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# CHERN CLASSES OF SPLAYED INTERSECTIONS

PAOLO ALUFFI AND ELEONORE FABER

ABSTRACT. We generalize the Chern class relation for the transversal intersection of two nonsingular varieties to a relation for possibly singular varieties, under a *splayedness* assumption. The relation is shown to hold for both the Chern–Schwartz–MacPherson class and the Chern–Fulton class. The main tool is a formula for Segre classes of splayed subschemes. We also discuss the Chern class relation under the assumption that one of the varieties is a general very ample divisor.

## 1. INTRODUCTION

1.1. Let  $X, Y$  be nonsingular subvarieties of a nonsingular complex variety  $V$ . If  $X$  and  $Y$  intersect properly and transversally, then the intersection  $X \cap Y$  is nonsingular, and an elementary Chern class computation proves that

$$(1) \quad c(X) \cdot c(Y) = c(TV) \cap c(X \cap Y) \quad ,$$

where  $c(X)$ , etc. denote the push-forward to  $V$  of the total (homology) Chern class of the tangent bundle of  $X$ , etc., and  $\cdot$  is the intersection product in  $V$ . It is natural to ask whether (1) holds if  $X, Y, X \cap Y$  are allowed to be singular. In [AF13], §3, we proposed the following generalization of (1):

**Scholium.** *Let  $X, Y$  be (possibly singular) subvarieties of a nonsingular variety  $V$ . Assume that  $X$  and  $Y$  are splayed. Then*

$$(2) \quad c_{\text{SM}}(X) \cdot c_{\text{SM}}(Y) = c(TV) \cap c_{\text{SM}}(X \cap Y) \quad .$$

In (2),  $c_{\text{SM}}(-)$  denotes the *Chern-Schwartz-MacPherson* class; this is a natural generalization of the total Chern class to singular varieties, and we silently push this class forward to the ambient variety  $V$ . The  $c_{\text{SM}}$  class is defined for more general schemes— $X$  and  $Y$  could be reducible, and should not be required to be pure dimensional. The purpose of this note is to investigate (2) at this level of generality. For example, we will prove that the Scholium holds for arbitrary splayed hypersurfaces, and more generally for subschemes satisfying a hypothesis of ‘strong’ splayedness. We also prove (2) for splayed subschemes for a different notion of Chern class defined for arbitrary subschemes of a nonsingular variety. Finally, we will discuss a ‘Bertini’ statement, according to which (2) holds if  $X$  is a sufficiently general very ample divisor.

The notion of *splayedness* was introduced and studied in the hypersurface case by the second author in [Fab], and it is explored further in [AF13]:  $X$  and  $Y$  are splayed if at each point  $p$  of the intersection there exist analytic coordinates  $(x_1, \dots, x_r, y_1, \dots, y_s)$  such that  $X$  may be defined by an ideal generated by functions in the coordinates  $x_i$  and  $Y$  by an ideal generated by functions in the coordinates  $y_j$ .

We also say that two *sets*  $\{X_1, X_2, \dots\}$  and  $\{Y_1, Y_2, \dots\}$  are splayed if there are local analytic splittings so that all the  $X_i$  are defined in the first set of coordinates, and all the  $Y_j$  are defined in the second set of coordinates. These notions generalize to possibly singular varieties and subschemes the notion of proper, transversal intersection of nonsingular varieties.

The reader who is not too familiar with characteristic classes may view (2) as a very general form of identities involving the topological Euler characteristics of  $X$ ,  $Y$ ,  $X \cap Y$ . For example, let  $X$  and  $Y$  be splayed surfaces in  $\mathbb{P}^3$ , of degrees  $d$ ,  $e$  resp.; assume that the Euler characteristic of a general hyperplane section of  $X$ , resp.  $Y$  is  $a$ , resp.  $b$ . Then it may be checked that the Euler characteristic of the curve  $X \cap Y$  is  $ea + db - 2de$ . Similarly explicit formulas relate the Euler characteristics of general linear sections of  $X$ ,  $Y$ ,  $X \cap Y$  if these are subsets of projective space and  $X$ ,  $Y$  are splayed (cf. [Alu13]). The Scholium reveals the underlying structure of all such identities, and generalizes them to splayed subsets in arbitrary nonsingular algebraic varieties.

Note that some transversality hypothesis is certainly needed for (2) to hold, as the following example shows.

*Example 1.1.* Let  $X$  be a nonsingular quadric in  $V = \mathbb{P}^3$ , and let  $Y$  be a plane tangent to  $X$ . Then  $c_{\text{SM}}(X) = 2[\mathbb{P}^2] + 4[\mathbb{P}^1] + 4[\mathbb{P}^0]$  and  $c_{\text{SM}}(Y) = [\mathbb{P}^2] + 3[\mathbb{P}^1] + 3[\mathbb{P}^0]$  (since  $X$  and  $Y$  are nonsingular, these are simply the push-forward to  $\mathbb{P}^3$  of the total Chern classes of their tangent bundles). Thus, the left-hand side of (2) is

$$(2[\mathbb{P}^2] + 4[\mathbb{P}^1] + 4[\mathbb{P}^0]) \cdot ([\mathbb{P}^2] + 3[\mathbb{P}^1] + 3[\mathbb{P}^0]) = 2[\mathbb{P}^1] + 10[\mathbb{P}^0] \quad .$$

On the other hand, denoting by  $H$  the hyperplane class,  $c(T\mathbb{P}^3) = 1 + 4H + 6H^2 + 4H^3$ ; and  $X \cap Y$  consists of two lines meeting at a point, a curve of degree 2 and topological Euler characteristic 3, and hence  $c_{\text{SM}}(X \cap Y) = 2[\mathbb{P}^1] + 3[\mathbb{P}^0]$ . Thus the right-hand side of (2) equals

$$c(TV) \cap c_{\text{SM}}(X \cap Y) = 2[\mathbb{P}^1] + 11[\mathbb{P}^0] \quad ,$$

verifying that (2) does *not* hold in this case. ┘

1.2. Several particular cases of the Scholium are proven in [AF13]. In this paper we prove (2) under a hypothesis generalizing all those particular cases, but possibly more restrictive than splayedness. We say that  $X$  and  $Y$  are ‘strongly splayed’ if  $X = D'_1 \cap \dots \cap D'_r$ ,  $Y = D''_1 \cap \dots \cap D''_s$  where  $\{D'_1, \dots, D'_r\}$ ,  $\{D''_1, \dots, D''_s\}$  are splayed sets of hypersurfaces. For example, two hypersurfaces are strongly splayed if and only if they are splayed. We do not know if splayed subschemes of higher codimension are necessarily strongly splayed, and this seems an interesting question.

**Theorem I.** *Let  $X, Y$  be strongly splayed subschemes of a nonsingular variety  $V$ . Then*

$$(3) \quad c_{\text{SM}}(X) \cdot c_{\text{SM}}(Y) = c(TV) \cap c_{\text{SM}}(X \cap Y) \quad .$$

*Example 1.2.* Let  $X$  be the union of a  $\mathbb{P}^4$  and a transversal  $\mathbb{P}^3$  in  $V = \mathbb{P}^5$ ; we choose coordinates  $(x_0 : \dots : x_5)$  so that  $X$  has ideal  $(x_0(x_1, x_2))$ . Let  $Y$  be the quadric cone

with ideal  $(x_3^2 + x_4^2 + x_5^2)$ . Both  $X$  and  $Y$  are singular, and  $X$  is reducible and not pure-dimensional;  $X$  and  $Y$  are strongly splayed.

