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Chern classes of tropical vector bundles

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Abstract. We introduce tropical vector bundles, morphisms and rational sections of these bundles and define the pull-back of a tropical vector bundle and of a rational section along a morphism. Most of the definitions presented here for tropical vector bundles will be contained in Torchiani, C., *Line Bundles on Tropical Varieties*, Diploma thesis, Technische Universität Kaiserslautern, Kaiserslautern, 2010, for the case of line bundles. Afterwards we use the bounded rational sections of a tropical vector bundle to define the Chern classes of this bundle and prove some basic properties of Chern classes. Finally we give a complete classification of all vector bundles on an elliptic curve up to isomorphisms.

1. Tropical vector bundles

In this section we will introduce our basic objects such as tropical vector bundles, morphisms of tropical vector bundles and rational sections.

Definition 1.1. (Tropical matrices) A tropical matrix is an ordinary matrix with entries in the tropical semi-ring

$$(\mathbb{T}=\mathbb{R}\cup\{-\infty\},\oplus,\odot),$$

where $a\oplus b=\max\{a,b\}$ and $a\odot b=a+b$. We denote by $\mathrm{Mat}(m\times n,\mathbb{T})$ the set of tropical $m\times n$ matrices. Let $A\in\mathrm{Mat}(m\times n,\mathbb{T})$ and $B\in\mathrm{Mat}(n\times p,\mathbb{T})$. We can form a tropical matrix product $A\odot B:=(c_{ij})\in\mathrm{Mat}(m\times p,\mathbb{T})$, where $c_{ij}=\bigoplus_{k=1}^{m} a_{ik}\odot b_{kj}$. Moreover, let $G(r\times s)\subseteq\mathrm{Mat}(r\times s,\mathbb{T})$ be the subset of tropical matrices with at most one finite entry in every row. Let $G(r)$ be the subset of $G(r\times r)$ containing all tropical matrices with exactly one finite entry in every row and every column. This set $G(r)$ together with tropical matrix multiplication is a group whose neutral element is the tropical unit matrix, i.e. the matrix with zeros on the diagonal and all other entries equal to $-\infty$.

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Notation 1.2. For an element σ of the symmetric group S_r we denote by A_σ the tropical matrix $A_\sigma=(a_{ij})\in\mathrm{Mat}(r\times r,\mathbb{T})$ given by

$$a_{ij}:=\begin{cases}0, & \text{if } i=\sigma(j),\\ -\infty, & \text{otherwise.}\end{cases}$$

Moreover, for $a_1,\dots,a_r\in\mathbb{R}$ we denote by $D(a_1,\dots,a_r)$ the tropical diagonal matrix $D(a_1,\dots,a_r)=(d_{ij})\in\mathrm{Mat}(r\times r,\mathbb{T})$ given by

$$d_{ij}:=\begin{cases}a_i, & \text{if } i=j,\\ -\infty, & \text{otherwise.}\end{cases}$$

Lemma 1.3. *Every element $M\in G(r)$ can be written as $M=A_\sigma\odot D(a_1,\dots,a_r)$ for some $\sigma\in S_r$ and some numbers $a_1,\dots,a_r\in\mathbb{R}$.*

Proof. Let $M\in G(r)$. By definition there exists exactly one finite entry in every column. Let $a_i\in\mathbb{R}$ be this finite entry in column i, situated in row p_i, $i=1,\dots,r$. Hence we can define a permutation $\sigma:\{1,\dots,r\}\to\{1,\dots,r\}$ by $\sigma(i):=p_i$, $i=1,\dots,r$, and obtain $A_\sigma\odot D(a_1,\dots,a_r)=M$. $\square$

Lemma 1.4. *$G(r)$ is precisely the set of invertible tropical matrices, i.e.*

$$G(r)=\{A\in\mathrm{Mat}(r\times r,\mathbb{T})\mid A\odot A'=A'\odot A=E \textit{ for some } A'\in\mathrm{Mat}(r\times r,\mathbb{T})\}.$$

Proof. The inclusion

$$G(r)\subseteq\{A\in\mathrm{Mat}(r\times r,\mathbb{T})\mid A\odot A'=A'\odot A=E \text{ for some } A'\in\mathrm{Mat}(r\times r,\mathbb{T})\}$$

is obvious. Thus, let $A,A'\in\mathrm{Mat}(r\times r,\mathbb{T})$ be given such that $A\odot A'=A'\odot A=E$. Assume that $A=(a_{ij})$ contains more than one finite entry in a row or column. For simplicity we assume that $a_{11},a_{12}\neq-\infty$. As $A\odot A'=E$ we can conclude that the first two rows of A' look as

$$A'=\begin{pmatrix}\alpha & -\infty & \dots & -\infty\\ \beta & -\infty & \dots & -\infty\\ & & * & \end{pmatrix}\quad\text{for some } \alpha,\beta\in\mathbb{R}.$$

As moreover $A'\odot A=E$, we can conclude from the second row of A' and the first column of A that

$$a_{11}+\beta=-\infty,$$

which is a contradiction to $a_{11},\beta\in\mathbb{R}$. $\square$

Remark 1.5. Note that a matrix $A\in G(r\times s)$ does, in general, not induce a map $f_A\colon \mathbb{R}^s\to\mathbb{R}^r$, $x\mapsto A\odot x$, as the vector $A\odot x$ may contain entries that are $-\infty$. To obtain a map $f_A\colon \mathbb{R}^s\to\mathbb{R}^r$ anyway we use the following definition: For $x\in\mathbb{R}^s$ we set $f_A(x):=(y_1,\dots,y_r)$, where

$$y_i=\begin{cases}\bigoplus_{k=1}^r a_{ik}\odot x_k, & \text{if } \bigoplus_{k=1}^r a_{ik}>-\infty,\\ 0, & \text{otherwise.}\end{cases}$$

We have all requirements now to state our main definition.

Definition 1.6. (Tropical vector bundles) Let X be a tropical cycle (cf. [AR1, Definition 5.12]). A *tropical vector bundle* over X of rank r is a tropical cycle F together with a morphism $\pi\colon F\to X$ (cf. [AR1, Definition 7.1]) and a finite open covering $\{U_1,\dots,U_s\}$ of X as well as a homeomorphism $\Phi_i\colon \pi^{-1}(U_i)\xrightarrow{\cong}U_i\times\mathbb{R}^r$ for every $i\in\{1,\dots,s\}$ such that

(a) for all i we obtain a commutative diagram

$$\begin{array}{ccc}\pi^{-1}(U_i) & \xrightarrow{\Phi_i} & U_i\times\mathbb{R}^r\\ & \searrow^{\pi} & \downarrow P^{(i)}\\ & & U_i,\end{array}$$

where $P^{(i)}\colon U_i\times\mathbb{R}^r\to U_i$ is the projection on the first factor;

(b) for all i and j the composition $p_j^{(i)}\circ\Phi_i\colon \pi^{-1}(U_i)\to\mathbb{R}$ is a regular invertible function (cf. [AR1, Definition 6.1]), where $p_j^{(i)}\colon U_i\times\mathbb{R}^r\to\mathbb{R}$, $(x,(a_1,\dots,a_r))\mapsto a_j$;

(c) for every $i,j\in\{1,\dots,s\}$ there exists a *transition map* $M_{ij}\colon U_i\cap U_j\to G(r)$ such that

$$\Phi_j\circ\Phi_i^{-1}\colon (U_i\cap U_j)\times\mathbb{R}^r\longrightarrow(U_i\cap U_j)\times\mathbb{R}^r$$

is given by $(x,a)\mapsto(x,M_{ij}(x)\odot a)$ and the entries of M_{ij} are regular invertible functions on $U_i\cap U_j$ or constantly $-\infty$;

(d) there exist representatives F_0 of F and X_0 of X such that $F_0=\{\pi^{-1}(\tau)\mid \tau\in X_0\}$ and $\omega_{F_0}(\pi^{-1}(\tau))=\omega_{X_0}(\tau)$ for all maximal polyhedra $\tau\in X_0$.

An open set U_i together with the map $\Phi_i:\pi^{-1}(U_i)\xrightarrow{\cong}U_i\times\mathbb{R}^r$ is called a *local trivialization* of F. Tropical vector bundles of rank one are called *tropical line bundles.*

Remark 1.7. Let $V_1,\dots,V_t$ be any open covering of X. Then the covering $\{U_i\cap V_j\}$ together with the restricted homeomorphisms $\Phi_i|_{\pi^{-1}(U_i\cap V_j)}$ and transition maps $M_{ij}|_{(U_i\cap V_k)\cap(U_j\cap V_l)}$ fulfills all requirements of Definition 1.6 too, and hence defines again a vector bundle. As the open covering, the homeomorphisms and the transition maps are part of the data of Definition 1.6 this new bundle is (according to our definition) different from our initial one even though they are "the same" in some sense. Hence, if two vector bundles arise one out of the other by such a construction, we will identify those two vector bundles.

