the boxes of $\lambda \backslash \mu$. We have

$$\langle \lambda; f_{1,l_1} \cdots f_{1,l_n}; \mu \rangle = \sum_{\sigma \in \mathfrak{S}_n} \big\langle \lambda^{\sigma,0}; f_{1,l_1}; \lambda^{\sigma,1} \big\rangle \cdots \big\langle \lambda^{\sigma,n-1}; f_{1,l_1}; \lambda^{\sigma,n} \big\rangle \tag{D.11}$$

where $\lambda^{\sigma,0} = \lambda$ and $\lambda^{\sigma,i} = \lambda \backslash \{s_{\sigma(1)}, \dots, s_{\sigma(i)}\}$ for $i = 1, \dots, n$. We set $\langle \lambda^{\sigma,i-1}; f_{1,l_i}; \lambda^{\sigma,i} \rangle = 0$ if $\lambda^{\sigma,i-1}$ or $\lambda^{\sigma,i}$ is not a multipartition. We say that σ is *admissible* if $\lambda^{\sigma,1}, \dots, \lambda^{\sigma,n-1}$ are all multipartitions. If $n = 1$ then we have

$$\langle \lambda; f_{1,l}; \mu \rangle = \mathrm{c}_1(\tau_{\mu,\lambda})^l\, \mathrm{eu}\big(\mathrm{N}^*_{\lambda,\mu} - \mathrm{T}^*_\lambda\big) = \mathrm{c}_1(\tau_{\mu,\lambda})^l a_{\mu,\lambda}. \tag{D.12}$$

Hence, if σ is admissible then

$$\prod_{i=1}^n \big\langle \lambda^{\sigma,i-1}; f_{1,l_i}; \lambda^{\sigma,i} \big\rangle = \mathrm{c}_1(\tau_{s_{\sigma(1)}})^{l_1} \cdots \mathrm{c}_1(\tau_{s_{\sigma(n)}})^{l_n}\, \mathrm{eu}\bigg(\sum_{i=1}^n \mathrm{N}^*_{\lambda^{\sigma,i-1},\lambda^{\sigma,i}} - \mathrm{T}^*_{\lambda^{\sigma,i-1}}\bigg).$$

Now let $\sigma \in \mathfrak{S}_n$ be arbitrary. Using (3.10), (3.11), we get after a straightforward computation

$$\begin{aligned}\sum_{i=1}^n \big(\mathrm{N}^*_{\lambda^{\sigma,i-1},\lambda^{\sigma,i}} - \mathrm{T}^*_{\lambda^{\sigma,i-1}}\big) &= \big((1-q)(1-t)\big(\tau^*_{\mu,\lambda} \otimes \tau_\lambda\big) - \tau^*_{\mu,\lambda} \otimes \mathrm{W} - nv^{-1}\big) \\ &\quad - (1-q)(1-t) \sum_{i>j} \tau^*_{\lambda^{\sigma,i},\lambda^{\sigma,i-1}} \otimes \tau_{\lambda^{\sigma,j},\lambda^{\sigma,j-1}}.\end{aligned} \tag{D.13}$$

We have already seen in the proof of Proposition 3.13 that

$$a_{\mu,\lambda} = \mathrm{eu}\big((1-q)(1-t)\big(\tau^*_{\mu,\lambda} \otimes \tau_\lambda\big) - \tau^*_{\mu,\lambda} \otimes \mathrm{W} - nv^{-1}\big)$$

is nonzero and well-defined. A similar reasoning shows that

$$\mathrm{eu}\bigg(-(1-q)(1-t) \sum_{i>j} \tau^*_{\lambda^{\sigma,i},\lambda^{\sigma,i-1}} \otimes \tau_{\lambda^{\sigma,j},\lambda^{\sigma,j-1}}\bigg) \tag{D.14}$$

is well-defined and vanishes if σ is not admissible. Now note that

$$\mathrm{eu}\bigg(-(1-q)(1-t) \sum_{i>j} \tau^*_{\lambda^{\sigma,i},\lambda^{\sigma,i-1}} \otimes \tau_{\lambda^{\sigma,j},\lambda^{\sigma,j-1}}\bigg) = g\big(\mathrm{c}_1(\tau_{s_{\sigma(1)}}), \dots, \mathrm{c}_1(\tau_{s_{\sigma(n)}})\big). \tag{D.15}$$

It follows that

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_n} \mathrm{c}_1(\tau_{s_{\sigma(1)}})^{l_1}\cdots \mathrm{c}_1(\tau_{s_{\sigma(n)}})^{l_n}\,\mathrm{eu}\Big(-(1-q)(1-t)\sum_{i>j}\tau^*_{\lambda^{\sigma,i-1}\backslash\lambda^{\sigma,i}}\otimes\tau_{\lambda^{\sigma,j-1}\backslash\lambda^{\sigma,j}}\Big)\\
&\quad=\sum_{\sigma\in\mathfrak{S}_n} \mathrm{c}_1(\tau_{s_{\sigma(1)}})^{l_1}\cdots \mathrm{c}_1(\tau_{s_{\sigma(n)}})^{l_n} g\big(\mathrm{c}_1(\tau_{s_{\sigma(1)}}),\dots,\mathrm{c}_1(\tau_{s_{\sigma(n)}})\big)\\
&\quad=\varpi_n\big(z_1^{l_1}\cdots z_n^{l_n}\big)(\tau_{\mu,\lambda}).
\end{aligned}$$

Lemma 3.12 is an easy consequence.

Appendix E: The Heisenberg subalgebra

In this section we prove the formula in Section 1.11.

Lemma **E.1.** — *For* $k, l\geq 0$ *we have* $[\mathrm{D}_{l,0},\mathrm{D}_{k,0}]=[\mathrm{D}_{-l,0},\mathrm{D}_{-k,0}]=0.$

Proof. — Follows from Remark 1.29. □

Lemma **E.2.** — *For* $l\geq 0$ *we have* $[\mathrm{D}_{l,0},\mathrm{D}_{-1,0}]=-\mathrm{E}_0\,\delta_{l,1}$ *and* $[\mathrm{D}_{1,0},\mathrm{D}_{-l,0}]=-\mathrm{E}_0\,\delta_{l,1}$.

