Chern characters for proper equivariant homology theories and applications to K- and L-theory

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April 27, 2000

Abstract

We construct for an equivariant homology theory for proper equivariant CW-complexes an equivariant Chern character under certain conditions. This applies for instance to the sources of the assembly maps in the Farrell-Jones Conjecture with respect to the family $\mathcal{F}$ of finite subgroups and in the Baum-Connes Conjecture. Thus we get an explicit calculation of $\mathbb{Q} \otimes_{\mathbb{Z}} K_n(RG)$ and $\mathbb{Q} \otimes_{\mathbb{Z}} L_n(RG)$ for a commutative ring R with $\mathbb{Q} \subset R$ and of $\mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\text{top}}(C_r^*(G, F))$ for $F = \mathbb{R}, \mathbb{C}$ in terms of group homology, provided the Farrell-Jones Conjecture with respect to $\mathcal{F}$ resp. the Baum-Connes Conjecture is true.

Key words: equivariant homology theory, Chern character, K- and L-groups.
1991 mathematics subject classification: 55N91, 19D50, 19G24, 19K99.

0. Introduction and statements of results

In this paper we want to achieve the following two goals. Firstly, we want to construct an equivariant Chern character for a proper equivariant homology theory $\mathcal{H}_*^?$ which takes values in R-modules for a commutative ring R with $\mathbb{Q} \subset R$. The Chern character identifies $\mathcal{H}_n^G(X)$ with the associated Bredon homology which is much easier to handle and can often be simplified further. Secondly, we apply it to the sources of the assembly maps appearing in the Farrell-Jones Conjecture with respect to the family $\mathcal{F}$ of finite subgroups and in the Baum-Connes Conjecture. The target of these assembly maps are the groups we are interested in, namely, the rationalized algebraic K- and L-groups $\mathbb{Q} \otimes_{\mathbb{Z}} K_n(RG)$ and $\mathbb{Q} \otimes_{\mathbb{Z}} L_n(RG)$ of the group ring RG of a (discrete) group G with coefficients in R and the rationalized topological K-groups $\mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\text{top}}(C_r^*(G, F))$ of the reduced group C^*-algebra of G over $F = \mathbb{R}, \mathbb{C}$. These conjectures say that these assembly maps are isomorphisms. Thus combining them with our equivariant Chern character yields explicit computations of these rationalized K- and L-groups in terms of group homology and the K-groups and L-groups of the coefficient ring R resp. F. As an example of such a computation we state

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Theorem 0.1 *Let G be a (discrete) group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram*

$$\begin{array}{ccc} \oplus_{p+q=n} \oplus_{(g)\in T} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & \mathbb{C} \otimes_{\mathbb{Z}} K_n(\mathbb{C}G) \\ \downarrow & & \downarrow \\ \oplus_{p+q=n} \oplus_{(g)\in T} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & \mathbb{C} \otimes_{\mathbb{Z}} K_n^{\mathrm{top}}(C_r^*(G)) \end{array}$$

where $C_G\langle g\rangle$ is the centralizer of the cyclic group generated by g in G and the vertical arrows come from the obvious change of ring and of K-theory maps $K_q(\mathbb{C}) \to K_q^{\mathrm{top}}(\mathbb{C})$ and $K_n(\mathbb{C}G) \to K_n^{\mathrm{top}}(C_r^*(G))$. The horizontal arrows can be identified with the assembly maps occuring in the Farrell-Jones Conjecture with respect to $\mathcal{F}$ for $K_n(\mathbb{C}G)$ and in the Baum-Connes Conjecture for $K_n^{\mathrm{top}}(C_r^*(G))$ after applying $\mathbb{C} \otimes_{\mathbb{Z}} -$. If these conjectures are true for G, then the horizontal arrows are isomorphisms.

Throughout this paper all groups are discrete and R will denote a commutative associative ring with unit. A proper G-homology theory $\mathcal{H}_*^G$ assigns to any G-CW-pair (X,A) which is proper, i.e. all isotropy groups are finite, a $\mathbb{Z}$-graded R-module $\mathcal{H}_*^G(X,A)$ such that G-homotopy invariance, excision and the disjoint union axiom hold and there is a long exact sequence of a proper G-CW-pair. An equivariant proper homology theory $\mathcal{H}_*^?$ assigns to any group G a proper G-homology theory $\mathcal{H}_*^G$, and these are linked for the various groups G by an induction structure. An example is equivariant bordism for smooth oriented manifolds with proper orientation preserving group actions whose orbit spaces are compact. The main examples for us will be given by the sources of the assembly maps appearing in the Farrell-Jones Conjecture with respect to $\mathcal{F}$ and the Baum-Connes Conjecture. These notions will be explained in Section 1.

To any equivariant proper homology theory $\mathcal{H}_*^?$ we will construct in Section 3 another equivariant proper homology theory, the associated Bredon homology $\mathcal{BH}_*^?$. The point is that $\mathcal{BH}_*^?$ is much easier to handle than $\mathcal{H}_*^?$. We will construct an isomorphism of equivariant homology theories

$$\mathrm{ch}_*^? : \mathcal{BH}_?^* \xrightarrow{\cong} \mathcal{H}_*^?$$

in Section 4, provided that a certain technical assumption is fullfilled, namely, that the covariant $R\mathsf{Sub}(G,\mathcal{F})$-module $\mathcal{H}_q^G(G/?) \cong \mathcal{H}_q^?(*)$ is flat for all $q \in \mathbb{Z}$ and all groups G. There are some favourite situations, where this condition is automatically satisfied, and the Bredon homology $\mathcal{BH}_*^?$ can be computed further. Let FGINJ be the category of finite groups with injective group homomorphisms as morphisms. The equivariant homology theory defines a covariant functor $\mathcal{H}_q^?(*) : \mathsf{FGINJ} \to R - \mathrm{MOD}$ which sends H to $\mathcal{H}_q^H(*)$. Functoriality comes from the induction structure. Suppose that this functor can be extended to a Mackey functor. This essentially means that we also get a contravariant structure by restriction and the induction and restriction structures are related by a double coset formula (see Section 5). An important example of a Mackey functor is given by sending H to the rational, real or complex representation ring. Define for a finite group H

$$S_H\mathcal{H}_q^H(*) := \mathrm{coker}\left(\oplus_{K\subset H, K\neq H} \mathrm{ind}_K^H : \oplus_{K\subset H,K\neq H}\mathcal{H}_q^K(*) \to \mathcal{H}_q^H(*)\right).$$

For a subgroup $H \subset G$ we denote by N_GH the normalizer, by C_GH the centralizer of H in G and by W_GH the quotient N_GH/C_GH. Notice that W_GH is finite if H is finite.

Theorem 0.2 *Let F be a field of characteristic 0. Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in F-modules. Suppose that the covariant functor $\mathcal{H}_q^?(*) : \mathsf{FGINJ} \to F - \mathrm{MOD}$ extends to a Mackey functor for all $q \in \mathbb{Z}$. Let I be the set of conjugacy classes (H) of finite subgroups H of G. Then there is an isomorphism of proper homology theories*

$$\mathrm{ch}_*^? : \mathcal{BH}_*^? \xrightarrow{\cong} \mathcal{H}_*^?$$

such that for a group G and a proper G-CW-pair (X,A)

$$\mathcal{BH}_n^G(X,A) \;=\; \oplus_{p+q=n} \oplus_{(H)\in I} H_p(C_GH\backslash(X^H,A^H);F) \otimes_{F[W_GH]} S_H\mathcal{H}_q^H(*).$$

This theorem reduces the computation of $\mathcal{H}_n^G(X,A)$ to the computation of the singular or cellular homology F-modules $H_p(C_GH\backslash(X^H,A^H);F)$ of the CW-pairs $C_GH\backslash(X^H,A^H)$ including the obvious right W_GH-operation and of the left $F[W_GH]$-modules $S_H\mathcal{H}_q^H(*)$ which only involve the values $\mathcal{H}_q^G(G/H)=\mathcal{H}_q^H(*)$. Suppose that $\mathcal{H}_*^?$ comes with a restriction structure as explained in Section 6. Then it induces a Mackey structure on $\mathcal{H}_q^?(*)$ for all $q\in\mathbb{Z}$ and a preferred restriction structure on $\mathcal{BH}_*^?$ so that Theorem 0.2 applies and the equivariant Chern character is compatible with these restriction structures. If $\mathcal{H}_*^?$ comes with a multiplicative structure as explained in Section 6, then $\mathcal{BH}_*^?$ inherits a multiplicative structure and the equivariant Chern character is compatible with these multiplicative structures (see Theorem 6.3).

If we have the following additional structure which will be available in the examples we are interested in, we can simplify the Bredon homology further. Namely we assume that the Mackey functor $\mathcal{H}_q^H(*)$ is a module over the Green functor $\mathbb{Q}\otimes_{\mathbb{Z}}R_{\mathbb{Q}}(?)$ which assigns to a finite group H the rationalized ring of rational H-representations. This notion is explained in Section 7. In particular it yields for any finite group H the structure of a $\mathbb{Q}\otimes_{\mathbb{Z}}R_{\mathbb{Q}}(H)$-module on $\mathcal{H}_q^H(*)$. Let $\operatorname{class}_{\mathbb{Q}}(H)$ be the ring of functions $f:H\to\mathbb{Q}$ which satisfy $f(h_1)=f(h_2)$ if the cyclic subgroups $\langle h_1\rangle$ and $\langle h_2\rangle$ generated by h_1 and h_2 are conjugated in H. Taking characters yields an isomorphism of rings

$$\chi:\mathbb{Q}\otimes_{\mathbb{Z}}R_{\mathbb{Q}}(H)\xrightarrow{\cong}\operatorname{class}_{\mathbb{Q}}(H).$$

Given a finite cyclic group C, there is the idempotent $\theta_C^C\in\operatorname{class}_{\mathbb{Q}}(C)$ which assigns 1 to a generator of C and 0 to the other elements. This element acts on $\mathcal{H}_q^C(*)$. The image $\operatorname{im}\left(\theta_C^C:\mathcal{H}_q^C(*)\to\mathcal{H}_q^C(*)\right)$ of the map given by multiplication with θ_C^C is a direct summand in $\mathcal{H}_q^C(*)$.

Theorem 0.3 *Let F be a field of characteristic 0. Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in F-modules. Suppose that the covariant functor $\mathsf{FGINJ}\to R-\mathrm{MOD}$ sending H to $\mathcal{H}_q^H(*)$ extends to a Mackey functor for all $q\in\mathbb{Z}$ which is a module over the Green functor $\mathbb{Q}\otimes_{\mathbb{Z}}R_{\mathbb{Q}}(?)$ with respect to the inclusion $\mathbb{Q}\to F$. Let J be the set of conjugacy classes (C) of finite cyclic subgroups C of G. Then there is an isomorphism of proper homology theories*

$$\operatorname{ch}_*^?:\mathcal{BH}_*^?\xrightarrow{\cong}\mathcal{H}_*^?$$

such that

$$\mathcal{BH}_n^G(X,A)\quad=\quad\oplus_{p+q=n}\oplus_{(C)\in J}H_p(C_GC\backslash(X^C,A^C);F)\otimes_{F[W_GC]}\operatorname{im}\left(\theta_C^C:\mathcal{H}_q^C(*)\to\mathcal{H}_q^C(*)\right).$$

Since $K_q(R?)$, $L_q(R?)$ and $K_q^{\mathrm{top}}(C_r^*(?,F))$ are Mackey functors and come with module structures over the Green functor $\mathbb{Q}\otimes_{\mathbb{Z}}R_{\mathbb{Q}}(?)$ as explained in Section 8, Theorem 0.3 implies

Theorem 0.4 *Let R be an associative commutative ring with unit such that $\mathbb{Q}\subset R$. Denote by F the field $\mathbb{R}$ or $\mathbb{C}$. Let G be a (discrete) group. Let J be the set of conjugacy classes (C) of finite cyclic subgroups C of G. Then the rationalized assembly map in the Farrell-Jones Conjecture with respect to $\mathcal{F}$ for the algebraic K-groups $K_n(RG)$ and the algebraic L-groups $L_n(RG)$ and in the Baum Connes Conjecture for the topological K-groups $K_n^{\mathrm{top}}(C_r^*(G,F))$ can be identified with the homomorphisms*

$$\oplus_{p+q=n}\oplus_{(C)\in J}H_p(C_GC;\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\operatorname{im}\left(\theta_C^C:\mathbb{Q}\otimes_{\mathbb{Z}}K_q(RC)\to\mathbb{Q}\otimes_{\mathbb{Z}}K_q(RC)\right)\quad\to\quad\mathbb{Q}\otimes_{\mathbb{Z}}K_n(RG);$$

$$\oplus_{p+q=n}\oplus_{(C)\in J}H_p(C_GC;\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\operatorname{im}\left(\theta_C^C:\mathbb{Q}\otimes_{\mathbb{Z}}L_q(RC)\to\mathbb{Q}\otimes_{\mathbb{Z}}L_q(RC)\right)\quad\to\quad\mathbb{Q}\otimes_{\mathbb{Z}}L_n(RG);$$

$$\oplus_{p+q=n}\oplus_{(C)\in J}H_p(C_GC;\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\operatorname{im}\left(\theta_C^C:\mathbb{Q}\otimes_{\mathbb{Z}}K_q^{\mathrm{top}}(C_r^*(C,F))\to\mathbb{Q}\otimes_{\mathbb{Z}}K_q^{\mathrm{top}}(C_r^*(C,F))\right)\quad\to\quad\mathbb{Q}\otimes_{\mathbb{Z}}K_q^{\mathrm{top}}(C_r^*(G,F)).$$

In the L-theory case we assume that R comes with an involution $R\to R,\quad r\mapsto\overline{r}$ and that we use on RG the involution which sends $\sum_{g\in G}r_g\cdot g$ to $\sum_{g\in G}\overline{r_g}\cdot g^{-1}$.

If the Farrell-Jones Conjecture with respect to $\mathcal{F}$ resp. the Baum-Connes Conjecture is true, then these maps are isomorphisms.

Notice that in Theorem 0.3 and hence in Theorem 0.4 only cyclic groups occur. The basic input in the proof is essentially the same as in the proof of Artin's theorem that any character in the complex representation ring of a finite group H is rationally a linear combination of characters induced from cyclic subgroups. Moreover, we emphasize that all the splitting results are obtained after tensoring with $\mathbb{Q}$, no roots of unity are needed in our construction. In the special situation that the coefficient ring R is a field F of characteristic zero and we tensor with $\overline{F}\otimes_{\mathbb{Z}}?$ for an algebraic closure $\overline{F}$ of F, one can simplify the expressions further as carried out in Section 8. We have already mentioned a particular nice situation in Theorem 0.1. The computations of K- and L-groups integrally and with $R = \mathbb{Z}$ as coefficients are much harder (see for instance [18]).

Notice that Theorem 0.1 and the results of Section 8 show that the computation of the K- and L-theory of RG seems to split into a piece, which involves only the group and consists essentially of group homology, and a part, which involves only the coefficient ring and consists essentially of its K-theory. Moreover, a change of rings or change of K-theory map involves only the coefficient ring R and not the part involving the group. This seems to suggest to look for a proof of the Farrell-Jones Conjecture which works for all coefficients simultaneously. We refer to Example 1.5 and to [3], [9], [12], [13], [14] and [15] for more information about the Farrell-Jones and the Baum-Connes Conjectures and about the classes of groups, for which they have been proven.

We mention that a different construction of an equivariant Chern character has been given in [2] in the case, where $\mathcal{H}_*^G$ is equivariant K-homology after applying $\mathbb{C} \otimes_{\mathbb{Z}} -$. Moreover, the lower horizontal arrow in Theorem 0.1 has already been discussed there.

The paper is organized as follows

I would like to thank Tom Farrell for a lot of fruitful discussions of the Farrell-Jones Conjecture and related topics.

1. Equivariant homology theories

In this section we describe the axioms of a (proper) equivariant homology theory. The main examples for us are the source of the assembly map appearing in the Farrell-Jones Conjecture with respect to the family $\mathcal{F}$ of finite subgroups for algebraic K- and L-theory and the equivariant K-homology theory which appears as the source of the Baum-Connes assembly map and is defined in terms of Kasparov's equivariant KK-theory.

Fix a discrete group G and an associative commutative ring R with unit. A G-CW-pair (X, A) is a pair of G-CW-complexes. It is called *proper* if all isotropy groups of X are finite. Basic informations about

G-CW-pairs can be found for instance in [16, Section 1 and 2]. A *G-homology theory $\mathcal{H}_*^G$ with values in R-modules* is a collection of covariant functors $\mathcal{H}_n^G$ from the category of G-CW-pairs to the category of R-modules indexed by $n \in \mathbb{Z}$ together with natural tranformations $\partial_n^G(X,A) : \mathcal{H}_n^G(X,A) \to \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A,\emptyset)$ for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

(a) G-homotopy invariance
If f_0 and f_1 are G-homotopic maps $(X,A) \to (Y,B)$ of G-CW-pairs, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for $n \in \mathbb{Z}$;

(b) Long exact sequence of a pair
Given a pair (X,A) of G-CW-complexes, there is a long exact sequence

$$\ldots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X,A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X,A) \xrightarrow{\partial_n^G} \ldots,$$

where $i : A \to X$ and $j : X \to (X,A)$ are the inclusions;

(c) Excision
Let (X,A) be a G-CW-pair and let $f : A \to B$ be a cellular G-map of G-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G-CW-pair. Then the canonical map $(F,f) : (X,A) \to (X \cup_f B, B)$ induces an isomorphism

$$\mathcal{H}_n^G(F,f) : \mathcal{H}_n^G(X,A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B);$$

(d) Disjoint union axiom
Let $\{X_i \mid i \in I\}$ be a family of G-CW-complexes. Denote by $j_i : X_i \to \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\oplus_{i \in I} \mathcal{H}_n^G(j_i) : \oplus_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\left(\coprod_{i \in I} X_i\right)$$

is bijective.

If $\mathcal{H}_*^G$ is defined or considered only for proper G-CW-pairs (X,A), we call it a *proper G-homology theory $\mathcal{H}_*^G$ with values in R-modules.*

Let $\alpha : H \to G$ be a group homomorphism. Given an H-space X, define the *induction of X with f* to be the G-space $\operatorname{ind}_\alpha X$ which is the quotient of $G \times X$ by the right H-action $(g,x) \cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g,x) \in G \times X$. If $\alpha : H \to G$ is an inclusion, we also write ind_H^G instead of $\operatorname{ind}_\alpha$.

A *(proper) equivariant homology theory $\mathcal{H}_*^?$ with values in R-modules* is an assignment which associates to each discrete group G a (proper) G-homology theory $\mathcal{H}_*^G$ with values in R-modules together with the following so called *induction structure*. This induction structure links the various homology theories for different groups G. It will play a key role in the construction of the equivariant Chern character even if we want to carry it out only for a fixed group G.