Remark and Definition 1.8. Let $\pi\colon F\to X$ together with the open covering $U_1,...,U_s$, homeomorphisms Φ_i and transition maps M_{ij} and $\pi\colon F\to X$ together with the open covering $V_1,...,V_t$, homeomorphisms Ψ_i and transition maps N_{ij} be two tropical vector bundles according to Definition 1.6. We will identify these vector bundles if the vector bundles $\pi\colon F\to X$ with open covering $\{U_i\cap V_j\}$ and restricted homeomorphisms $\Phi_i|_{\pi^{-1}(U_i\cap V_j)}$ respectively $\Psi_j|_{\pi^{-1}(U_i\cap V_j)}$ and transition maps $M_{ij}|_{(U_i\cap V_k)\cap(U_j\cap V_l)}$ respectively $N_{kl}|_{(U_i\cap V_k)\cap(U_j\cap V_l)}$ are equal.

Remark 1.9. Let $\pi_1\colon F_1\to X$ and $\pi_2\colon F_2\to X$ be two vector bundles on X. By Definition 1.8 we can always assume that F_1 and F_2 satisfy Definition 1.6 with the same open covering.

Remark 1.10. Let $\pi\colon F\to X$ be a vector bundle with open covering $U_1,...,U_s$ and transition maps M_{ij} as in Definition 1.6. On the common intersection $U_i\cap U_j\cap U_k$ we obviously have $M_{ij}(x)=M_{kj}(x)\odot M_{ik}(x)$. This last equation is called the *cocycle condition*. Conversely, this data, the open covering $U_1,...,U_s$ together with transition maps M_{ij} fulfilling the cocycle condition, is enough to construct a vector bundle as we will see in the following proposition.

Proposition 1.11. *Let $U_1,...,U_s$ be an open covering of X and let*

$$M_{ij}\colon U_i\cap U_j\longrightarrow G(r)$$

be maps such that the entries of $M_{ij}(x)$ are regular invertible functions on $U_i\cap U_j$ or constantly $-\infty$ and the cocycle condition $M_{ij}(x)=M_{kj}(x)\odot M_{ik}(x)$ holds on $U_i\cap U_j\cap U_k$. Then there exists a vector bundle $\pi\colon F\to X$ with open covering $U_1,...,U_s$ and transition functions M_{ij}.

Proof. We take the disjoint union $\coprod_{i=1}^{s}(U_i\times\mathbb{R}^r)$ and identify points $(x,y)\sim(x,M_{ij}(x)\odot y)$ to obtain the topological space $|F|$. We have to equip this space with the structure of a tropical cycle. As this construction is exactly the same as for tropical line bundles, we only sketch it here and refer to [T] for more details. Let $(((X_0,|X_0|,\{\varphi_\sigma\}),\omega_{X_0}),\{\Phi_\sigma\})$ be a representative of X. We define $F_0:=\{\pi^{-1}(\sigma)|\sigma\in X_0\}$ and $\omega_{F_0}(\pi^{-1}(\sigma)):=\omega_{X_0}(\sigma)$ for all maximal polyhedra $\sigma\in X_0$. Our next step is to construct the polyhedral charts $\widetilde{\varphi}_{\pi^{-1}(\sigma)}$ for F_0: Let $\sigma\in X_0$ be given and let $U_{i_1},...,U_{i_t}$ be all open sets having non-empty intersection with σ.

Moreover, let $\{V_i | i \in I\}$ be the set of all connected components of all $\sigma \cap U_{i_k}$. Every such set V_i comes from a set $U_{j(i)}$ of the given open covering. Hence, for every pair $k, l \in I$ we have a restricted transition map $N_{kl} \colon = M_{j(k),j(l)}|_{V_k \cap V_l}$. This implies that for all $k, l \in I$ the entries of $N_{kl} \circ \Phi_\sigma^{-1}$ are (globally) integer affine linear functions on $V_k \cap V_l$. As σ is simply connected, for every such entry $h \in \mathcal{O}^*(V_k \cap V_l)$ of N_{kl} there exists a unique continuation $\tilde{h} \in \mathcal{O}^*(\sigma)$. Hence we can extend all transition maps $N_{kl} \colon V_k \cap V_l \to G(r)$ to maps $N'_{kl} \colon \sigma \to G(r)$. Now we choose for every $i \in I$ a point $P_i \in V_i$ and for all pairs $k, l \in I$ a path $\gamma_{kl} \colon [0,1] \to \sigma$ from P_k to P_l. Let $k, l \in I$ be given. As the image of γ_{kl} is compact there exists a finite covering $V_{\mu_1}, ..., V_{\mu_c}$ of $\gamma_{kl}([0,1])$. For $x \in V_l$ we set

$$S(\gamma_{kl})(x) := N'_{\mu_1,\mu_2}(x)^{-1} \odot ... \odot N'_{\mu_{c-1},\mu_c}(x)^{-1} \in G(r).$$

Now fix some $k_0 \in I$. For all $l \in I$ we define maps

$$\widetilde{\varphi}^{(l)}_{\pi^{-1}(\sigma)} \colon V_l \times \mathbb{R}^r \cong \pi^{-1}(V_l) \longrightarrow \mathbb{R}^{n_\sigma + r},$$

$$(x, a) \longmapsto (\varphi_\sigma(x), S(\gamma_{k_0 l})(x) \odot a).$$

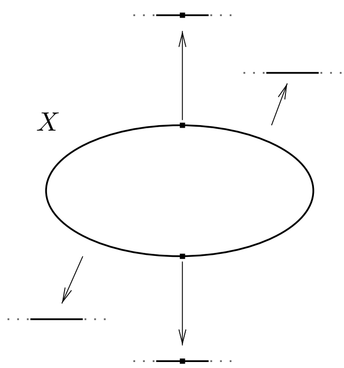
These maps agree on overlaps and hence glue together to an embedding

$$\widetilde{\varphi}_{\pi^{-1}(\sigma)} \colon \pi^{-1}(\sigma) \longrightarrow \mathbb{R}^{n_\sigma + r}.$$

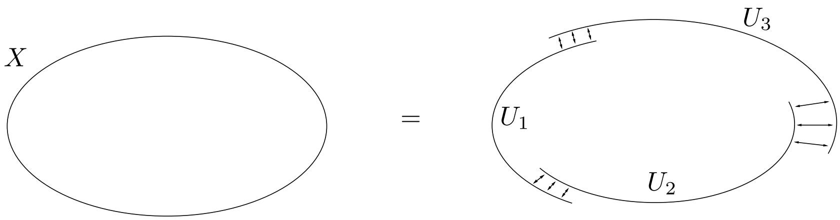
In the same way we can construct the fan charts $\widetilde{\Phi}_{\pi^{-1}(\sigma)}$. Then we define F to be the equivalence class

$$F := [(((F_0, |F_0|, \{\widetilde{\varphi}_{\pi^{-1}(\sigma)}\}), \omega_{F_0}), \{\widetilde{\Phi}_{\pi^{-1}(\sigma)}\})]. \quad \square$$

Example 1.12. Throughout the chapter, the curve $X := X_2$ from [AR1, Example 5.5] will serve us as a central example. Recall that X arises by gluing open fans as drawn in the following figure.



Moreover, recall from [AR1, Definition 5.4] that the transition functions between these open fans composing X are integer affine linear maps. This implies that the curve X has a well-defined lattice length L. We can cover X by open sets U_1, U_2 and U_3 as drawn in the following figure.



The easiest way to construct a (non-trivial) vector bundle of rank r on X is fixing a (non-trivial) transition map $M_{12}\colon U_1\cap U_2\to G(r)$ and defining $M_{23}\colon U_2\cap U_3\to G(r)$ and $M_{31}\colon U_3\cap U_1\to G(r)$ to be the trivial maps $x\mapsto E$ for all x. We will see later that in fact every vector bundle of rank r on X arises in this way.

Knowing what tropical vector bundles are, there are a few notions related to this definition we want to introduce.

Definition 1.13. (Direct sums of vector bundles) Let $\pi_1\colon F_1\to X$ and $\pi_2\colon F_2\to X$ be two vector bundles of rank r and r', respectively, with a common open covering $U_1,\dots,U_s$ and transition maps $M_{ij}^{(1)}$ and $M_{ij}^{(2)}$, respectively, satisfying Definition 1.6 (see Remark 1.9). We define the *direct sum bundle* $\pi\colon F_1\oplus F_2\to X$ to be the vector bundle of rank $r+r'$ we obtain from the gluing data

$$U_1,\dots,U_s \quad\text{and}\quad M_{ij}^{(1)}\times M_{ij}^{(2)}\colon U_i\cap U_j\longrightarrow G(r+r'),$$
$$x\longmapsto\begin{pmatrix} M_{ij}^{(1)}(x) & -\infty\\ -\infty & M_{ij}^{(2)}(x)\end{pmatrix}.$$

Definition 1.14. (Subbundles) Let $\pi\colon F\to X$ be a vector bundle with open covering $U_1,\dots,U_s$ and homeomorphisms Φ_i according to Definition 1.6. A subcycle $E\in Z_l(F)$ is called a *subbundle* of rank r' of F if $\pi|_E\colon E\to X$ is a vector bundle of rank r' such that we have for all $i=1,\dots,s$:

$$\Phi_i|_{(\pi|_E)^{-1}(U_i)}\colon (\pi|_E)^{-1}(U_i)\xrightarrow{\cong} U_i\times\langle e_{j_1},\dots,e_{j_{r'}}\rangle_{\mathbb{R}}$$

for some $1\le j_1<\dots<j_{r'}\le r$, where the e_j are the standard basis vectors in $\mathbb{R}^r$.