Proof. — The proof is by induction on l. From (1.69) we have $[\mathrm{D}_{1,0},\mathrm{D}_{-1,0}]=-\mathrm{E}_0$. Next, we have

$$[\mathrm{D}_{1,1},\mathrm{D}_{l,0}]=l\mathrm{D}_{l+1,0},\qquad [\mathrm{D}_{-1,1},\mathrm{D}_{-l,0}]=-l\mathrm{D}_{-l-1,0}. \tag{E.1}$$

Thus, using the induction hypothesis and (1.69), we get

$$l[\mathrm{D}_{l+1,0},\mathrm{D}_{-1,0}]=\big[[\mathrm{D}_{1,1},\mathrm{D}_{l,0}],\mathrm{D}_{-1,0}\big]=-\big[\mathrm{D}_{l,0},[\mathrm{D}_{1,1},\mathrm{D}_{-1,0}]\big]=[\mathrm{D}_{l,0},\mathrm{E}_1]. \tag{E.2}$$

Now, by definition, E_1 is a central element. Thus, we have

$$[\mathrm{D}_{l,0},\mathrm{D}_{-1,0}]=0,\qquad l\geq 2.$$

The second identity follows from the first one by applying the anti-involution π. □

Lemma **E.3.** — *For* $l\in\mathbf{Z}$ *we have* $[\mathrm{D}_{0,1},\mathrm{D}_{l,0}]=l\mathrm{D}_{l,0}$.

Proof. — From the faithful representation of $\mathbf{SH}^+$ in the Fock space, see Proposition 1.20, we get

$$[\mathrm{D}_{0,1},\mathrm{D}_{l,0}]=l\mathrm{D}_{l,0},\qquad l\geq 1.$$

Now apply the anti-automorphism π. □

Lemma **E.4.** — *For* $l \geq 2$ *we have* $[\mathrm{D}_{1,1}, \mathrm{D}_{-l,0}] = -\kappa l \mathrm{D}_{1-l,0}$ *and* $[\mathrm{D}_{-1,1}, \mathrm{D}_{l,0}] = \kappa l \mathrm{D}_{l-1,0}$.

Proof. — The second relation follows from the first one and the anti-involution π of $\mathbf{SH^c}$. Let us concentrate on the first relation. The proof is by induction on l. By (1.69) and the induction hypothesis we have

$$\begin{aligned}
[\mathrm{D}_{1,1}, \mathrm{D}_{-l,0}] &= \big[\mathrm{D}_{1,1}, [\mathrm{D}_{-1,1}, \mathrm{D}_{1-l,0}]\big]/(1-l) \\
&= \big[[\mathrm{D}_{1,1}, \mathrm{D}_{-1,1}], \mathrm{D}_{1-l,0}\big]/(1-l) + \big[\mathrm{D}_{-1,1}, [\mathrm{D}_{1,1}, \mathrm{D}_{1-l,0}]\big]/(1-l) \\
&= -[\mathrm{E}_2, \mathrm{D}_{1-l,0}]/(1-l) + \kappa [\mathrm{D}_{-1,1}, \mathrm{D}_{2-l,0}].
\end{aligned}$$

Now, using (A.4), we get that the element $\mathrm{E}_2 - 2\kappa \mathrm{D}_{0,1}$ is central. Therefore, by (E.1) and Lemma E.3, we have

$$\begin{aligned}
[\mathrm{D}_{1,1}, \mathrm{D}_{-l,0}] &= -2\kappa [\mathrm{D}_{0,1}, \mathrm{D}_{1-l,0}]/(1-l) + \kappa(2-l)\mathrm{D}_{1-l,0} \\
&= -2\kappa \mathrm{D}_{1-l,0} + \kappa(2-l)\mathrm{D}_{1-l,0} \\
&= -\kappa l \mathrm{D}_{1-l,0}.
\end{aligned}$$

□

We now prove the formula (1.90), which is equivalent to $(\mathrm{I}_{l,k})$ below

$$(\mathrm{I}_{l,k}) \qquad [\mathrm{D}_{l,0}, \mathrm{D}_{-k,0}] = \mathrm{A}_l \mathrm{E}_0 \, \delta_{l,k}, \qquad \mathrm{A}_l = -l\kappa^{l-1}, \qquad k, l \geq 1.$$

For $k = 1$ or $l = 1$ this is Lemma E.2. Let us prove the formula $(\mathrm{I}_{l,k+1})$, assuming that we have already proved $(\mathrm{I}_{\bullet,k})$ and $(\mathrm{I}_{l',k+1})$ for $l' \leq l$. Using $(\mathrm{I}_{l,k})$ and Lemma E.4 we get

$$\begin{aligned}
[\mathrm{D}_{l+1,0}, \mathrm{D}_{-k,0}] &= \big[[\mathrm{D}_{1,1}, \mathrm{D}_{l,0}], \mathrm{D}_{-k,0}\big]/l \\
&= -\big[\mathrm{D}_{l,0}, [\mathrm{D}_{1,1}, \mathrm{D}_{-k,0}]\big]/l \\
&= \kappa k [\mathrm{D}_{l,0}, \mathrm{D}_{1-k,0}]/l \\
&= \kappa(l+1)\mathrm{A}_l \mathrm{E}_0 \delta_{l+1,k}/l.
\end{aligned}$$

We deduce that $\mathrm{A}_{l+1} = \kappa(l+1)\mathrm{A}_l/l$. Since $\mathrm{A}_1 = -1$ this shows that $\mathrm{A}_l = -l\kappa^{l-1}$ as wanted. □

Appendix F: Relation to $\mathbf{W}_{1+\infty}$

Let $\mathrm{W}_{1+\infty}$ be the universal central extension of the Lie algebra, over $\mathbf{C}$, of regular differential operators on the circle. To unburden the notation we'll abbreviate $\mathfrak{W} = \mathrm{W}_{1+\infty}$. The aim of this section is to prove that the specialization at $\kappa = 1$ of $\mathbf{SH^c}$ is isomorphic to the enveloping algebra of $\mathfrak{W}$.