Let $\alpha : H \to G$ be a group homomorphism and (X,A) be a H-CW-pair such that $\ker(\alpha)$ acts freely on X. Then there are for each $n \in \mathbb{Z}$ natural isomorphisms

$$\operatorname{ind}_\alpha : \mathcal{H}_n^H(X,A) \xrightarrow{\cong} \mathcal{H}_n^G(\operatorname{ind}_\alpha(X,A)). \tag{1.1}$$

We require

(a) Compatibility with the boundary homomorphisms
$\partial_n^G \circ \operatorname{ind}_\alpha = \operatorname{ind}_\alpha \circ \partial_n^H$;

(b) Functoriality

Let $\beta : G \to K$ be another group homomorphism such that $\ker(\beta \circ \alpha)$ acts freely on X. Then we have for $n \in \mathbb{Z}$

$$\mathrm{ind}_{\beta\circ\alpha} \;=\; \mathcal{H}_n^K(f_1) \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha : \mathcal{H}_n^H(X,A) \to \mathcal{H}_n^K(\mathrm{ind}_{\beta\circ\alpha}(X,A)),$$

where $f_1 : \mathrm{ind}_\beta \mathrm{ind}_\alpha(X,A) \xrightarrow{\cong} \mathrm{ind}_{\beta\circ\alpha}(X,A), \quad (k,g,x) \mapsto (k\beta(g),x)$ is the natural K-homeomorphism;

(c) Compatibility with conjugation

For $n \in \mathbb{Z}$, $g \in G$ and a (proper) G-CW-pair (X,A) the homomorphism $\mathrm{ind}_{c(g):G\to G} : \mathcal{H}_n^G(X,A) \to \mathcal{H}_n^G(\mathrm{ind}_{c(g):G\to G}(X,A))$ agrees with $\mathcal{H}_n^G(f_2)$ for the G-homeomorphism $f_2 : (X,A) \to \mathrm{ind}_{c(g):G\to G}(X,A)$ which sends x to $(1,g^{-1}x)$ in $G \times_{c(g)} (X,A)$.

We will later need

Lemma 1.2 *Consider finite subgroups $H,K \subset G$ and an element $g \in G$ with $gHg^{-1} \subset K$. Let $R_{g^{-1}} : G/H \to G/K$ be the G-map sending $g'H$ to $g'g^{-1}K$ and $c(g) : H \to K$ be the homomorphism sending h to ghg^{-1}. Let $\mathrm{pr} : \mathrm{ind}_{c(g):H\to K} * \to *$ be the projection. Then the following diagram commutes*

$$\begin{array}{ccc} \mathcal{H}_n^H(*) & \xrightarrow{\mathcal{H}_n^K(\mathrm{pr})\circ\mathrm{ind}_{c(g)}} & \mathcal{H}_n^K(*) \\ {\scriptstyle \mathrm{ind}_H^G}\downarrow{\scriptstyle\cong} & & {\scriptstyle \mathrm{ind}_K^G}\downarrow{\scriptstyle\cong} \\ \mathcal{H}_n^G(G/H) & \xrightarrow{\mathcal{H}_n^G(R_{g^{-1}})} & \mathcal{H}_n^G(G/K) \end{array}$$

Proof : Define a bijective G-map $f_1 : \mathrm{ind}_{c(g):G\to G} \mathrm{ind}_H^G * \to \mathrm{ind}_K^G \mathrm{ind}_{c(g):H\to K} *$ by sending $(g_1,g_2,*)$ in $G \times_{c(g)} G \times_H *$ to $(g_1gg_2g^{-1},1,*)$ in $G \times_K K \times_{c(g)} *$. The condition that induction is compatible with composition of group homomorphisms means precisely that the composition

$$\mathcal{H}_n^H(*) \xrightarrow{\mathrm{ind}_H^G} \mathcal{H}_n^G(\mathrm{ind}_H^G *) \xrightarrow{\mathrm{ind}_{c(g):G\to G}} \mathcal{H}_n^G(\mathrm{ind}_{c(g):G\to G} \mathrm{ind}_H^G *) \xrightarrow{\mathcal{H}_n^G(f_1)} \mathcal{H}_n^G(\mathrm{ind}_K^G \mathrm{ind}_{c(g):H\to K} *)$$

agrees with the composition

$$\mathcal{H}_n^H(*) \xrightarrow{\mathrm{ind}_{c(g):H\to K}} \mathcal{H}_n^K(\mathrm{ind}_{c(g):H\to K} *) \xrightarrow{\mathrm{ind}_K^G} \mathcal{H}_n^G(\mathrm{ind}_K^G \mathrm{ind}_{c(g):H\to K} *).$$

Naturality of induction implies $\mathcal{H}_n^G(\mathrm{ind}_K^G \mathrm{pr}) \circ \mathrm{ind}_K^G = \mathrm{ind}_K^G \circ \mathcal{H}_n^K(\mathrm{pr})$. Hence the following diagram commutes

$$\begin{array}{ccc} \mathcal{H}_n^H(*) & \xrightarrow{\mathcal{H}_n^K(\mathrm{pr})\circ\mathrm{ind}_{c(g):H\to K}} & \mathcal{H}_n^K(*) \\ {\scriptstyle \mathrm{ind}_H^G}\downarrow & & \downarrow{\scriptstyle \mathrm{ind}_K^G} \\ \mathcal{H}_n^G(G/H)) & \xrightarrow[\mathcal{H}_n^G(\mathrm{ind}_K^G \mathrm{pr})\circ\mathcal{H}_n^G(f_1)\circ\mathrm{ind}_{c(g):G\to G}]{} & \mathcal{H}_n^G(G/K) \end{array}$$

By the axioms the homomorphism $\mathrm{ind}_{c(g):G\to G} : \mathcal{H}_n^G(G/H) \to \mathcal{H}_n^G(\mathrm{ind}_{c(g):G\to G} G/H)$ agrees with $\mathcal{H}_n^G(f_2)$ for the map $f_2 : G/H \to \mathrm{ind}_{c(g):G\to G} G/H$ which sends $g'H$ to $(g'g^{-1},1H)$ in $G \times_{c(g)} G/H$. Since the composition $(\mathrm{ind}_K^G \mathrm{pr}) \circ f_1 \circ f_2$ is just $R_{g^{-1}}$, Lemma 1.2 follows. ■

Example 1.3 Let $\mathcal{K}_*$ be a homology theory for (non-equivariant) CW-pairs with values in R-modules. Examples are singular homology, oriented bordism theory or topological K-homology. Then we obtain two equivariant homology theories with values in R-modules by the following constructions

$$\begin{aligned}\mathcal{H}_n^G(X,A) &= \mathcal{K}_n(G\backslash X, G\backslash A);\\ \mathcal{H}_n^G(X,A) &= \mathcal{K}_n(EG\times_G(X,A)).\end{aligned}$$

The second one is called the *equivariant Borel homology associated to* $\mathcal{K}$. In both cases $\mathcal{H}_*^G$ inherits the structure of a G-homology theory from the homology structure on $\mathcal{K}_*$. Let $a : H\backslash X \xrightarrow{\cong} G\backslash(G\times_\alpha X)$ be the bijection sending Hx to $G(1,x)$. Define $b : EH\times_H X \to EG\times_G G\times_\alpha X$ by sending (e,x) to $(E\alpha(e),1,x)$ for $e\in EH$, $x\in X$ and $E\alpha : EH\to EG$ the α-equivariant map induced by α. Induction for a group homomorphism $\alpha : H\to G$ is induced by these maps a and b. If the kernel $\ker(\alpha)$ acts freely on X, the map b is a homotopy equivalence and hence in both cases $\operatorname{ind}_\alpha$ is bijective.

Example 1.4 Given a proper G-CW-pair (X,A), one can define the G-bordism group $\Omega_n^G(X,A)$ as the abelian group of G-bordism classes of maps $f : (M,\partial M)\to(X,A)$ whose sources are oriented smooth manifolds with orientation preserving proper smooth G-actions such that $G\backslash M$ is compact. The definition is analogous to the one in the non-equivariant case. This is also true for the proof that this defines a proper G-homology theory. There is an obvious induction structure coming from induction of H-spaces. It is well-defined because of the following fact. Let $\alpha : H\to G$ be a group homomorphism. Let M be an oriented smooth H-manifold with orientation preserving proper smooth H-action such that $H\backslash M$ is compact and $\ker(\alpha)$ acts freely. Then $\operatorname{ind}_\alpha M$ is an oriented smooth G-manifold with orientation preserving proper smooth G-action such that $G\backslash M$ is compact. The boundary of $\operatorname{ind}_\alpha M$ is $\operatorname{ind}_\alpha \partial M$.

Our main example will be

Example 1.5 Let R be a commutative associative ring with unit. There are equivariant homology theories $\mathcal{H}_*^?$ such that $\mathcal{H}_n^G(*)$ is the rationalized algebraic K-group $\mathbb{Q}\otimes_{\mathbb{Z}} K_n(RG)$ or the rationalized algebraic L-group $\mathbb{Q}\otimes_{\mathbb{Z}} L_n(RG)$ of the group ring RG or such that $\mathcal{H}_n^G(*)$ is the rationalized topological K-theory $\mathbb{Q}\otimes_{\mathbb{Z}} K_n(C_*^r(G;\mathbb{R}))$ resp. $\mathbb{Q}\otimes_{\mathbb{Z}} K_n(C_*^r(G;\mathbb{C}))$ of the reduced real resp. complex C^*-algebra of G. Let $\mathcal{F}$ resp. $\mathcal{VC}$ be the family of finite resp. of virtually cyclic subgroups of G. Denote by $E(G,\mathcal{F})$ resp. $E(G,\mathcal{VC})$ the classifying space of G with respect to the family $\mathcal{F}$ resp. $\mathcal{VC}$. This is a G-CW-complex whose H-fixed point set is contractible for $H\in\mathcal{F}$ resp. $H\in\mathcal{VC}$ and is empty otherwise. It is unique up to G-homotopy because it is characterized by the property that for any G-CW-complex X, whose isotropy groups belong to $\mathcal{F}$ resp. $\mathcal{VC}$, there is up to G-homotopy precisely one G-map from X to $E(G,\mathcal{F})$ resp. $E(G,\mathcal{VC})$. The G-space $E(G,\mathcal{F})$ agrees with the classifying space $\underline{E}G$ for proper G-actions. The assembly map in the Farrell-Jones Conjecture with respect to $\mathcal{F}$ resp. the Baum-Connes Conjecture are the maps induced by the projection $E(G,\mathcal{F})\to *$

$$\mathcal{H}_n^G(E(G,\mathcal{F})) \quad\to\quad \mathcal{H}_n^G(*), \tag{1.6}$$

where one has to choose the appropriate homology theory among the ones mentioned above. The Baum-Connes Conjecture says that this map is an isomorphism (even without rationalizing) for the topological K-theory of the reduced group C^*-algebra. The Farrell-Jones Conjecture with respect to $\mathcal{F}$ is the analogous statement.

It is important to notice that the situation in the Farrell-Jones Conjecture is more complicated. The Farrell-Jones Conjecture itself is formulated with respect to the family $\mathcal{VC}$, i.e. it says that the projection $E(G,\mathcal{VC})\to *$ induces an isomorphism (even without rationalizing)

$$\mathcal{H}_n^G(E(G,\mathcal{VC}))\to\mathcal{H}_n^G(*). \tag{1.7}$$

For the version of the Farrell-Jones Conjecture with respect to $\mathcal{VC}$ no counterexamples are known, whereas the version for $\mathcal{F}$ is not true in general. In other words, the canonical map $E(G,\mathcal{F})\to E(G,\mathcal{VC})$ does not necessarily induce an isomorphism $\mathcal{H}_n^G(E(G,\mathcal{F})) \to \mathcal{H}_n^G(E(G,\mathcal{VC}))$. This is due to the existence of Nil-groups. However, if for instance R is a field of characteristic zero, this map is bijective for algebraic K-theory. Hence the Farrell-Jones Conjecture for $\mathbb{Q} \otimes_{\mathbb{Z}} K_n(FG)$ for a field F of characteristic zero is true with respect to $\mathcal{F}$ if and only if it is true with respect to $\mathcal{VC}$. At the time of writing not much is known about this conjecture for $K_n(FG)$ for a field F of characteristic zero, since most of the known results are for the algebraic K-theory for $\mathbb{Z}G$. The situation in L-theory is better since the change of rings map $\mathbb{Q} \otimes_{\mathbb{Z}} L_n(\mathbb{Z}G) \to \mathbb{Q} \otimes_{\mathbb{Z}} L_n(\mathbb{Q}G)$ is bijective for any group G. The Farrell-Jones Conjecture for both $\mathbb{Q} \otimes_{\mathbb{Z}} L_n(\mathbb{Z}G)$ and $\mathbb{Q} \otimes_{\mathbb{Z}} L_n(\mathbb{Q}G)$ is true with respect to both $\mathcal{F}$ and $\mathcal{VC}$ if G is a cocompact discrete subgroup of a Lie group with finitely many path components [9], if G is a discrete subgroup of $GL_n(\mathbb{C}G)$ [10], or if G is an elementary amenable group [11].

The target of the assembly map for $\mathcal{F}$ in (1.6) is $\mathbb{Q} \otimes_{\mathbb{Z}} K_n(RG)$, $\mathbb{Q} \otimes_{\mathbb{Z}} L_n(RG)$ resp. $\mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\mathrm{top}}(C_r^*(G,F))$ for $F = \mathbb{R}, \mathbb{C}$. These are the groups we would like to compute. The source of the assembly map for $\mathcal{F}$ in (1.6) is the part which is better accessible for computations. We will apply the equivariant Chern character for proper equivariant homology theories to it which is possible since $E(G,\mathcal{F})$ is proper (in contrast to $E(G,\mathcal{VC})$ and $*$). Thus we get computations of the rationalized K- and L-groups, provided the Farrell-Jones Conjecture with respect to $\mathcal{F}$ resp. the Baum-Connes Conjecture is true.

For more informations about the relevant G-homology theories $\mathcal{H}_*^G$ mentioned above we refer to [3], [5], [9]. It is not hard to construct the relevant induction structures so that they yield equivariant homology theories $\mathcal{H}_*^?$. We remark that one can construct for them also restriction structures and multiplicative structures in the sense of Section 6.

We refer to [3], [9], [12], [13], [14] and [15] for more information about the Farrell-Jones and the Baum-Connes Conjectures and about the classes of groups, for which they have been proven.

2. Modules over a category

In this section we give a brief summary about modules over a category as far as needed for this paper. They will appear in the definition of the source of the equivariant Chern character.

Let $\mathcal{C}$ be a small category and let R be a commutative associative ring with unit. A *covariant resp. contravariant $R\mathcal{C}$-module* is a covariant resp. contravariant functor from $\mathcal{C}$ to the category $R-\mathrm{MOD}$ of R-modules. Morphisms of $R\mathcal{C}$-modules are natural transformations. Given a group G, let $\widehat{G}$ be the category with one object whose set of morphisms is given by G. Then a covariant resp. contravariant $R\widehat{G}$-module is the same as a left resp. right RG-module. All the constructions, which we will introduce for $R\mathcal{C}$-modules below, reduce in the special case $\mathcal{C} = \widehat{G}$ under the identification above to their classical versions for RG-modules. The reader should have this example in mind.

The category $R\mathcal{C}-\mathrm{MOD}$ of covariant resp. contravariant $R\mathcal{C}$-modules inherits the structure of an abelian category from $R-\mathrm{MOD}$ in the obvious way, namely objectwise. For instance a sequence $0 \to M \to N \to P \to 0$ of $R\mathcal{C}$-modules is called exact if its evaluation at each object in $\mathcal{C}$ is an exact sequence in $R-\mathrm{MOD}$. The notion of a projective $R\mathcal{C}$-module is now clear. Given a family $B = (c_i)_{i\in I}$ of objects of $\mathcal{C}$, the *free $R\mathcal{C}$-module with basis B* is

$$R\mathcal{C}(B) \quad := \quad \oplus_{i\in I} R\operatorname{mor}_{\mathcal{C}}(c_i, ?).$$

The name free with basis B refers to the following basic property. Given a covariant $R\mathcal{C}$-module N, there is a natural bijection

$$\hom_{R\mathcal{C}}(R\mathcal{C}(B), N) \quad \xrightarrow{\cong} \quad \prod_{i\in I} N(c_i), \qquad f \mapsto (f(c_i)(\mathrm{id}_{c_i}))_{i\in I}. \tag{2.1}$$

Obviously $R\mathcal{C}(B)$ is a projective $R\mathcal{C}$-module. Any $R\mathcal{C}$-module is a quotient of some free $R\mathcal{C}$-module. For instance, there is an obvious epimorphism from $R\mathcal{C}(B)$ to M if we take B to be the family of objects indexed by $\coprod_{c\in \mathrm{Ob}(\mathcal{C})} M(c)$, where we assign c to $m \in M(c)$. Therefore a $R\mathcal{C}$-module M is projective if and only if it is a direct summand in a free $R\mathcal{C}$-module. The analogous considerations apply to the contravariant case.

Given a contravariant $R\mathcal{C}$-module M and a covariant $R\mathcal{C}$-module N, one can define a R-module $M\otimes_{R\mathcal{C}}N$ called *tensor product over* $R\mathcal{C}$ as follows. It is given by

$$M \otimes_{R\mathcal{C}} N \ = \oplus_{c\in \mathrm{Ob}(\mathcal{C})} M(c) \otimes_R N(c)/\sim,$$

where $\sim$ is the typical tensor relation $mf \otimes n = m \otimes fn$, i.e. for each morphism $f : c \to d$ in $\mathcal{C}$, $m \in M(d)$ and $n \in N(c)$ we introduce the relation $M(f)(m) \otimes n - m \otimes N(f)(n) = 0$. The main property of this construction is that it is adjoint to the $\hom_R$-functor in the sense that for any R-module L there are natural isomorphisms of R-modules

$$\hom_R(M \otimes_{R\mathcal{C}} N, L) \xrightarrow{\cong} \hom_{R\mathcal{C}}(M, \hom_R(N, L)); \tag{2.2}$$

$$\hom_R(M \otimes_{R\mathcal{C}} N, L) \xrightarrow{\cong} \hom_{R\mathcal{C}}(N, \hom_R(M, L)). \tag{2.3}$$

Consider a functor $F : \mathcal{C} \to \mathcal{D}$. Given a covariant resp. contravariant $R\mathcal{D}$-module M, define the covariant resp. contravariant $R\mathcal{C}$-module $\mathrm{res}_F M$ called *restriction with* F to be $M \circ F$. Given a covariant resp. contravariant $R\mathcal{C}$-module M, one can define a covariant resp. contravariant $R\mathcal{D}$-module $\mathrm{ind}_F M$ called *induction with* F. In the covariant case it is defined by

$$\mathrm{ind}_F M(??) := R\,\mathrm{mor}_{\mathcal{D}}(F(?), ??) \otimes_{R\mathcal{C}} M(?).$$

Restriction with F can be written as $\mathrm{res}_F N(?) = \hom_{R\mathcal{D}}(R\,\mathrm{mor}_{\mathcal{D}}(F(?), ??), N(??))$ because of (2.1). We conclude from (2.3) that induction and restriction form an adjoint pair, i.e. for a covariant resp. contravariant $R\mathcal{C}$-module M and a covariant resp. contravariant $R\mathcal{D}$-module N there is a natural isomorphism of R-modules

$$\hom_{R\mathcal{D}}(\mathrm{ind}_F M, N) \xrightarrow{\cong} \hom_{R\mathcal{C}}(M, \mathrm{res}_F N). \tag{2.4}$$

Given a contravariant $R\mathcal{C}$-module M and a covariant $R\mathcal{D}$-module N, there is a natural R-isomorphism

$$\mathrm{ind}_F M \otimes_{R\mathcal{D}} N \xrightarrow{\cong} M \otimes_{R\mathcal{C}} \mathrm{res}_F N. \tag{2.5}$$

It is explicitly given by $(f :?? \to F(?)) \otimes m \otimes n \ \mapsto \ m \otimes N(f)(n)$ or can be obtained formally from (2.2) and (2.4). One easily checks

$$\mathrm{ind}_F R\,\mathrm{mor}_{\mathcal{C}}(c, ?) = R\,\mathrm{mor}_{\mathcal{D}}(F(c), ??) \tag{2.6}$$

for $c \in \mathrm{Ob}(\mathcal{C})$. This shows that ind_F respects direct sums and the properties free and projective.