The Macaulay2 code from [Alu03] may be used to compute the  $c_{\text{SM}}$  classes of  $X$  and  $Y$ :

$$\begin{aligned} c_{\text{SM}}(X) &= [\mathbb{P}^4] + 6[\mathbb{P}^3] + 13[\mathbb{P}^2] + 13[\mathbb{P}^1] + 6[\mathbb{P}^0] \\ c_{\text{SM}}(Y) &= 2[\mathbb{P}^4] + 8[\mathbb{P}^3] + 13[\mathbb{P}^2] + 11[\mathbb{P}^1] + 5[\mathbb{P}^0] \end{aligned}$$

According to Theorem I,

$$c(T\mathbb{P}^5) \cap c_{\text{SM}}(X \cap Y) = c_{\text{SM}}(X) \cdot c_{\text{SM}}(Y) = 2[\mathbb{P}^3] + 20[\mathbb{P}^2] + 87[\mathbb{P}^1] + 219[\mathbb{P}^0]$$

from which

$$c_{\text{SM}}(X \cap Y) = 2[\mathbb{P}^3] + 8[\mathbb{P}^2] + 9[\mathbb{P}^1] + 5[\mathbb{P}^0] \quad .$$

This can be verified by again using [Alu03]. ┘

In fact, the proper level of generality for the result is that of *constructible functions*: a  $c_{\text{SM}}$  class in  $A_*V$  is defined for every constructible function on  $V$ ; if  $X$  is a subvariety of  $V$ ,  $c_{\text{SM}}(X) = c_{\text{SM}}(\mathbb{1}_X)$ , where  $\mathbb{1}_X$  is the indicator function of  $X$ . Intersection of varieties corresponds naturally to the *product* of the corresponding constructible functions.

**Theorem II.** *Let  $\varphi, \psi$  be constructible functions on a nonsingular variety  $V$ , and assume that  $\varphi$  and  $\psi$  are strongly splayed. Then*

$$(4) \quad c_{\text{SM}}(\varphi) \cdot c_{\text{SM}}(\psi) = c(TV) \cap c_{\text{SM}}(\varphi \cdot \psi) \quad .$$

The precise definition of ‘strongly splayed’ in the context of constructible functions is given in Definition 2.19; it generalizes naturally the notion for subvarieties. Note that (4) amounts to the statement that the assignment

$$\varphi \mapsto c(TV)^{-1} \cap c_{\text{SM}}(\varphi)$$

of a class in  $A_*V$  from a constructible function  $\varphi$  ‘preserves multiplication’ for strongly splayed constructible functions. Again, this is clearly false without some kind of transversality condition on the constructible functions. It would be interesting to determine weaker conditions than ‘strong splayedness’ guaranteeing that this multiplicativity property holds.

Our proofs of Theorems I and II rely on intersection-theoretic considerations based on a formula for the Chern-Schwartz-MacPherson class of a hypersurface from [Alu99], and on a general statement about Segre classes proven in this note (Theorem 2.9, which should be of independent interest). A proof of the Scholium for *splayed* (rather than *strongly splayed*) subvarieties should result as a particular case of Jörg Schürmann’s Verdier-Riemann-Roch theorem for Chern-Schwartz-MacPherson classes ([Sch]).

1.3. It is natural to ask whether a version of the Scholium holds for other characteristic classes for singular varieties. Substantial work has been carried out comparing the Chern-Schwartz-MacPherson class to the *Chern-Fulton* class, another class agreeing with the Chern class of the tangent bundle for nonsingular varieties. See Example 4.2.6 (a) in [Ful84] for the definition (reproduced here in §3). We denote

this class by  $c_F$ . The difference  $c_{SM}(X) - c_F(X)$  is called the *Milnor class* of  $X$ , since it generalizes Milnor numbers of isolated hypersurface singularities to arbitrary singularities.

**Theorem III.** *Let  $X, Y$  be splayed subschemes of a nonsingular variety  $V$ . Then*

$$(5) \quad c_F(X) \cdot c_F(Y) = c(TV) \cap c_F(X \cap Y) \quad .$$

The proof of this result also follows from Segre class considerations, in fact of a simpler nature than those leading to Theorem 2.9.

It is also natural to ask whether (2) and (5) hold when  $X$  and  $Y$  are ‘in general position’. The following is a prototype situation where this can be established.

**Theorem IV.** *Let  $V$  be a nonsingular variety, and let  $X \subseteq V$  be a general very ample divisor on  $V$ . Then for all subschemes  $Y \subseteq V$ ,*

$$\begin{aligned} c(X) \cdot c_{SM}(Y) &= c(TV) \cap c_{SM}(X \cap Y) \\ c(X) \cdot c_F(Y) &= c(TV) \cap c_F(X \cap Y) \end{aligned}$$

Theorem IV hints that a condition analogous to splayedness may satisfy results along the lines of the Bertini or Kleiman-Bertini theorems. It would be interesting to establish a precise result of this type. In general, however, a ‘splayed’ Bertini theorem cannot hold, as the following example illustrates:

*Example 1.3.* Let  $X$  be the so-called *4-lines divisor* in  $\mathbb{C}^3$ , given by the polynomial  $xy(x+y)(x+yz)$ . It is well-known that  $X$  is not analytically trivial along the  $z$ -axis, i.e., there is no analytic isomorphism between two hyperplane sections  $X_{t_1}$  and  $X_{t_2}$ , where  $X_t := X \cap (\mathbb{C}^2 \times \{t\})$ . If  $X$  were splayed with a general hyperplane at a general point of the  $z$ -axis, then it would be possible to write the equation of the divisor only using two coordinates at that point. This would imply that nearby sections are analytically isomorphic.  $\lrcorner$

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## 2. PROOFS OF THEOREMS I AND II

**2.1. Splayed blow-ups.** Throughout the paper,  $V$  will denote a smooth complex algebraic variety; several results extend without change to the context of nonsingular algebraic varieties over an algebraically closed field of characteristic 0. (See e.g., [Ken90] for a treatment of Chern-Schwartz-MacPherson classes in this generality.) We call two subschemes  $Z_1, Z_2 \subseteq V$  *splayed* if at every point  $p$  in the intersection of  $Z_1$  and  $Z_2$  there is a local analytic isomorphism  $\varphi : V \rightarrow V' \times V''$ , and subschemes  $Z'_1 \subseteq V', Z''_2 \subseteq V''$  such that  $Z_1 = \varphi^{-1}(Z'_1 \times V'')$  and  $Z_2 = \varphi^{-1}(V' \times Z''_2)$ . Equivalently,

there are analytic coordinates for  $V$  at  $p$  such that  $Z_1$  and  $Z_2$  are defined in different sets of variables.

More generally, we will say that two sets of subschemes are splayed in  $V$  if at each point there is a local analytic isomorphism  $\varphi$  as above, such that the schemes in the first set are inverse images from the first factor of the product  $V' \times V''$ , and the schemes in the second sets are inverse images from the second factor.

Denote by  $\pi_i : \tilde{V}_i \rightarrow V$  the blowup of  $V$  along  $Z_i$ . Denote further by  $\tilde{V}_{12}$  the blowup of  $\tilde{V}_1$  along the inverse image  $\pi_1^{-1}Z_2$  of  $Z_2$  and by  $\tilde{V}_{21}$  the blowup of  $\tilde{V}_2$  along the inverse image  $\pi_2^{-1}Z_1$  of  $Z_1$ . We begin by recalling the following fact, for which the splayedness assumption on  $Z_1, Z_2$  or the smoothness of  $V$  are not needed.

**Proposition 2.1.** *The blow-ups  $\tilde{V}_{12}$  and  $\tilde{V}_{21}$  are isomorphic, and they are isomorphic to the blow-up of  $V$  along  $Z_1 \cup Z_2$ , where the defining ideal sheaf of  $Z_1 \cup Z_2$  in  $V$  is the product of the ideal sheaves defining  $Z_1$  and  $Z_2$ .*

*Proof.* Both statements follow from the universal property of blow-ups; see for example [EH00], Lemma IV-41 and [Li09], Lemma 3.2.  $\square$

We want to compare  $\tilde{V}_{12}$  to the fiber product of  $\tilde{V}_1$  and  $\tilde{V}_2$ .