F.1 *The integral form of* $\mathbf{SH^c}$. — Let $\mathbf{SH}^{\mathbf{c}}_{\mathrm{A}}$ be the A-subalgebra of $\mathbf{SH^c}$ generated by the set $\{\mathbf{c}_l, \mathrm{D}_{\pm 1,0}, \mathrm{D}_{0,l};\, l \geq 0\}$. From (1.33) and (1.56) it follows that $\mathbf{SH}^{\mathbf{c}}_{\mathrm{A}}$ contains the elements $\mathrm{D}_{l,0}, \mathrm{D}_{\pm 1,l}$ for any $l \geq 0$. Let $\mathbf{SH}^{>}_{\mathrm{A}}$, $\mathbf{SH}^{\mathbf{c},0}_{\mathrm{A}}$ and $\mathbf{SH}^{<}_{\mathrm{A}}$ be the A-subalgebras generated by $\{\mathrm{D}_{1,l};\, l \geq 0\}$, $\{\mathbf{c}_l, \mathrm{D}_{0,l};\, l \geq 0\}$ and $\{\mathrm{D}_{-1,l};\, l \geq 0\}$ respectively. Recall (see Remark 1.4) that we denoted by A_1 the localization of A at the ideal $(\kappa - 1)$. Replacing everywhere A by A_1 we obtain the A_1-algebras $\mathbf{SH}^{>}_{\mathrm{A}_1}$, $\mathbf{SH}^{\mathbf{c},0}_{\mathrm{A}_1}$, $\mathbf{SH}^{<}_{\mathrm{A}_1}$ and $\mathbf{SH}_{\mathrm{A}_1}$.

Proposition **F.1.** — (a) *The* A_1*-module* $\mathbf{SH}^{\mathbf{c}}_{\mathrm{A}_1}$ *is free and we have* $\mathbf{SH}^{\mathbf{c}}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathrm{F} = \mathbf{SH^c}$.
(b) *We have a triangular decomposition* $\mathbf{SH}^{\mathbf{c}}_{\mathrm{A}_1} = \mathbf{SH}^{>}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathbf{SH}^{\mathbf{c},0}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathbf{SH}^{<}_{\mathrm{A}_1}$.

Proof. — We claim that $\mathbf{SH}^{>}_{\mathrm{A}_1}$, $\mathbf{SH}^{\mathbf{c},0}_{\mathrm{A}_1}$ and $\mathbf{SH}^{<}_{\mathrm{A}_1}$ are free over A_1, and that

$$\mathbf{SH}^{>}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathrm{F} = \mathbf{SH}^{>}, \qquad \mathbf{SH}^{\mathbf{c},0}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathrm{F} = \mathbf{SH}^{\mathbf{c},0}, \qquad \mathbf{SH}^{<}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathrm{F} = \mathbf{SH}^{<}.$$

Thus, we have an isomorphism

$$\big(\mathbf{SH}^{>}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathbf{SH}^{\mathbf{c},0}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathbf{SH}^{<}_{\mathrm{A}_1}\big) \otimes_{\mathrm{A}_1} \mathrm{F} = \mathbf{SH}^{>} \otimes \mathbf{SH}^{\mathbf{c},0} \otimes \mathbf{SH}^{<}.$$

Therefore, the multiplication map

$$\mathbf{SH}^{>}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathbf{SH}^{\mathbf{c},0}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathbf{SH}^{<}_{\mathrm{A}_1} \to \mathbf{SH}^{\mathbf{c}}_{\mathrm{A}_1},$$

being the restriction of a similar map over F, it is injective by Proposition 1.37. We only need to show its surjectivity. The proof is the same as for $\mathbf{SH^c}$ in Proposition 1.37. It is based on the fact that $\mathrm{D}_{-1,l}, \mathrm{D}_{1,l} \in \mathbf{SH}^{\mathbf{c}}_{\mathrm{A}_1}$ for $l \geq 0$. Then, using the triangular decomposition, we get that $\mathbf{SH}^{\mathbf{c}}_{\mathrm{A}_1}$ is free as an A_1-module and that

$$\mathbf{SH}^{\mathbf{c}}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathrm{F} = \mathbf{SH^c}.$$

Now, we prove the claim. It is clear for $\mathbf{SH}^{\mathbf{c},0}_{\mathrm{A}_1}$. The remaining two cases are similar, we only deal with the first one. Recall that $\mathbf{SH}^{>}$ carries an $\mathbf{N}$-grading and an $\mathbf{N}$-filtration, with finite-dimensional pieces $\mathbf{SH}^{>}[r, \leq l]$. Consider the A_1-module

$$\mathbf{SH}^{>}_{\mathrm{A}_1}[r, \leq l] = \mathbf{SH}^{>}_{\mathrm{A}_1} \cap \mathbf{SH}^{>}[r, \leq l].$$

Since the tensor product commutes with direct limits, it is enough to check that

$$\mathbf{SH}^{>}_{\mathrm{A}_1}[r, \leq l] \otimes_{\mathrm{A}_1} \mathrm{F} = \mathbf{SH}^{>}[r, \leq l] \tag{\textbf{F.1}}$$

and that the inclusion of A_1-modules

$$\mathbf{SH}^{>}_{\mathrm{A}_1}[r, < l] \subset \mathbf{SH}^{>}_{\mathrm{A}_1}[r, \leq l] \tag{\textbf{F.2}}$$

is a direct summand. Now, for n large enough the map Φ_n yields an isomorphism

$$\mathbf{SH}^{>}[r, < l] \to \mathbf{SH}^{>}_{n}[r, < l].$$

By Remark 1.4, this map restricts to an isomorphism of A_1-modules

$$\mathbf{SH}^{>}_{A_1}[r, <l] \to \mathbf{SH}^{>}_{n,A_1}[r, <l].$$

In particular, the left hand side is finitely generated and torsion free. Hence it is free. Further, (F.1) holds by (1.11). Finally, to prove (F.2) it is enough to check that the inclusion of A_1-modules

$$\mathbf{SH}^{>}_{n,A_1}[r, <l] \subset \mathbf{SH}^{>}_{n,A_1}[r, \leq l]$$

is a direct summand. This follows from the fact that the inclusions of A_1-modules

$$\mathbf{H}^{>}_{n,A_1}[r, <l] \subset \mathbf{H}^{>}_{n,A_1}[r, \leq l], \qquad \mathbf{SH}^{>}_{n,A_1}[r, \leq l] \subset \mathbf{H}^{>}_{n,A_1}[r, \leq l]$$

are direct summands, by the PBW theorem and formula

$$\mathbf{SH}^{>}_{n,A_1}[r, \leq l] = \mathbf{S} \cdot \mathbf{H}^{>}_{n,A_1}[r, \leq l] \cdot \mathbf{S}. \qquad \square$$