Next we explain how one can reduce the study of projective R$\mathcal{C}$-modules to the study of projective $R\,\mathrm{aut}(c)$-modules, where $\mathrm{aut}(c)$ is the group of automorphism of an object c in $\mathcal{C}$. Given a covariant $R\mathcal{C}$-module M, we obtain for each object c in $\mathcal{C}$ a left $R\,\mathrm{aut}(c)$-module $R_cM := M(c)$. Given a left $R\,\mathrm{aut}(c)$-module N, we obtain a covariant $R\mathcal{C}$-module E_cN by

$$E_cN(?) \quad := \quad R\,\mathrm{mor}_{\mathcal{C}}(c, ?) \otimes_{R\,\mathrm{aut}(c)} N. \tag{2.7}$$

Notice that E_c resp. R_c is induction resp. restriction with the obvious inclusion of categories $\widehat{\mathrm{aut}(c)} \to \mathcal{C}$. Hence E_c and R_c form an adjoint pair by (2.4). In particular we get for any covariant $R\mathcal{C}$-module M an in M natural homomorphism

$$i_c(M) : E_cM(c) \quad \to M \tag{2.8}$$

by the adjoint of $\mathrm{id} : R_c M \to R_c M$. Explicitly $i_c(M)$ maps $(f : c \to ?) \otimes_R m$ to $M(f)(m)$. Given a covariant $R\mathcal{C}$-module M, define $M(c)_s$ to be the R-submodule of $M(c)$ which is spanned by the images of all R-maps $M(f) : M(b) \to M(c)$, where f runs through all morphisms $f : b \to c$ with target c which are not isomorphisms in $\mathcal{C}$. Obviously $M(c)_s$ is an $R\operatorname{aut}(c)$-submodule of $M(c)$. Define a left $R\operatorname{aut}(c)$-module $S_c M$ by

$$S_c M \quad := \quad M(c)/M(c)_s. \tag{2.9}$$

Notice that E_c maps $R\operatorname{aut}(c)$ to $R\operatorname{mor}_{\mathcal{C}}(c,?)$ and that $S_c R\operatorname{mor}_{\mathcal{C}}(d,?) \cong_{R\operatorname{aut}(c)} R\operatorname{aut}(c)$, if $c \cong d$, and $S_c R\operatorname{mor}_{\mathcal{C}}(d,?) = 0$ otherwise. This implies for a free $R\mathcal{C}$-module $M = \oplus_{i \in I} R\operatorname{mor}_{\mathcal{C}}(c_i,?)$

$$\oplus_{(c) \in \operatorname{Is}(\mathcal{C})} E_c S_c M \cong_{R\mathcal{C}} M,$$

where $\operatorname{Is}(\mathcal{C})$ is the set of isomorphism classes (c) of objects c in $\mathcal{C}$. This splitting can be extended to projective modules as follows.

Let M be an $R\mathcal{C}$-module. We want to check whether it is projective or not. Since S_c is compatible with direct sums and each projective module is a direct sum in a free $R\mathcal{C}$-module, a necessary (but not sufficient) condition is that $S_c M$ is a projective $R\operatorname{aut}(c)$-module. Assume that $S_c M$ is $R\operatorname{aut}(c)$-projective for all objects c in $\mathcal{C}$. We can choose a $R\operatorname{aut}(c)$-splitting $\sigma_c : S_c M \to M(c)$ of the canonical projection $M(c) \to S_c M = M(c)/M(c)_s$. Then we obtain after a choice of representatives $c \in (c)$ for any $(c) \in \operatorname{Is}(\mathcal{C})$ a morphism of $R\mathcal{C}$-modules

$$T : \oplus_{(c) \in \operatorname{Is}(\mathcal{C})} E_c S_c M \xrightarrow{\oplus_{(c) \in \operatorname{Is}(\mathcal{C})} E_c \sigma_c} \oplus_{(c) \in \operatorname{Is}(\mathcal{C})} E_c M(c) \xrightarrow{\oplus_{(c) \in \operatorname{Is}(\mathcal{C})} i_c(M)} M, \tag{2.10}$$

where $i_c(M)$ has been introduced in (2.8).

We call $\mathcal{C}$ an *EI-category* if any endomorphism in $\mathcal{C}$ is an isomorphism. The *length* $l(c) \in \mathbb{N} \cup \{\infty\}$ of an object c is the supremum over all natural numbers l for which there exists a sequence of morphisms $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} \ldots \xrightarrow{f_l} c_l$ such that no f_i is an isomorphism and $c_l = c$. If each object c has length $l(c) < \infty$, we say that $\mathcal{C}$ *has finite length.*

Theorem 2.11 *Let $\mathcal{C}$ be an EI-category of finite length. Let M be a covariant $R\mathcal{C}$-module such that the $R\operatorname{aut}(c)$-module $S_c M$ is projective for all objects c in $\mathcal{C}$. Let $\sigma_c : S_c M \to M(c)$ be an $R\operatorname{aut}(c)$-section of the canonical projection $M(c) \to S_c M$. Then the map introduced in* (2.10)

$$T : \oplus_{(c) \in \operatorname{Is}(\mathcal{C})} E_c S_c M \to M$$

is surjective. It is bijective if and only if M is a projective $R\mathcal{C}$-module.

Proof : We show by induction over the length $l(c)$ that $T(c)$ is surjective. For any object c and $R\operatorname{aut}(c)$-module N there is an in N natural $\operatorname{aut}(c)$-isomorphism $N(c) \xrightarrow{\cong} S_c E_c N$ which sends n to the class of $(\mathrm{id} : c \to c) \otimes n$. If b and c are non-isomorphic objects in $\mathcal{C}$, then $S_b E_c N = 0$. This implies that $S_c T$ is an isomorphism for all objects $c \in \mathcal{C}$. Hence it suffices for the proof of surjectivity of $T(c)$ to show that each element of $M(c)_s$ is in the image of $T(c)$. It is enough to verify this for an element of the form $M(f)(x)$ for $x \in M(b)$ and a morphism $f : b \to c$ which is not an isomorphism in $\mathcal{C}$. Since $\mathcal{C}$ is an EI-category, $l(b) < l(c)$. By induction hypothesis $T(b)$ is surjective and the claim follows.

Suppose that T is injective. Then T is an isomorphism of $R\mathcal{C}$-modules. Its source is projective since E_c sends projective $R\operatorname{aut}(c)$-modules to projective $R\mathcal{C}$-modules. Therefore M is projective. We will not need the other implication that for projective M the map T is bijective in this paper. Therefore we omit its proof but refer to [16, Theorem 3.39 and Corollary 9.40]. ■

Given a contravariant $R\mathcal{C}$-module M and a left $R\operatorname{aut}(c)$-module N, there is a natural isomorphism

$$M \otimes_{R\mathcal{C}} E_c N \quad \cong \quad M(c) \otimes_{R\operatorname{aut}(c)} N. \tag{2.12}$$

It is explicitly given by $m \otimes (f : c \to ?) \otimes n \mapsto M(f)(m) \otimes n$. It is due to the fact that tensor products are associative. For more details about modules over a category we refer to [16, Section 9A].

3. The associated Bredon homology theory

Given a (proper) G-homology theory resp. equivariant homology theory with values in R-modules, we can associate to it another (proper) G-homology theory resp. equivariant homology theory with values in R-modules called Bredon homology, which is much simpler. The equivariant Chern character will identify this simpler homology theory with the given one.

Before we give the construction we have to organize the coefficients of a G-homology theory $\mathcal{H}_*^G$. The smallest bulding blocks of G-CW-complexes or G-spaces in general are the homogeneous spaces G/H. The book keeping of all the values $\mathcal{H}_*^G(G/H)$ is organized using the following two categories.

The *orbit category* $\mathsf{Or}(G)$ has as objects homogeneous spaces G/H and as morphisms G-maps. Let $\mathsf{Sub}(G)$ be the category whose objects are subgroups H of G. For two subgroups H and K of G denote by $\operatorname{conhom}_G(H,K)$ the set of group homomorphisms $f: H \to K$, for which there exists an element $g \in G$ with $gHg^{-1} \subset K$ such that f is given by conjugation with g, i.e. $f = c(g): H \to K, \quad h \mapsto ghg^{-1}$. Notice that $c(g) = c(g')$ holds for two elements $g, g' \in G$ with $gHg^{-1} \subset K$ and $g'H(g')^{-1} \subset K$ if and only if $g^{-1}g'$ lies in the centralizer $C_GH = \{g \in G \mid gh = hg \text{ for all } h \in H\}$ of H in G. The group of inner automorphisms of K acts on $\operatorname{conhom}_G(H,K)$ from the left by composition. Define the set of morphisms $\operatorname{mor}_{\mathsf{Sub}(G)}(H,K)$ by $\operatorname{Inn}(K)\backslash \operatorname{conhom}_G(H,K)$.

There is a natural projection $\operatorname{pr}: \mathsf{Or}(G) \to \mathsf{Sub}(G)$ which sends a homogeneous space G/H to H. Given a G-map $f: G/H \to G/K$, we can choose an element $g \in G$ with $gHg^{-1} \subset K$ and $f(g'H) = g'g^{-1}K$. Then $\operatorname{pr}(f)$ is represented by $c(g): H \to K$. Notice that $\operatorname{mor}_{\mathsf{Sub}(G)}(H,K)$ can be identified with the quotient $\operatorname{mor}_{\mathsf{Or}(G)}(G/H, G/K)/C_GH$, where $g \in C_GH$ acts on $\operatorname{mor}_{\mathsf{Or}(G)}(G/H, G/K)$ by composition with $R_{g^{-1}}: G/H \to G/H, \quad g'H \mapsto g'g^{-1}H$. We mention as illustration that for abelian G $\operatorname{mor}_{\mathsf{Sub}(G)}(H,K)$ is empty if H is not a subgroup of K, and consists of precisely one element given by the inclusion $H \to K$ if H is a subgroup in K.

Denote by $\mathsf{Or}(G,\mathcal{F}) \subset \mathsf{Or}(G)$ and $\mathsf{Sub}(G,\mathcal{F}) \subset \mathsf{Sub}(G)$ the full subcategories whose objects G/H resp. H are given by finite subgroups $H \subset G$. Both $\mathsf{Or}(G,\mathcal{F})$ and $\mathsf{Sub}(G,\mathcal{F})$ are EI-categories of finite length.

Given a proper G-homology theory $\mathcal{H}_*^G$ with values in R-modules we obtain for $n \in \mathbb{Z}$ a covariant $R\mathsf{Or}(G,\mathcal{F})$-module

$$\mathcal{H}_n^G(G/?): \mathsf{Or}(G,\mathcal{F}) \to R-\mathrm{MOD}, \qquad G/H \mapsto \mathcal{H}_n^G(G/H). \tag{3.1}$$

Let (X,A) be a pair of proper G-CW-complexes. Then there is a canonical identification $X^H = \operatorname{map}(G/H,X)^G$. Thus we obtain contravariant functors

$$\begin{aligned} \mathsf{Or}(G,\mathcal{F}) &\to CW-\mathsf{PAIRS}, \qquad & G/H &\mapsto (X^H, A^H); \\ \mathsf{Sub}(G,\mathcal{F}) &\to CW-\mathsf{PAIRS}, \qquad & G/H &\mapsto C_GH\backslash(X^H, A^H), \end{aligned}$$

where $CW-\mathsf{PAIRS}$ is the category of pairs of CW-complexes. Composing them with the covariant functor $CW-\mathsf{PAIRS} \to R-\mathsf{CHCOM}$ sending (Z,B) to its cellular chain complex with coefficients in R yields the contravariant $R\mathsf{Or}(G,\mathcal{F})$-chain complex $C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A)$ and the contravariant $R\mathsf{Sub}(G,\mathcal{F})$-chain complex $C_*^{\mathsf{Sub}(G,\mathcal{F})}(X,A)$. Both chain complexes are free. Namely, if X_n is obtained from X_{n-1} by attaching the equivariant cells $G/H_i \times D^n$ for $i \in I$, then

$$C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) = \oplus_{i \in I} R \operatorname{mor}_{\mathsf{Or}(G,\mathcal{F})}(G/?, G/H_i); \tag{3.2}$$

$$C_*^{\mathsf{Sub}(G,\mathcal{F})}(X,A) = \oplus_{i \in I} R \operatorname{mor}_{\mathsf{Sub}(G,\mathcal{F})}(?, H_i). \tag{3.3}$$

Given a covariant $R\mathsf{Or}(G,\mathcal{F})$-module M, the *equivariant Bredon homology* (see [4]) of a pair of proper G-CW-complexes (X,A) with coefficients in M is defined by

$$H_n^{\mathsf{Or}(G,\mathcal{F})}(X,A;M) := H_n(C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) \otimes_{R\mathsf{Or}(G,\mathcal{F})} M). \tag{3.4}$$

This is indeed a proper G-homology theory. Hence we can assign to a proper G-homology theory $\mathcal{H}_*^G$ another proper G-homology theory which we call the *associated Bredon homology*

$$\mathcal{BH}_n^G(X,A) \quad := \quad \oplus_{p+q=n} H_p^{\mathsf{Or}(G,\mathcal{F})}(X,A;\mathcal{H}_q^G(G/?)). \tag{3.5}$$

There is a canonical homomorphism $\mathrm{ind}_{\mathrm{pr}} C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) \xrightarrow{\cong} C_*^{\mathsf{Sub}(G,\mathcal{F})}(X,A)$ which is bijective (see (2.6), (3.2), (3.3)). Given a covariant $R\mathsf{Sub}(G,\mathcal{F})$-module M, it induces using (2.5) natural isomorphisms

$$H_n^{\mathsf{Or}(G,\mathcal{F})}(X,A;\mathrm{res}_{\mathrm{pr}} M) \quad \xrightarrow{\cong} \quad H_n(C_*^{\mathsf{Sub}(G,\mathcal{F})}(X,A) \otimes_{R\mathsf{Sub}(G,\mathcal{F})} M). \tag{3.6}$$

This will allows to view modules over the category $\mathsf{Sub}(G;\mathcal{F})$ which is smaller than the orbit category and has nicer properties from the homological algebra point of view. In particular we will exploit the following elementary lemma.

Lemma 3.7 *Suppose that the covariant $R\mathsf{Sub}(G,\mathcal{F})$-module M is flat, i.e. for any exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of contravariant $R\mathsf{Sub}(G,\mathcal{F})$-modules the induced sequence of R-modules $0 \to N_1 \otimes_{R\mathsf{Sub}(G,\mathcal{F})} M \to N_2 \otimes_{R\mathsf{Sub}(G,\mathcal{F})} M \to N_3 \otimes_{R\mathsf{Sub}(G,\mathcal{F})} M \to 0$ is exact. Then the natural map*

$$H_n(C_*^{\mathsf{Sub}(G,\mathcal{F})}(X,A)) \otimes_{R\mathsf{Sub}(G,\mathcal{F})} M \xrightarrow{\cong} H_n(C_*^{\mathsf{Sub}(G,\mathcal{F})}(X,A) \otimes_{R\mathsf{Sub}(G,\mathcal{F})} M)$$

is bijective.

Suppose, we are given a proper equivariant homology theory $\mathcal{H}_*^?$ with values in R-modules. We get from (3.1) for each group G and $n \in \mathbb{Z}$ a covariant $R\mathsf{Sub}(G,\mathcal{F})$-module

$$\mathcal{H}_n^G(G/?) : \mathsf{Sub}(G,\mathcal{F}) \to R-\mathrm{MOD}, \qquad H \mapsto \mathcal{H}_n^G(G/H). \tag{3.8}$$

We have to show that for $g \in C_G H$ the G-map $R_{g^{-1}} : G/H \to G/H, \quad g'H \to g'g^{-1}H$ induces the identity on $\mathcal{H}_n^G(G/H)$. This follows from Lemma 1.2. We will denote the covariant $\mathsf{Or}(G,\mathcal{F})$-module obtained by restriction with $\mathrm{pr} : \mathsf{Or}(G,\mathcal{F}) \to \mathsf{Sub}(G,\mathcal{F})$ from the $\mathsf{Sub}(G,\mathcal{F})$-module $\mathcal{H}_n^G(G/?)$ of (3.8) again by $\mathcal{H}_n^G(G/?)$ as introduced already in (3.1).

Next we show that the collection of the G-homology theories $\mathcal{BH}_*^G(X,A;\mathcal{H}_q^G(G/?))$ defined in (3.5) inherits the structure of a proper equivariant homology theory. We have to specify the induction structure.

Let $\alpha : H \to G$ be a group homomorphism and (X,A) be a H-CW-pair such that $\ker(\alpha)$ acts freely on X. We only explain the case, where α is injective. In the general case one has to replace $\mathcal{F}$ by the smaller family $\mathcal{F}(X)$ of subgroups of H which occur as subgroups of isotropy groups of X. Induction with α yields a functor denoted in the same way

$$\alpha : \mathsf{Or}(H,\mathcal{F}) \to \mathsf{Or}(G,\mathcal{F}), \qquad H/K \mapsto \mathrm{ind}_\alpha(H/K) = H/\alpha(K).$$

There is a natural isomorphism of $\mathsf{Or}(G,\mathcal{F})$-chain complexes

$$\mathrm{ind}_\alpha C_*^{\mathsf{Or}(H,\mathcal{F})}(X,A) \xrightarrow{\cong} C_*^{\mathsf{Or}(G,\mathcal{F})}(\mathrm{ind}_\alpha(X,A))$$

and a natural isomorphism (see (2.5))

$$\mathrm{ind}_\alpha C_*^{\mathsf{Or}(H,\mathcal{F})}(X,A) \otimes_{R\mathsf{Or}(G,\mathcal{F})} \mathcal{H}_q^G(G/?) \xrightarrow{\cong} C_*^{\mathsf{Or}(H,\mathcal{F})}(X,A) \otimes_{R\mathsf{Or}(H,\mathcal{F})} \mathrm{res}_\alpha \mathcal{H}_q^G(G/?).$$

The induction structure on $\mathcal{H}_*^?$ yields a natural equivalence of $R\mathsf{Or}(H,\mathcal{F})$-modules

$$\mathcal{H}_q^H(H/?) \xrightarrow{\cong} \mathrm{res}_\alpha \mathcal{H}_q^G(G/?).$$

The last three maps can be composed to a chain isomorphism

$$C_*^{\mathsf{Or}(H,\mathcal{F})}(X,A) \otimes_{R\mathsf{Or}(H,\mathcal{F})} \mathcal{H}_q^H(H/?) \xrightarrow{\cong} C_*(\mathrm{ind}_\alpha(X,A)) \otimes_{R\mathsf{Or}(G,\mathcal{F})} \mathcal{H}_q^G(G/?),$$

which induces a natural isomorphism

$$\mathrm{ind}_\alpha : H_p^{\mathsf{Or}(H,\mathcal{F})}(X,A,\mathcal{H}_q^H(H/?)) \xrightarrow{\cong} H_p^{\mathsf{Or}(G,\mathcal{F})}(\mathrm{ind}_\alpha(X,A),\mathcal{H}_q^G(G/?)).$$

Thus we obtain the required induction structure.