Remark 1.15. If $\pi\colon F\to X$ is a vector bundle of rank r with subbundle E of rank r' like in Definition 1.14 this implies that there exists another subbundle E'

of rank $r-r'$ with

$$\Phi_i|_{(\pi|_{E'})^{-1}(U_i)}\colon (\pi|_{E'})^{-1}(U_i) \xrightarrow{\cong} U_i \times \langle e_j | j \notin \{j_1, ..., j_{r'}\}\rangle_{\mathbb{R}}$$

and hence that $F=E\oplus E'$ holds.

Definition 1.16. (Decomposable bundles) Let $\pi\colon F\to X$ be a vector bundle of rank r. We say that F is *decomposable* if there exists a subbundle $\pi|_E\colon E\to X$ of F of rank $1\leq r'<r$. Otherwise we call F an *indecomposable vector bundle.*

As announced in the very beginning of this section we also want to talk about morphisms and, in particular, isomorphisms of tropical vector bundles.

Definition 1.17. (Morphisms of vector bundles) A *morphism of vector bundles* $\pi_1\colon F_1\to X$ of rank r and $\pi_2\colon F_2\to X$ of rank r' is a morphism $\Psi\colon F_1\to F_2$ of tropical cycles such that

(a) $\pi_1=\pi_2\circ\Psi$;

(b) there exist an open covering $U_1, ..., U_s$ according to Definition 1.6 for both F_1 and F_2 (see Remark 1.9) and maps $A_i\colon U_i\to G(r'\times r)$ for all i such that

$$\Phi_i^{F_2}\circ\Psi\circ(\Phi_i^{F_1})^{-1}\colon U_i\times\mathbb{R}^r \longrightarrow U_i\times\mathbb{R}^{r'}$$

is given by $(x,a)\mapsto(x, f_{A_i(x)}(a))$ (cf. Remark 1.5) and the entries of A_i are regular invertible functions on U_i or constantly $-\infty$.

An *isomorphism of tropical vector bundles* is a morphism of vector bundles $\Psi\colon F_1\to F_2$ such that there exists a morphism of vector bundles $\Psi'\colon F_2\to F_1$ with $\Psi'\circ\Psi=\mathrm{id}=\Psi\circ\Psi'$.

Lemma 1.18. *Let $\pi_1\colon F_1\to X$ and $\pi_2\colon F_2\to X$ be two vector bundles of rank r over X. Then the following are equivalent*:

(a) *There exists an isomorphism of vector bundles $\Psi\colon F_1\to F_2$;*

(b) *There exist a common open covering $U_1, ..., U_s$ of X and transition maps $M_{ij}^{(1)}$ for F_1 and $M_{ij}^{(2)}$ for F_2 satisfying Definition 1.6 (cf. Remark 1.9) and maps $E_i\colon U_i\to G(r)$ for $i=1, ..., s$ such that*

– *the entries of E_i are regular invertible functions on U_i or constantly $-\infty$;*

– *$E_j(x)\odot M_{ij}^{(1)}(x)=M_{ij}^{(2)}(x)\odot E_i(x)$ for all $x\in U_i\cap U_j$ and all i and j.*

Proof. (a)$\Rightarrow$(b) We claim that the maps $A_i\colon U_i\to G(r\times r)$ of Definition 1.17 are the wanted maps E_i. As Ψ is an isomorphism we can conclude that $A_i(x)$ is an invertible matrix for all $x\in U_i$, i.e. that $A_i\colon U_i\to G(r)$. Hence it remains to check that $A_j(x)\odot M_{ij}^{(1)}(x)=M_{ij}^{(2)}(x)\odot A_i(x)$ holds for all $x\in U_i\cap U_j$: Let i and j be given.

As $\Psi\colon F_1\to F_2$ is an isomorphism, the diagram

$$\begin{array}{ccc}
(U_i\cap U_j)\times\mathbb{R}^r & \xrightarrow{\Phi_i^{F_2}\circ\Psi\circ(\Phi_i^{F_1})^{-1}} & (U_i\cap U_j)\times\mathbb{R}^r \\
{\scriptstyle \Phi_j^{F_1}\circ(\Phi_i^{F_1})^{-1}}\downarrow & & \downarrow{\scriptstyle \Phi_j^{F_2}\circ(\Phi_i^{F_2})^{-1}} \\
(U_i\cap U_j)\times\mathbb{R}^r & \xrightarrow[\Phi_j^{F_2}\circ\Psi\circ(\Phi_j^{F_1})^{-1}]{} & (U_i\cap U_j)\times\mathbb{R}^r
\end{array}$$

commutes. Hence $A_j(x)\odot M_{ij}^{(1)}(x)=M_{ij}^{(2)}(x)\odot A_i(x)$.

(b)$\Rightarrow$(a) Conversely, let the maps $E_i\colon U_i\to G(r)$ be given. The equation

$$E_j(x)\odot M_{ij}^{(1)}(x)=M_{ij}^{(2)}(x)\odot E_i(x)$$

for all $x\in U_i\cap U_j$ ensures that the maps

$$\begin{aligned} U_i\times\mathbb{R}^r &\longrightarrow U_i\times\mathbb{R}^r, \\ (x,a) &\longmapsto (x, E_i(x)\odot a), \end{aligned}$$

on the local trivializations can be glued to a globally defined map $\Psi\colon |F_1|\to|F_2|$. Moreover, this map is a morphism as π_1 and π_2 are morphisms and the maps $p_j^{(i)}\circ\Phi_i^{F_1}$, $p_j^{(i)}\circ\Phi_i^{F_2}$ and the finite entries of E_i are regular invertible functions (cf. Definition 1.6). The equation $E_j(x)\odot M_{ij}^{(1)}(x)=M_{ij}^{(2)}(x)\odot E_i(x)$ implies that

$$E_j^{-1}(x)\odot M_{ij}^{(2)}(x)=M_{ij}^{(1)}(x)\odot E_i^{-1}(x)$$

holds for all $x\in U_i\cap U_j$, where $E_k^{-1}(x):=E_k(x)^{-1}$ for all $x\in U_k$. As the finite entries of $E_k^{-1}\colon U_k\to G(r)$ are again regular invertible functions we can also glue the maps

$$\begin{aligned} U_i\times\mathbb{R}^r &\longrightarrow U_i\times\mathbb{R}^r, \\ (x,a) &\longmapsto (x, E_i^{-1}(x)\odot a), \end{aligned}$$

on the local trivializations to obtain the inverse morphism $\Psi'\colon |F_2|\to|F_1|$, which proves that Ψ is an isomorphism. $\square$

The morphisms we have just introduced admit another important operation, namely the pull-back of a vector bundle.

Definition 1.19. (Pull-back of vector bundles) Let $\pi\colon F\to X$ be a vector bundle of rank r with open covering $U_1,\ldots,U_s$ and transition maps M_{ij} as in Definition 1.6, and let $f\colon Y\to X$ be a morphism of tropical cycles. Then the *pull-back bundle* $\pi'\colon f^*F\to Y$ is the vector bundle we obtain by gluing the patches $f^{-1}(U_1)\times\mathbb{R}^r,\ldots,f^{-1}(U_s)\times\mathbb{R}^r$ along the transition maps $M_{ij}\circ f$. Hence we obtain the commutative diagram

$$\begin{array}{ccc} f^*F & \overset{f'}{\dashrightarrow} & F \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \underset{f}{\longrightarrow} & X \end{array},$$

where f' and π' are locally given by

$$\begin{aligned} f'\colon f^{-1}(U_i)\times\mathbb{R}^r &\longrightarrow U_i\times\mathbb{R}^r, & \text{and}\quad \pi'\colon f^{-1}(U_i)\times\mathbb{R}^r &\longrightarrow f^{-1}(U_i),\\ (y,a) &\longmapsto (f(y),a), & (y,a) &\longmapsto y. \end{aligned}$$

To be able to define Chern classes in the second section we need the notion of a rational section of a vector bundle.