F.2 *The Lie algebra* $W_{1+\infty}$. — The Lie algebra $\mathfrak{W}$ has the basis $\{C, w_{l,k};\, l \in \mathbf{Z},\, k \in \mathbf{N}\}$ and, given formal variables α, β, the relations are given by

$$\begin{aligned} w_{l,k} &= t^l D^k, \\ \big[t^l \exp(\alpha D), t^k \exp(\beta D)\big] &= \big(\exp(k\alpha) - \exp(l\beta)\big)\, t^{l+k} \exp(\alpha D + \beta D) \\ &\quad + \delta_{l,-k} \frac{\exp(-l\alpha) - \exp(-k\beta)}{1 - \exp(\alpha + \beta)} C. \end{aligned} \tag{F.3}$$

Example **F.2.** — The elements $b_l = w_{l,0}$ with $l \in \mathbf{Z}$ satisfy the relations of the Heisenberg algebra with central charge C, i.e., we have $[b_l, b_{-k}] = l\delta_{l,k} C$. Let $\mathscr{H}$ be this Lie subalgebra.

Example **F.3.** — For $\beta \in \mathbf{C}$, the elements $L^{\beta}_l = -w_{l,1} - \beta(l+1)b_l$ with $l \in \mathbf{Z}$ satisfy the relations of the Virasoro algebra, i.e., we have

$$\big[L^{\beta}_l, L^{\beta}_k\big] = (l-k)L^{\beta}_{l+k} + \frac{l^3 - l}{12}\,\delta_{l,-k}\, C_{\beta}, \qquad C_{\beta} = \big(-12\beta^2 + 12\beta - 2\big)\, C.$$

In particular $\{L^{1/2}_l;\, l \in \mathbf{Z}\}$ generates a Virasoro algebra of central charge C such that

$$\big[L^{1/2}_l, b_k\big] = -k b_{l+k}.$$

Example **F.4.** — We have the following formulas

$$[w_{0,2}, w_{l,k}] = 2l w_{l,k+1} + l^2 w_{l,k}, \qquad [b_1, w_{l,k}] = -\sum_{h=0}^{k-1} \binom{k}{h} w_{l+1,h} + \delta_{l,-1}\delta_{k,0} C.$$

In particular, we have $[w_{0,2}, b_l] = -2l L^{1/2}_l - l b_l$.

Let $\mathfrak{W}^+, \mathfrak{W}^>, \mathfrak{W}^0 \subset \mathfrak{W}$ be the Lie subalgebras spanned by

$$\{\mathrm{C}, w_{l,k};\, l, k \in \mathbf{N}\}, \qquad \{w_{l,k};\, l \geq 1,\, k \in \mathbf{N}\}, \qquad \{\mathrm{C}, w_{0,l};\, l \geq 0\}.$$

We define $\mathfrak{W}^-$ and $\mathfrak{W}^<$ in a similar fashion. The enveloping algebra $\mathrm{U}(\mathfrak{W})$ carries a $\mathbf{Z}$-grading, called the *rank grading*, in which $w_{l,k}$ is placed in degree l and C is placed in degree zero, and an $\mathbf{N}$-filtration, called the *order filtration*, in which $w_{l,k}$ is placed in degree $\leq k$. The order filtration may alternatively be described as follows: an element u is of order $\leq k$ if

$$\mathrm{ad}(p_1) \circ \cdots \circ \mathrm{ad}(p_k)(u) \in \mathbf{C}[\mathrm{C}, w_{l,0};\, l \in \mathbf{Z}], \quad \forall p_1, \ldots, p_k \in \mathrm{U}(\mathscr{H}).$$

Let $\mathrm{U}(\mathfrak{W})[r, \leq k]$ stands for the piece of degree r and order $\leq k$. The graded pieces $\mathrm{U}(\mathfrak{W}^>)[r, \leq k]$ and $\mathrm{U}(\mathfrak{W}^<)[r, \leq k]$ are finite-dimensional and the Poincaré polynomials of $\mathrm{U}(\mathfrak{W}^>)$ and $\mathrm{U}(\mathfrak{W}^<)$ with respect to this grading and filtration are given by

$$\text{(F.4)} \qquad \mathrm{P}_{\mathfrak{W}^>}(t, q) = \prod_{r>0} \prod_{k \geq 0} \frac{1}{1 - t^r q^k}, \qquad \mathrm{P}_{\mathfrak{W}^<}(t, q) = \prod_{r<0} \prod_{k \geq 0} \frac{1}{1 - t^r q^k}.$$

The proof of the following result is left to the reader.

Lemma **F.5.** — *The following holds*
(a) $\mathfrak{W}$ *is generated by* b_{-1}, b_1 *and* $w_{0,2}$,
(b) $\mathfrak{W}^>, \mathfrak{W}^<$ *are generated by* $\{w_{1,l};\, l \geq 0\}$, $\{w_{-1,l};\, l \geq 0\}$ *respectively.*

F.3 *The Fock space representation of* $\mathrm{W}_{1+\infty}$. — For $c, d \in \mathbf{C}$ we set

$$\mathrm{U}_{c,d}(\mathfrak{W}) = \mathrm{U}(\mathfrak{W})/(\mathrm{C}-c,\, b_0-d), \qquad \mathrm{U}_{c,d}(\mathscr{H}) = \mathrm{U}(\mathscr{H})/(\mathrm{C}-c,\, b_0-d).$$

Let $\mathrm{S}_{c,d}$ be the *irreducible vacuum module with level* (c, d), see [17, Sect. 1]. It is the top of the *Verma module*

$$\mathrm{M}_{c,d} = \mathrm{Ind}_{\mathfrak{W}^+}^{\mathfrak{W}}(\mathbf{C}_{c,d}),$$

where $\mathbf{C}_{c,d}$ is the one-dimensional $\mathfrak{W}^+$-module given by

$$w_{l,k} \mapsto 0, \quad l, k \geq 0,\ (l, k) \neq (0, 0), \qquad \mathrm{C} \mapsto c, \qquad b_0 \mapsto d.$$

We will mainly be interested in the pair η given by $(c, d) = (1, -1/2)$.