Remark 3.9 For any G-homology theory $\mathcal{H}_*^G$ with values in R-modules for an associative commutative ring R with unit there is an equivariant version of the Atiyah-Hirzebruch spectral sequence. It converges to $\mathcal{H}_{p+q}^G(X,A)$ and its E^2-term is $E^2_{p,q} = H_p^{\mathsf{Or}(G)}(X,A;\mathcal{H}_q^G(G/?))$. If (X,A) is proper, the E^2-term reduces to $H_p^{\mathsf{Or}(G,\mathcal{F})}(X,A;\mathcal{H}_q^G(G/?))$. The existence of a bijective equivariant Chern character says that this spectral sequence collapses completely for proper G-CW-pairs (X,A).

4. The construction of the equivariant Chern character

Let (X,A) be a proper G-CW-pair. Let R be an associative commutative ring with unit satisfying $\mathbb{Q} \subset R$. We want to construct an R-homomorphism

$$\underline{\mathrm{ch}}^G_{p,q}(X,A)(H) : H_p(C_GH\backslash(X^H,A^H);R) \otimes_R \mathcal{H}_q^G(G/H) \quad\to\quad \mathcal{H}^G_{p+q}(X,A), \tag{4.1}$$

where $H_p(C_GH\backslash(X^H,A^H);R)$ is the cellular homology of the CW-pair $C_GH\backslash(X^H,A^H)$ with R-coefficients. For (notational) simplicity we give the details only for $A=\emptyset$. The map will be defined by the following composition

$$\begin{array}{c}
H_p(C_GH\backslash X^H;R)\otimes_R \mathcal{H}_q^G(G/H)\\
{\scriptstyle H_p(\mathrm{pr}_1;R)\otimes_R \mathrm{id}}\big\uparrow \cong\\
H_p(EG\times_{C_GH} X^H;R)\otimes_R \mathcal{H}_q^G(G/H)\\
{\scriptstyle \mathrm{hur}(EG\times_{C_GH}X^H)\otimes_R \mathrm{ind}_H^G}\big\uparrow \cong\\
\pi_p^s((EG\times_{C_GH} X^H)_+)\otimes_{\mathbb{Z}} R\otimes_R \mathcal{H}_q^H(*)\\
{\scriptstyle D^H_{p,q}(EG\times_{C_GH}X^H)}\big\downarrow\\
\mathcal{H}^H_{p+q}(EG\times_{C_GH} X^H)\\
{\scriptstyle \mathrm{ind}_{\mathrm{pr}:C_GH\times H\to H}}\big\uparrow \cong\\
\mathcal{H}^{C_GH\times H}_{p+q}(EG\times X^H)\\
{\scriptstyle \mathrm{ind}_{m_H}}\big\downarrow \cong\\
\mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H} EG\times X^H)\\
{\scriptstyle \mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H}\mathrm{pr}_2)}\big\downarrow\\
\mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H} X^H)\\
{\scriptstyle \mathcal{H}^G_{p+q}(v_H)}\big\downarrow\\
\mathcal{H}^G_{p+q}(X)
\end{array}$$

Some explanations are in order. We have a left C_GH-action on $EG \times X^H$ by $g(e,x) = (eg^{-1},gx)$ for $g \in C_GH$, $e \in EG$ and $x \in X^H$. The map $\mathrm{pr}_1 : EG\times_{C_GH} X^H \to C_GH\backslash X^H$ is the canonical projection. It induces an isomorphism

$$H_p(\mathrm{pr}_1;R) : H_p(EG\times_{C_GH} X^H;R) \xrightarrow{\cong} H_p(X^H/C_GH;R)$$

by the following argument. Each isotropy group of the C_GH-space X^H is finite. The projection induces an isomorphism $H_p(BL;R) \cong H_p(*;R)$ for $p \in \mathbb{Z}$ and any finite group L because by assumption the order of L is invertible in R. Hence $H_p(\mathrm{pr}_1;R)$ is bijective if $X^H = C_GH/L$ for some finite $L \subset C_GH$. Now apply the usual Mayer-Vietoris argument.

For any space Y let $\operatorname{hur}(Y) : \pi_p^s(Y_+) \otimes_{\mathbb{Z}} R \to H_p(Y;R)$ be the Hurewicz homomorphism. It is bijective since $\mathbb{Q} \subset R$ and therefore hur is a natural tranformation of (non-equivariant) homology theories which induces for the one-point space $Y = *$ an isomorphism $\pi_p^s(*_+) \otimes_{\mathbb{Z}} R \cong H_p(*;R)$ for $p \in \mathbb{Z}$.

Given a space Z and a finite group H, consider Z as an H-space by the trivial action and define a map

$$D_{p,q}^H(Z) : \pi_p^s(Z_+) \otimes_{\mathbb{Z}} \mathcal{H}_q^H(*) = \pi_p^s(Z_+) \otimes_{\mathbb{Z}} R \otimes_R \mathcal{H}_q^H(*) \to \mathcal{H}_{p+q}^H(Z)$$

as follows. For an element $a \otimes b \in \pi_p^s(Z_+) \otimes_{\mathbb{Z}} \mathcal{H}_q^H(*)$ choose a representative $f : S^{p+k} \to S^k \wedge Z_+$ of a. Define $D_{p,q}^H(Z)(a \otimes b)$ to be the image of b under the composition

$$\mathcal{H}_q^H(*) \xrightarrow{\sigma} \mathcal{H}_{p+q+k}^H(S^{p+k},*) \xrightarrow{f} \mathcal{H}_{p+q+k}^H(S^k \wedge Z_+,*) \xrightarrow{\sigma^{-1}} \mathcal{H}_{p+q}^H(Z),$$

where σ denotes the suspension isomorphism. Notice that H is finite so that any H-CW-complex is proper.

The group homomorphism $\operatorname{pr} : C_GH \times H \to H$ is the obvious projection and the group homomorphism $m_H : C_GH \times H \to G$ sends (g,h) to gh. Notice that the $C_GH \times H$-action on $EG \times X^H$ comes from the given C_GH-action and the trivial H-action and that the kernels of the two group homomorphisms above act freely on $EG \times X^H$. So the induction isomorphisms on homology for these group homomorphisms exists for the $C_GH \times H$-space $EG \times X^H$.

We denote by $\operatorname{pr}_2 : EG \times X^H \to X^H$ the canonical projection. The G-map $v_H : \operatorname{ind}_{m_H} X^H = G \times_{m_H} X^H \to X$ sends (g,x) to gx.

Lemma 4.2 *Let G be a group and let X be a proper G-CW-complex. Then*

(a) The map $\underline{\operatorname{ch}}_{p,q}^G(X)(H)$ is natural in X;

(b) Consider $H,K \subset G$ and $g \in G$ with $gHg^{-1} \subset K$. Let $L_{g^{-1}} : X^K \to X^H$ and $\overline{L_{g^{-1}}} : C_GK\backslash X^K \to C_GH\backslash X^H$ be the map induced by left multiplication with g^{-1}. Then following two maps agree

$$H_p(C_GK\backslash X^K;R)\otimes_R\mathcal{H}_q^G(G/H) \xrightarrow{H_p(\overline{L_{g^{-1}}};R)\otimes_R\operatorname{id}} H_p(C_GH\backslash X^H;R)\otimes_R\mathcal{H}_q^G(G/H) \xrightarrow{\underline{\operatorname{ch}}_{p,q}^G(X)(H)} \mathcal{H}_{p+q}^G(X)$$

and

$$H_p(C_GK\backslash X^K;R)\otimes_R\mathcal{H}_q^G(G/H) \xrightarrow{\operatorname{id}\otimes_R\mathcal{H}_q^G(R_{g^{-1}})} H_p(C_GK\backslash X^K;R)\otimes_R\mathcal{H}_q^G(G/K) \xrightarrow{\underline{\operatorname{ch}}_{p,q}^G(X)(K)} \mathcal{H}_{p+q}^G(X);$$

(c) Consider a G-map $f : G/H \to X$. Then $f(eH) \in X^H$ represents an element u in the set $\pi_0(C_GH\backslash X^H)$, which is a R-basis for $H_0(C_GH\backslash X^H;R)$, and the map

$$\mathcal{H}_q^G(G/H) \to \mathcal{H}_q^G(X), \qquad v \mapsto \underline{\operatorname{ch}}_{0,q}^G(X)(H)(u \otimes_R v)$$

agrees with the map $\mathcal{H}_q^G(f)$.

Proof : (a) is obvious

(b) Since $gHg^{-1} \subset K$ we can define a group homomorphism $c(g^{-1}) : C_GK \to C_GH$ by mapping g' to $g^{-1}g'g$. The map

$$R_g \times L_{g^{-1}} : EG \times X^K \to EG \times X^H, \qquad (e,x) \mapsto (eg, g^{-1}x)$$

is $c(g^{-1}) : C_GK \to C_GH$ equivariant with respect to the C_GK-action on $EG \times X^K$ given by $g' \cdot (e,x) = (eg'^{-1}, g'x)$ and the analogous C_GH-action on $EG \times X^H$. It induces a map

$$\overline{R_g \times L_{g^{-1}}} : EG \times_{C_GK} X^K \to EG \times_{C_GH} X^H.$$

If we extend the C_GH- resp. C_GK-action on $EG \times X^H$ resp. $EG \times X^K$ to a $C_GH \times H$- resp. $C_GK \times H$-action in the trivial way, we also get $C_GH \times H$-maps

$$\begin{array}{rcl} \widetilde{R_g \times L_{g^{-1}}} : \operatorname{ind}_{c(g^{-1} \times \mathrm{id}): C_GK \times H \to C_GH \times H} EG \times X^K = C_GH \times H \times_{c(g^{-1}) \times \mathrm{id}} EG \times X^K & \to & EG \times X^H \\ (c,h,e,x) & \mapsto & (egc^{-1}, cg^{-1}x) \end{array}$$

and

$$\widetilde{L_{g^{-1}}} : \operatorname{ind}_{c(g^{-1}) \times \mathrm{id}: C_GK \times H \to C_GH \times H} X^K = C_GK \times H \times_{c(g^{-1}) \times \mathrm{id}} X^K \to X^H, \quad (c,h,x) \mapsto (cg^{-1}x).$$

In the sequel the maps p_i denote the canonical projections. They are of the shape $Y \times K/gHg^{-1} \to Y$. The maps f_i denote canonical equivariant homeomorphisms which describe the natural identifications of $\operatorname{ind}_{\beta \circ \alpha} Z$ with $\operatorname{ind}_\beta \operatorname{ind}_\alpha Z$. One easily checks using the axioms of an induction structure that the following three diagrams commute:

$$\begin{array}{ccc} H_p(C_GK\backslash X^K; R) & \xrightarrow{H_p(\overline{L_{g^{-1}}}; R)} & H_p(C_GH\backslash X^H; R) \\ \cong \uparrow H_p(\mathrm{pr}_1; R) & & H_p(\mathrm{pr}_1; R) \uparrow \cong \\ H_p(EG \times_{C_GK} X^K; R) & \xrightarrow{H_p(\overline{R_g \times L_{g^{-1}}}; R)} & H_p(EG \times_{C_GH} X^H; R) \\ \cong \uparrow \mathrm{hur}(EG \times_{C_GK} X^K) & & \mathrm{hur}(EG \times_{C_GH} X^H) \uparrow \cong \\ \pi_p^s((EG \times_{C_GK} X^K)_+) \otimes_{\mathbb{Z}} R & \xrightarrow{\pi_p^s(\overline{R_g \times L_{g^{-1}}})} & \pi_p^s((EG \times_{C_GH} X^H)_+) \otimes_{\mathbb{Z}} R \end{array}$$

and

$$\begin{array}{ccc} \pi_p^s((EG \times_{C_GK} X^K)_+) \otimes_{\mathbb{Z}} \mathcal{H}_q^K(*) & \xleftarrow{\mathrm{id} \otimes \mathcal{H}_q^K(p_1) \circ \operatorname{ind}_{c(g): H \to K}} & \pi_p^s((EG \times_{C_GK} X^K)_+) \otimes_{\mathbb{Z}} \mathcal{H}_q^H(*) \\ \downarrow D_{p,q}^K & & D_{p,q}^H \downarrow \\ \mathcal{H}_{p+q}^K(EG \times_{C_GK} X^K) & \xleftarrow{\mathcal{H}_q^K(p_2) \circ \operatorname{ind}_{c(g): H \to K}} & \mathcal{H}_{p+q}^H(EG \times_{C_GK} X^K) \\ \cong \uparrow \operatorname{ind}_{\mathrm{pr}: C_GK \times K \to K} & & \operatorname{ind}_{\mathrm{pr}: C_GK \times H \to H} \uparrow \cong \\ \mathcal{H}_{p+q}^{C_GK \times K}(EG \times X^K) & \xleftarrow{\mathcal{H}_q^K(p_3) \circ \operatorname{ind}_{\mathrm{id} \times c(g)}} & \mathcal{H}_{p+q}^{C_GK \times H}(EG \times X^K) \\ \downarrow \operatorname{ind}_{m_K} & & \operatorname{ind}_{m_K \circ \mathrm{id} \times c(g)} \downarrow \\ \mathcal{H}_{p+q}^G(\operatorname{ind}_{m_K} EG \times X^K) & \xleftarrow{\mathcal{H}_{p+q}^G(\operatorname{ind}_{m_K} p_3) \circ \mathcal{H}_{p+q}^G(f_1)} & \mathcal{H}_{p+q}^G(\operatorname{ind}_{m_K \circ \mathrm{id} \times c(g)} EG \times X^K) \\ \downarrow \mathcal{H}_{p+q}^G(\operatorname{ind}_{m_K} \mathrm{pr}_2) & & \mathcal{H}_{p+q}^G(\operatorname{ind}_{m_K \circ \mathrm{id} \times c(g)} \mathrm{pr}_2) \downarrow \\ \mathcal{H}_{p+q}^G(\operatorname{ind}_{m_K} X^K) & \xleftarrow{\mathcal{H}_{p+q}^G(\operatorname{ind}_{m_K} p_4) \circ \mathcal{H}_{p+q}^G(f_2)} & \mathcal{H}_{p+q}^G(\operatorname{ind}_{m_K \circ \mathrm{id} \times c(g)} X^K) \end{array}$$

and

$$\begin{array}{ccc}
\pi_p^s((EG \times_{C_GK} X^K)_+) \otimes_R \mathcal{H}_q^H(*) & \xrightarrow{\pi_p^s(\overline{R_g \times L_{g^{-1}}})\otimes \mathrm{id}} & \pi_p^s((EG \times_{C_GK} X^H)_+) \otimes_R \mathcal{H}_q^H(*) \\
\downarrow D^H_{p,q} & & D^H_{p,q} \downarrow \\
\mathcal{H}^H_{p+q}(EG \times_{C_GK} X^K) & \xrightarrow{\mathcal{H}^G_{p+q}(\overline{R_g \times L_{g^{-1}}})} & \mathcal{H}^H_{p+q}(EG \times_{C_GK} X^H) \\
\cong \uparrow \mathrm{ind}_{\mathrm{pr}: C_GK \times H \to H} & & \mathrm{ind}_{\mathrm{pr}: C_GH \times H \to H} \uparrow \cong \\
\mathcal{H}^{C_GK \times H}_{p+q}(EG \times X^K) & \xrightarrow{\mathcal{H}^G_{p+q}(\widetilde{R_g \times L_{g^{-1}}}) \circ \mathrm{ind}_{c(g^{-1}) \times \mathrm{id}}} & \mathcal{H}^{C_GH \times H}_{p+q}(EG \times X^H) \\
\downarrow \mathrm{ind}_{m_K \circ \mathrm{id} \times c(g)} & & \mathrm{ind}_{m_H} \downarrow \\
\mathcal{H}^G_{p+q}(\mathrm{ind}_{m_K \circ \mathrm{id} \times c(g)} EG \times X^K) & \xrightarrow{\mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H} \widetilde{R_g \times L_{g^{-1}}}) \circ \mathcal{H}^G_{p+q}(f_3) \circ \mathrm{ind}_{c(g^{-1})}} & \mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H} EG \times X^H) \\
\downarrow \mathcal{H}^G_{p+q}(\mathrm{ind}_{m_K \circ \mathrm{id} \times c(g)} \mathrm{pr}_2) & & \mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H} \mathrm{pr}_2) \downarrow \\
\mathcal{H}^G_{p+q}(\mathrm{ind}_{m_K \circ \mathrm{id} \times c(g)} X^K) & \xrightarrow{\mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H} \widetilde{L_{g^{-1}}}) \circ \mathcal{H}^G_{p+q}(f_4) \circ \mathrm{ind}_{c(g^{-1})}} & \mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H} X^H) \\
\downarrow \mathcal{H}^G_{p+q}(\mathrm{ind}_{m_K} p_4) \circ \mathcal{H}^G_{p+q}(f_2) & & \mathcal{H}^G_{p+q}(v_H) \downarrow \\
\mathcal{H}^G_{p+q}(\mathrm{ind}_{m_K} X^K) & \xrightarrow{\mathcal{H}^G_{p+q}(v_K)} & \mathcal{H}^G_{p+q}(X)
\end{array}$$

Now assertion (b) follows from an easy diagram chase in the three commutative diagrams above and Lemma 1.2.

(c) Its proof is similar to the one of (b) but much easier and hence left to the reader. This finishes the proof of Lemma 4.2. ■

Theorem 4.3 *Let R be an associative commutative ring with unit for which $\mathbb{Q} \subset R$ holds. Let $\mathcal{H}^?_*$ be a proper equivariant homology theory with values in R-modules. Suppose for any group G that the $R\mathsf{Sub}(G,\mathcal{F})$-module $\mathcal{H}^G_q(G/?)$ is flat for all $q \geq 0$. Then there is an isomorphism, called* equivariant Chern character, *of proper equivariant homology theories*

$$\mathrm{ch}^?_* : \mathcal{BH}^?_* \xrightarrow{\cong} \mathcal{H}^?_*,$$

i.e. for any group G and any proper G-CW-pair (X, A) there is an in (X, A) natural isomorphism

$$\mathrm{ch}^G_n(X, A) : \oplus_{p+q=n} H_p^{\mathsf{Or}(G,\mathcal{F})}(X, A; \mathcal{H}^G_q(G/?)) \xrightarrow{\cong} \mathcal{H}^G_n(X, A)$$

such that the obvious compatibility conditions for the boundary homomorphisms of pairs and the induction structures hold.

Proof : We get for a pair of proper G-CW-complexes (X, A) from the collection of the homomorphisms of (4.1), the identification (3.6), Lemma 3.7 and Lemma 4.2 (which holds for pairs (X, A) also) a natural R-homomorphism

$$\mathrm{ch}^G_{p,q}(X, A) : H_p^{\mathsf{Or}(G,\mathcal{F})}(X, A; \mathcal{H}^G_q(G/?)) = H_p(C_*^{\mathsf{Sub}(G,\mathcal{F})}(X, A)) \otimes_{R\mathsf{Sub}(G,\mathcal{F})} \mathcal{H}^G_q(G/?) \quad \to \quad \mathcal{H}^G_{p+q}(X).$$

Taking their direct sum for $p + q = n$ yields an in (X, A)-natural homomorphism

$$\mathrm{ch}^G_n(X, A) : \mathcal{BH}^G_n(X, A) \quad \to \quad \mathcal{H}^G_n(X). \tag{4.4}$$

One easily checks that $\mathrm{ch}^G_* : \mathcal{BH}^G_* \to \mathcal{H}^G_*$ is a transformation of G-homology theories. Essentially one has to check that it is compatible with the boundary maps in the long exact sequences of pairs.