By the universal property of fiber products, there is a unique morphism  $\alpha : \tilde{V}_{12} \rightarrow \tilde{V}_1 \times_V \tilde{V}_2$ :

$$\begin{array}{ccccc}
 \tilde{V}_{12} & & & & \\
 \searrow^{\exists! \alpha} & & & & \\
 & \tilde{V}_1 \times_V \tilde{V}_2 & \longrightarrow & \tilde{V}_2 & \\
 & \downarrow & & \downarrow \pi_2 & \\
 & \tilde{V}_1 & \xrightarrow{\pi_1} & V & \\
 & & & & \\
 & & & & 
 \end{array}$$

**Proposition 2.2.** *Let  $V$  be an irreducible variety. Then  $\alpha$  induces an isomorphism from  $\tilde{V}_{12}$  to the unique irreducible component  $\tilde{V}_1 \hat{\times}_V \tilde{V}_2$  of  $\tilde{V}_1 \times_V \tilde{V}_2$  mapping dominantly to  $V$ .*

*Remark 2.3.* M. Kwieciński ([Kwi94]) calls this irreducible component the ‘modified fiber product’, and observes that it is a product in the category of proper birational morphisms from varieties to  $V$ .  $\lrcorner$

*Proof.* Let  $\hat{\alpha}$  be the induced morphism  $\tilde{V}_{12} \rightarrow \tilde{V}_1 \hat{\times}_V \tilde{V}_2$ . Since  $\pi_1^{-1}(Z_1)$  and  $\pi_2^{-1}(Z_2)$  are Cartier divisors, it follows that their inverse images in  $\tilde{V}_1 \hat{\times}_V \tilde{V}_2$  are Cartier divisors, and hence so is the inverse image of  $Z_1 \cup Z_2$ . By the universal property of blow-ups, we obtain a morphism  $\tilde{V}_1 \hat{\times}_V \tilde{V}_2 \rightarrow \tilde{V}_{12}$ , which is immediately checked to be the inverse of  $\hat{\alpha}$ .  $\square$

**Corollary 2.4.** *Assume that  $V$  is nonsingular and  $Z_1$  and  $Z_2$  are splayed in  $V$ . Then  $\alpha$  is an isomorphism  $\tilde{V}_{12} \rightarrow \tilde{V}_1 \times_V \tilde{V}_2$ .*

*Proof.* We claim that  $\tilde{V}_1 \times_V \tilde{V}_2$  is irreducible. Indeed, it suffices to verify this fact locally analytically over every  $p$  in  $V$ , so by the splayedness condition we may assume

$Z_1 = Z'_1 \times V''$ ,  $Z_2 = V' \times Z'_2$  in  $V = V' \times V''$ . In this situation  $\tilde{V}_1 \times_V \tilde{V}_2 \cong Bl_{Z'_1} V' \times Bl_{Z'_2} V''$ .

Since  $Bl_{Z'_1} V'$  and  $Bl_{Z'_2} V''$  are irreducible, this product is irreducible.

With notation as in Proposition 2.2, this shows that  $\tilde{V}_1 \times_V \tilde{V}_2 = \tilde{V}_1 \hat{\times}_V \tilde{V}_2$ , and the result follows then from the proposition.  $\square$

*Remark 2.5.* It is worth pointing out that  $\alpha$  is not an isomorphism in general. For example, let  $V = \mathbb{A}^2$  and let  $Z_1 = Z_2$  be the origin  $p = (0, 0)$ . Then  $\tilde{V}_1 \times_V \tilde{V}_2$  consists of two components: an isomorphic copy of the blow-up of  $V$  at  $p$ , and a component isomorphic to  $E \times E$ , where  $E$  is the exceptional divisor in  $\tilde{V}_1 = \tilde{V}_2$ . (This is easily verified by a computation with charts.)

Tracing the proof of Proposition 2.2, the problem is that while the inverse image of e.g.,  $Z_1$  in  $\tilde{V}_1 \times_V \tilde{V}_2$  is locally principal, it contains a whole component of the fiber product (i.e., local generators of its ideal are zero-divisors), so this subscheme is not a Cartier divisor of the fiber product. It is however a Cartier divisor in the *modified* fiber product.

On the other hand,  $\alpha$  may be an isomorphism even if  $Z_1$  and  $Z_2$  are not splayed. For instance, if  $Z_1$  and  $Z_2$  are any Cartier divisors, then all blow-ups are isomorphisms, and so is the fiber product. For a more substantive example, take two coordinate axes  $Z_1, Z_2$  in  $V = \mathbb{A}^3$ . It can easily be seen via computation in charts that  $\tilde{V}_1 \times_V \tilde{V}_2$  is irreducible and isomorphic to the blow-up of  $V$  along  $Z_1 \cup Z_2$ . Thus in this case  $\alpha$  is an isomorphism, although  $Z_1$  and  $Z_2$  are not splayed according to our definition.  $\square$

**Corollary 2.6.** *Let  $Z_1, Z_2$  be splayed in  $V$ , and consider the blow-ups along  $Z_1, Z_2$ , and  $Z_1 \cup Z_2$  as above.*

$$\begin{array}{ccc} \tilde{V}_{12} & \xrightarrow{\tilde{\pi}_1} & \tilde{V}_2 \\ \tilde{\pi}_2 \downarrow & & \downarrow \pi_2 \\ \tilde{V}_1 & \xrightarrow{\pi_1} & V \end{array}$$

*Then the homomorphisms  $\tilde{\pi}_{2*} \tilde{\pi}_1^*$  and  $\pi_1^* \pi_{2*}$  from  $A_* \tilde{V}_2$  to  $A_* \tilde{V}_1$  coincide.*

*Proof.* The maps are all proper l.c.i. morphisms, and the diagram is a fiber square by Corollary 2.4. By [Ful84], Example 17.4.1 (a),

$$\pi_1^* \pi_{2*}(\alpha) = \tilde{\pi}_{2*}(c_e(\mathcal{E}) \cap \tilde{\pi}_1^*(\alpha))$$

for all  $\alpha \in A_*(\tilde{V}_2)$ , where  $\mathcal{E}$  is an excess bundle and  $e$  is the difference in the codimensions of  $\pi_1$  and  $\tilde{\pi}_1$ . Here both  $\pi_1$  and  $\tilde{\pi}_1$  are birational, so  $e = 0$ , and  $c_e(\mathcal{E}) = 1$ , hence the equality follows.  $\square$

**2.2. A Segre class formula.** The  $c_{SM}$  class of a hypersurface  $D$  in a nonsingular variety may be expressed in terms of the *Segre class* of the singularity subscheme  $JD$  in  $V$ . The precise relationship (from [Alu99]) will be recalled below. The hypersurface case of Theorem I will then follow from a statement on Segre classes of singularity subschemes of splayed hypersurfaces. In this subsection we prove a more general form of this statement (Theorem 2.9).

*Reminder.* Segre classes are one of the ingredients of Fulton-MacPherson intersection theory, and the reader is addressed to Chapter 4 of [Ful84] for a thorough treatment of these classes. The following summary should suffice for the purpose of this paper. The Segre class  $s(S, X)$  of a proper subscheme  $S$  of a scheme  $X$  is the class in the Chow group of  $S$  determined by the following properties:

- Birational invariance: If  $f : X' \rightarrow X$  is a proper birational morphism, then  $s(S, X) = f_*s(f^{-1}(S), X')$  (Proposition 4.2(a) in [Ful84]);
- If  $S$  is a Cartier divisor in  $X$ , then  $s(S, X) = [S] - [S]^2 + [S]^3 - \dots$  (Corollary 4.2.2 in [Ful84]).

We use the shorthand  $\frac{[S]}{1+S}$  for the class  $[S] - [S]^2 + [S]^3 - \dots$ .

By the first property, blowing-up  $X$  along  $S$  reduces the computation of  $s(S, X)$  to the computation of the Segre class for the exceptional divisor in the blow-up, which may be performed by using the second property. In practice it is often very difficult to carry out this process, but useful formulas for Segre classes may be proven by using this strategy. The second property is a particular case of the following fact:

- If  $S$  is regularly embedded in  $X$ , with normal bundle  $N_S X$ , then  $s(S, X) = c(N_S X)^{-1} \cap [S]$  (Corollary 4.2.1 in [Ful84]).