Proposition **F.6.** — (a) *The restriction of* S_η *to* $\mathscr{H}$ *is the level one Fock space of* $\mathscr{H}$.
(b) *The action of* $\mathrm{U}_\eta(\mathfrak{W})$ *on* S_η *is faithful.*

Proof. — See [17, Thm. 5.1] for (a). Now, we prove part (b). Let $\mathrm{I} \subset \mathrm{U}_\eta(\mathfrak{W})$ be the annihilator of S_η. Since $\mathrm{U}_\eta(\mathscr{H})$ acts faithfully on S_η we have $\mathrm{I} \cap \mathrm{U}_\eta(\mathscr{H}) = \{0\}$. The proposition is a consequence of the following lemma.

Lemma **F.7.** — *Let* I *be an ideal of* $\mathrm{U}(\mathfrak{W})$ *such that* $\mathrm{I} \cap \mathrm{U}(\mathscr{H}) = \{0\}$. *Then* $\mathrm{I} = \{0\}$.

Proof. — Let I be as above, and let $\mathrm{I}_0 \subseteq \mathrm{I}_1 \subseteq \cdots$ be the filtration on I induced from the order filtration on $\mathrm{U}(\mathfrak{W})$. Assuming that $\mathrm{I} \neq \{0\}$, let n be minimal such that $\mathrm{I}_n \neq \{0\}$. Since $\mathrm{I}_0 = \mathrm{I} \cap \mathrm{U}(\mathscr{H}) = \{0\}$, we have $n \geq 1$. Moreover, since

$$\mathrm{ad}(b_l)\big(\mathrm{U}(\mathfrak{W})[\leq n]\big) \subset \mathrm{U}(\mathfrak{W})[< n],$$

we have $[\mathrm{I}_n, \mathrm{U}(\mathscr{H})] = 0$. This contradicts the following claim.

Claim. — *The centralizer of* $\mathscr{H}$ *in* $\mathrm{U}(\mathfrak{W})$ *is* $\mathbf{C}\mathrm{C} \oplus \mathbf{C}b_0$.

Proof. — For $l \in \mathbf{Z}$ we consider the map

$$\sigma_l = \mathrm{ad}(b_l) : \mathrm{U}(\mathfrak{W})[\leq n]/\mathrm{U}(\mathfrak{W})[< n] \longrightarrow \mathrm{U}(\mathfrak{W})[\leq n-1]/\mathrm{U}(\mathfrak{W})[< n-1].$$

The space $\mathrm{U}(\mathfrak{W})[\leq n]/\mathrm{U}(\mathfrak{W})[< n]$ is identified with the degree $(\bullet, n)$ part of the polynomial ring $\mathbf{C}[\overline{w}_{h,k}; h, k]$. One checks from the definition of $\mathfrak{W}$ that σ_l acts as the derivation satisfying

$$\sigma_l(\overline{w}_{h,k}) = \begin{cases} -kl\overline{w}_{l+h,k-1} & \text{if } k \geq 1 \\ 0 & \text{if } k = 0 \end{cases}$$

From this it is easy to check that $\bigcap_l \mathrm{Ker}(\sigma_l) = \{0\}$ if $n \geq 1$. This implies that the centralizer of $\mathscr{H}$ in $\mathrm{U}(\mathfrak{W})$ is contained into $\mathrm{U}(\mathfrak{W})[\leq 0] = \mathrm{U}(\mathscr{H})$. The claim now follows from the fact that the center of $\mathrm{U}(\mathscr{H})$ is $\mathbf{C}\mathrm{C} \oplus \mathbf{C}b_0$. □

This finishes the proof of the lemma and of the proposition. □

Lemma **F.8.** — *The element* $w_{0,2}/2$ *acts in* S_η *as the Laplace-Beltrami operator specialized at* $\kappa = 1$, *i.e., we have*

$$\rho(w_{0,2}) = 2\,\square^1 = \sum_{k,l>0} (b_{-l}b_{-k}b_{l+k} + b_{-l-k}b_l b_k),$$

where $\rho : \mathrm{U}_\eta(\mathfrak{W}) \to \mathrm{End}(\mathrm{S}_\eta) = \mathrm{End}(\mathbf{C}[b_l;\, l < 0])$ *is the Fock representation.*

Proof. — The free field formula for $\square^1$ is obtained by setting $\kappa = 1$ in Proposition 8.14. Because S_η is cyclic over $\mathrm{U}(\mathscr{H})$, the action of $w_{0,2}$ on S_η is completely determined by the commutation relations of $w_{0,2}$ with $\{b_l;\, l \in \mathbf{Z}\}$ and by the equation $w_{0,2} \cdot 1 = 0$. Hence it is enough to check that $[\rho(w_{0,2}), b_l]/2 = [\square^1, b_l]$ for all l, because $\square^1 \cdot 1 = 0$. Likewise, the actions of the Virasoro operators $\mathrm{L}_l^{1/2}$ on S_η are fully determined by their commutation relations with the Heisenberg operators. More precisely, from the relations, see Example F.3,

$$\big[\mathrm{L}_l^{1/2}, b_k\big] = -kb_{l+k}, \qquad \mathrm{L}_0^{1/2}\cdot 1 = 1/4$$

it follows that

$$\rho\big(\mathrm{L}_0^{1/2}\big) = \sum_{k\geq 0} b_{-k}b_k, \qquad \rho\big(\mathrm{L}_l^{1/2}\big) = \sum_{k\in\mathbf{Z}} b_{l-k}b_k/2, \quad l\neq 0.$$

Now, one checks by a direct computation using Example F.4 that

$$\begin{aligned}\big[\rho(w_{0,2}), b_l\big]/2 &= \rho\big([w_{0,2}, b_l]\big)/2 = -l\rho\big(\mathrm{L}_l^{1/2} + b_l/2\big)\\ &= -l\Big(\sum_{k\in\mathbf{Z}} b_{l-k}b_k + b_l\Big)/2 = \big[\square^1, b_l\big]\end{aligned}$$