Next we show that ch^G_* is a natural equivalence, i.e. $\mathrm{ch}^G_n(X,A)$ is bijective for all $n \in \mathbb{Z}$ and all proper G-CW-pairs (X,A). The disjoint union axiom implies that both G-homology theories are compatible with colimits over directed systems indexed by the natural numbers such as the system given by the skeletal filtration $X_0 \subset X_1 \subset X_2 \ldots \cup_{n \geq 0} X_n = X$. The argument for this claim is analogous to the one in [24, 7.53] or [26, Theorem XIII.1.1 on page 604]. Hence it suffices to prove the bijectivity of $\mathrm{ch}^G_n(X,A)$ for finite-dimensional pairs. By excision, the exact sequence of pairs, the disjoint union axiom and the five-lemma one reduces the proof of the bijectivity of $\mathrm{ch}^G_n(X,A)$ to the special case $(X,A) = (G/H, \emptyset)$ for finite $H \subset G$. In this case the bijectivity follows from the consequence of Lemma 4.2 (b) that $\mathrm{ch}^G_n(G/H)$ is the identity under the obvious identification of its source with $\mathcal{H}^G_n(G/H)$ coming from (3.2). ■

Remark 4.5 Suppose that G is trivial and we consider a (non-equivariant) homology theory $\mathcal{H}_*$ with values in R-modules for $\mathbb{Q} \subset R$. Then the construction of the equivariant Chern character reduces to the following composition

$$\begin{aligned}\mathrm{ch}_n : \oplus_{p+q=n} H_p(X,A;\mathcal{H}_q(*)) \xleftarrow[\alpha]{\cong} & \oplus_{p+q=n} H_p(X,A;R) \otimes_R \mathcal{H}_q(*) \\ \xleftarrow[\cong]{\mathrm{hur}\otimes\mathrm{id}} & \pi^s_p(X_+, A_+) \otimes_{\mathbb{Z}} R \otimes_R \mathcal{H}_q(*) \xrightarrow{D_{p,q}} \mathcal{H}_n(X,A).\end{aligned}$$

Here the canonical map α is bijective, since any R-module is flat over $\mathbb{Z}$ because of the assumption $\mathbb{Q} \subset R$. The second bijective map comes from the Hurewicz homomorphism. This construction is due to Dold [7].

Example 4.6 Given a homology theory $\mathcal{K}$ with values in R-modules for $\mathbb{Q} \subset R$, we can associate to it an equivariant homology theory $\mathcal{H}^?_*$ in two ways as explained in Example 1.3. There is an obvious equivariant Chern character coming from the non-equivariant one of Remark 4.5. Our general construction reduces to it by the following elementary observation. For any finite group H the natural map $\mathcal{K}_q(BH) \to \mathcal{K}_q(*)$ is an isomorphism by the Atiyah-Hirzebruch spectral sequence since $H_p(BH;\mathbb{Q}) \to H_q(*;\mathbb{Q})$ is bijective. Hence in both cases the $R\mathsf{Sub}(G,\mathcal{F})$-module $\mathcal{H}^G_q(G/?) = \mathcal{H}^?_q(*)$ is constant with value $\mathcal{K}_q(*)$. Therefore it is isomorphic to $\mathbb{Q}\,\mathrm{mor}_{\mathsf{Sub}(G,\mathcal{F})}(1,?) \otimes_{\mathbb{Q}} \mathcal{K}_q(*)$ which is obviously a projective $R\mathsf{Sub}(G,\mathcal{F})$-module. By (2.12) the source of our equivariant Chern character reduces in this special case to

$$\oplus_{p+q=n} H^{\mathsf{Or}(G,\mathcal{F})}_p(X,A;\mathcal{H}^G_q(G/?)) \;\cong\; \oplus_{p+q=n} H_p(G\backslash(X,A);\mathcal{K}_q(*)).$$

Remark 4.7 Let $\mathcal{H}^*_G$ be an equivariant proper cohomology theory with values in F-modules for a field F of characteristic zero. It is defined axiomatically in the obvious way analogous to the definition of an equivariant homology theory. Suppose that $\mathcal{H}^n_H(*)$ is a finite-dimensional F-vector space for all finite groups H and $n \in \mathbb{Z}$. Put $\mathcal{H}^G_n(X,A) := \mathrm{hom}_F(\mathcal{H}^n_G(X,A),F)$. This defines an equivariant homology theory for proper finite G-CW-pairs (X,A). We can rediscover $\mathcal{H}^n_G(X,A)$ by $\mathrm{hom}_F(\mathcal{H}^G_n(X,A),F)$ for proper finite G-CW-pairs (X,A). If one obtains a bijective Chern character for $\mathcal{H}^G_*$ for proper finite G-CW-pairs, dualizing yields a bijective Chern character from $\mathcal{H}^*_G$ to the associated equivariant Bredon cohomology for proper finite G-CW-pairs.

This applies for instance to equivariant K-cohomology after tensoring with $\mathbb{Q}$ over $\mathbb{Z}$. Equivariant Chern characters for equivariant K-cohomology have been constructed after tensoring with $\mathbb{C}$ resp. $\mathbb{Q}$ over $\mathbb{Z}$ in [2] resp. [17]. Our construction of an equivariant Chern character for proper equivariant homology theories is motivated by [17].

5. Mackey functors

In order to apply Theorem 4.3, we have to check the flatness condition about the $R\mathsf{Sub}(G,\mathcal{F})$-module $\mathcal{H}^G_q(G/?)$. We will see that the existence of a Mackey structure will guarantee that it is projective and hence flat. This would not work if we would consider $\mathcal{H}_q^G(G/?)$ over the orbit category. Recall that we can consider it over $\mathsf{Sub}(G,\mathcal{F})$ because of Lemma 1.2 which is a consequence of the induction structure. The desired Mackey structures do exist in all relevant examples.

Let R be an associative commutative ring with unit. Let FGINJ be the category of finite groups with injective group homomorphisms as morphisms. Let $M : \mathsf{FGINJ} \to R-\mathrm{MOD}$ be a bifunctor, i.e. a pair (M_*, M^*) consisting of a covariant functor M_* and a contravariant functor M^* from FGINJ to $R-\mathrm{MOD}$ which agree on objects. We will often denote for an injective group homomorphism $f : H \to G$ the map $M_*(f) : P(H) \to P(G)$ by ind_f and the map $M^*(f) : P(G) \to P(H)$ by res_f and write $\mathrm{ind}_H^G = \mathrm{ind}_f$ and $\mathrm{res}_G^H = \mathrm{res}_f$ if f is an inclusion of groups. We call such a bifunctor M a *Mackey functor* with values in R-modules if

(a) For an inner automorphism $c(g) : G \to G$ we have $M_*(c(g)) = \mathrm{id} : M(G) \to M(G)$;

(b) For an isomorphism of groups $f : G \xrightarrow{\cong} H$ the compositions $\mathrm{res}_f \circ \mathrm{ind}_f$ and $\mathrm{ind}_f \circ \mathrm{res}_f$ are the identity;

(c) Double coset formula

We have for two subgroups $H, K \subset G$

$$\mathrm{res}_G^K \circ \mathrm{ind}_H^G \quad = \sum_{KgH \in K\backslash G/H} \mathrm{ind}_{c(g): H\cap g^{-1}Kg \to K} \circ \mathrm{res}_H^{H\cap g^{-1}Kg},$$

where $c(g)$ is conjugation with g, i.e. $c(g)(h) = ghg^{-1}$.

Our main examples of Mackey functors will be $R_{\mathbb{Q}}(H)$, $K_q(RH)$, $L_q(RH)$ and $K_q^{\mathrm{top}}(C_*^r(H,F))$. Recall that for a subgroup $H \subset G$ we denote by N_GH and C_GH the normalizer and the centralizer of H in G and by W_GH the quotient N_GH/C_GH. In the sequel we will use the identification $W_GH \cong \mathrm{aut}_{\mathsf{Sub}(G,\mathcal{F})}(H)$ which sends the class of $n \in N_GH$ to the class of $c(n) : H \to H$. We have introduced $S_HP = P(H)/P(H)_s$ for a covariant $\mathsf{Sub}(G,\mathcal{F})$-module P in (2.9). Notice for the sequel that

$$P(H)_s \quad = \quad \mathrm{im}\left(\oplus_{K\subset H, K\neq H} \mathrm{ind}_K^H : \oplus_{K\subset H, K\neq H} P(K) \to P(H)\right). \tag{5.1}$$

Given a left $R[W_GH]$-module Q, we have defined the covariant $R\mathsf{Sub}(G,\mathcal{F})$-module E_HQ in (2.7). Recall that (H) has two meanings, namely, the set of subgroups of G which are conjugated to H and the isomorphism class of objects in $\mathsf{Sub}(G,\mathcal{F})$. One easily checks that these two interpretations give the same.

Theorem 5.2 *Let F be a field of characteristic* 0. *Let M be a Mackey functor with values in F-modules. It induces a covariant $F\mathsf{Sub}(G,\mathcal{F})$-module denoted in the same way*

$$M : \mathsf{Sub}(G,\mathcal{F}) \to F-\mathrm{MOD}, \qquad (f : H \to L) \mapsto (M_*(f) : M(H) \to M(K)).$$

Each $F[W_GH]$-module S_HM is projective. For any finite subgroup $H \subset G$ choose a section $\sigma_H : S_HM \to M(H)$ of the canonical projection $M(H) \to S_HM$. Then the homomorphism defined in (2.10)

$$T : \oplus_{(H)\in I} E_H \circ S_HM \to M$$

is an isomorphism and the $F\mathsf{Sub}(G,\mathcal{F})$-module M is projective and hence flat.

Proof : Since W_GH is finite, any $F[W_GH]$-module is projective. Because of Theorem 2.11 it suffices to show for any finite subgroup $K \subset G$ that $T(K)$ is injective. Consider an element u in the kernel of $T(K)$. Put $J(H) = \mathrm{mor}_{\mathsf{Sub}(G,\mathcal{F})}(H,K)/(W_GH)$ and $I = \mathrm{Is}(\mathsf{Sub}(G,\mathcal{F}))$. Choose for any $(H) \in I$ a representative $H \in (H)$. Then fix for any element $\overline{f} \in J(H)$ a representative $f : H \to K$ in $\mathrm{mor}_{\mathsf{Sub}(G,\mathcal{F})}(H,K)$. We can find elements $x_{H,f} \in S_H M$ for $(H) \in I$ and $\overline{f} \in J(H)$ such that only finitely many are different from zero and u can be written as

$$u = \sum_{(H)\in I} \sum_{\overline{f}\in J(H)} (f : H \to K) \otimes_{R[W_G H]} x_{H,f}.$$

We want to show that all elements $x_{H,f}$ are zero. Suppose that this is not the case. Let (H_0) be maximal among those elements $(H) \in I$ for which there is $\overline{f} \in J(H)$ with $x_{H,f} \neq 0$, i.e. if for $(H) \in I$ the element $x_{H,f}$ is different from zero for some morphism $f : H \to K$ in $\mathsf{Sub}(G, \mathcal{F})$ and there is a morphism $H_0 \to H$ in $\mathsf{Sub}(G, \mathcal{F})$, then $(H_0) = (H)$. In the sequel we choose for any of the morphisms $f : H \to K$ in $\mathsf{Sub}(G, \mathcal{F})$ a group homomorphism denoted in the same way $f : H \to K$ representing it. Recall that $f : H \to K$ is given by conjugation with an appropriate element $g \in G$. Fix $f_0 : H_0 \to K$ with $x_{H_0 \cdot f_0} \neq 0$. We claim that the composition

$$A : \oplus_{(H)\in I} E_H \circ S_H M(K) \xrightarrow{T(K)} M(K) \xrightarrow{\operatorname{res}_K^{\operatorname{im}(f_0)}} M(\operatorname{im}(f_0)) \xrightarrow{\operatorname{ind}_{f_0^{-1}:\operatorname{im}(f_0)\to H_0}} M(H_0) \xrightarrow{\operatorname{pr}_{H_0}} S_{H_0} M$$

maps u to $m \cdot x_{H_0,f_0}$ for some integer $m > 0$. This would lead to a contradiction because of $T(K)(u) = 0$ and $x_{H_0,f_0} \neq 0$.

Consider $(H) \in I$ and $\overline{f} \in J(H)$. It suffices to show that $A\left((f : H \to K) \otimes_{F[W_G H]} x_{H,f}\right)$ is $[K \cap N_G \operatorname{im}(f_0) : \operatorname{im}(f_0)] \cdot x_{H,f}$ if $(H) = (H_0)$ and $\overline{f} = \overline{f_0}$, and is zero otherwise. One easily checks that $A((f : H \to K) \otimes_{F[W_G H]} x_{H,f})$ is the image of $x_{H,f}$ under the composition

$$a(H, f) : S_H M \xrightarrow{\sigma_H} M(H) \xrightarrow{\operatorname{ind}_{f:H\to\operatorname{im}(f)}} M(\operatorname{im}(f)) \xrightarrow{\operatorname{ind}_{\operatorname{im}(f)}^K} M(K) \xrightarrow{\operatorname{res}_K^{\operatorname{im}(f_0)}} M(\operatorname{im}(f_0))$$
$$\xrightarrow{\operatorname{ind}_{f_0^{-1}:\operatorname{im}(f_0)\to H_0}} M(H_0) \xrightarrow{\operatorname{pr}_{H_0}} S_{H_0} M.$$

The Double Coset formula implies

$$\operatorname{res}_K^{\operatorname{im}(f_0)} \circ \operatorname{ind}_{\operatorname{im}(f)}^K = \sum_{k\in\operatorname{im}(f_0)\backslash K/\operatorname{im}(f)} \operatorname{ind}_{c(k):\operatorname{im}(f)\cap k^{-1}\operatorname{im}(f_0)k\to\operatorname{im}(f_0)} \circ \operatorname{res}_{\operatorname{im}(f)}^{\operatorname{im}(f)\cap k^{-1}\operatorname{im}(f_0)k}.$$

The composition $\operatorname{pr}_{H_0} \circ \operatorname{ind}_{f_0^{-1}:\operatorname{im}(f_0)\to H_0} \circ \operatorname{ind}_{c(k):\operatorname{im}(f)\cap k^{-1}\operatorname{im}(f_0)k\to\operatorname{im}(f_0)}$ is trivial, if $c(k) : \operatorname{im}(f) \cap k^{-1}\operatorname{im}(f_0)k \to \operatorname{im}(f_0)$ is not an isomorphism. Suppose that $c(k) : \operatorname{im}(f) \cap k^{-1}\operatorname{im}(f_0)k \to \operatorname{im}(f_0)$ is an isomorphism. Then $k^{-1}\operatorname{im}(f_0)k \subset \operatorname{im}(f)$. Since H_0 has been choosen maximal among the H for which $x_{H,f} \neq 0$ for some morphism $f : H \to K$, this implies $x_{H,f} = 0$ or that $k^{-1}\operatorname{im}(f_0)k = \operatorname{im}(f)$. Suppose $k^{-1}\operatorname{im}(f_0)k = \operatorname{im}(f)$. Then $(H) = (H_0)$ which implies $H = H_0$. Moreover, the homomorphisms in $\mathsf{Sub}(G, \mathcal{F})$ represented by f_0 and f agree. Hence the group homomorphisms f_0 and f agree themselves and we get $k \in N_G \operatorname{im}(f_0) \cap K$. This implies that $a(H, f) = [K \cap N_G \operatorname{im}(f_0) : \operatorname{im}(f_0)] \cdot \operatorname{id}$ if $(H) = (H_0)$ and $\overline{f} = \overline{f_0}$, and that otherwise $a(H, f) = 0$ or $x_{H,f} = 0$ holds. Hence the map T is injective. This finishes the proof of Theorem 5.2. ∎

Now Theorem 0.2 follows from Theorem 4.3 and Theorem 5.2 using (2.12).

6. Restriction structures and multiplicative structures

Before we simplify the source of the equivariant Chern character further in the presence of a module structure over the Green functor $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$ on $\mathcal{H}_q^?(*)$ in Section 7, we introduce an additional structure on an equivariant homology theory called restriction structure. It will guarantee that the Mackey structure appearing in Theorem 0.2 exists. This restriction structure is canonically given in all relevant examples.

We also briefly deal with multiplicative structures. The material of this section is not needed for the following sections.

A *restriction structure* on an equivariant homology theory $\mathcal{H}_*^?$ consists of the following data. For any injective group homomorphism $\alpha : H \to G$, whose image has finite index in G, we require in (X, A) natural homomorphisms

$$\operatorname{res}_\alpha : \mathcal{H}_n^G(X, A) \to \mathcal{H}_n^H(\operatorname{res}_\alpha(X, A)),$$

where (X, A) is a pair of G-CW-complexes and $\operatorname{res}_\alpha(X, A)$ is the H-CW-pair obtained from (X, A) by restriction with α. If α is an inclusion of a subgroup $H \subset G$, we also write res_G^H instead of $\operatorname{res}_\alpha$. We require

(a) Compatibility with the boundary homomorphisms
The restriction homomorphism $\operatorname{res}_\alpha$ is compatible with the boundary homomorphism δ_n^G and δ_n^H;

(b) Functoriality
If $\beta : G \to K$ is another injective group homomorphism whose image has finite index in K, then $\operatorname{res}_{\beta\circ\alpha} = \operatorname{res}_\alpha \circ \operatorname{res}_\beta$;

(c) Compatibility of induction and restriction for isomorphisms
If $\alpha : H \xrightarrow{\cong} G$ is an isomorphism of groups, then the composition

$$\mathcal{H}_n^G(X) \xrightarrow{\operatorname{res}_\alpha} \mathcal{H}_n^H(\operatorname{res}_\alpha X) \xrightarrow{\operatorname{ind}_\alpha} \mathcal{H}_n^G(\operatorname{ind}_\alpha \operatorname{res}_\alpha X) \xrightarrow{T(X)} \mathcal{H}_n^G(X)$$

is the identity, where $T(X) : \operatorname{ind}_\alpha \operatorname{res}_\alpha X \to X$ is the canonical G-homeomorphism;

(d) Double Coset formula
Let $H, K \subset G$ be subgroups such that K has finite index in G. Let (X, A) be a H-CW-pair. (Notice for the sequel that $K\backslash G/H$ is finite.) Denote by

$$f : \coprod_{KgH\in K\backslash G/H} \operatorname{ind}_{c(g):H\cap g^{-1}Kg\to K} \operatorname{res}_H^{H\cap g^{-1}Kg}(X, A) \xrightarrow{\cong} \operatorname{res}_G^K \operatorname{ind}_H^G(X, A)$$

the canonical K-homeomorphism.
Then the following two compositions agree for all $q \in \mathbb{Z}$

$$\mathcal{H}_q^H(X) \xrightarrow{\prod_{KgH\in K\backslash G/H} \operatorname{ind}_{c(g):H\cap g^{-1}Kg\to K} \circ \operatorname{res}_H^{H\cap g^{-1}Kg}}$$
$$\prod_{KgH\in K\backslash G/H} \mathcal{H}_q^K\left(\operatorname{ind}_{c(g):H\cap g^{-1}Kg\to K} \operatorname{res}_H^{H\cap g^{-1}Kg}(X, A)\right)$$
$$\xrightarrow{\cong} \mathcal{H}_q^K\left(\coprod_{KgH\in K\backslash G/H} \operatorname{ind}_{c(g):H\cap g^{-1}Kg\to K} \operatorname{res}_H^{H\cap g^{-1}Kg}(X, A)\right) \xrightarrow{\mathcal{H}_q^K(f)} \mathcal{H}_q^K(\operatorname{res}_G^K \operatorname{ind}_H^G(X, A))$$

and

$$\operatorname{res}_G^K \circ \operatorname{ind}_H^G : \mathcal{H}_q^H(X, A) \to \mathcal{H}_q^K(\operatorname{res}_G^K \operatorname{ind}_H^G(X, A)).$$