Below, this will be used in order to compute the Segre class of the complete intersection of two hypersurfaces.

Let  $Z_1, Z_2, V$ , etc. be as in §2.1. By the birational invariance of Segre classes recalled above,

$$\pi_{1*}s(\pi_1^{-1}(Z_2), \tilde{V}_1) = s(Z_2, V) \quad .$$

In the splayed situation, a stronger statement holds.

**Lemma 2.7.** *Let  $Z_1, Z_2$  be splayed in  $V$  (as in §2.1). Then*

$$s(\pi_1^{-1}(Z_2), \tilde{V}_1) = \pi_1^*s(Z_2, V) \quad .$$

*Proof.* Consider the diagram

$$\begin{array}{ccc} \tilde{V}_{12} & \xrightarrow{\tilde{\pi}_1} & \tilde{V}_2 \\ \tilde{\pi}_2 \downarrow & & \downarrow \pi_2 \\ \tilde{V}_1 & \xrightarrow{\pi_1} & V \end{array}$$

as in §2.1. Let  $E_2 = \pi_2^{-1}(Z_2)$  be the exceptional divisor in  $\tilde{V}_2$ , and let  $E'_2 = \tilde{\pi}_1^{-1}(E_2) = \tilde{\pi}_2^{-1}(\pi_1^{-1}(Z_2))$ . By the birational invariance of Segre classes,

$$s(\pi_1^{-1}(Z_2), \tilde{V}_1) = \tilde{\pi}_{2*} \left( \frac{[E'_2]}{1+E'_2} \right) = \tilde{\pi}_{2*} \tilde{\pi}_1^* \left( \frac{[E_2]}{1+E_2} \right) \quad .$$

Since  $Z_1$  and  $Z_2$  are splayed, by Corollary 2.6 this equals

$$\pi_1^* \pi_{2*} \left( \frac{[E_2]}{1+E_2} \right) = \pi_1^*s(Z_2, V)$$

as claimed. □



*Remark 2.8.* The equality stated in Lemma 2.7 does not hold in general:  $V = \mathbb{A}^2$ ,  $Z_1 = Z_2 =$  the origin give a simple counterexample. It does hold whenever the fiber product  $\tilde{V}_1 \times_V \tilde{V}_2$  is irreducible, as the arguments given above show, and this may occur even if  $Z_1$  and  $Z_2$  are not splayed. For example, if  $Z_1$  is a hypersurface of  $V$ , then this condition is trivially satisfied regardless of splayedness. For a more interesting example, two lines  $Z_1, Z_2$  meeting at a point in  $V = \mathbb{P}^3$  are not splayed according to our definition, yet  $\tilde{V}_1 \times_V \tilde{V}_2$  is irreducible (cf. Remark 2.5).  $\lrcorner$

We will use Lemma 2.7 in the proof of the following more general Segre class formula, which is the key technical result needed for the first proof of Theorem I.

Let  $D_1, D_2$  be hypersurfaces in  $V$ , and let  $Z_1 \subseteq D_1, Z_2 \subseteq D_2$  be subschemes. At the level of ideal sheaves, we have

$$\mathcal{I}_{D_1, V} \subseteq \mathcal{I}_1 \quad , \quad \mathcal{I}_{D_2, V} \subseteq \mathcal{I}_2$$

where  $\mathcal{I}_1 = \mathcal{I}_{Z_1, V}, \mathcal{I}_2 = \mathcal{I}_{Z_2, V}$ . We consider the subscheme  $W$  of  $V$  defined by the ideal sheaf

$$\mathcal{I}_{W, V} := \mathcal{I}_{D_1, V} \cdot \mathcal{I}_2 + \mathcal{I}_{D_2, V} \cdot \mathcal{I}_1 \quad .$$

This subscheme is supported on  $(D_1 \cup Z_2) \cap (D_2 \cup Z_1) = (D_1 \cap D_2) \cup (Z_1 \cup Z_2)$ , with a scheme structure depending subtly on  $Z_1$  and  $Z_2$ . Under a splayedness assumption, we will obtain a relation between the Segre classes of  $Z_1, Z_2$ , and  $W$ . The relation is best expressed in terms of the following notation: for  $\iota : Z \subset V$  an embedding of schemes, let

$$\hat{s}(Z, V) = [V] - \iota_* s(Z, V)^\vee \quad .$$

Here, the dual  $(\cdot)^\vee$  changes the sign of components of odd codimension in  $V$ .

**Theorem 2.9.** *Let  $D_1, D_2$  be hypersurfaces of a smooth variety  $V$ , and  $Z_1 \subseteq D_1, Z_2 \subseteq D_2, W$  as above. Assume that  $\{Z_1, D_1\}$  and  $\{Z_2, D_2\}$  are splayed. Then*

$$(6) \quad \frac{\hat{s}(W, V) \otimes \mathcal{O}(D_1 + D_2)}{1 + D_1 + D_2} = \left( \frac{\hat{s}(Z_1, V) \otimes \mathcal{O}(D_1)}{1 + D_1} \right) \cdot \left( \frac{\hat{s}(Z_2, V) \otimes \mathcal{O}(D_2)}{1 + D_2} \right)$$

*in the Chow group of  $V$ .*

*Remark 2.10.* In this statement we use the notation introduced in §2 of [Alu94]: if  $\mathcal{L}$  is a line bundle on  $V$  and  $A = \sum a^{(i)}$  is a class in the Chow group, where  $a^{(i)}$  has codimension  $i$  in  $V$ , then

$$A \otimes \mathcal{L} := \sum \frac{a^{(i)}}{c(\mathcal{L})^i} \quad .$$

The formula given in Theorem 2.9 is a good example of the usefulness of this notation: the formula (and its proof) would look unintelligibly complicated if it were written out without adopting this shorthand. The notation satisfies simple properties, see Propositions 1 and 2 in [Alu94]; these will be used liberally in what follows. It is also useful to observe that if  $A$  and  $B$  are classes in  $A_*V$ , then

$$(A \cdot B) \otimes \mathcal{L} = (A \otimes \mathcal{L}) \cdot (B \otimes \mathcal{L})$$

(this is evident from the definition).  $\lrcorner$

*Proof of Theorem 2.9.* We consider a sequence of three blow-ups over  $V$ :

$$\tilde{V} \xrightarrow{\tilde{\pi}} \tilde{V}_{12} \xrightarrow{\tilde{\pi}_2} \tilde{V}_1 \xrightarrow{\pi_1} V \quad :$$

the blow-up  $\pi_1$  of  $V$  along  $Z_1$ , with exceptional divisor  $E_1$ ; the blow-up  $\tilde{\pi}_2$  of  $\tilde{V}_1$  along  $\pi_1^{-1}(Z_2)$ , with exceptional divisor  $E'_2$ ; and the blow-up  $\tilde{\pi}$  of  $\tilde{V}_{12}$  along the intersection of the residual subschemes of  $\tilde{\pi}_2^{-1}(E_1)$ , resp.,  $E'_2$  in the inverse images of  $D_1$ , resp.,  $D_2$ . Note that under our splayedness hypothesis this last center is a complete intersection of codimension 2. We let  $\tilde{E}$  be the exceptional divisor of  $\tilde{\pi}$ . For notational convenience, we often use the same notation for an object and for its inverse image to a variety in the sequence: for instance,  $E_1$  will also denote its inverse image  $\tilde{\pi}^{-1}\tilde{\pi}_2^{-1}E_1$  in  $\tilde{V}$ . Finally,  $\pi$  will denote the composition  $\pi_1 \circ \tilde{\pi}_2 \circ \tilde{\pi} : \tilde{V} \rightarrow \tilde{V}_{12} \rightarrow \tilde{V}_1 \rightarrow V$ .