(recall that $b_0 = -1/2$). The lemma is proved. □

F.4 *The isomorphism at the level* 1. — Let $\mathbf{SH}^{(1)}_{\mathrm{A}_1}$ be the specialization of $\mathbf{SH}^{\mathbf{c}}_{\mathrm{A}_1}$ at $\mathbf{c} = (1, 0, 0, \ldots)$. Recall the representation $\rho : \mathbf{SH}^{(1)}_{\mathrm{A}_1} \to \mathrm{End}(\Lambda_{\mathrm{A}_1})$. We set

$$\mathbf{SH}^{(1)}_1 = \mathbf{SH}^{(1)}_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathbf{C}, \qquad \Lambda_1 = \Lambda_{\mathrm{A}_1} \otimes_{\mathrm{A}_1} \mathbf{C} \tag{F.5}$$

where A_1 acts on $\mathbf{C}$ via $\kappa \mapsto 1$. We identify S_η and Λ_1 via the assignment

$$b_{-l_1}\cdots b_{-l_r}\cdot 1 \mapsto p_{l_1}\cdots p_{l_r}\cdot 1, \quad l_1, \ldots, l_r \geq 1. \tag{F.6}$$

This identification intertwines the actions of the Heisenberg generators b_l in $\mathrm{U}_\eta(\mathfrak{W})$ with the Heisenberg generators $\mathrm{D}_{-l,0}$ in $\mathbf{SH}^{(1)}_1$ for $l \in \mathbf{Z}$. It intertwines also the action of $w_{0,2}/2$ with that of $\mathrm{D}_{0,2}$ by Lemma F.8 and Remark 1.23, see also Proposition 2.6. Since, by Lemma F.5 and Proposition 1.35, the algebras $\mathrm{U}_\eta(\mathfrak{W})$ and $\mathbf{SH}^{(1)}_1$ are respectively generated by $\{b_{-1}, w_{0,2}, b_1\}$ and $\{\mathrm{D}_{-1,0}, \mathrm{D}_{0,2}, \mathrm{D}_{1,0}\}$, and since by Proposition F.6 the representation on S_η is faithful we obtain in this way a canonical surjective algebra homomorphism

$$\Theta^1 : \mathbf{SH}^{(1)}_1 \to \mathrm{U}_\eta(\mathfrak{W}), \qquad \mathrm{D}_{-l,0} \mapsto b_l, \qquad \mathrm{D}_{0,2} \mapsto w_{0,2}/2, \quad l \in \mathbf{Z}. \tag{F.7}$$

Proposition **F.9.** — *The map* Θ^1 *is an algebra isomorphism.*

Proof. — Set $\mathbf{SH}^{(1),0}_1 = \mathbf{SH}^0_1/(\mathbf{c} - c)$. We first show that Θ^1 restricts to an isomorphism

$$\mathbf{SH}^0_1 \to \mathrm{U}_\eta\big(\mathfrak{W}^0\big). \tag{F.8}$$

By (1.67) we have $\mathrm{D}_{1,l} = \mathrm{ad}(\mathrm{D}_{0,2})^l(\mathrm{D}_{1,0})$ for $l \geq 0$. So, a direct computation proves that

$$\Theta^1(\mathrm{D}_{1,l}) = 2^{-l}\mathrm{ad}(w_{0,2})^l(b_{-1}) \in (-1)^l w_{-1,l} \oplus \bigoplus_{k=0}^{l-1} \mathbf{C} w_{-1,k}, \quad l \geq 0. \tag{F.9}$$

Thus, since $\mathrm{E}_l = [\mathrm{D}_{-1,0}, \mathrm{D}_{1,l}]$, we get

$$\Theta^1(\mathrm{E}_l) = \big[b_1, \Theta^1(\mathrm{D}_{1,l})\big] \in (-1)^{l+1} l w_{0,l-1} \oplus \bigoplus_{k=1}^{l-2} \mathbf{C} w_{0,k} \oplus \mathbf{C}, \quad l \geq 0. \tag{F.10}$$

Next, from (1.69) and (E.1), we have the following formula in $\mathbf{SH}_1^{(1)}$

$$\mathrm{E}_l \in l(l-1)\mathrm{D}_{0,l-1} + \mathbf{C}[\mathrm{D}_{0,1}, \dots, \mathrm{D}_{0,l-2}], \quad l \geq 2. \tag{F.11}$$

It follows that

$$\Theta^1(\mathrm{D}_{0,l}) \in (-1)^l w_{0,l}/l + \mathbf{C}[w_{0,1}, \dots, w_{0,l-1}], \quad l \geq 1. \tag{F.12}$$

Thus Θ^χ restricts to an isomorphism $\mathbf{SH}_1^{\chi,0} \to \mathrm{U}_\eta(\mathfrak{W}^0)$. Next, observe that

$$\Theta^1\big(\mathbf{SH}_1^>\big) \subset \mathrm{U}\big(\mathfrak{W}^<\big), \qquad \Theta^1\big(\mathbf{SH}_1^<\big) \subset \mathrm{U}\big(\mathfrak{W}^>\big). \tag{F.13}$$

Moreover, since

$$\begin{aligned} &\mathbf{SH}_1^{(1)} = \mathbf{SH}_1^> \otimes \mathbf{SH}_1^{(1),0} \otimes \mathbf{SH}_1^<, \\ &\mathrm{U}_\eta(\mathfrak{W}) = \mathrm{U}\big(\mathfrak{W}^<\big) \otimes \mathrm{U}_\eta\big(\mathfrak{W}^0\big) \otimes \mathrm{U}\big(\mathfrak{W}^>\big), \end{aligned} \tag{F.14}$$

by Proposition 1.37. and the PBW theorem, and since Θ^1 is surjective we deduce that

$$\Theta^1 : \mathbf{SH}_1^> \to \mathrm{U}\big(\mathfrak{W}^<\big), \quad \Theta^1 : \mathbf{SH}_1^< \to \mathrm{U}\big(\mathfrak{W}^>\big) \tag{F.15}$$

are surjective as well. It only remains to prove that they are isomorphisms. Both $\mathbf{SH}_1^>$ and $\mathrm{U}(\mathfrak{W}^<)$ carry a $\mathbf{Z}$-grading and an $\mathbf{N}$-filtration. The map Θ^1 is compatible with these gradings and filtrations, i.e., we have

$$\Theta^1\big(\mathbf{SH}_1^>[r, \leq l]\big) = \mathrm{U}\big(\mathfrak{W}^<\big)[-r, \leq l]. \tag{F.16}$$