If $\mathcal{H}_*^?$ is an equivariant homology theory with a restriction structure, $\mathcal{BH}_*^?$ inherits a restriction structure as follows. For $K \subset H$ we get a natural map $H/K \to \operatorname{res}_\alpha \operatorname{ind}_\alpha H/K$ as the adjoint of the identity on $\operatorname{ind}_\alpha H/K$. It induces $\mathcal{H}_q^H(H/K) \to \mathcal{H}_q^H(\operatorname{res}_\alpha \operatorname{ind}_\alpha H/K)$. We get a $R\mathsf{Or}(G, \mathcal{F})$-module $\mathcal{H}_q^H(\operatorname{res}_\alpha G/?)$ which assigns to G/K the R-module $\mathcal{H}_q^H(\operatorname{res}_\alpha G/K)$. Thus we obtain a transformation of covariant $R\mathsf{Or}(H, \mathcal{F})$-modules $\mathcal{H}_q^H(H/?) \to \operatorname{res}_\alpha \mathcal{H}_q^H(\operatorname{res}_\alpha G/?)$. Its adjoint is a map of $R\mathsf{Or}(G, \mathcal{F})$-modules

$$i_q : \operatorname{ind}_\alpha \mathcal{H}_q^H(H/?) \xrightarrow{\cong} \mathcal{H}_q^H(\operatorname{res}_\alpha G/?),$$

which turns out to be bijective. This can be seen from its more explicit description as the composition of isomorphisms

$$\begin{aligned}\operatorname{ind}_\alpha \mathcal{H}_q^H(H/?) &= R\operatorname{mor}_{\mathsf{Or}(G,\mathcal{F})}(\operatorname{ind}_\alpha H/?, G/??) \otimes_{R\mathsf{Or}(H,\mathcal{F})} \mathcal{H}_q^H(H/?)\\ &\xrightarrow{\mu} R\operatorname{mor}_{\mathsf{Or}(H,\mathcal{F})}(H/?, \operatorname{res}_\alpha G/??) \otimes_{R\mathsf{Or}(H,\mathcal{F})} \mathcal{H}_q^H(H/?) \xrightarrow{\nu} \mathcal{H}_q^G(G/??),\end{aligned}$$

where μ comes from the adjunction of $\operatorname{ind}_\alpha$ and $\operatorname{res}_\alpha$ and ν sends $(f : H/? \to \operatorname{res}_\alpha G/??) \otimes x$ to $\mathcal{H}_q^H(f)(x)$. The restriction structure on $\mathcal{H}_*^?$ induces a map of $\mathsf{Or}(G,\mathcal{F})$-modules

$$\mathcal{H}_q^G(G/?) \to \mathcal{H}_q^H(\operatorname{res}_\alpha G/?).$$

There is a natural isomorphism of $R\mathsf{Or}(H,\mathcal{F})$-chain complexes

$$C_*^{\mathsf{Or}(H,\mathcal{F})}(\operatorname{res}_\alpha(X,A)) \xrightarrow{\cong} \operatorname{res}_\alpha C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A).$$

There is a natural isomorphism of R-modules (compare (2.5))

$$\operatorname{res}_\alpha C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A)) \otimes_{R\mathsf{Or}(H,\mathcal{F})} \mathcal{H}_q^H(H/?) \xrightarrow{\cong} C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) \otimes_{R\mathsf{Or}(G,\mathcal{F})} \operatorname{ind}_\alpha \mathcal{H}_q^H(H/?).$$

The last four maps together can be combined to a map of R-chain complexes

$$C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) \otimes_{R\mathsf{Or}(G,\mathcal{F})} \mathcal{H}_q^G(G/?) \to C_*^{\mathsf{Or}(H,\mathcal{F})}(\operatorname{res}_\alpha(X,A)) \otimes_{R\mathsf{Or}(H,\mathcal{F})} \mathcal{H}_q^H(H/?).$$

It induces on homology homomorphisms

$$\operatorname{res}_\alpha : H_p^{\mathsf{Or}(G,\mathcal{F})}(X,A; \mathcal{H}_q^G(X,A)) \to H_p^{\mathsf{Or}(H,\mathcal{F})}(\operatorname{res}_\alpha(X,A); \mathcal{H}_q^H(H/?)).$$

Their direct sum yields the desired natural homomorphism

$$\operatorname{res}_\alpha : \mathcal{BH}^G(X,A) \to \mathcal{BH}^H(\operatorname{res}_\alpha(X,A)).$$

We leave it to the reader to check that the axioms of a restriction structure are fullfilled.

Next we introduce *multiplicative structures*. An *external product* on $\mathcal{H}_*^G$ assigns to any two groups G and G' and pairs of (proper) G- resp. G'-CW-complexes (X,A) resp. (X',A') an in (X,A) and (X',A') natural homomorphism

$$\times : \mathcal{H}_n^G(X,A) \otimes_R \mathcal{H}_{n'}^{G'}(X',A') \to \mathcal{H}_{n+n'}^{G\times G'}((X,A)\times(X',A')), \tag{6.1}$$

where $(X,A)\times(X',A')$ is the pair of (proper) $G\times G'$-CW-complexes $(X\times X', X\times A' \cup A\times X')$. We mention that we work in the category of compactly generated spaces (see [25], [26, I.4]) so that $(X,A)\times(X',A')$ is indeed a (proper) $G\times G'$-CW-pair. These parings are required to be compatible with the boundary homomorphisms, namely, for $u \in \mathcal{H}_p^G(X,A)$ and $v \in \mathcal{H}_q^G(Y,B)$ we have

$$\partial(u \times v) = \partial(u) \times v + (-1)^p \cdot u \times \partial(v).$$

We also assume that these pairings are compatible with induction, i.e. for group homomorphisms $\alpha : H \to G$ and $\alpha' : H' \to G'$ and $u \in \mathcal{H}_p^H(X,A)$ and $u' \in \mathcal{H}_q^{H'}(Y,B)$ we require

$$\mathcal{H}_{p+p'}^{G\times G'}(f)\,(\operatorname{ind}_\alpha(u) \times \operatorname{ind}_{\alpha'}(u')) \quad = \quad \operatorname{ind}_{\alpha\times\alpha'}(u \times v)$$

for $f : \operatorname{ind}_\alpha(X,A) \times \operatorname{ind}_{\alpha'}(X',A') \xrightarrow{\cong} \operatorname{ind}_{\alpha\times\alpha'}((X,A)\times(X',A'))$ the canonical $G\times G'$-homeomorphism. Furthermore we require that the external product $\times$ is associative, graded commutative and has a unit element 1 in $\mathcal{H}_0^{\{1\}}(*)$.

If $\mathcal{H}_*^?$ comes with an external product, we call it a *multiplicative (proper) equivariant homology theory with values in R-modules.* If $\mathcal{H}_*^?$ comes with a restriction structure, we will require that the multiplicative structure and restriction structure are compatible. Namely, for injective group homomorphisms $\alpha : H \to G$ and $\alpha' : H' \to G'$, whose images have finite index, and $u \in \mathcal{H}_p^G(X,A)$ and $u' \in \mathcal{H}_q^{G'}(Y,B)$ we require

$$\mathrm{res}_\alpha(u) \times \mathrm{res}_{\alpha'}(u') \;=\; \mathrm{res}_{\alpha\times\alpha'}(u \times v).$$

Next we explain how a multiplicative structure on $\mathcal{H}_*^?$ induces a multiplicative structure on the associated Bredon homology $\mathcal{BH}_*^?$. Let (X,A) be a proper G-CW-pair and let (X',A') be a proper G'-CW-pair. Let $C_*(X,A) \otimes_R C_*(X',A')$ be the obvious $R\mathsf{Or}(G,\mathcal{F}) \times \mathsf{Or}(G',\mathcal{F})$-chain complex. Denote by $I : \mathsf{Or}(G,\mathcal{F}) \times \mathsf{Or}(G',\mathcal{F}) \to \mathsf{Or}(G \times G',\mathcal{F})$ the functor sending $(G/H, G'/H')$ to $G \times G'/H \times H'$. There is a natural isomorphism of $\mathsf{Or}(G \times H,\mathcal{F})$-chain complexes

$$\mathrm{ind}_I \left(C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) \otimes_R C_*^{\mathsf{Or}(G',\mathcal{F})}(X',A')\right) \xrightarrow{\cong} C_*^{\mathsf{Or}(G\times G',\mathcal{F})}((X,A)\times(X',A')),$$

which comes from the adjunction (2.4) and the natural isomorphism of the cellular chain complex of a product of two (non-equivariant) CW-complexes with the tensor product of the individual cellular chain complexes. The multiplicative structure on $\mathcal{H}_*^?$ induces a natural transformation of $\mathsf{Or}(G,\mathcal{F}) \times \mathsf{Or}(H,\mathcal{F})$-modules

$$\mathcal{H}_p^G(G/?) \otimes_R \mathcal{H}_q^{G'}(G'/?') \to \mathrm{res}_I\, \mathcal{H}_{p+q}^{G\times G'}(G \times G'/??).$$

There are natural isomorphisms of R-chain complexes

$$\begin{aligned}&\left(C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) \otimes_{R\mathsf{Or}(G,\mathcal{F})} \mathcal{H}_p^G(G/?)\right) \otimes_R \left(C_*^{\mathsf{Or}(G',\mathcal{F})}(X',A') \otimes_{R\mathsf{Or}(G',\mathcal{F})} \mathcal{H}_p^{G'}(G'/?')\right)\\ &\quad\xrightarrow{\cong}\; \left(C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) \otimes_R C_*^{\mathsf{Or}(G',\mathcal{F})}(X',A')\right) \otimes_{R\mathsf{Or}(G,\mathcal{F})\times\mathsf{Or}(G',\mathcal{F})} \left(\mathcal{H}_p^G(G/?) \otimes_R \mathcal{H}_p^{G'}(G'/?')\right)\end{aligned}$$

and (see (2.5))

$$\begin{aligned}&\mathrm{ind}_I \left(C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) \otimes_R C_*^{\mathsf{Or}(G',\mathcal{F})}(X',A')\right) \otimes_{R\mathsf{Or}(G\times G',\mathcal{F})} \mathcal{H}_{p+q}^{G\times G'}(G \times G'/??)\\ &\quad\xrightarrow{\cong}\; C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) \otimes_R C_*^{\mathsf{Or}(G',\mathcal{F})}(X',A')) \otimes_{R\mathsf{Or}(G,\mathcal{F})\times\mathsf{Or}(G',\mathcal{F})} \mathrm{res}_I\, \mathcal{H}_{p+q}^{G\times G'}(G \times G'/??).\end{aligned}$$

Combining the last four maps yields a chain map

$$\begin{aligned}&\left(C_*^{\mathsf{Or}(G,\mathcal{F})}(X,A) \otimes_{R\mathsf{Or}(G,\mathcal{F})} \mathcal{H}_p^G(G/?)\right) \otimes_R \left(C_*^{\mathsf{Or}(G',\mathcal{F})}(X',A') \otimes_{R\mathsf{Or}(G',\mathcal{F})} \mathcal{H}_p^{G'}(G'/?')\right)\\ &\qquad\to\; C_*^{\mathsf{Or}(G\times G',\mathcal{F})}((X,A)\times(X',A')) \otimes_{R\mathsf{Or}(G\times G',\mathcal{F})} \mathcal{H}_{p+q}^{G\times G'}(G \times G'/??).\end{aligned}$$

It induces the required multiplicative structure

$$\mathcal{BH}_m^G(X,A) \otimes_R \mathcal{BH}_n^{G'}(X',A') \to \mathcal{BH}_{m+n}^{G\times G'}((X,A)\times(X',A')). \tag{6.2}$$

We leave it to the reader to verify the axioms of a multiplicative proper equivariant homology theory for $\mathcal{BH}_*^?$.

Theorem 6.3 *Let F be a field of characteristic 0. Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in F-modules. Suppose that $\mathcal{H}_*^?$ possesses a restriction structure. Let I be the set of conjugacy classes (H) of finite subgroups H of G. Then there is an isomorphism of proper homology theories*

$$\mathrm{ch}_*^? : \mathcal{BH}_*^? \xrightarrow{\cong} \mathcal{H}_*^?$$

such that

$$\mathcal{BH}_n^G(X,A) \;=\; \oplus_{p+q=n} \oplus_{(H)\in I} H_p(C_GH\backslash(X^H,A^H);F) \otimes_{F[W_GH]} S_H\mathcal{H}_q^G(G/?).$$

The isomorphism $\mathrm{ch}_*^?$ is compatible with the given restriction structure on $\mathcal{H}_*^?$ and the induced restriction structure on $\mathcal{BH}_*^?$. If $\mathcal{H}^*$ comes with a multiplicative structure and we equip $\mathcal{BH}_*^?$ with the associated multiplicative structure, $\mathrm{ch}_*^?$ is also compatible with the multiplicative structures.

Proof : Given a proper equivariant homology theory $\mathcal{H}^?_*$ with values in F-modules together with restriction structure, then $\mathcal{H}^?_q(*)$ inherits a Mackey structure in the obvious way. Given an injective group homomorphism $f : H \to K$ of finite groups, induction is given by the composition $\mathcal{H}^H(*) \xrightarrow{\mathrm{ind}_f} \mathcal{H}^K_q(\mathrm{ind}_f *) \xrightarrow{\mathcal{H}^K_q(\mathrm{pr})} \mathcal{H}^K_q(*)$ and restriction by $\mathrm{res}_f : \mathcal{H}^K_q(*) \to \mathcal{H}^H_q(*)$. Now apply Theorem 0.2. We leave the lengthy but straighforward verification that the equivariant Chern character is compatible with the restriction structures and multiplicative structures to the reader. ■

Example 6.4 Equivariant bordism as introduced in Example 1.4 has an obvious restriction structure coming from restriction of spaces and an obvious multiplicative structure coming from the cartesian product. Hence Theorem 6.3 applies to it and yields an isomorphism of multiplicative proper equivariant homology theories with restriction structure

$$\mathrm{ch}^G_n(X,A) : \oplus_{p+q=n} \oplus_{(H)\in I} H_p(C_GH\backslash(X^H,A^H);\mathbb{Q}) \otimes_{\mathbb{Q}[W_GH]} S_H\mathbb{Q} \otimes_{\mathbb{Z}} \Omega^G_q(G/?) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} \Omega^G_n(X,A),$$

where $S_H\mathbb{Q} \otimes_{\mathbb{Z}} \Omega^G_q(G/?) = \mathrm{coker}\left(\oplus_{K\subset H, K\neq H}\mathbb{Q} \otimes_{\mathbb{Z}} \Omega^K_q(*) \to \mathbb{Q} \otimes_{\mathbb{Z}} \Omega^H_q(*)\right)$.

7. Green functors

Next we simplify the source of the equivariant Chern character further in the presence of a module structure over the Green functor $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$ on $\mathcal{H}^?_q(*)$. Such additional structure is given in the situation of our main Example 1.5.

Let $\phi : R \to S$ be a homomorphism of associative commutative rings with unit. Let M be a Mackey functor with values in R-modules and let N and P be Mackey functors with values in S-modules. A *pairing* with respect to ϕ is a family of maps

$$m(H) : M(H) \times N(H) \to P(H), \qquad (x,y) \mapsto m(H)(x,y) =: x \cdot y,$$

where H runs through the finite groups and we require the following properties for all injective group homomorphisms $f : H \to K$ of finite groups:

$$\begin{array}{lll}
(x_1 + x_2) \cdot y = x_1 \cdot y + x_2 \cdot y & \text{for} & x_1, x_2 \in M(H), y \in N(H); \\
x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2 & \text{for} & x \in M(H), y_1, y_2 \in N(H); \\
(rx) \cdot y = \phi(r)(x \cdot y) & \text{for} & r \in R, x \in M(H), y \in N(H); \\
x \cdot sy = s(x \cdot y) & \text{for} & s \in S, x \in M(H), y \in N(H); \\
\mathrm{res}_f(x \cdot y) = \mathrm{res}_f(x) \cdot \mathrm{res}_f(y) & \text{for} & x \in M(K), y \in N(K); \\
\mathrm{ind}_f(x) \cdot y = \mathrm{ind}_f\,(x \cdot \mathrm{res}_f(y)) & \text{for} & x \in M(H), y \in N(K); \\
x \cdot \mathrm{ind}_f(y) = \mathrm{ind}_f\,(\mathrm{res}_f(x) \cdot y) & \text{for} & x \in M(K), y \in N(H).
\end{array}$$

A *Green functor* with values in R-modules is a Mackey functor U together with a pairing with respect to $\mathrm{id} : R \to R$ and elements $1_H \in U(H)$ for each finite group H such that for each finite group H the pairing $U(H) \times U(H) \to U(H)$ induces the structure of an R-algebra on $U(H)$ with unit 1_H and for any morphism $f : H \to K$ in FGINJ the map $U^*(f) : U(K) \to U(H)$ is a homomorphism of R-algebras with unit. Let U be a Green functor with values in R-modules and M be a Mackey functor with values in S-modules. A (left) U-module structure on M with respect to the ring homomorphism $\phi : R \to S$ is a pairing such that any of the maps $U(H) \times M(H) \to M(H)$ induces the structure of a (left) module over the R-algebra $U(H)$ on the R-module $\phi^*M(H)$ which is obtained from the S-module $M(H)$ by $rx := \phi(r)x$ for $r \in R$ and $x \in M(H)$.

Lemma 7.1 *Let $\phi : R \to S$ be a homomorphism of associative commutative rings with unit. Let U be a Green functor with values in R-modules and let M be a Mackey functor with values in S-modules such that M comes with a U-module structure with respect to ϕ. Let $\mathcal{S}$ be a set of subgroups of the finite group H. Suppose that the map*

$$\oplus_{K\in\mathcal{S}} \operatorname{ind}_K^H : \oplus_{K\in\mathcal{S}} U(K) \to U(H)$$

is surjective. Then the map

$$\oplus_{K\in\mathcal{S}} \operatorname{ind}_K^H : \oplus_{K\in\mathcal{S}} M(K) \to M(H)$$

is surjective.

Proof : By hypothesis there are elements $u_K \in U(K)$ for $K \in \mathcal{S}$ satisfying $1_H = \sum_{K\in\mathcal{S}} \operatorname{ind}_K^H u_K$ in $U(H)$. This implies for $x \in M(H)$.

$$x = 1_H \cdot x = \left(\sum_{K\in\mathcal{S}} \operatorname{ind}_K^H u_K\right) \cdot x = \sum_{K\in\mathcal{S}} \operatorname{ind}_K^H \left(u_K \cdot \operatorname{res}_H^K x\right). \qquad \blacksquare$$

Our main example of a Green functor with values in $\mathbb{Q}$-modules $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$ assigns to a finite group H the $\mathbb{Q}$-module $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(H)$, where $R_{\mathbb{Q}}(H)$ denotes the rational representation ring. Notice that $R_{\mathbb{Q}}(H)$ is the same as the projective class group $K_0(\mathbb{Q}H)$. The Mackey structure comes from induction and restriction of representations. The pairing $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(H) \times \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(H) \to \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(H)$ comes from the tensor product $P \otimes_{\mathbb{Q}} Q$ of two $\mathbb{Q}H$-modules P and Q equipped with the diagonal H-action. The unit element is the class of $\mathbb{Q}$ equipped with the trivial H-action.