Definition 1.20. (Rational sections of vector bundles) Let $\pi\colon F\to X$ be a vector bundle of rank r. A *rational section* $s\colon X\to F$ of F is a continuous map $s\colon |X|\to|F|$ such that

(a) $\pi(s(x))=x$ for all $x\in|X|$;

(b) there exist an open covering $U_1,\ldots,U_s$ and homeomorphisms Φ_i satisfying Definition 1.6 (cf. Definition 1.8) such that the maps $p_j^{(i)}\circ\Phi_i\circ s{:}U_i\to\mathbb{R}$ are rational functions on U_i for all i and j, where $p_j^{(i)}\colon U_i\times\mathbb{R}^r\to\mathbb{R}$, $(x,(a_1,\ldots,a_r))\mapsto a_j$.

A rational section $s\colon X\to F$ is called *bounded* if the above maps $p_j^{(i)}\circ\Phi_i\circ s$ are bounded for all i and j.

Remark 1.21. Let $\pi\colon L\to X$ be a line bundle and $s\colon X\to L$ be a rational section. By definition, the map $p^{(i)}\circ\Phi_i\circ s$ is a rational function on U_i for all i. Moreover, on $U_i\cap U_j$ the maps $p^{(i)}\circ\Phi_i\circ s$ and $p^{(j)}\circ\Phi_j\circ s$ differ by a regular invertible function only. Hence s defines a Cartier divisor $\mathcal{D}(s)\in\mathrm{Div}(X)$.

There is a useful statement on these Cartier divisors $\mathcal{D}(s)$ in [T] that we want to cite here including its proof.

Lemma 1.22. *Let* $\pi\colon L\to X$ *be a line bundle and let* $s_1, s_2\colon X\to L$ *be two bounded rational sections. Then* $\mathcal{D}(s_1)-\mathcal{D}(s_2)=h$ *for some bounded rational function* $h\in\mathcal{K}^*(X)$*, i.e.* $\mathcal{D}(s_1)$ *and* $\mathcal{D}(s_2)$ *are rationally equivalents.*

Proof. Let $U_1, ..., U_s$ be an open covering of X with transition maps M_{ij} and homeomorphisms Φ_i according to Definition 1.6 such that for all i both $s_1^{(i)}:=p_1^{(i)}\circ\Phi_i\circ s_1$ and $s_2^{(i)}:=p_1^{(i)}\circ\Phi_i\circ s_2$ are rational functions on U_i (cf. Definition 1.20). We define $h_i:=s_1^{(i)}-s_2^{(i)}\in\mathcal{K}^*(U_i)$. Since we have that $s_1^{(i)}-s_1^{(j)}=s_2^{(i)}-s_2^{(j)}=M_{ij}\in\mathcal{O}^*(U_i\cap U_j)$ for all i and j, these maps h_i glue together to $h\in\mathcal{K}^*(X)$. Hence we have that

$$\begin{aligned}\mathcal{D}(s_1)-\mathcal{D}(s_2) &= [\{(U_i, s_1^{(i)})\}]-[\{(U_i, s_2^{(i)})\}]\\ &= [\{(U_i, s_1^{(i)}-s_2^{(i)})\}]=[\{(U_i, h_i)\}]=[\{(|X|, h)\}]. \qquad \square\end{aligned}$$

Remark 1.23. Lemma 1.22 implies that we can associate with any line bundle L admitting a bounded rational section s a Cartier divisor class $\mathcal{D}(F):=[\mathcal{D}(s)]$ that only depends on the bundle L and not on the choice of the rational section s.

Combining both the notion of morphism of vector bundles and the notion of rational section we can give the following definition.

Definition 1.24. (Pull-back of rational sections) Let $\pi\colon F\to X$ be a vector bundle of rank r and $f\colon Y\to X$ be a morphism of tropical varieties. Moreover, let $s\colon X\to F$ be a rational section of F with open covering $U_1, ..., U_s$ and homeomorphisms $\Phi_1, ..., \Phi_s$ as in Definition 1.20. Then we can define a rational section $f^*s\colon Y\to f^*F$ of f^*F, the *pull-back section* of s, as follows: On $f^{-1}(U_i)$ we define

$$\begin{aligned}f^*s\colon f^{-1}(U_i) &\longrightarrow f^{-1}(U_i)\times\mathbb{R}^r,\\ y &\longmapsto (y, (p_i\circ\Phi_i\circ s\circ f)(y)),\end{aligned}$$

where $p_i\colon U_i\times\mathbb{R}^r\to\mathbb{R}^r$ is the projection on the second factor. Note that for $y\in f^{-1}(U_i)\cap f^{-1}(U_j)$ the points $(y, (p_i\circ\Phi_i\circ s\circ f)(y))$ and $(y, (p_j\circ\Phi_j\circ s\circ f)(y))$ are identified in f^*F if and only if $(f(y), (p_i\circ\Phi_i\circ s\circ f)(y))$ and $(f(y), (p_j\circ\Phi_j\circ s\circ f)(y))$ are identified in F. But this is the case as

$$(f(y), (p_i\circ\Phi_i\circ s\circ f)(y))=(\Phi_i\circ s)(f(y))\sim(\Phi_j\circ s)(f(y))=(f(y), (p_j\circ\Phi_j\circ s\circ f)(y)).$$

Hence we can glue our locally defined map f^*s to obtain a map $f^*s\colon Y\to f^*F$.

We finish this section with the following statement on vector bundles on simply connected tropical cycles which will be of use for us later on.

Theorem 1.25. *Let* $\pi\colon F\to X$ *be a vector bundle of rank* r *on the simply connected tropical cycle* X*. Then* F *is a direct sum of line bundles, i.e. there exist line bundles* $L_1,...,L_r$ *on* X *such that* $F=L_1\oplus...\oplus L_r$*.*

Proof. We show that every vector bundle of rank $r\geq 2$ on X is decomposable. Let $U_1,...,U_s$ be an open covering of X and let

$$M_{ij}(x)=D(\varphi_{i,j}^{(1)},...,\varphi_{i,j}^{(r)})(x)\odot A_{\sigma_{ij}}(x)=:D_{ij}(x)\odot A_{\sigma_{ij}}(x),\quad x\in U_i\cap U_j,$$

with $\varphi_{i,j}^{(1)},...,\varphi_{i,j}^{(r)}\in\mathcal{O}^*(U_i\cap U_j)$ and $\sigma_{ij}(x)\in S_r$ being transition functions according to Definition 1.6. We only have to show that it is possible to track the first coordinate of the $\mathbb{R}^r$-factor in $U_1\times\mathbb{R}^r$ consistently along the transition maps: Let $\gamma\colon[0,1]\to|X|$ be a closed path starting and ending in $P\in U_1$. Decomposing γ into several paths if necessary, we may assume that γ has no self-intersections, i.e. that $\gamma|_{[0,1)}$ is injective. As $\gamma([0,1])$ is compact we can choose an open covering $V_1,...,V_t$ of $\gamma([0,1])$ such that for all j we have $V_j\subseteq U_i$ for some index $i=i(j)$, $P\in V_1=V_t\subseteq U_1$, all sets V_j and all intersections $V_j\cap V_{j+1}$ are connected and all intersections $V_j\cap V_{j'}$ for non-consecutive indices are empty. For sets V_j and $V_{j'}$ with non-empty intersection we have restricted transition maps $\widetilde{M}_{V_j,V_{j'}}(x)=\widetilde{D}_{V_j,V_{j'}}(x)\odot A_{\sigma_{V_j,V_{j'}}}$ induced by the transition maps between $U_{i(j)}\supseteq V_j$ and $U_{i(j')}\supseteq V_{j'}$. Note that the permutation parts $A_{\sigma_{V_j,V_{j'}}}$ of the transition maps do not depend on x as all intersections $V_j\cap V_{j'}$ are connected and the permutations have to be locally constant. We define $I_\gamma:=\sigma_{V_{t-1},V_t}\circ...\circ\sigma_{V_1,V_2}(1)$. We have to check that $I_\gamma=1$ holds. First we show that I_γ does not depend on the choice of the covering $V_1,...,V_t$. Hence, let $V_1',...,V_{t'}'$ be another covering as above. We may assume that all intersections $V_j\cap V_{j'}'$ are connected, too. Between any two sets $A,B\in\{V_1,...,V_t,V_1',...,V_{t'}'\}$ with non-empty intersection we have restricted transition maps $\widetilde{M}_{A,B}(x)=\widetilde{D}_{A,B}(x)\odot A_{\sigma_{A,B}}$ as above. Moreover, let $0=\alpha_0<...<\alpha_p=1$ be a decomposition of $[0,1]$ such that for all i we have $\gamma([\alpha_i,\alpha_{i+1}])\subseteq V_j\cap V_{j'}'$ for some indices j and j'. Let i_0 be the maximal index such that $\gamma([\alpha_{i_0},\alpha_{i_0+1}])\subseteq V_a\cap V_b'$ and