But by Corollary 1.27 and (F.4) these spaces have the same dimension. It follows that

$$\Theta^1 : \mathbf{SH}_1^> \to \mathrm{U}\big(\mathfrak{W}^<\big) \tag{F.17}$$

is an isomorphism. The same holds for $\mathbf{SH}_1^<$. We are done. □

F.5 *The isomorphism for a general level.* — Now we construct a Z-algebra isomorphism

$$\Theta : \mathbf{SH}^{\mathbf{c}}_1 \to \mathrm{U}(\mathfrak{W}) \otimes \mathrm{Z}, \quad \mathrm{Z} = \mathbf{C}[\mathbf{c}_l;\, l \geq 1]. \tag{F.18}$$

The construction of Θ is inspired by Θ^1. Recall that (1.67), (1.68) yield

$$\mathrm{D}_{1,l} = \mathrm{ad}(\mathrm{D}_{0,2})^l(\mathrm{D}_{1,0}), \qquad \mathrm{D}_{-1,l} = (-1)^l \mathrm{ad}(\mathrm{D}_{0,2})^l(\mathrm{D}_{-1,0}), \quad l \geq 0. \tag{F.19}$$

Thus, by Proposition F.9, the assignments

$$\mathrm{D}_{1,l} \mapsto 2^{-l}\mathrm{ad}(w_{0,2})^l(b_{-1}), \qquad \mathrm{D}_{-1,l} \mapsto (-2)^{-l}\mathrm{ad}(w_{0,2})^l(b_1), \quad l \geq 0 \tag{F.20}$$

extend to algebra isomorphisms

$$\Theta : \mathbf{SH}^{>}_1 \to \mathrm{U}\big(\mathfrak{W}^{<}\big), \qquad \Theta : \mathbf{SH}^{<}_1 \to \mathrm{U}\big(\mathfrak{W}^{>}\big). \tag{F.21}$$

They coincide with the restrictions of Θ^1. Next, we lift the map

$$\Theta^1 : \mathbf{SH}^{(1),0}_1 \to \mathrm{U}_\eta\big(\mathfrak{W}^0\big) \tag{F.22}$$

to a Z-algebra isomorphism

$$\Theta : \mathbf{SH}^{\mathbf{c},0}_1 \to \mathrm{U}\big(\mathfrak{W}^0\big) \otimes \mathrm{Z}. \tag{F.23}$$

For $l \geq 2$ we have

$$\begin{gathered}\mathrm{E}_l \in l(l-1)\mathrm{D}_{0,l-1} + \mathrm{Z}[\mathbf{c}_0, \mathrm{D}_{0,1}, \dots, \mathrm{D}_{0,l-2}],\\ \big[b_1, \mathrm{ad}(w_{0,2})^l(b_{-1})\big] \in -(-2)^l l w_{0,l-1} \oplus \bigoplus_{k=1}^{l-2} \mathbf{C} w_{0,k} \oplus \mathbf{C}\mathrm{C} \oplus \mathbf{C} b_0.\end{gathered} \tag{F.24}$$

In particular, we have $\mathbf{SH}^{\mathbf{c},0}_1 = \mathrm{Z}[\mathbf{c}_0, \mathrm{E}_l;\, l \geq 1]$. Thus, there is a unique Z-algebra isomorphism Θ as in (F.23) such that

$$\Theta(\mathbf{c}_0) = \mathrm{C}, \qquad \Theta(\mathrm{E}_l) = 2^{-l}\big[b_1, \mathrm{ad}(w_{0,2})^l(b_{-1})\big], \quad l \geq 2. \tag{F.25}$$

We claim that the maps (F.21), (F.23) glue together into a Z-algebra isomorphism

$$\Theta : \mathbf{SH}^{\mathbf{c}}_1 \to \mathrm{U}(\mathfrak{W}) \otimes \mathrm{Z}. \tag{F.26}$$

By the triangular decomposition argument, it is enough to prove that Θ is an algebra morphism, i.e., that relations (1.67)–(1.69) hold in $\mathrm{U}(\mathfrak{W}) \otimes \mathrm{Z}$. This is clear for (1.67), (1.68), because Θ^1 is an algebra morphism and Θ is a lift of Θ^1. The relation (1.69) holds by construction, because

$$\begin{gathered}\Theta\big([\mathrm{D}_{-1,0}, \mathrm{D}_{1,l}]\big) = \Theta(\mathrm{E}_l) = \big[b_1, \Theta(\mathrm{D}_{1,l})\big], \quad l \geq 2,\\ \Theta\big([\mathrm{D}_{-1,0}, \mathrm{D}_{1,0}]\big) = \Theta(\mathbf{c}_0) = \mathrm{C} = [b_1, b_{-1}],\\ \Theta\big([\mathrm{D}_{-1,0}, \mathrm{D}_{1,1}]\big) = \Theta(-\mathbf{c}_1) = b_0 + \mathrm{C}/2 = \big[b_1, \mathrm{L}^{1/2}_{-1} + b_{-1}/2\big] = \big[b_1, \Theta(\mathrm{D}_{1,1})\big].\end{gathered} \tag{F.27}$$

Therefore, we have proved the following.