**Claim 2.11.**  $\pi^{-1}(W) = E_1 \cup E'_2 \cup \tilde{E}$ .

The statement of this claim is that the ideal of  $W$  pulls back to the product of the ideals of (the inverse images of)  $E_1$ ,  $E'_2$ , and  $\tilde{E}$  in  $\tilde{V}$ . In  $\tilde{V}_1$ ,

$$\begin{aligned} \mathcal{I}_{W,V} \cdot \mathcal{O}_{\tilde{V}_1} &= \mathcal{I}_{\tilde{D}_1, \tilde{V}_1} \mathcal{I}_{E_1, \tilde{V}_1} \mathcal{I}_{\pi_1^{-1}(Z_2), \tilde{V}_1} + \mathcal{I}_{\pi_1^{-1}(D_2), \tilde{V}_1} \cdot \mathcal{I}_{E_1, \tilde{V}_1} \\ &= \mathcal{I}_{E_1, \tilde{V}_1} \cdot (\mathcal{I}_{\tilde{D}_1, \tilde{V}_1} \mathcal{I}_{\pi_1^{-1}(Z_2), \tilde{V}_1} + \mathcal{I}_{\pi_1^{-1}(D_2), \tilde{V}_1}) \quad , \end{aligned}$$

where  $\tilde{D}_1$  is the residual of  $E_1$  in  $\pi_1^{-1}(D_1)$ . In  $\tilde{V}_{12}$ ,

$$\mathcal{I}_{W,V} \cdot \mathcal{O}_{\tilde{V}_{12}} = \mathcal{I}_{\tilde{\pi}_2^{-1}(E_1), \tilde{V}_{12}} \cdot \mathcal{I}_{E'_2, \tilde{V}_{12}} \cdot (\mathcal{I}_{\tilde{\pi}_2^{-1}(\tilde{D}_1), \tilde{V}_{12}} + \mathcal{I}_{\tilde{D}_2, \tilde{V}_{12}}) \quad ,$$

where  $\tilde{D}_2$  is the residual of  $E'_2$  in the inverse image of  $D_2$ . The ideal  $\mathcal{I}_{\tilde{\pi}_2^{-1}(\tilde{D}_1), \tilde{V}_{12}} + \mathcal{I}_{\tilde{D}_2, \tilde{V}_{12}}$  defines the center of the third blow-up, so this shows that

$$\mathcal{I}_{W,V} \cdot \mathcal{O}_{\tilde{V}} = \mathcal{I}_{\tilde{\pi}^{-1}\tilde{\pi}_2^{-1}(E_1), \tilde{V}} \cdot \mathcal{I}_{\tilde{\pi}^{-1}(E'_2), \tilde{V}} \cdot \mathcal{I}_{\tilde{E}, \tilde{V}}$$

as claimed.

By the birational invariance of Segre classes,

$$s(W, V) = \pi_* \frac{[E_1] + [E'_2] + [\tilde{E}]}{1 + E_1 + E'_2 + \tilde{E}} \quad ,$$

and therefore

$$[V] - s(W, V) = \pi_* \left( \frac{1}{1 + E_1 + E'_2 + \tilde{E}} \cap [\tilde{V}] \right) \quad .$$

Using the  $\otimes$  notation recalled after the statement of the proposition,

$$\begin{aligned} \frac{1}{1 + E_1 + E'_2 + \tilde{E}} \cap [\tilde{V}] &= \left( \frac{1 - E_1 - E'_2}{1 + \tilde{E}} \cap [\tilde{V}] \right) \otimes \mathcal{O}(E_1 + E'_2) \\ &= \left( (1 - E_1 - E'_2) \left( 1 - \frac{\tilde{E}}{1 + \tilde{E}} \right) \cap [\tilde{V}] \right) \otimes \mathcal{O}(E_1 + E'_2) \\ &= \frac{1}{1 + E_1 + E'_2} \cap \left( [\tilde{V}] - \frac{[\tilde{E}]}{1 + \tilde{E}} \otimes \mathcal{O}(E_1 + E'_2) \right) \quad . \end{aligned}$$

The term  $[\tilde{E}]/(1 + \tilde{E})$  pushes forward to the Segre class of the center of the third blow-up, which is the intersection of the (inverse images of the) residual of  $E_1$  in  $D_1$ , with class  $D_1 - E_1$ , and of the residual of  $E'_2$  in  $D_2$ , with class  $D_2 - E'_2$ . The intersection is regularly embedded in  $\tilde{V}_{12}$ , as noted earlier, and its Segre class equals the inverse Chern class of its normal bundle (by the third property of Segre classes recalled above):

$$\tilde{\pi}_* \left( \frac{\tilde{E}}{1 + \tilde{E}} \right) = \frac{[D_1 - E_1] \cdot [D_2 - E'_2]}{(1 + D_1 - E_1)(1 + D_2 - E'_2)} \quad ,$$

where evident pull-backs are omitted for notational simplicity. Using this fact, properties of the  $\otimes$  notation, and the projection formula,

$$\begin{aligned} & \tilde{\pi}_* \left( \frac{1}{1 + E_1 + E'_2 + \tilde{E}} \cap [\tilde{V}] \right) \\ &= \frac{1}{1 + E_1 + E'_2} \cap \left( [\tilde{V}_{12}] - \frac{[D_1 - E_1] \cdot [D_2 - E'_2]}{(1 + D_1 - E_1)(1 + D_2 - E'_2)} \otimes \mathcal{O}(E_1 + E'_2) \right) \\ &= \frac{1}{1 + E_1 + E'_2} \cap \left( [\tilde{V}_{12}] - \frac{[D_1 - E_1] \cdot [D_2 - E'_2]}{(1 + D_1 + E'_2)(1 + D_2 + E_1)} \right) \end{aligned}$$

A remarkable cancellation (and again the projection formula) now gives

$$\begin{aligned} & \tilde{\pi}_* \left( \frac{1}{(1 + D_1 + D_2)(1 + E_1 + E'_2 + \tilde{E})} \cap [\tilde{V}] \right) \\ &= \frac{1}{(1 + D_1 + D_2)(1 + E_1 + E'_2)} \cap \left( 1 - \frac{(D_1 - E_1) \cdot (D_2 - E'_2)}{(1 + D_1 + E'_2)(1 + D_2 + E_1)} \right) \cap [\tilde{V}_{12}] \\ &= \frac{[\tilde{V}_{12}]}{(1 + D_1 + E'_2)(1 + D_2 + E_1)} \quad . \end{aligned}$$

Summarizing, we have shown that

$$(7) \quad \frac{[V] - s(W, V)}{1 + D_1 + D_2} = \pi_{1*} \tilde{\pi}_{2*} \left( \frac{[\tilde{V}_{12}]}{(1 + D_1 + E'_2)(1 + D_2 + E_1)} \right) \quad .$$

In order to evaluate the right-hand side, note that

$$\frac{[\tilde{V}_{12}]}{1 + D_1 + E'_2} = \frac{1}{1 + D_1} \left( \frac{[\tilde{V}_{12}]}{1 + E'_2} \otimes \mathcal{O}(D_1) \right) = \frac{1}{1 + D_1} \left( [\tilde{V}_{12}] - \frac{[E'_2]}{1 + E'_2} \otimes \mathcal{O}(D_1) \right)$$

Pushing this forward by  $\tilde{\pi}_2$  shows that

$$\tilde{\pi}_{2*} \left( \frac{[\tilde{V}_{12}]}{1 + D_1 + E'_2} \right) = \frac{[\tilde{V}_1] - s(\pi_1^{-1} Z_2, \tilde{V}_1) \otimes \mathcal{O}(D_1)}{1 + D_1} \quad .$$

Since  $Z_1$  and  $Z_2$  are splayed, by Lemma 2.7 this may be rewritten as

$$\tilde{\pi}_{2*} \left( \frac{[\tilde{V}_{12}]}{1 + D_1 + E'_2} \right) = \pi_1^* \left( \frac{[V] - s(Z_2, V) \otimes \mathcal{O}(D_1)}{1 + D_1} \right).$$

By the projection formula and (7) we have

$$\frac{[V] - s(W, V)}{1 + D_1 + D_2} = \pi_{1*} \left( \frac{[\tilde{V}_1]}{1 + D_2 + E_1} \right) \cdot \frac{[V] - s(Z_2, V) \otimes \mathcal{O}(D_1)}{1 + D_1}.$$