$$\sigma_{V_{a-1},V_a}\circ...\circ\sigma_{V_1,V_2}=\sigma_{V_b',V_a}\circ\sigma_{V_{b-1}',V_b'}\circ...\circ\sigma_{V_1',V_2'}$$

is still fulfilled. Assume that $i_0<p-1$. Let $\gamma([\alpha_{i_0+1},\alpha_{i_0+2}])\subseteq V_{a'}\cap V_{b'}'$. Hence $\gamma(\alpha_{i_0+1})\in V_a\cap V_b'\cap V_{a'}\cap V_{b'}'$ and we can conclude using the cocycle condition that

$$\begin{aligned}\sigma_{V_a,V_{a'}}\circ\sigma_{V_{a-1},V_a}\circ...\circ\sigma_{V_1,V_2}&=\sigma_{V_a,V_{a'}}\circ\sigma_{V_b',V_a}\circ\sigma_{V_{b-1}',V_b'}\circ...\circ\sigma_{V_1',V_2'}\\&=\sigma_{V_a,V_{a'}}\circ\sigma_{V_{b'}',V_a}\circ\sigma_{V_b',V_{b'}'}\circ\sigma_{V_{b-1}',V_b'}\circ...\circ\sigma_{V_1',V_2'}\\&=\sigma_{V_{b'}',V_{a'}}\circ\sigma_{V_b',V_{b'}'}\circ\sigma_{V_{b-1}',V_b'}\circ...\circ\sigma_{V_1',V_2'},\end{aligned}$$

a contradiction to our assumption. Hence $i_0=p-1$ and we conclude that I_γ is independent of the chosen covering.

If γ and γ' are paths that pass through exactly the same open sets U_i in the same order, then we can conclude that $I_\gamma = I_{\gamma'}$ holds as exactly the same transition functions are involved. Hence, a continuous deformation of γ does not change I_γ. As X is simply connected we can contract γ to a point. This implies that $I_\gamma = I_{\gamma_0}$, where γ_0 is the constant path $\gamma_0(t) = P$ for all t. Thus $I_\gamma = I_{\gamma_0} = 1$. This proves the claim. □

There is a related theorem in [T] which we want to state here. As we will not need the result in this work, we will omit the proof and refer to [T] instead.

Theorem 1.26. *Let $\pi\colon L \to X$ be a line bundle on the simply connected tropical cycle X. Then L is trivial, i.e. $L \cong X \times \mathbb{R}$ as a vector bundle.*

Combining Theorems 1.25 and 1.26 we can conclude the following result.

Corollary 1.27. *Let $\pi\colon F \to X$ be a vector bundle of rank r on the simply connected tropical cycle X. Then F is trivial, i.e. $F \cong X \times \mathbb{R}^r$ as a vector bundle.*

2. Chern classes

In this section we will introduce Chern classes of tropical vector bundles and prove some basic properties. To be able to do this we need some preparation.

Definition 2.1. Let $\pi\colon F \to X$ be a vector bundle of rank r and let $s\colon X \to F$ be a rational section with open covering $U_1, \dots, U_s$ as in Definition 1.20. We fix a natural number $1 \le k \le r$ and a subcycle $Y \in Z_l(X)$. By definition, $s_{ij} := p_j^{(i)} \circ \Phi_i \circ s\colon U_i \to \mathbb{R}$ is a rational function on U_i for all i and j. Hence, for all i we can take local intersection products

$$(s^{(k)} \cdot Y) \cap U_i := \sum_{1 \le j_1 < \dots < j_k \le r} s_{ij_1} \dots s_{ij_k} \cdot (Y \cap U_i).$$

Since $s_{i'j} = s_{i\sigma(j)} + \varphi_j$ on $U_i \cap U_{i'}$ for some $\sigma \in S_r$ and some regular invertible map $\varphi_j \in \mathcal{O}^*(U_i \cap U_{i'})$, the intersection products $(s^{(k)} \cdot Y) \cap U_i$ and $(s^{(k)} \cdot Y) \cap U_{i'}$ coincide on $U_i \cap U_{i'}$ and we can glue them to obtain a global intersection cycle $s^{(k)} \cdot Y \in Z_{l-k}(X)$.

Lemma 2.2. *Let $\pi\colon F \to X$ be a vector bundle of rank r, fix $k \in \{1, \dots, r\}$ and let $s\colon X \to F$ be a rational section. Moreover, let $Y \in Z_l(X)$ be a cycle and let $\varphi \in \mathcal{K}^*(Y)$ be a bounded rational function on Y. Then*

$$s^{(k)} \cdot (\varphi \cdot Y) = \varphi \cdot (s^{(k)} \cdot Y).$$

Proof. The claim follows immediately from the definition of the product $s^{(k)} \cdot Y$. □

Lemma 2.3. *Let $\pi\colon F \to X$ and $\pi'\colon F' \to X$ be two isomorphic vector bundles of rank r with isomorphism $f\colon F \to F'$. Moreover, fix $k \in \{1, \dots, r\}$, let $s\colon X \to F$ be a rational section and let $Y \in Z_l(X)$ be a cycle. Then*

$$s^{(k)} \cdot Y = (f \circ s)^{(k)} \cdot Y \in Z_{l-k}(X).$$

Proof. Let $U_1, \dots, U_s$ be an open covering of X satisfying Definition 1.6 for both F and F' and let $s_{ij} := p_j^{(i)} \circ \Phi_i \circ s\colon U_i \to \mathbb{R}$ and $(f \circ s)_{ij} := p_j^{(i)} \circ \Phi_i \circ f \circ s\colon U_i \to \mathbb{R}$ as in Definition 2.1. By Lemma 1.18 the isomorphism f can be described on $U_i \times \mathbb{R}^r$ by $(x, a) \mapsto (x, E_i(x) \odot a)$ with $E_i(x) = D(\varphi_1, \dots, \varphi_r) \odot A_\sigma$ for some regular invertible functions $\varphi_1, \dots, \varphi_r \in \mathcal{O}^*(U_i)$ and a permutation $\sigma \in S_r$. Hence $(f \circ s)_{ij} = s_{i\sigma(j)} + \varphi_j$ on U_i and thus

$$\sum_{1 \le j_1 < \dots < j_k \le r} s_{ij_1} \cdots s_{ij_k} \cdot (Y \cap U_i) = \sum_{1 \le j_1 < \dots < j_k \le r} (f \circ s)_{ij_1} \cdots (f \circ s)_{ij_k} \cdot (Y \cap U_i),$$

which proves the claim. □

To be able to prove the next theorem, which will be essential for defining Chern classes, we first need some generalizations of our previous definitions:

Definition 2.4. (Infinite tropical cycle) We define an *infinite tropical polyhedral complex* to be a tropical polyhedral complex according to [AR1, Definition 5.4] but we do not require the set of polyhedra X to be finite. In particular, all open fans F_σ still have to be open tropical fans according to [AR1, Definition 5.3]. Then an *infinite tropical cycle* is an infinite tropical polyhedral complex modulo refinements analogous to [AR1, Definition 5.12].

Remark 2.5. Definition 2.4 implies that an infinite tropical polyhedral complex is locally finite, i.e. there are only finitely many polyhedra adjacent to any single polyhedron. We can therefore think of infinite tropical cycles to be infinitely many ordinary tropical fans glued together.

Definition 2.6. (Infinite rational functions and infinite Cartier divisors) Let C be an infinite tropical cycle and let U be an open set in $|C|$. As in [AR1, Definition 6.1] an *infinite rational function* on U is a continuous function $\varphi\colon U \to \mathbb{R}$ such that there exists a representative $(((X, |X|, \{m_\sigma\}_{\sigma \in X}), \omega_X), \{M_\sigma\}_{\sigma \in X})$ of C, which may now be an infinite tropical polyhedral complex, such that for each face

$\sigma \in X$ the map $\varphi \circ m_\sigma^{-1}$ is locally integer affine linear (where defined). Analogously it is possible to define *infinite regular invertible functions* on U.

A *representative of an infinite Cartier divisor* on C is a set $\{(U_i, \varphi_i) | i \in I\}$, where $\{U_i\}_i$ is an open covering of $|C|$ and φ_i is an infinite rational function on U_i. An *infinite Cartier divisor* on C is a representative of an infinite Cartier divisor modulo the equivalence relation given in [AR1, Definition 6.1].

Remark 2.7. Using these basic definitions it is possible to generalize many other concepts to the infinite case. In particular, as our infinite objects are locally finite, it is possible to perform intersection theory as before.

Definition 2.8. (Tropical vector bundles on infinite cycles) Let X be an infinite tropical cycle. A *tropical vector bundle* over X of rank r is an infinite tropical cycle F together with a morphism $\pi\colon F \to X$ such that properties (a)–(d) given in Definition 1.6 are fulfilled with the difference that the open covering $\{U_i\}_i$ of X may now be infinite.

Now we are ready to prove the following announced theorem.