Theorem **F.10.** — *There is a unique* $\mathbf{Z}$*-algebra isomorphism* $\Theta : \mathbf{SH}^{\mathbf{c}}_1 \to \mathrm{U}(\mathfrak{W}) \otimes \mathbf{Z}$ *satisfying*

$$\Theta(\mathbf{c}_0) = \mathrm{C}, \qquad \Theta(\mathrm{D}_{-l,0}) = b_l, \qquad \Theta(\mathrm{D}_{0,2}) = w_{0,2}/2, \quad l \neq 0. \tag{F.28}$$

Appendix G: Complements on Section 9

We freely use the notations of Appendices B, C and D. We begin by explicitly computing $f_{-1,d}(\mathrm{G})$. By definition, we have $f_{-1,d}[\mathrm{M}_{r,n}] = c[\mathrm{M}_{r,n-1}]$ if and only if the quantity

$$c_\mu = \sum_{\substack{\lambda \supset \mu \\ |\lambda \setminus \mu| = 1}} \mathrm{eu}_\mu\, \mathrm{eu}_\lambda^{-1} \langle \mu; f_{-1,d}; \lambda \rangle \tag{G.1}$$

is equal to c for any r-partition μ. Using (D.1) and (D.6) we have

$$\begin{aligned} \mathrm{eu}_\mu\, \mathrm{eu}_\lambda^{-1} \langle \mu; f_{-1,d}; \lambda \rangle &= c_1(\tau_{\lambda,\mu})^d\, \mathrm{eu}\big(\mathrm{N}^*_{\lambda,\mu} - \mathrm{T}^*_\lambda\big) \\ &= (xy)^{-1} c_1(\tau_{\lambda,\mu})^d\, \mathrm{eu}\big(\tau^*_{\lambda,\mu}\mathrm{H}_\mu + 1\big). \end{aligned} \tag{G.2}$$

Furthermore, by Lemma D.1,

$$\tau^*_{\lambda,\mu}\mathrm{H}_\mu + 1 = qt \sum_{\sigma \subset \mu} \tau^*_{\lambda,\mu}\tau_{\sigma,\mu} - \sum_{\substack{\lambda' \supset \mu \\ \lambda' \neq \lambda}} \tau^*_{\lambda,\mu}\tau_{\lambda',\mu}, \tag{G.3}$$

where in first sum $|\mu \setminus \sigma| = 1$ while in the second $|\lambda' \setminus \mu| = 1$. Setting $a_\lambda = c_1(\tau_{\lambda,\mu})$ and $b_\sigma = c_1(\tau_{\mu,\sigma}) + x + y$ we obtain

$$c_\mu = (xy)^{-1} \sum_\lambda a_\lambda^d \frac{\prod_\sigma (b_\sigma - a_\sigma)}{\prod_{\lambda' \neq \lambda}(a_{\lambda'} - a_\lambda)}. \tag{G.4}$$

Lemma **G.1.** — *Let* $m \geq 0$, $n = m + r$ *and* $d \geq 0$. *Let* $z_1, \ldots, z_n, y_1, \ldots, y_m$ *be formal variables. Then*

$$\sum_i z_i^d \frac{\prod_k (y_k - z_i)}{\prod_{j \neq i}(z_j - z_i)} = \begin{cases} 0 & \text{if } d < r-1 \\ (-1)^{r-1} & \text{if } d = r-1 \\ (-1)^r \sum_k y_k - \sum_i z_i & \text{if } d = r. \end{cases} \tag{G.5}$$

Proof. — Let $\mathrm{P}_d(z,y)$ be the left hand side of the above expression. It is a rational function of degree $d - r + 1$ with at most simple poles along the divisors $z_i = z_j$. It is easy to see that the residue of $\mathrm{P}_d(z,y)$ along each of these divisors is in fact equal to zero, so that $\mathrm{P}_d(z,y)$ is a homogeneous polynomial of degree $d - r + 1$. This proves that $\mathrm{P}_d(z,y) = 0$

if $d < r-1$. To compute the scalar $\mathrm{P}_{r-1}(z,y)$ we may set $y_k = 0$ for all k and let $z_1 \mapsto \infty$. To compute $\mathrm{P}_r(z,y)$ we may likewise consider the limits $\mathrm{P}_r(z,y)/z_i$, $\mathrm{P}_r(z,y)/y_k$ as $z_i \mapsto \infty$ and $y_k \mapsto \infty$ respectively. $\square$

Using Lemma G.1 together with the fact that for a given r-partition μ,

$$\sum_\sigma b_\sigma - \sum_\lambda a_\lambda = -(e_1 + \cdots + e_r) \tag{G.6}$$

we deduce

$$c_\mu = \begin{cases} 0 & \text{if } d < r-1 \\ (-1)^{r-1} & \text{if } d = r-1 \\ (-1)^r(e_1 + \cdots + e_r) & \text{if } d = r \end{cases} \tag{G.7}$$

and thus that, for $d < r-1$,

$$\begin{aligned} &f_{-1,d}(\mathrm{G}) = 0, \qquad f_{-1,r-1}(\mathrm{G}) = -(1)^{r-1}(xy)^{-1}\mathrm{G}, \\ &f_{-1,r}(\mathrm{G}) = (-1)^r\Big(\sum_i e_i\Big)(xy)^{-1}\mathrm{G}. \end{aligned} \tag{G.8}$$

This proves (9.3) for $l = 1$, the first part of (9.4) and (9.5). Relation (9.3) for $l \geq 1$ and the second part of (9.4) follow since $\mathrm{D}_{-l,d}$ is obtained from $\mathrm{D}_{-1,d}$ and $\mathrm{D}_{-1,d+1}$ by iterated commutators with $\mathrm{D}_{-1,0}$ or $\mathrm{D}_{-1,1}$. Proposition 9.1 is proved. $\square$

Remark **G.2.** — The operator $\rho^{(r)}(x^{d-1+2l}y^l\mathrm{D}_{-l,d})$ preserves the lattice $\mathbf{L}^{(r)}$ by Remark 3.5, and it has the cohomological degree $2(2l - rl + d - 1)$ by Remark 3.6. Thus for $l \geq 1$ and $r \geq 2$ we may deduce directly that

$$\rho^{(r)}(\mathrm{D}_{-l,d})\big([\mathrm{M}_{r,n}]\big) = 0, \qquad \rho^{(r)}(\mathrm{D}_{-l,r-1})\big([\mathrm{M}_{r,n}]\big) \in \mathrm{K}_r[\mathrm{M}_{r,n-l}], \quad d < r-1. \tag{G.9}$$

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Manuscrit reçu le 30 mai 2012
Manuscrit accepté le 13 mars 2013
publié en ligne le 9 mai 2013.