The last push-forward is handled similarly to the previous one, giving

$$\pi_{1*} \left( \frac{[\tilde{V}_1]}{1 + D_2 + E_1} \right) = \frac{[V] - s(Z_1, V) \otimes \mathcal{O}(D_2)}{1 + D_2}.$$

Therefore,

$$\frac{[V] - s(W, V)}{1 + D_1 + D_2} = \frac{([V] - s(Z_1, V)) \otimes \mathcal{O}(D_2)}{1 + D_2} \cdot \frac{([V] - s(Z_2, V)) \otimes \mathcal{O}(D_1)}{1 + D_1}.$$

The stated formula follows from this by taking duals and tensoring by  $\mathcal{O}(D_1 + D_2)$ .  $\square$

The argument shows that the formula in Theorem 2.9 holds as soon as  $\tilde{V}_1 \times_V \tilde{V}_2$  is irreducible (cf. Remark 2.8) and the residuals of  $E_1$  in  $\pi_1^{-1}(D_1)$  and  $E'_2$  in  $\tilde{\pi}_2^{-1}\pi_1^{-1}(D_2)$  have no common components. While we focus on splayedness in this paper, the formula in Theorem 2.9 has a substantially more general scope.

*Example 2.12.* Let  $Z_1, Z_2$  be two lines in  $V = \mathbb{P}^3$  intersecting at a point. Then  $Z_1$  and  $Z_2$  are not splayed according to our definition, but  $\tilde{V}_1 \times_V \tilde{V}_2$  is irreducible (Remark 2.5). Choosing coordinates  $(x_0 : \dots : x_3)$ , we may assume that  $Z_1$  has the ideal  $(x_0, x_1)$  and  $Z_2$  has the ideal  $(x_0, x_2)$ . Then  $Z_i$  is contained in  $D_i = \{x_i = 0\}$ ; a computation shows that the relevant residuals have no common components. A direct computation of Segre classes, which may for example be carried out using [Alu03], confirms that formula (6) does hold.  $\lrcorner$

*Example 2.13.* If  $Z_1 = Z_2 = \emptyset$ , then  $W = D_1 \cap D_2$ . Assume that  $D_1$  and  $D_2$  have no common components, so that  $W$  is a codimension 2 local complete intersection with normal bundle  $\mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$ . This is of course the case if  $D_1$  and  $D_2$  are splayed, and considerably more generally. We have

$$\begin{aligned} \hat{s}(W, V) &= [V] - \left( \frac{D_1 \cdot D_2}{(1 + D_1)(1 + D_2)} \cap [V] \right)^\vee = \left( 1 - \frac{D_1 \cdot D_2}{(1 - D_1)(1 - D_2)} \right) \cap [V] \\ &= \frac{1 - D_1 - D_2}{(1 - D_1)(1 - D_2)} \cap [V] \end{aligned}$$

and hence

$$\hat{s}(W, V) \otimes \mathcal{O}(D_1 + D_2) = \frac{1 + D_1 + D_2}{(1 + D_1)(1 + D_2)} \cap [V]$$

(use Proposition 1 from [Alu94]). Formula (6) follows immediately in this case.

The reader is encouraged to consider the opposite extreme  $Z_1 = D_1, Z_2 = D_2$ , and verify that (6) reduces to  $[V] = [V] \cdot [V]$  in this case (regardless of splayedness).  $\lrcorner$

**2.3. Chern classes of hypersurface complements.** For a rapid review of *Chern-Schwartz-MacPherson* ( $c_{\text{SM}}$ ) classes, we address the reader to §3.1 of [AF13] and references therein. Briefly, every locally closed subset  $U$  of a complete variety  $V$  determines a class  $c_{\text{SM}}(U)$  in the Chow group of  $V$ , such that if  $U = Z$  is a nonsingular closed subvariety, then  $c_{\text{SM}}(Z)$  equals the push-forward to  $V$  of the total Chern class of the tangent bundle to  $Z$ . This notion is functorial in a strong sense, and satisfies an inclusion-exclusion property: if  $U_1, U_2$  are locally closed in  $V$ , then

$$c_{\text{SM}}(U_1 \cup U_2) = c_{\text{SM}}(U_1) + c_{\text{SM}}(U_2) - c_{\text{SM}}(U_1 \cap U_2) \quad .$$

The classes arose in seminal work of Marie-Hélène Schwartz ([Sch65a], [Sch65b]) and Robert MacPherson ([Mac74]). See Example 19.1.7 in [Ful84] for an efficient statement of MacPherson's definition and result.

We will use the following formula computing the  $c_{\text{SM}}$  class of a hypersurface  $D$  in a nonsingular variety  $V$  in terms of the Segre class of the *singularity subscheme*  $JD$ , locally defined (as a subscheme of  $D$ ) by the partial derivatives of a local equation for  $D$ .

**Lemma 2.14** ([Alu99], Theorem I.4). *Let  $D$  be a hypersurface of a nonsingular variety  $V$ , with singularity subscheme  $JD$ . Then*

$$(8) \quad c_{\text{SM}}(D) = c(TV) \cap (s(D, V) + c(\mathcal{O}(D))^{-1} \cap (s(JD, V)^\vee \otimes \mathcal{O}(D))) \quad .$$

This statement again uses the operations  $\cdot^\vee$ ,  $\otimes$  employed in §2.2. In terms of the notation introduced before the statement of Theorem 2.9, (8) is equivalent to

$$(9) \quad c_{\text{SM}}(V \setminus D) = c(TV) \cap \frac{\hat{s}(JD, V) \otimes \mathcal{O}(D)}{1 + D} \quad .$$

Now suppose that  $D_1$  and  $D_2$  are splayed divisors in  $V$ , and let  $D = D_1 \cup D_2$ . Note that this implies that  $\{D_1, JD_1\}$  and  $\{D_2, JD_2\}$  are splayed. It is clear set-theoretically that  $JD$  is supported on  $(D_1 \cap D_2) \cup (JD_1 \cup JD_2)$ . The splayedness condition implies that the scheme structure of  $JD$  on this union matches the one studied in §2.2 *vis-a-vis*  $W, Z_1, Z_2$ .

**Lemma 2.15.** *With notation as above,  $D_1$  and  $D_2$  are splayed if and only if*

$$\mathcal{I}_{JD, V} = \mathcal{I}_{D_1, V} \cdot \mathcal{I}_{JD_2, V} + \mathcal{I}_{D_2, V} \cdot \mathcal{I}_{JD_1, V} \quad .$$

This is a restatement of Corollary 2.6 in [AF13]. Only the 'only if' part will be needed here.

**Corollary 2.16.** *Let  $D_1, D_2$  be splayed hypersurfaces in a smooth variety  $V$ , and let  $D$  be  $D_1 \cup D_2$ . Then*

$$\frac{\hat{s}(JD, V) \otimes \mathcal{O}(D)}{1 + D} = \left( \frac{\hat{s}(JD_1, V) \otimes \mathcal{O}(D_1)}{1 + D_1} \right) \cdot \left( \frac{\hat{s}(JD_2, V) \otimes \mathcal{O}(D_2)}{1 + D_2} \right)$$

in  $A_*V$ .

*Proof.* This follows from Lemma 2.15 and Theorem 2.9, since if  $D_1$  and  $D_2$  are splayed, then so are  $\{D_1, JD_1\}$  and  $\{D_2, JD_2\}$ .  $\square$

**Corollary 2.17.** *Let  $D_1, D_2$  be splayed hypersurfaces in a smooth variety  $V$ , and let  $D$  be  $D_1 \cup D_2$ . Then*

$$(10) \quad c(TV) \cap c_{\text{SM}}(V \setminus D) = c_{\text{SM}}(V \setminus D_1) \cdot c_{\text{SM}}(V \setminus D_2) \quad .$$

*Proof.* This follows from Corollary 2.16 and (9).  $\square$

Formula (10) was proposed in [AF13], where it was observed that under a strong freeness assumption on the divisors it follows from an analogous formula for Chern classes of sheaves of logarithmic differentials (Proposition 3.2 in [AF13]). Several other particular instances of the formula are studied in §3 of [AF13]. Corollary 2.17 proves the formula without extraneous assumptions.