Theorem 2.9. *Let $\pi\colon F \to X$ be a vector bundle of rank r and $s_1, s_2\colon X \to F$ be two bounded rational sections. Then $s_1^{(k)} \cdot Y$ and $s_2^{(k)} \cdot Y$ are rationally equivalent, i.e.*

$$[s_1^{(k)} \cdot Y] = [s_2^{(k)} \cdot Y] \in A_*(X)$$

for all subcycles $Y \in Z_l(X)$.

Proof. Let $p\colon |\widetilde{X}| \to |X|$ be the universal covering space of $|X|$. We can locally equip $|\widetilde{X}|$ with the tropical structure inherited form X and obtain an infinite tropical cycle $\widetilde{X}$ according to Definition 2.4. Moreover, pulling back F along p, we obtain a tropical vector bundle p^*F on $\widetilde{X}$ according to Definition 2.8. As $\widetilde{X}$ is simply connected we can conclude by Lemma 1.25 that $p^*F = L_1 \oplus \ldots \oplus L_r$ for some infinite tropical line bundles $L_1, \ldots, L_r$ on $\widetilde{X}$. Hence, the bounded rational sections p^*s_1 and p^*s_2 correspond to r infinite tropical Cartier divisors as in Definition 2.6 each, which we will denote by $\varphi_1, \ldots, \varphi_r$ and $\psi_1, \ldots, \psi_r$, respectively. By Lemma 1.22 we can conclude that for all i these Cartier divisors differ by bounded infinite rational functions only, i.e. $\varphi_i - \psi_i = h_i$ for some bounded infinite rational function h_i on $\widetilde{X}$. In particular,

$$\left(\sum_{1 \le j_1 < \ldots < j_k \le r} \varphi_{j_1} \ldots \varphi_{j_k} - \sum_{1 \le j_1 < \ldots < j_k \le r} \psi_{j_1} \ldots \psi_{j_k} \right) \cdot \widetilde{X} = \tilde{h} \cdot \tilde{\xi}_2 \ldots \tilde{\xi}_k \cdot \widetilde{X}$$

with a bounded infinite rational function $\tilde{h}$ and infinite Cartier divisors $\tilde{\xi}_i$. Then we can define a rational function h, which is then also bounded, and Cartier divisors ξ_i on X as follows: Let $U \subseteq |X|$ and $\widetilde{U} \subseteq |\widetilde{X}|$ be open subsets such that $p|_{\widetilde{U}} \colon \widetilde{U} \to U$ is bijective with inverse map $p' \colon U \to \widetilde{U}$. Then we locally define $h|_U := (p')^* \tilde{h}|_{\widetilde{U}}$ and $\xi_i|_U := (p')^* \tilde{\xi}_i|_{\widetilde{U}}$. Note that h and ξ_i are well-defined as the Cartier divisors φ_i and ψ_i are the same on every possible set $\widetilde{U} \overset{\cong}{\to} U$. As we locally have

$$(s_1^{(k)} \cdot Y) \cap U = p_* \Bigg(\sum_{1 \le j_1 < \ldots < j_k \le r} \varphi_{j_1} \ldots \varphi_{j_k} \cdot (p')_*(Y \cap U) \Bigg)$$

and

$$(s_2^{(k)} \cdot Y) \cap U = p_* \Bigg(\sum_{1 \le j_1 < \ldots < j_k \le r} \psi_{j_1} \ldots \psi_{j_k} \cdot (p')_*(Y \cap U) \Bigg)$$

we conclude that

$$(s_1^{(k)} - s_2^{(k)}) \cdot Y = h \cdot \xi_2 \ldots \xi_k \cdot Y,$$

which proves the claim. □

Now we are ready to give a definition of Chern classes.

Definition 2.10. (Chern classes) Let $\pi \colon F \to X$ be a vector bundle of rank r admitting bounded rational sections. For $k \in \{1, \ldots, r\}$ we define the *k-th Chern class* of F to be the endomorphism

$$\begin{aligned} c_k(F) \colon A_*(X) &\longrightarrow A_*(X), \\ [Y] &\longmapsto [s^{(k)} \cdot Y], \end{aligned}$$

where $A_*(X) = \bigoplus_i A_i(X)$ and $s \colon X \to F$ is any bounded rational section. Note that the map $c_k(F)$ is well defined by Lemma 2.2 and independent of the choice of the rational section s by Theorem 2.9. Moreover, we define $c_0(F) \colon A_*(X) \to A_*(X)$ to be the identity map and $c_k(F) \colon A_*(X) \to A_*(X)$ to be the zero map for all $k \notin \{0, \ldots, r\}$. To stress the character of an intersection product of $c_k(F)$ we usually write $c_k(F) \cdot Y$ instead of $c_k(F)(Y)$ for $Y \in A_*(X)$.

As announced at the beginning of this section we finish this section with proving some basic properties of Chern classes.

Theorem 2.11. (Properties of Chern classes) *Let $\pi \colon F \to X$ and $\pi' \colon F' \to X$ be vector bundles of rank r and r', respectively, admitting bounded rational sections. Moreover, let $f \colon Y \to X$ be a morphism of tropical cycles. Then the following holds:*

(a) $c_i(F)=0$ *for all* $i\notin\{0,\dots,\mathrm{rank}(F)\}$;
(b) $c_i(F)\cdot(c_j(F')\cdot Z_X)=c_j(F')\cdot(c_i(F)\cdot Z_X)$ *for all* $Z_X\in A_*(X)$;
(c) $f_*(c_i(f^*F)\cdot Z_Y)=c_i(F)\cdot f_*(Z_Y)$ *for all* $Z_Y\in A_*(Y)$;
(d) $c_i(f^*F)\cdot f^*(Z_X)=f^*(c_i(F)\cdot Z_X)$ *for all* $Z_X\in A_*(X)$ *if* X *and* Y *are smooth varieties*;
(e) $c_k(F\oplus F')=\sum_{i+j=k} c_i(F)\cdot c_j(F')$;
(f) $c_1(F)\cdot Z_X=\mathcal{D}(F)\cdot Z_X$ *for all* $Z_X\in A_*(X)$ *if* $r=\mathrm{rank}(F)=1$, *where* $\mathcal{D}(F)$ *is the Cartier divisor class associated with* F.

Proof. Properties (a) and (e) follow immediately from Definition 2.10, property (b) follows from the fact that the intersection product is commutative and property (f) follows from Remark 1.23.

(c) The projection formula implies that

$$f_*(c_i(f^*F)\cdot Z_Y)=f_*([(f^*s)^{(i)}\cdot Z_Y])=[s^{(i)}\cdot f_*Z_Y]=c_i(F)\cdot f_*Z_Y,$$

where s is any bounded rational section of F.

(d) Applying [A, Theorem 3.2(c) and (f)] we obtain that

$$c_i(f^*F)\cdot f^*Z_X=[(f^*s)^{(i)}\cdot f^*Z_X]=[f^*(s^{(i)}\cdot Z_X)]=f^*[s^{(i)}\cdot Z_X]=f^*(c_i(F)\cdot Z_X),$$

where s is again any bounded rational section of F. □

Remark 2.12. In "classical" algebraic geometry even the following, generalized version of property (e) is true: Let $0\to F'\to F\to F''\to 0$ be an exact sequence of vector bundles, then $c_k(F)=\sum_{i+j=k} c_i(F')\cdot c_j(F'')$. In the tropical world it is not entirely clear what an exact sequence of tropical vector bundles should be. Nevertheless, in some sense the "classical" statement is true in tropical geometry as well: Let $\pi_1\colon F_1\to X$ and $\pi_2\colon F_2\to X$ be tropical vector bundles of rank r_1 and r_2, respectively, and let $U_1,\dots,U_s$ be an open covering of X such that all requirements of Definition 1.6 are fulfilled for F_1 and F_2 simultaneously. Moreover, let $f\colon F_1\to F_2$ be an injective morphism of tropical vector bundles such that $(\Phi_i^{F_2}\circ f\circ(\Phi_i^{F_1})^{-1})(U_i\times\mathbb{R}^{r_1})=U_i\times\langle e_{i_1},\dots,e_{i_{r_1}}\rangle_{\mathbb{R}}$ for all i, i.e. such that the image of F_1 under f is a subbundle F' of F_2 (cf. Definition 1.14). Then we conclude by Remark 1.15 that F_2 is decomposable into $F_2=F'\oplus F''$ for some other subbundle F'' of F_2. Hence we conclude by Theorem 2.11 that $c_k(F_2)=\sum_{i+j=k} c_i(F')\cdot c_j(F'')$.

3. Vector bundles on an elliptic curve

In this section we will give a complete classification of all vector bundles on an elliptic curve up to isomorphism. One characteristic to distinguish different bundles will be the following result.