Let $\operatorname{class}_{\mathbb{Q}}(H)$ be the $\mathbb{Q}$-vector space of functions $H \to \mathbb{Q}$ which are invariant under $\mathbb{Q}$-conjugation, i.e. we have $f(h_1) = f(h_2)$ for two elements $h_1, h_2 \in H$ if the cyclic subgroups $\langle h_1\rangle$ and $\langle h_2\rangle$ generated by h_1 and h_2 are conjugated in H. Elementwise multiplication defines the structure of a $\mathbb{Q}$-algebra on $\operatorname{class}_{\mathbb{Q}}$ with the function which is constant 1 as unit element. Taking the character of a rational representation yields an isomorphism of $\mathbb{Q}$-algebras [23, Theorem 29 on page 102]

$$\chi^H : \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(H) \xrightarrow{\cong} \operatorname{class}_{\mathbb{Q}}(H). \tag{7.2}$$

We define a Mackey structure on $\operatorname{class}_{\mathbb{Q}}(?)$ as follows. Let $f : H \to K$ be an injective group homomorphism. For a character $\chi \in \operatorname{class}_{\mathbb{Q}}(H)$ define its induction with f to be the character $\operatorname{ind}_f(\chi) \in \operatorname{class}_{\mathbb{Q}}(K)$ given by

$$\operatorname{ind}_f(\chi)(k) = \frac{1}{|H|} \cdot \sum_{l\in K, h\in H, f(h)=l^{-1}kl} \chi(h).$$

For a character $\chi \in \operatorname{class}_{\mathbb{Q}}(H)$ define its restriction with f to be the character $\operatorname{res}_f(\chi) \in \operatorname{class}_{\mathbb{Q}}(H)$ given by

$$\operatorname{res}_f(\chi)(h) := \chi(f(h)).$$

One easily checks that this yields the structure of a Green functor on $\operatorname{class}_{\mathbb{Q}}(?)$ and that the family of isomorphisms χ^H defined in (7.2) yields an isomorphism of Green functors from $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$ to $\operatorname{class}_{\mathbb{Q}}(?)$.

For a finite group H and any cyclic subgroup $C \subset H$, define

$$\theta_C^H \in \operatorname{class}_{\mathbb{Q}}(H) \tag{7.3}$$

to be the function which sends $h \in H$ to 1 if $\langle h\rangle$ and C are conjugated in H and to 0 otherwise.

Lemma 7.4 *Let $\phi : \mathbb{Q} \to R$ be a homomorphism of associative commutative rings with unit. Let M be a Mackey functor with values in R-modules which is a module over the Green functor $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(H)$ with respect to ϕ. Then* *(a) For a finite group H the map*

$$\oplus_{C \subset H, C \text{ cyclic}} \operatorname{ind}_C^H : \oplus_{C \subset H, C \text{ cyclic}} M(C) \to M(H)$$

is surjective;

(b) Let C be a finite cyclic group. Let

$$\theta_C^C : M(C) \to M(C)$$

be the map induced by the $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(C)$-module structure and multiplication with preimage of the element $\theta_C^C \in \operatorname{class}_{\mathbb{Q}}(C)$ under the isomorphism $\chi^H : \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(C) \cong \operatorname{class}_{\mathbb{Q}}(C)$ of (7.2). Then the image resp. cokernel of

$$\oplus_{D \subset C, D \neq C} \operatorname{ind}_D^C : \oplus_{D \subset C, D \neq C} M(D) \to M(C)$$

is equal resp. isomorphic to the kernel resp. image of the map $\theta_C^C : M(C) \to M(C)$.

Proof : Let $C \subset H$ be a cyclic subgroup of the finite group H. Then we get for $h \in H$

$$\frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \theta_C^C(h) \; = \; \frac{1}{[H:C]} \cdot \frac{1}{|C|} \cdot \sum_{l \in H, l^{-1}hl \in C} \theta_C^C(l^{-1}hl) \; = \; \frac{1}{|H|} \cdot \sum_{l \in H, \langle l^{-1}hl \rangle = C} 1.$$

This implies in $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(H) \cong \operatorname{class}_{\mathbb{Q}}(H)$

$$1_H \quad = \sum_{C \subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \theta_C^C \tag{7.5}$$

since for any $l \in H$ and $h \in H$ there is precisely one cyclic subgroup $C \subset H$ with $C = \langle l^{-1}hl \rangle$. Now assertion (a) follows from the following calculation for $x \in M(H)$

$$x = 1_H \cdot x = \left(\sum_{C \subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \theta_C^C \right) \cdot x = \sum_{C \subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H (\theta_C^C \cdot \operatorname{res}_H^C x).$$

It remains to prove assertion (b). Obviously θ_C^C is an idempotent for any cyclic group C. We get for $x \in M(C)$ from (7.5)

$$(1_C - \theta_C^C) \cdot x = \left(\sum_{D \subset C, D \neq C} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C \theta_D^D \right) \cdot x = \sum_{D \subset C, D \neq C} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C (\theta_D^D \cdot \operatorname{res}_C^D x)$$

and for $D \subset C, D \neq C$ and $y \in M(D)$

$$\theta_C^C \cdot \operatorname{ind}_D^C y = \operatorname{ind}_D^C (\operatorname{res}_C^D \theta_C^C \cdot y) = \operatorname{ind}_D^C (0 \cdot y) = 0.$$

This finishes the proof of Lemma 7.4. ■

Now Theorem 0.3 follows from Theorem 0.2 and Lemma 7.4. For more information about Mackey and Green functors and induction theorems we refer for instance to [6, Section 6] and [8].

8. Applications to K- and L-theory

In this section we apply Theorem 0.3 to the equivariant homology theories of Example 1.5. Thus we obtain explicit computations of the rationalized source of the assembly map (1.6). These give explicit computations of the rationalized algebraic K- and L-groups of RG and of the topological K-groups of the real and complex reduced group C^*-algebras of G, provided that the Farrell-Jones Conjecture with respect to the family $\mathcal{F}$ of finite subgroups resp. the Baum Connes Conjecture is true for G. Before we carry out this program, we mention the following facts. Notice for the sequel that all various versions of L-groups, symmetric, quadratic or decorated L-groups, differ only by 2-torsion and hence agree after inverting 2.

Theorem 8.1 *There are natural isomorphisms*

$$\begin{array}{rcl} L_n(\mathbb{Z}G)[1/2] & \xrightarrow{\cong} & L_n(\mathbb{Q}G)[1/2]; \\ K_n(C_r^*(G,\mathbb{R}))[1/2] & \xrightarrow{\cong} & L_n(C_r^*(G,\mathbb{R}))[1/2]; \\ K_n(C_r^*(G,\mathbb{C}))[1/2] & \xrightarrow{\cong} & L_n(C_r^*(G,\mathbb{C}))[1/2]. \end{array}$$

Proof : The proof of the first isomorphism can be found in [20, page 376]. The other two isomorphism are explained in [22, Theorem 1.8 and 1.11], where they are attributed to Karoubi, Miller and Mishchenko. ■

Next we introduce a Mackey structure and then a module structure over the Green functor $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$ on the various K- and L-groups. Let R be an associative commutative ring with unit satisfying $\mathbb{Q} \subset R$ and let F be $\mathbb{R}$, $\mathbb{C}$. Induction and restriction yield obvious Mackey functors

$$\begin{array}{rll} \mathbb{Q} \otimes_{\mathbb{Z}} K_q(R?) : \mathsf{FGINJ} & \to \mathbb{Q} - \mathrm{MOD}, & H \mapsto \mathbb{Q} \otimes_{\mathbb{Z}} K_q(RH); \\ \mathbb{Q} \otimes_{\mathbb{Z}} L_q(R?) : \mathsf{FGINJ} & \to \mathbb{Q} - \mathrm{MOD}, & H \mapsto \mathbb{Q} \otimes_{\mathbb{Z}} L_q(RH); \\ \mathbb{Q} \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(C_r^*(?,F)) : \mathsf{FGINJ} & \to \mathbb{Q} - \mathrm{MOD}, & H \mapsto \mathbb{Q} \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(C_r^*(H,F)). \end{array}$$

The tensor product over R resp. F with the diagonal action induces on $\mathbb{Q} \otimes_{\mathbb{Z}} K_0(R?)$, $\mathbb{Q} \otimes_{\mathbb{Z}} L_0(R?)$ and $\mathbb{Q} \otimes_{\mathbb{Z}} K_0^{\mathrm{top}}(C^*(?,F))$ the structure of a Green functor with values in $\mathbb{Q}$-modules and the structure of a module over these Green functors on $\mathbb{Q} \otimes_{\mathbb{Z}} K_q(R?)$, $\mathbb{Q} \otimes_{\mathbb{Z}} L_q(R?)$ and $\mathbb{Q} \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(C^*(?,F))$ for all $q \in \mathbb{Z}$. The change of ring maps

$$\begin{array}{rcl} \mathbb{Q} \otimes_{\mathbb{Z}} K_0(\mathbb{Q}?) & \to & \mathbb{Q} \otimes_{\mathbb{Z}} K_0(R?); \\ \mathbb{Q} \otimes_{\mathbb{Z}} L_0(\mathbb{Q}?) & \to & \mathbb{Q} \otimes_{\mathbb{Z}} L_0(R?); \\ \mathbb{Q} \otimes_{\mathbb{Z}} K_0^{\mathrm{top}}(C_r^*(?,\mathbb{R})) & \to & \mathbb{Q} \otimes_{\mathbb{Z}} K_0^{\mathrm{top}}(C_r^*(?,\mathbb{C})) \end{array}$$

induce maps of Green functors. Since $\mathbb{Q} \otimes_{\mathbb{Z}} K_0(\mathbb{Q}?) = \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$, we get a module structure over the Green functor $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$ on each Mackey functor $\mathbb{Q} \otimes_{\mathbb{Z}} K_q(R?)$. The change of rings map

$$\mathbb{Q} \otimes_{\mathbb{Z}} L_0(\mathbb{Q}?) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} L_0(\mathbb{R}?)$$

is known to be an isomorphism (see [21, Proposition 22.19 on page 237]. There is an isomorphism of Green functors (see Theorem 8.1 or [21, Proposition 22.33 on page 252])

$$\mathbb{Q} \otimes_{\mathbb{Z}} K_0(\mathbb{R}?) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} L_0(\mathbb{R}?).$$

Thus we get a morphism of Green functors

$$\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} L_0(\mathbb{Q}?).$$

Hence we obtain a module structure over the Green functor $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$ on the Mackey functor $\mathbb{Q} \otimes_{\mathbb{Z}} L_q(R?)$. Since $K_0(\mathbb{R}?) = K_0^{\mathrm{top}}(C_r^*(?,\mathbb{R}))$, we finally obtain also a module structure over the Green functor $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$ on the Mackey functor $\mathbb{Q} \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(C_r^*(?,F))$. If $\mathbb{Q} \subset R$, then the cellular $R[C_G H]$-chain complex $C_*(E(G,\mathcal{F})^H)$ is a projective resolution of the trivial $R[C_G H]$-module R and we obtain for any finite group $H \subset G$ an identification

$$H_p(C_G H \backslash E(G,\mathcal{F})^H; R) \quad \cong \quad H_p(C_G H; R). \qquad (8.2)$$

Notice that now Theorem 0.4 follows from Theorem 0.3 and Example 1.5. The homomorphisms appearing in Theorem 0.4 are compatible with the various change of rings or of K-theory maps since these maps are compatible with the relevant module structures over the Green functor $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$.

If the ring R is a field of characteristic zero and we are willing to extend $\mathbb{Q}$ to a larger field, then we can simplify the right side of the various maps appearing in Theorem 0.4 as follows. Let F be a field of characteristic zero. Fix an integer $m \geq 1$. Let $F(\zeta_m) \supset F$ be the Galois extension given by adjoining the primitive m-th root of unity ζ_m to F. Denote by $G(m, F)$ the Galois group of this extension of fields, i.e. the group of automorphisms $\sigma : F(\zeta_m) \to F(\zeta_m)$ which induce the identity on F. It can be identified with a subgroup of $\mathbb{Z}/m^*$ by sending σ to the unique element $u(\sigma) \in \mathbb{Z}/m^*$ for which $\sigma(\zeta_m) = \zeta_m^{u(\sigma)}$ holds. Given a finite cyclic group C of order $|C|$, the Galois group $G(|C|, F)$ acts on C by sending c to $c^{u(\sigma)}$, and thus on the set $\operatorname{Gen}(C)$ of generators of C. Let V be a $F(\zeta_{|C|})$-module. Denote by $\operatorname{res}_\sigma V$ for $\sigma \in G(|C|, F)$ the $F(\zeta_{|C|})$-module obtained from V by restriction with σ, i.e. the underlying abelian groups of $\operatorname{res}_\sigma V$ and V agree and multiplication with $x \in F(\zeta_m)$ on $\operatorname{res}_\sigma V$ is given by multiplication with $\sigma(x)$ on V. Thus we obtain an action of $G(|C|, F)$ on $K_q(F(\zeta_{|C|}))$ by sending $\sigma \in G(|C|, F)$ to the automorphism $\operatorname{res}_\sigma : K_q(F(\zeta_{|C|})) \to K_q(F(\zeta_{|C|}))$ coming from the functor $V \mapsto \operatorname{res}_\sigma V$. This action extends to an action of the Galois group $G(|C|, F)$ on $F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|}))$ by $\sigma \cdot (v \otimes w) := v \otimes \operatorname{res}_\sigma(w)$. Equip $\operatorname{map}\left(\operatorname{Gen}(C), F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|}))\right)^{G(|C|,F)}$ and $F(\zeta_{|C|}) \otimes_{\mathbb{Z}} S_C K_q(F[C])$ with the obvious $F(\zeta_{|C|})$-module structures.

Lemma 8.3 *Let C be a finite cyclic group. Then there is an isomorphism of $F(\zeta_{|C|})$-modules*

$$\operatorname{map}\left(\operatorname{Gen}(C), F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|}))\right)^{G(|C|,F)} \xrightarrow{\cong} F(\zeta_{|C|}) \otimes_{\mathbb{Q}} \operatorname{im}\left(\theta_C^C : \mathbb{Q} \otimes_{\mathbb{Z}} K_q(F[C]) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_q(F[C])\right)$$

which is natural with respect to automorphisms of C.

Its proof needs some preparation. Let G be a group. Given a positive integer m and a $F(\zeta_m)[G]$-module V, we define an in V natural isomorphism of $F(\zeta_m)[G]$-modules

$$\Phi : \operatorname{ind}_F^{F(\zeta_m)} \operatorname{res}_{F(\zeta_m)}^F V = F(\zeta_m) \otimes_F V \quad \xrightarrow{\cong} \quad \oplus_{\sigma \in G(m,F)} \operatorname{res}_\sigma V, \qquad x \otimes v \mapsto (\sigma(x)v)_{\sigma \in G(m,F)}.$$

Obviously ϕ is natural in V and $F(\zeta_m)[G]$-linear. We claim that an inverse of ϕ is given by

$$\begin{array}{rcl} \Phi^{-1} : \oplus_{\sigma \in G} \operatorname{res}_\sigma V & \to & \operatorname{ind}_F^{F(\zeta_m)} \operatorname{res}_{F(\zeta_m)}^F V = F(\zeta_m) \otimes_F V, \\ (v_\sigma)_{\sigma \in G(m,F)} & \mapsto & \frac{1}{m} \cdot \sum_{i=1}^m \sum_{\sigma \in G(m,F)} \zeta_m^{-i} \otimes_F \sigma(\zeta_m)^i v_\sigma. \end{array}$$

This follows from an easy calculation using the facts that for a m-th root of unity ζ the sum $\sum_{i=1}^m \zeta^i$ is zero, if $\zeta \neq 1$, and is m, if $\zeta = 1$, and that an element $x \in F(\zeta_m)$ belongs to F if and only if $\sigma(x) = x$ for all $\sigma \in G(m, F)$ holds. Fix a F- basis $\{b_\sigma \mid \sigma \in G(m, F)\}$ for $F(\zeta_m)$. Given a FG-module W, we obtain an in W natural FG-isomorphism

$$\Psi : \oplus_{G(m,F)} W \quad \xrightarrow{\cong} \quad \operatorname{res}_{F(\zeta_m)}^F \operatorname{ind}_F^{F(\zeta_m)} W = F(\zeta) \otimes_F W, \quad (w_\sigma)_{\sigma \in G(m,F)} \mapsto \sum_{\sigma \in G(m,F)} b_\sigma \otimes_F w_\sigma$$

and an in W natural $F(\zeta_m)[G]$-isomorphism for $\sigma \in G(m, F)$

$$\Lambda : \operatorname{ind}_F^{F(\zeta_m)} W = F(\zeta) \otimes_F W \quad \to \quad \operatorname{res}_\sigma \operatorname{ind}_F^{F(\zeta_m)} W, \qquad x \otimes_F w \mapsto \sigma(x) \otimes_F w.$$

From the existence of the natural isomorphisms Φ,Ψ and Λ above we conclude for the homomorphisms

$$\begin{array}{rcl} \operatorname{ind}_F^{F(\zeta_m)} : K_q(FG) & \to & K_q(F(\zeta_m)[G]); \\ \operatorname{res}_{F(\zeta_m)}^F : K_q(F(\zeta_m)[G]) & \to & K_q(FG); \\ \operatorname{res}_\sigma : K_q(F(\zeta_m)[G]) & \to & K_q(F(\zeta_m)[G]), \end{array}$$

that $\operatorname{res}_{F(\zeta_m)}^F \circ \operatorname{ind}_F^{F(\zeta_m)} = |G(m, F)| \cdot \operatorname{id}$, $\operatorname{ind}_F^{F(\zeta_m)} \circ \operatorname{res}_{F(\zeta_m)}^F = \sum_{\sigma \in G(m,F)} \operatorname{res}_\sigma$ and $\operatorname{res}_\sigma \circ \operatorname{ind}_F^{F(\zeta_m)} = \operatorname{ind}_F^{F(\zeta_m)}$ holds for $\sigma \in G(m, F)$. The various maps $\operatorname{res}_\sigma$ induce a $G(m, F)$-action on $K_q(F(\zeta_m)[G])$. We conclude

Lemma 8.4 *Induction induces an isomorphism*

$$\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{ind}_F^{F(\zeta_m)} : \mathbb{Q} \otimes_{\mathbb{Z}} K_q(FG) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} K_q(F(\zeta)[G])^{G(m,F)}.$$

Let C be a finite cyclic group of order $|C|$. Then all irreducible $F(\zeta_{|C|})$-representations of C are 1-dimensional. The number of isomorphism classes of irreducible $F(\zeta_{|C|})$-representations is equal to $|C|$. Given a finite-dimensional $F(\zeta_{|C|})$-representation V of C, we obtain a functor from the category of finitely generated projective $F(\zeta_{|C|})$-modules to the category of finitely generated projective $F(\zeta_{|C|})[C]$-modules by tensoring with V over $F(\zeta_m)$ and thus a map $K_q(F(\zeta_{|C|})[C]) \to K_q(F(\zeta_{|C|})[C])$. This yields a homomorphism

$$\alpha : K_0(F(\zeta_{|C|})[C]) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|})) \quad \xrightarrow{\cong} \quad K_q(F(\zeta_{|C|})[C]), \tag{8.5}$$

which is an isomorphism by the following elementary facts. Given a $F(\zeta_{|C|})[C]$-module U and an irreducible $F(\zeta_{|C|})[C]$-module V, denote by U_V the V-isotypical summand. This is the $F(\zeta_{|C|})[C]$-submodule of U generated by all element $u \in U$ for which there exists a $F(\zeta_{|C|})[C]$-submodule $U' \subset U$ which contains u and is $F(\zeta_{|C|})[C]$-isomorphic to V. For any homomorphism $f : U \to W$ of finitely generated projective $F(\zeta_{|C|})[C]$-modules there are natural splittings $U = \oplus_V U_V$ and $W = \oplus_V W_V$, where V runs over the irreducible representations, f maps U_V to W_V and $\operatorname{aut}_{F(\zeta_{|C|})[C]}(V) = \{x \cdot \operatorname{id}_V \mid x \in F(\zeta_{|C|})\}$.