For splayed divisors, Theorem I follows immediately from Corollary 2.17 and the inclusion-exclusion property of  $c_{\text{SM}}$  classes.

**Theorem 2.18** (Theorem I, hypersurface case). *Let  $D_1, D_2$  be splayed divisors in a smooth variety  $V$ . Then*

$$c_{\text{SM}}(D_1) \cdot c_{\text{SM}}(D_2) = c(TV) \cap c_{\text{SM}}(D_1 \cap D_2)$$

in  $A_*V$ .

*Proof.* With  $D = D_1 \cup D_2$  as in Corollary 2.17, and noting that  $c(TV) \cap \alpha = c(V) \cdot \alpha$  for all  $\alpha \in A_*V$ ,

$$\begin{aligned} c_{\text{SM}}(D_1) \cdot c_{\text{SM}}(D_2) &= (c(V) - c_{\text{SM}}(V \setminus D_1)) \cdot (c(V) - c_{\text{SM}}(V \setminus D_2)) \\ &= c(TV) \cap (c(V) - c_{\text{SM}}(V \setminus D_1) - c_{\text{SM}}(V \setminus D_2)) \\ &\quad + c_{\text{SM}}(V \setminus D_1) \cdot c_{\text{SM}}(V \setminus D_2) \\ &= c(TV) \cap (c_{\text{SM}}(D_1) + c_{\text{SM}}(D_2) - c(V)) + c(TV) \cap c_{\text{SM}}(V \setminus D) \\ &= c(TV) \cap (c_{\text{SM}}(D_1) + c_{\text{SM}}(D_2) - c_{\text{SM}}(D)) \\ &= c(TV) \cap c_{\text{SM}}(D_1 \cap D_2) \end{aligned}$$

where the last equality follows by inclusion-exclusion.  $\square$

**2.4. Strongly splayed varieties and constructible functions.** We say that two subvarieties  $Z_1, Z_2$  of  $V$  are *strongly splayed* if  $Z_1 = D'_1 \cap \cdots \cap D'_r$ ,  $Z_2 = D''_1 \cap \cdots \cap D''_s$  where  $D'_i, D''_j$  are hypersurfaces, and  $\{D'_1, \dots, D'_r\}, \{D''_1, \dots, D''_s\}$  are splayed in the sense of §2.1. We do not know if splayed subvarieties are necessarily strongly splayed; the distinction is of course immaterial for hypersurfaces.

We can also consider this notion for *constructible functions*. By definition, every constructible function can be written as a linear combination of indicator functions of closed subvarieties. Since every subvariety is an intersection of hypersurfaces, it follows that every constructible function may be written as an integer linear combination of indicator functions of *hypersurfaces*.

**Definition 2.19.** Two constructible functions  $\varphi, \psi$  are *strongly splayed* if they admit representations

$$(11) \quad \varphi = \sum_i a'_i D'_i \quad , \quad \psi = \sum_j a''_j D''_j$$

with  $\{D'_1, \dots, D'_r\}, \{D''_1, \dots, D''_s\}$  splayed sets of hypersurfaces and  $a'_i, a''_j \in \mathbb{Z}$ .  $\lrcorner$

Thus, if  $Z_1$  and  $Z_2$  are strongly splayed, then so are the corresponding indicator functions  $\mathbb{1}_{Z_1}, \mathbb{1}_{Z_2}$ .

**Theorem II.** *Let  $\varphi, \psi$  be strongly splayed constructible functions on a nonsingular variety  $V$ . Then*

$$c_{\text{SM}}(\varphi) \cdot c_{\text{SM}}(\psi) = c(TV) \cap c_{\text{SM}}(\varphi \cdot \psi) \quad .$$

*Proof.* We will prove this statement by induction on the number of splayed hypersurfaces needed to define  $\varphi, \psi$  (as in (11)). More precisely, assume that the equality

$$c_{\text{SM}}(\varphi) \cdot c_{\text{SM}}(\psi) = c(TV) \cap c_{\text{SM}}(\varphi \cdot \psi) \quad .$$

is known whenever  $\varphi = \sum_{i=1}^r a'_i D'_i, \psi = \sum_{j=1}^s a''_j D''_j$  for a given pair  $(r, s)$  of positive integers, with  $\{D'_1, \dots, D'_r\}$  and  $\{D''_1, \dots, D''_s\}$  splayed, and for all pairs preceding  $(r, s)$  in the lexicographic order. We will show that the equality is then also true for  $(r+1, s)$ . Since the statement is true for  $(r, s) = (1, 1)$  by Theorem 2.18 (and  $\cdot$  is symmetric), the general case follows by induction. Thus we are reduced to showing that

$$c_{\text{SM}}(a\mathbb{1}_D + \varphi) \cdot c_{\text{SM}}(\psi) = c(TV) \cap c_{\text{SM}}((\mathbb{1}_D + \varphi) \cdot \psi)$$

with  $\varphi$  and  $\psi$  as above, under the assumption that  $\{D, D'_1, \dots, D'_r\}$  and  $\{D''_1, \dots, D''_s\}$  are splayed. Since  $c_{\text{SM}}$  is linear,

$$\begin{aligned} c_{\text{SM}}(a\mathbb{1}_D + \varphi) \cdot c_{\text{SM}}(\psi) &= a c_{\text{SM}}(\mathbb{1}_D) \cdot c_{\text{SM}}(\psi) + c_{\text{SM}}(\varphi) \cdot c_{\text{SM}}(\psi) \\ &= a c(TV) \cap c_{\text{SM}}(\mathbb{1}_D \cdot \psi) + c(TV) \cap c_{\text{SM}}(\varphi \cdot \psi) \end{aligned}$$

by the induction hypothesis

$$\begin{aligned} &= c(TV) \cap (a c_{\text{SM}}(\mathbb{1}_D \cdot \psi) + c_{\text{SM}}(\varphi \cdot \psi)) \\ &= c(TV) \cap c_{\text{SM}}((a\mathbb{1}_D + \varphi) \cdot \psi) \end{aligned}$$

as needed. □

Theorem II implies the full statement of Theorem I from the introduction. Indeed, for  $\varphi = \mathbb{1}_X, \psi = \mathbb{1}_Y$ , under the assumption that  $X$  and  $Y$  (and hence  $\varphi, \psi$ ) are strongly splayed, Theorem II gives

$$c_{\text{SM}}(\mathbb{1}_X) \cdot c_{\text{SM}}(\mathbb{1}_Y) = c(TV) \cap c_{\text{SM}}(\mathbb{1}_X \cdot \mathbb{1}_Y) \quad ,$$

which gives (3) as  $\mathbb{1}_X \cdot \mathbb{1}_Y = \mathbb{1}_{X \cap Y}$ .

### 3. PROOF OF THEOREM III

If  $X$  is a subscheme of a nonsingular variety, the *Chern-Fulton* class of  $X$ ,  $c_{\text{F}}(X)$ , is defined by

$$c_{\text{F}}(X) := c(TV) \cap s(X, V) \quad .$$

W. Fulton introduced this class in [Ful84], Example 4.2.6 (a), and proved that it is in fact independent of the choice of the ambient nonsingular variety  $V$ . If  $X$  is itself nonsingular, then  $s(X, V) = c(N_X V)^{-1} \cap [X]$  (§2.2), so that  $c_{\text{F}}(X) = c(X) = c_{\text{SM}}(X)$  in this case. The classes  $c_{\text{SM}}(X)$  and  $c_{\text{F}}(X)$  differ in general; for example,  $c_{\text{F}}(X)$  is sensitive to the scheme structure of  $X$ , while  $c_{\text{SM}}(X)$  only depends on the support of  $X$ .

Theorem III is a straightforward consequence of the following multiplicative formula for Segre classes of splayed subschemes.