Definition 3.1. (Degree of a vector bundle) Let $X := X_2$ be the curve from [AR1, Example 5.5] and let $\pi: F \to X$ be a vector bundle of rank r. We define the *degree* of F to be the number

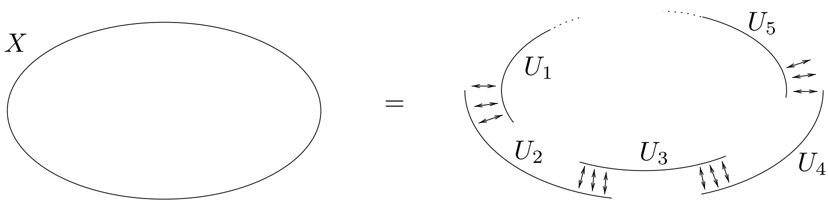
$$\deg(F) := \deg(c_1(F) \cdot X).$$

Remark 3.2. Note that Lemma 2.3 implies that isomorphic vector bundles on X have the same Chern classes and hence have the same degree.

As already advertised in Example 1.12 vector bundles on the elliptic curve X can be described by a single transition function. We will prove this fact in the following lemma.

Lemma 3.3. *Again, let $X := X_2$ be the curve from* [AR1, Example 5.5] *and let $\pi: F \to X$ be a vector bundle of rank r. Then F is isomorphic to a vector bundle $\pi': F' \to X$ that admits an open covering $U'_1, \ldots, U'_s$ and transition maps M'_{ij} such that at most one transition map is non-trivial.*

Proof. Let $U_1, \ldots, U_s$ be the open covering with transition maps M_{ij} for F according to Definition 1.6. We may assume that all the sets U_i are connected and that for all i and j the intersections $U_i \cap U_j$ are connected as well. Moreover, we may assume that the sets U_i are numbered consecutively as shown in the figure. For simplicity of notation we will consider our indices modulo s.



We can write every map $M_{i,i+1}$, $i = 1, \ldots, s$, as

$$M_{i,i+1}(x) = D(\varphi^{(1)}_{i,i+1}, \ldots, \varphi^{(r)}_{i,i+1})(x) \odot A_{\sigma_{i,i+1}} =: D_i(x) \odot P_i$$

for some regular invertible functions $\varphi^{(k)}_{i,i+1} \in \mathcal{O}^*(U_i \cap U_{i+1})$ and some permutations $\sigma_{i,i+1} \in S_r$. We will show that we can successively replace all the transition maps $M_{i,i+1}$ but one by the constant map $M'_{i,i+1}: U_i \cap U_{i+1} \to G(r)$, $x \mapsto E$, and the resulting vector bundle F' is isomorphic to F: Choose $j_0 \in \{2, \ldots, s\}$. Note that if we are given a regular invertible function $\varphi \in \mathcal{O}^*(U_i \cap U_j)$ there is a unique regular invertible function $\widetilde{\varphi} \in \mathcal{O}^*(U_i)$ such that $\widetilde{\varphi}|_{U_i \cap U_j} = \varphi$. As they are regular invertible functions, too, we can extend in exactly the same way the finite entries of the matrix D_{j_0} along the chain $U_{j_0-1}, U_{j_0-2}, \ldots, U_{i+1}$ to any set U_{i+1} for $i \in \{2, \ldots, j_0-1\}$. By abuse of notation we will denote this continuation of D_{j_0} as well by D_{j_0}. Now, we take $U'_i := U_i$ for all $i = 1, \ldots, s$ and

$$M'_{i,i+1}(x) := \begin{cases} P_{j_0} \odot D_{j_0}(x) \odot M_{i,i+1}(x) \odot D_{j_0}(x)^{-1} \odot P_{j_0}^{-1}, & \text{if } i \in \{2, \ldots, j_0-1\}, \\ M_{i,i+1}(x), & \text{if } i \in \{j_0+1, \ldots, s\}. \end{cases}$$

Moreover, we set $M'_{12}(x) := P_{j_0} \odot D_{j_0}(x) \odot D_1(x) \odot P_1$ and $M'_{j_0,j_0+1}(x) := E$. To check that the vector bundle F' we obtain from this gluing data is isomorphic to F we apply Lemma 1.18: We set

$$E_i(x) := \begin{cases} D_{j_0}(x) \odot P_{j_0}, & \text{if } i \in \{2, \ldots, j_0\}, \\ E, & \text{otherwise}, \end{cases}$$

and get that

$$\begin{aligned} (D_{j_0} \odot P_{j_0}) \odot (D_1 \odot P_1) &= (D_{j_0} \odot P_{j_0} \odot D_1 \odot P_1) \odot E, \\ (D_{j_0} \odot P_{j_0}) \odot (D_2 \odot P_2) &= (D_{j_0} \odot P_{j_0} \odot D_2 \odot P_2 \odot D_{j_0}^{-1} \odot P_{j_0}^{-1}) \odot (D_{j_0} \odot P_{j_0}), \\ &\vdots \\ E \odot (D_{j_0} \odot P_{j_0}) &= E \odot (D_{j_0} \odot P_{j_0}). \end{aligned}$$

This finishes our proof. □

To classify all vector bundles on our elliptic curve X we now give a non-redundant parametrization of all indecomposable vector bundles on X. Arbitrary vector bundles are then just direct sums of these building blocks.

Theorem 3.4. (Vector bundles on elliptic curves) *Let $X := X_2$ be the curve from* [AR1, Example 5.5]. *Then the set of indecomposable vector bundles of rank r and degree d is in bijection with $\gcd(r,d) \cdot X$, i.e. with points of the curve X stretched to $\gcd(r,d)$ times the original length.*

Remark 3.5. Note that the claim of Theorem 3.4 coincides with the equivalent result in "classical" algebraic geometry (see [At, Theorem 7]).

To prove the theorem we need the following lemmas.

Lemma 3.6. *Let $\pi\colon F\to X$ be an indecomposable vector bundle of rank r with open covering $U_1,...,U_s$ and transition maps M_{ij} according to Definition 1.6. If*

$$M_{12}(x)=D(\varphi_1,...,\varphi_r)(x)\odot A_\sigma$$

for some regular invertible functions $\varphi_1,...,\varphi_r\in\mathcal{O}^*(U_1\cap U_2)$ and a permutation $\sigma\in S_r$ and $M_{ij}(x)=E$ for all $\{i,j\}\neq\{1,2\}$, then there exists an isomorphic vector bundle $\pi\colon F'\to X$ with open covering $U_1,...,U_s$ and transition maps M'_{ij} according to Definition 1.6 such that

$$M'_{12}(x)=D(\varphi'_1,...,\varphi'_r)(x)\odot A_{(12...r)}$$

for some regular invertible functions $\varphi'_1,...,\varphi'_r\in\mathcal{O}^*(U_1\cap U_2)$ and $M'_{ij}(x)=E$ for all $\{i,j\}\neq\{1,2\}$.

Proof. As F is indecomposable σ must by a single cycle. Hence there exists $\varrho\in S_r$ such that $\varrho\sigma\varrho^{-1}=(12...r)$. We will apply Lemma 1.18 to show that we can replace $M_{12}(x)$ by $M'_{12}(x):=A_\varrho\odot D(x)\odot A_{\varrho^{-1}}\odot A_{(12...r)}$ without changing the isomorphism class of F: We set $E_i(x):=A_\varrho$ for all x and all i and obtain

$$\begin{aligned}A_\varrho\odot(D(x)\odot A_\sigma)&=(A_\varrho\odot D(x)\odot A_{\varrho^{-1}}\odot A_{(12...r)})\odot A_\varrho,\\ A_\varrho\odot E&=E\odot A_\varrho,\\ &\vdots\\ A_\varrho\odot E&=E\odot A_\varrho.\end{aligned}$$

This proves the claim. □

Lemma 3.7. *Let $\pi\colon F\to X$ be a vector bundle of rank r with open covering $U_1,...,U_s$ and transition maps M_{ij} according to Definition 1.6. Moreover, let all sets U_i be connected, let for all i and j the intersections $U_i\cap U_j$ be connected as well, and let the sets U_i be numbered consecutively. If*

$$M_{12}(x)=D(\varphi_1,...,\varphi_r)(x)\odot A_{(12...r)}$$

for some regular invertible functions $\varphi_1,...,\varphi_r\in\mathcal{O}^*(U_1\cap U_2)$ and $M_{ij}(x)=E$ for all $\{i,j\}\neq\{1,2\}$, then there exists an isomorphic vector bundle $\pi\colon F'\to X$ with open covering $U_1,...,U_s$ and transition maps M'_{ij} according to Definition 1.6 such that

$$M'_{12}(x)=D(\varphi',0,...,0)(x)\odot A_{(12...r)}$$

for some regular invertible function $\varphi'\in\mathcal{O}^*(U_1\cap U_2)$ and $M_{ij}(x)=E$ for all $\{i,j\}\neq\{1,2\}$.