An element $\sigma \in G(|C|, F)$ induces automorphisms $\operatorname{res}_\sigma$ of $K_q(F(\zeta_{|C|}))$ and of $K_q(F(\zeta_{|C|})[C])$ by restriction with $\sigma : F(\zeta_{|C|}) \to F(\zeta_{|C|})$ and $\sigma : F(\zeta_{|C|})[C] \to F(\zeta_{|C|})[C], \quad \sum_{c\in C} x_c \cdot c \mapsto \sum_{c \in C} \sigma(x_c) \cdot c$. We get for $\sigma \in G(|C|, F)$

$$\operatorname{res}_\sigma \circ \alpha \quad = \quad \alpha \circ (\operatorname{res}_\sigma \otimes_{\mathbb{Z}} \operatorname{res}_\sigma).$$

Taking the character of a representation yields an isomorphism

$$\chi : F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_0(F(\zeta_{|C|})[C]) \quad \xrightarrow{\cong} \quad \operatorname{map}(C, F(\zeta_{|C|})), \qquad x \otimes [V] \mapsto x \cdot \chi_V. \tag{8.6}$$

The operation of $G(|C|, F)$ on $K_0(F(\zeta_{|C|})[C])$ extends to an operation on $F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_0(F(\zeta_{|C|})[C])$ by taking the tensor product $\operatorname{id} \otimes_{\mathbb{Z}} ?$. We define a $G(|C|, F)$-operation on $\operatorname{map}(C, F(\zeta_{|C|})$ by assigning to $\sigma \in G(|C|, F)$ and $\chi \in \operatorname{map}(C, F(\zeta_{|C|})$ the element $\sigma \cdot \chi$ which sends $c \in C$ to $\chi(c^{u(\sigma)})$. The map χ is compatible with these $G(|C|, F)$-actions. It suffices to check this for $1 \otimes_{\mathbb{Z}} [V]$ if V is an irreducible $F(\zeta_{|C|})[C]$-representation. Its character is a homomorphism $\chi_V : C \to F(\zeta_{|C|})$ whose values are multiples of $\zeta_{|C|}$ and $c \in C$ acts on V by multiplication with $\chi_V(c)$. Hence $c \in C$ acts on $\operatorname{res}_\sigma V$ by multiplication with $\sigma(\chi_V(c))$ on V. This implies $\chi_{\operatorname{res}_\sigma V}(c) = \sigma(\chi_V(c)) = \chi_V(c)^{u(\sigma)} = \chi_V(g^{u(\sigma)})$. We have the obvious isomorphism

$$\beta : \operatorname{map}\left(C, F(\zeta_{|C|})\right) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|})) \quad \xrightarrow{\cong} \quad \operatorname{map}\left(C, F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|}))\right). \tag{8.7}$$

Now the maps α, χ and β defined in (8.5), (8.6) and (8.7) can be combined to an isomorphism

$$\gamma = \quad (\operatorname{id} \otimes \alpha) \circ (\chi \otimes \operatorname{id})^{-1} \circ \beta^{-1} : \operatorname{map}\left(C, F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|})\right) \xrightarrow{\cong} F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|})[C]). \tag{8.8}$$

It is $G(|C|, F)$-equivariant, where we use on the source the action given by $(\sigma \cdot \chi)(c) := (\operatorname{id} \otimes \sigma)(\chi(c^{u(\sigma)}))$ and on the target by $\operatorname{res}_\sigma \otimes \operatorname{id}$.

Next we treat the various $\mathbb{Q} \otimes_{\mathbb{Q}} R_{\mathbb{Q}}(C)$-module structures. The source of α and the source of χ inherit a module structure over $\mathbb{Q} \otimes_{\mathbb{Q}} R_{\mathbb{Q}}(C)$ by the obvious ring homomorphism $\operatorname{ind}_{\mathbb{Q}}^{F(\zeta_{|C|})} : R_{\mathbb{Q}}(C) = K_0(\mathbb{Q}[C]) \to K_0(F(\zeta_{|C|})[C])$. We equip the target of α with the $\mathbb{Q} \otimes_{\mathbb{Q}} R_{\mathbb{Q}}(C)$-module structure for which α becomes a $\mathbb{Q} \otimes_{\mathbb{Q}} R_{\mathbb{Q}}(C)$-homomorphism. We have introduced the isomorphism of $\mathbb{Q}$-algebras $\chi^H : \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(C) \xrightarrow{\cong} \operatorname{class}_{\mathbb{Q}}(C)$ in (7.2). The target of the isomorphism χ is a module over $\operatorname{class}_{\mathbb{Q}}(C)$ by the obvious inclusion of rings $\operatorname{class}_{\mathbb{Q}}(C) \to \operatorname{map}(C, F(\zeta_{|C|})$. Then χ is a $\mathbb{Q} \otimes_{\mathbb{Q}} R_{\mathbb{Q}}(C)$-homomorphism. Equip the source of the isomorphism β with the $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(C)$-module structure given by the one on the target of χ and the trivial one on $K_q(F(\zeta_{|C|}))$. Equip the target of β with the $\mathbb{Q} \otimes_{\mathbb{Q}} R_{\mathbb{Q}}(C)$-structure for which β becomes a $\mathbb{Q} \otimes_{\mathbb{Q}} R_{\mathbb{Q}}(C)$-homomorphism. Then the isomorphism γ is a $\mathbb{Q} \otimes_{\mathbb{Q}} R_{\mathbb{Q}}(C)$-homomorphism. Therefore we obtain a commutative diagram of $F(\zeta_{|C|})$-modules where all maps are $G(|C|, F)$-equivariant

$$\begin{array}{ccc}
\operatorname{map}\big(C, F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|}))\big) & \xrightarrow{\gamma} & K_q(F(\zeta_{|C|})[C]) \otimes_{\mathbb{Z}} F(\zeta_{|C|}) \\
\theta_C^C \big\downarrow & & \big\downarrow \theta_C^C \\
\operatorname{map}\big(C, F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|}))\big) & \xrightarrow[\gamma]{} & K_q(F(\zeta_{|C|})[C]) \otimes_{\mathbb{Z}} F(\zeta_{|C|})
\end{array}$$

By taking the fixed point sets, we obtain a commutative diagram of $F(\zeta_{|C|})$-modules

$$\begin{array}{ccc}
\operatorname{map}\big(C, F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|}))\big)^{G(|C|,F)} & \xrightarrow{\gamma} & K_q(F(\zeta_{|C|})[C])^{G(|C|,F)} \otimes_{\mathbb{Z}} F(\zeta_{|C|}) \\
\theta_C^C \big\downarrow & & \big\downarrow \theta_C^C \\
\operatorname{map}\big(C, F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|}))\big)^{G(|C|,F)} & \xrightarrow[\gamma]{} & K_q(F(\zeta_{|C|})[C])^{G(|C|,F)} \otimes_{\mathbb{Z}} F(\zeta_{|C|})
\end{array}$$

Thus we obtain an isomorphism from the image of the left vertical arrow in the diagram above to the the image of the right vertical arrow. Recall that θ_C^C is the character which sends a generator of C to 1 and all other elements to 0. Hence the image of the left vertical arrow is canonically isomorphic to $\operatorname{map}(\operatorname{Gen}(C), F(\zeta_{|C|}) \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|}))^{G(|C|,F)}$. The image of the right vertical arrow is by Lemma 8.4 canonically isomorphic to the image of $\theta_C^C : K_q(F[C]) \otimes_{\mathbb{Z}} F(\zeta_{|C|}) \to K_q(F[C]) \otimes_{\mathbb{Z}} F(\zeta_{|C|})$. This finishes the proof of Lemma 8.3. ■

We conclude from Theorem 0.4 and Lemma 8.3

Theorem 8.9 *Let F be a field of characteristic zero. Let $F \subset \overline{F}$ be a field extension such that for any finite cyclic subgroup $C \subset G$ the the primitive $|C|$-th root belongs to $\overline{F}$. Let G be a group. Let J be the set of conjugacy classes (C) of finite cyclic subgroups of G. Then the assembly map (1.6) in the Farrell-Jones Conjecture with respect to $\mathcal{F}$ for the algebraic K-groups $K_n(FG)$ can be identified after applying $\overline{F}\otimes_{\mathbb{Z}}?$ with*

$$\oplus_{p+q=n} \oplus_{(C)\in J} H_p(C_GC; \overline{F}) \otimes_{\overline{F}[W_GC]} \operatorname{map}\big(\operatorname{Gen}(C), \overline{F} \otimes_{\mathbb{Z}} K_q(F(\zeta_{|C|}))\big)^{G(|C|,F)} \to \overline{F} \otimes_{\mathbb{Z}} K_n(FG).$$

If the Farrell-Jones Conjecture with respect to $\mathcal{F}$ is true, then this maps is an isomorphism.

Example 8.10 If $F = \mathbb{C}$, then $F(\zeta_{|C|}) = \mathbb{C}$ and $G(|C|, \mathbb{C}) = 1$. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. The action of W_GC on $\operatorname{Gen}(C)$ is free. Then the assembly maps (1.6) in the Farrell-Jones Conjecture with respect to $\mathcal{F}$ and in the Baum-Connes conjecture can be identified after applying $\mathbb{C}\otimes_{\mathbb{Z}}?$ with

$$\begin{array}{rcl}
\oplus_{p+q=n} \oplus_{(g)\in T} H_p(C_G\langle g\rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \to & \mathbb{C} \otimes_{\mathbb{Z}} K_n(\mathbb{C}G); \\
\oplus_{p+q=n} \oplus_{(g)\in T} H_p(C_G\langle g\rangle; \mathbb{C}) \otimes_{\mathbb{Z}} L_q(\mathbb{C}) & \to & \mathbb{C} \otimes_{\mathbb{Z}} L_n(\mathbb{C}G); \\
\oplus_{p+q=n} \oplus_{(g)\in T} H_p(C_G\langle g\rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \to & \mathbb{C} \otimes_{\mathbb{Z}} K_n^{\mathrm{top}}(C_r^*(G, \mathbb{C})),
\end{array}$$

where we use in the definition of $L_q(\mathbb{C})$ and $L_n(\mathbb{C}G)$ the involutions coming from complex conjugation. We get the first one from Theorem 8.9. The proof for the third is completely analogous to the one of the first. The proof of the second can be reduced to the one of the third by Theorem 8.1. In particular this proves Theorem 0.1. We mention that the restriction of the upper horizontal arrow in Theorem 0.1 to the part for $q = 0$ has been shown to be split injective for all groups G using the Dennis trace map but not the Farell-Jones Conjecture in [19].

If we use the trivial involution on $\mathbb{C}$ in the definition of $L_n(\mathbb{C}G)$, then the Farrell-Jones Conjecture with respect to $\mathcal{F}$ implies $L_n(\mathbb{C}G)[1/2] = 0$ since $L_n(\mathbb{C}H)[1/2]) = 0$ is known for all finite groups H with respect to the trivial involution on $\mathbb{C}$ [21, Proposition 22.21 on page 239]. Notice that the Farrell-Jones Conjecture with respect to $\mathcal{F}$ and the Baum Connes Conjecture together with Theorem 8.1 imply that the change of ring maps $L_n(\mathbb{C}G) \to L_n(C_r^*(G,\mathbb{C}))$ becomes a bijection after inverting 2.

Example 8.11 Next we consider the case $F = \mathbb{R}$. Put $\overline{F} = \mathbb{C}$. We call g_1 and g_2 in G $\mathbb{R}$-conjugated if $(g_1) = (g_2)$ or $(g_1) = (g_2^{-1})$. Denote by $(g)_\mathbb{R}$ the $\mathbb{R}$-conjugacy class of $g \in G$. Denote by $T_\mathbb{R}$ the set of $\mathbb{R}$-conjugacy classes of elements of finite order in G. This splits as the disjoint union $T'_\mathbb{R} \coprod T''_\mathbb{R}$, where $T'_\mathbb{R}$ resp. $T''_\mathbb{R}$ consists of classes $(g)_\mathbb{R}$ with $(g) \neq (g^{-1})$ resp. $(g) = (g^{-1})$. For a class $(g)_\mathbb{R} \in T''_\mathbb{R}$ we can find an element $g' \in G$ such that the homomorphism $c(g') : G \to G$ given by conjugation with g' maps g to g^{-1}. Then $c(g')$ induces also an automorphism $C_G\langle g\rangle \to C_G\langle g\rangle$. The induced automorphism of $H_p(C_G\langle g\rangle;\mathbb{C})$ does not depend on the choice of g' and is of order two. Thus we obtain a $\mathbb{Z}/2$-action on $H_p(C_G\langle g\rangle;\mathbb{C})$. The Galois group of the field extension $\mathbb{R} \subset \mathbb{C}$ is $\mathbb{Z}/2$ with complex conjugation as generator. Complex conjugation induces a $\mathbb{Z}[\mathbb{Z}/2]$-structure on $K_q(\mathbb{C})$ and $K_q^{\mathrm{top}}(\mathbb{C})$. We obtain analogously to Example 8.10 an identification of the assembly maps (1.6) in the Farrell-Jones Conjecture with respect to $\mathcal{F}$ and in the Baum-Connes conjecture after applying $\mathbb{C}\otimes_\mathbb{Z}?$ with

$$\begin{aligned}
&\oplus_{p+q=n}\Big(\oplus_{(g)_\mathbb{R}\in T'_\mathbb{R}} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_\mathbb{Z} K_q(\mathbb{C}) \oplus \oplus_{(g)_\mathbb{R}\in T''_\mathbb{R}} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}[\mathbb{Z}/2]} K_q(\mathbb{C})\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\to \quad \mathbb{C}\otimes_\mathbb{Z} K_n(\mathbb{R}G),\\
&\oplus_{p+q=n}\Big(\oplus_{(g)_\mathbb{R}\in T'_\mathbb{R}} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_\mathbb{Z} L_q(\mathbb{C}) \oplus \oplus_{(g)_\mathbb{R}\in T''_\mathbb{R}} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}[\mathbb{Z}/2]} L_q(\mathbb{C})\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\to \quad \mathbb{C}\otimes_\mathbb{Z} L_n(\mathbb{R}G),\\
&\oplus_{p+q=n}\Big(\oplus_{(g)_\mathbb{R}\in T'_\mathbb{R}} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_\mathbb{Z} K_q^{\mathrm{top}}(\mathbb{C}) \oplus \oplus_{(g)_\mathbb{R}\in T''_\mathbb{R}} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}[\mathbb{Z}/2]} K_q^{\mathrm{top}}(\mathbb{C})\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\to \quad \mathbb{C}\otimes_\mathbb{Z} K_n^{\mathrm{top}}(C_r^*(G,\mathbb{R})),
\end{aligned}$$

where we use in the definition of $L_q(\mathbb{C})$ the involution coming from complex conjugation. Notice that the Farrell-Jones Conjecture with respect to $\mathcal{F}$ and the Baum Connes Conjecture together with Theorem 8.1 imply that the change of ring maps $L_n(\mathbb{Q}G) \to L_n(\mathbb{R}G)$ and $L_n(\mathbb{R}G) \to L_n(C_r^*(G,\mathbb{R}))$ become bijections after inverting 2 since $L_n(\mathbb{Q}H) \to L_n(\mathbb{R}H)$ is known to be bijective after inverting 2 for finite groups H [21, Proposition 22.33 on page 252].

Example 8.12 If $F = \mathbb{Q}$, then $G(|C|,\mathbb{Q}) = \mathbb{Z}/|C|^* = \mathrm{aut}(C)$. Since $G(|C|,\mathbb{Q})$ acts freely and transitively on $\mathrm{Gen}(C)$, we obtain after the choice of a generator $c \in C$ an isomorphism

$$\mathrm{map}\left(\mathrm{Gen}(C), \mathbb{Q}(\zeta_{|C|}) \otimes_\mathbb{Z} K_q(\mathbb{Q}(\zeta_{|C|}))\right)^{G(|C|,\mathbb{Q})} \cong \mathbb{Q}(\zeta_{|C|}) \otimes_\mathbb{Z} K_q(\mathbb{Q}(\zeta_{|C|})).$$

It is natural with respect to automorphisms of C, if $f \in \mathrm{aut}(C)$ acts on $\mathbb{Q}(\zeta_{|C|}) \otimes_\mathbb{Z} K_q(\mathbb{Q}(\zeta_{|C|}))$ by $\mathrm{id}\otimes \mathrm{res}_\sigma$ for the element σ in the Galois group $G(|C|,\mathbb{Q})$ for which $\sigma(\zeta) = \zeta^u$ and $f(c) = c^u$ holds. Let J be the set of conjugacy classes (C) of finite cyclic subgroups of G. We conclude from Theorem 8.9 that the assembly map (1.6) in the Farrell-Jones Conjecture with respect to $\mathcal{F}$ can be identified with

$$\oplus_{p+q=n} \oplus_{(C)\in J} H_p(C_GC;\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}[W_GC]} \overline{\mathbb{Q}} \otimes_\mathbb{Z} K_q(\mathbb{Q}(\zeta_{|C|})) \quad\to\quad \overline{\mathbb{Q}} \otimes_\mathbb{Z} K_n(\mathbb{Q}G).$$

Example 8.13 Let F be a field of characteristic zero and let G be a group. Let g_1 and g_2 be two elements of G of finite order. We call them F-conjugated if for some (and hence all) positive integers m with $g_1^m = g_2^m = 1$ there exists an element σ in the Galois group $G(m,F)$ with the property $(g_1^{u(\sigma)}) = (g_2)$. Denote by $\mathrm{con}_F(G)$ the set of F-conjugacy classes $(g)_F$ of elements $g \in G$ of finite order. Let $\mathrm{class}_F(G)$ be the F-vector space with the set $\mathrm{con}_F(G)$ as basis, or, equivalently, the F-vector space of functions $\mathrm{con}_F(G) \to F$ with finite support. Recall that for a finite group H taking characters yields an isomorphism [23, Corollary 1 on page 96]

$$\chi : F\otimes_\mathbb{Z} R_F(H) = F\otimes_\mathbb{Z} K_0(FH) \quad\xrightarrow{\cong}\quad \mathrm{class}_F(H). \tag{8.14}$$

By Theorem 0.4 and (8.14) the assembly map (1.6) of the Farrell-Jones Conjecture with respect to $\mathcal{F}$ for $K_0(FG)$ can be identified with a map

$$\mathrm{class}_F(G) \to F \otimes_{\mathbb{Z}} K_0(FG).$$

If the Farrell-Jones Conjecture with respect to $\mathcal{F}$ for $K_0(FG)$ is true, this map is an isomorphism. This generalizes (8.14) for finite groups to infinite groups. This example is related to the Hattori-Stalling rank and the Bass Conjecture [1].

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