**Lemma 3.1.** *Let  $Z_1, Z_2$  be splayed subschemes of a nonsingular variety  $V$ . Then*

$$(12) \quad s(Z_1 \cap Z_2, V) = s(Z_1, V) \cdot s(Z_2, V)$$

in  $A_*(Z_1 \cap Z_2)$ .

*Proof.* Consider again the fiber square of blow-ups

$$\begin{array}{ccc} \tilde{V}_{12} & \xrightarrow{\tilde{\pi}_1} & \tilde{V}_2 \\ \tilde{\pi}_2 \downarrow & & \downarrow \pi_2 \\ \tilde{V}_1 & \xrightarrow{\pi_1} & V \end{array}$$

as in §2.1. Let  $E_1$  be the exceptional divisor in  $\tilde{V}_1$ ,  $E_2$  the divisor in  $\tilde{V}_2$ . By splayedness, the inverse images  $\tilde{\pi}_2^{-1}(E_1)$  and  $\tilde{\pi}_1^{-1}(E_2)$  have no components in common. Thus the inverse image of  $Z_1 \cap Z_2$  in  $\tilde{V}_{12}$  is the complete intersection of  $\tilde{\pi}_2^{-1}(E_1)$  and  $\tilde{\pi}_1^{-1}(E_2)$ . Therefore

$$\begin{aligned} s(Z_1 \cap Z_2, V) &= (\pi_1 \circ \tilde{\pi}_2)_* \frac{\tilde{\pi}_2^*(E_1) \cdot \tilde{\pi}_1^*(E_2)}{(1 + \tilde{\pi}_2^*(E_1))(1 + \tilde{\pi}_1^*(E_2))} \\ &= \pi_{1*} \left( \frac{E_1}{1 + E_1} \cdot \tilde{\pi}_{2*} \tilde{\pi}_1^* \frac{E_2}{1 + E_2} \right) \end{aligned}$$

by the projection formula

$$= \pi_{1*} \left( \frac{E_1}{1 + E_1} \cdot \pi_1^* \pi_{2*} \frac{E_2}{1 + E_2} \right)$$

since the diagram is a fiber square

$$\begin{aligned} &= \pi_{1*} \left( \frac{E_1}{1 + E_1} \cdot \pi_1^* s(Z_2, V) \right) \\ &= s(Z_1, V) \cdot s(Z_2, V) \end{aligned}$$

again by the projection formula.  $\square$

*Remark 3.2.* Formula (12) also follows formally by setting  $D_1 = D_2 = 0$  in (6); note that if  $\mathcal{S}_{D_1, V}$  and  $\mathcal{S}_{D_2, V}$  are trivial, then the scheme  $W$  appearing in Theorem 2.9 equals  $Z_1 \cap Z_2$ . However, then the proof given for Theorem 2.9 does then not work: one has to assume that  $D_1$  and  $D_2$  are hypersurfaces containing  $Z_1, Z_2$  respectively, and this is in general incompatible with assuming that their classes vanish. Also, (6) only holds in  $A_*V$ , while Lemma 3.1 proves (12) in  $A_*(Z_1 \cap Z_2)$ .  $\lrcorner$



Theorem III, stated in the introduction, follows immediately from Lemma 3.1: assuming  $X$  and  $Y$  are splayed,

$$\begin{aligned} c_F(X) \cdot c_F(Y) &= (c(TV) \cap s(X, V)) \cdot (c(TV) \cap s(Y, V)) \\ &= c(TV) \cap (c(TV) \cap (s(X, V) \cdot s(Y, V))) \\ &= c(TV) \cap (c(TV) \cap s(X \cap Y, V)) \\ &= c(TV) \cap c_F(X \cap Y) \quad . \end{aligned}$$

*Example 3.3.* With  $X$  and  $Y$  as in Example 1.2, we have

$$\begin{aligned} c_F(X) &= [\mathbb{P}^4] + 6[\mathbb{P}^3] + 11[\mathbb{P}^2] + 12[\mathbb{P}^1] + 3[\mathbb{P}^0] \\ c_F(Y) &= 2[\mathbb{P}^4] + 8[\mathbb{P}^3] + 14[\mathbb{P}^2] + 12[\mathbb{P}^1] + 6[\mathbb{P}^0] \end{aligned}$$

(obtained using the code from [Alu03]). According to Theorem III,

$$c(T\mathbb{P}^5) \cap c_F(X \cap Y) = c_F(X) \cdot c_F(Y) = 2[\mathbb{P}^3] + 20[\mathbb{P}^2] + 84[\mathbb{P}^1] + 208[\mathbb{P}^0]$$

from which

$$c_{SM}(X \cap Y) = 2[\mathbb{P}^3] + 8[\mathbb{P}^2] + 6[\mathbb{P}^1] + 12[\mathbb{P}^0] \quad .$$

Again, this can be verified in this example by using the code in [Alu03].  $\square$

#### 4. PROOF OF THEOREM IV

We now assume that  $X$  is a general section of a very ample line bundle on  $V$ ; in particular,  $X$  is itself nonsingular. If a ‘Bertini theorem for splayedness’ held, then one would expect that for any  $Y \subseteq V$ , the formulas established in Theorem I and Theorem III would hold. We prove these formulas independently of such Bertini statements (and without invoking splayedness); as we pointed out in the introduction, a simple-minded ‘splayed Bertini’ statement in fact does *not* hold (Example 1.3).

Our main tool is again a formula for Segre classes, which we reproduce here for the convenience of the reader.

**Lemma 4.1.** *Let  $Z \subseteq W$  be schemes, and let  $D$  be a Cartier divisor in  $W$ , meeting properly the support of every component of the normal cone of  $Z$  in  $W$ . Then*

$$(13) \quad s(D \cap Z, D) = D \cdot s(Z, W) \quad .$$

*Proof.* Under the hypothesis of this statement, the blow-up of  $D$  along  $D \cap Z$  is the inverse image of  $D$  in the blow-up of  $W$  along  $Z$ . The statement follows then from the projection formula.  $\square$

The formula for the Chern-Fulton class in Theorem IV follows easily. Indeed, if  $X$  is general and very ample, then it can be chosen to intersect properly the components of the normal cone of  $Y$ ; further,  $X$  is nonsingular, so applying (13) and the definition of Chern-Fulton class,

$$c_F(X \cap Y) = c(TX) \cap s(X \cap Y, X) = c(TX) \cap (X \cdot s(Y, V)) = c(X) \cdot s(Y, V) \quad .$$

Thus

$$c(TV) \cap c_F(X \cap Y) = c(X) \cdot c(TV) \cap s(Y, V) = c(X) \cdot c_F(Y) \quad ,$$

as stated in Theorem IV.

For the proof of the corresponding statement about  $c_{\text{SM}}$  classes, after applying a Veronese embedding we may assume that  $V \subseteq \mathbb{P}^n$  and  $X$  is a general hyperplane section. In this situation,

$$(14) \quad c_{\text{SM}}(X \cap Y) = \frac{X}{1+X} \cdot c_{\text{SM}}(Y) \quad .$$

This follows from Proposition 2.6 in [Alu13]. (The proof of this proposition may be summarized as follows: by inclusion-exclusion it can be reduced to the case in which  $Y$  is a hypersurface; using Lemma 2.14, the formula amounts then to a relation for Segre classes that ultimately depends again on Lemma 4.1.) From (14),

$$c(TV) \cap c_{\text{SM}}(X \cap Y) = \left( c(TV) \cap \frac{X}{1+X} \right) \cdot c_{\text{SM}}(Y) = c(X) \cdot c_{\text{SM}}(Y) \quad ,$$

completing the proof of Theorem IV.

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MATHEMATICS DEPARTMENT, FLORIDA STATE UNIVERSITY, TALLAHASSEE FL 32306, U.S.A.  
*E-mail address:* `aluffi@math.fsu.edu`

DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES, UNIVERSITY OF TORONTO AT  
 SCARBOROUGH, TORONTO, ON M1A 1C4, CANADA  
*E-mail address:* `efaber@math.toronto.edu`