Proof. We apply Lemma 1.18. For $i=1,...,r$ let α_i be the slope of φ_i and let L be the (lattice) length of our curve X. For $i=2,...,r$ we set $\delta_i:=\sum_{j=i}^{r}(j-i+1)\cdot\alpha_j$. Moreover, we define $\varphi':=\varphi_1+...+\varphi_r-\delta_2 L$. Note that if we are given a regular invertible function $\psi\in\mathcal{O}^*(U_i\cap U_j)$ there is a unique regular invertible function $\widetilde{\psi}\in\mathcal{O}^*(U_i)$ such that $\widetilde{\varphi}|_{U_i\cap U_j}=\varphi$. Hence we can extend our regular invertible functions $\varphi_1,...,\varphi_r$ along the chain $U_2,U_3,...,U_s,U_1$ to any of the sets $U_1,...,U_s$. Note that on $U_1\cap U_2$ the extension of φ_i to U_2 and the extension of φ_i to U_1 differ exactly by $\alpha_i L$. We use these continuations to define the maps E_i as

$$E_i(x):=D(\widetilde{\varphi}_2+...+\widetilde{\varphi}_r-\delta_2 L,\widetilde{\varphi}_3+...+\widetilde{\varphi}_r-\delta_3 L,...,\widetilde{\varphi}_r-\delta_r L,0),$$

where for entries of E_i the map $\widetilde{\varphi}_j$ denotes the continuation of φ_j to U_i. Hence we obtain on $U_1\cap U_2$:

$$\begin{aligned}E_2\odot M_{12}&=D(\widetilde{\varphi}_2+...+\widetilde{\varphi}_r-\delta_2 L,...,\widetilde{\varphi}_r-\delta_r L,0)\odot(D(\varphi_1,...,\varphi_r)\odot A_{(12...r)})\\&=D(\varphi_2+...+\varphi_r-\delta_2 L,...,\varphi_r-\delta_r L,0)\odot(D(\varphi_1,...,\varphi_r)\odot A_{(12...r)})\\&=D(\varphi_1+...+\varphi_r-\delta_2 L,\varphi_2+...+\varphi_r-\delta_3 L,...,\varphi_{r-1}+\varphi_r-\delta_r L,\varphi_r)\odot A_{(12...r)}\end{aligned}$$

and

$$\begin{aligned}M'_{12}\odot E_1&=(D(\varphi_1+...+\varphi_r-\delta_2 L,0,...,0)\odot A_{(12...r)})\\&\quad\odot D(\widetilde{\varphi}_2+...+\widetilde{\varphi}_r-\delta_2 L,...,\widetilde{\varphi}_r-\delta_r L,0)\\&=(D(\varphi_1+...+\varphi_r-\delta_2 L,0,...,0)\odot A_{(12...r)})\\&\quad\odot D(\varphi_2+...+\varphi_r-\delta_3 L,...,\varphi_r-\delta_{r-1} L,0)\\&=D(\varphi_1+...+\varphi_r-\delta_2 L,\varphi_2+...+\varphi_r-\delta_3 L,...,\varphi_{r-1}+\varphi_r-\delta_r L,\varphi_r)\odot A_{(12...r)}.\end{aligned}$$

The other conditions are trivially fulfilled as $E_i|_{U_i\cap U_{i+1}}=E_{i+1}|_{U_i\cap U_{i+1}}$ for all $i\neq 1$. This proves the claim. □

Lemma 3.8. *Let $\pi:F\to X$ and $\pi:F'\to X$ be vector bundles of rank r and degree d with open covering $U_1,...,U_s$ and transition maps M_{ij} respectively M'_{ij} according to Definition 1.6. Moreover, let all sets U_i be connected, let for all i and j the intersections $U_i\cap U_j$ be connected as well, and let the sets U_i be numbered consecutively. If*

$$M_{12}(x)=D(\varphi,0,...,0)(x)\odot A_{(12...r)}\quad\textit{and}\quad M'_{12}(x)=D(\varphi+cL,0,...,0)(x)\odot A_{(12...r)}$$

for some regular invertible function $\varphi\in\mathcal{O}^*(U_1\cap U_2)$ and the (lattice) length L of our curve X, and $M_{ij}(x)=M'_{ij}(x)=E$ for all $\{i,j\}\neq\{1,2\}$, then F and F' are isomorphic if and only if c is an integer multiple of $\gcd(r,d)$.

Proof. By Lemma 1.18, F and F' are isomorphic if and only if for all $i=1,\dots,s$ there exists a map $E_i\colon U_i\to G(r)$ such that for all i the equation $E_{i+1}(x)\odot M_{i,i+1}(x)=M'_{i,i+1}(x)\odot E_i(x)$ holds for all $x\in U_i\cap U_{i+1}$. As $M_{i,i+1}$ is trivial for all $i\neq 1$ these equations imply that $E_i|_{U_i\cap U_{i+1}}=E_{i+1}|_{U_i\cap U_{i+1}}$ for all $i\neq 1$. Hence F and F' are isomorphic if and only if there exist a permutation $\tau\in S_r$ and regular invertible functions $\psi_1,\dots,\psi_r\in\mathcal{O}^*(U_1\cap U_2)$ with continuations $\tilde{\psi}_1,\dots,\tilde{\psi}_r$ to all sets $U_1,\dots,U_s$ along the chain $U_2,U_3,\dots,U_s,U_1$ such that

$$(D(\tilde{\psi}_1,\dots,\tilde{\psi}_r)\odot A_\tau)\odot(D(\varphi,0,\dots,0)\odot A_\sigma) \\ =(D(\varphi+cL,0,\dots,0)\odot A_\sigma)\odot(D(\tilde{\psi}_1,\dots,\tilde{\psi}_r)\odot A_\tau)$$

holds on $U_1\cap U_2$. In particular, the last equation implies that $A_\tau\odot A_\sigma=A_\sigma\odot A_\tau$ and hence $\tau=\sigma^k$ for some $k\in\mathbb{Z}$. Thus F and F' are isomorphic if and only if there exist $k\in\mathbb{Z}$ and $\psi_1,\dots,\psi_r$ as above such that

$$D(\tilde{\psi}_1,\dots,\tilde{\psi}_k,\tilde{\psi}_{k+1}+\varphi,\tilde{\psi}_{k+2},\dots,\tilde{\psi}_r)\odot A_{\sigma^{k+1}}=D(\varphi+cL+\tilde{\psi}_r,\tilde{\psi}_1,\dots,\tilde{\psi}_{r-1})\odot A_{\sigma^{k+1}}.$$

Let α_i be the slope of ψ_i. Then on $U_1\cap U_2$ the continuation of ψ_i to U_2 and the continuation of ψ_i to U_1 differ exactly by $\alpha_i L$. Hence we obtain the system of equations

$$\begin{aligned}
\psi_1&=\varphi+cL+\psi_r+\alpha_r L,\\
\psi_2&=\psi_1+\alpha_1 L,\\
&\vdots\\
\psi_k&=\psi_{k-1}+\alpha_{k-1}L,\\
\psi_{k+1}+\varphi&=\psi_k+\alpha_k L,\\
\psi_{k+2}&=\psi_{k+1}+\alpha_{k+1}L,\\
&\vdots\\
\psi_r&=\psi_{r-1}+\alpha_{r-1}L.
\end{aligned}$$

In particular, we can conclude that $\alpha_1=\dots=\alpha_k$ and $\alpha_{k+1}=\dots=\alpha_r$. Hence F and F' are isomorphic if and only if there exist α_1,α_r and $k\in\mathbb{Z}$ such that

$$-c=(r-k)\cdot\alpha_r+k\cdot\alpha_1 \quad\text{and}\quad \alpha_1=-d+\alpha_r,$$

or equivalently if and only if there exist α_r and $k\in\mathbb{Z}$ with

$$-c=r\alpha_r-k\cdot d.$$

This finishes the proof. □

Proof of Theorem 3.4. Let $\pi\colon F\to X$ and $\pi\colon F'\to X$ be indecomposable vector bundles of rank r and degree d. We may assume by Remark 1.9 that F and F' fulfill Definition 1.6 with the same open covering $U_1,\dots,U_s$ and transition maps M_{ij} respectively M'_{ij}. Again, we may assume that all sets U_i are connected, that for all i and j the intersections $U_i\cap U_j$ are connected as well, and that the sets U_i are numbered consecutively. Moreover, by Lemma 3.3 we may assume that M_{12} and M'_{12} are the only non-trivial transition maps. Applying Lemmas 3.6 and 3.7 consecutively we can furthermore assume that $M_{12}(x)=D(\varphi,0,\dots,0)(x)\odot A_{(12\dots r)}$ and $M'_{12}(x)=D(\varphi',0,\dots,0)(x)\odot A_{(12\dots r)}$. As F and F' are vector bundles of degree d the affine linear maps φ and φ' both must have slope $-d$. Hence we have $\varphi'=\varphi+cL$, where $c\in\mathbb{R}$ and L is the (lattice) length of the curve X. Thus F and F' are isomorphic if and only if c is an integer multiple of $\gcd(r,d)$ by Lemma 3.8. □

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