Chernoff-Type Bounds for the Gaussian Error Function

Seok-Ho Chang, *Member, IEEE,* Pamela C. Cosman, *Fellow, IEEE,* and Laurence B. Milstein, *Fellow, IEEE*

***Abstract*—We study single-term exponential-type bounds (also known as Chernoff-type bounds) on the Gaussian error function. This type of bound is analytically the simplest such that the performance metrics in most fading channel models can be expressed in a concise closed form. We derive the conditions for a general single-term exponential function to be an upper or lower bound on the Gaussian error function. We prove that there exists no tighter single-term exponential upper bound beyond the Chernoff bound employing a factor of one-half. Regarding the lower bound, we prove that the single-term exponential lower bound of this letter outperforms previous work. Numerical results show that the tightness of our lower bound is comparable to that of previous work employing eight exponential terms.**

***Index Terms*—Bounds, error function, exponential, Gaussian Q-function.**

I. Introduction

THE Gaussian Q-function, $Q(x)$, or, equivalently, the complementary error function, erfc(x), play an important role in the performance analysis of many communication systems. However, there is no known closed-form expression for $Q(x)$ [1, Ch. 4.1.1] and the analytical problems associated with it have provoked much interest in finding its bounds or approximations for decades [2]–[10].

System performance such as average bit, symbol, or block error probabilities in fading channels typically include the expectation of $Q(x)$ or its powers with regard to a random variable that characterizes the fading channel (i.e., $E[Q^N(x)]$, where N is a positive integer) [1]. Therefore, the bounds or approximations of $Q(x)$ need to be both tight and analytically simple enough to express the above performance metrics in closed-form. For example, the upper and lower bounds given in [2]–[5] are quite tight as shown in [6, Fig. 1], but they are not easily integrable with regard to random variables representing fading channels.

An exponential-type bound on $Q(x)$ was presented in [11]. The bound given in [11, eq. (8)] employs a series of exponential terms, and a sufficient condition for the series to be an upper or lower bound on the error function is presented. This bound has received attention [12]–[15] because exponential functions are easily integrable with regard to a wide variety of fading channel models, such as Rayleigh, Rician and Nakagami-m channels (m is the Nakagami fading parameter) [1, Table 2.2]. In fact, the Chernoff upper bound, which is known to be much tighter than the Chebyshev upper bound, also has a single exponential-type expression for $Q(x)$ [16].

This letter studies the single exponential-type or "Chernoff-type" [17] bound on $Q(x)$. We recognize that the bound in [11] can become arbitrarily close to the exact $Q(x)$ by increasing the number of terms in the series, as do other series representations of $Q(x)$ [7][8]. However, the efficiency of these series is determined by the tightness combined with the number of terms used in the series [8]. Note that the single exponential-type bound on the first-order Marcum Q-function is presented in [18, eqs. (3) and (4)], and to our knowledge, this type of bound on the Gaussian Q-function has not been studied.

Paper approved by A. Zanella, the Editor for Wireless Systems of the IEEE Communications Society. Manuscript received February 27, 2010; revised July 6, 2010.

The authors are with the Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, 92093–0407, USA (e-mail: {s4chang, pcosman}@ucsd.edu, milstein@ece.ucsd.edu).

This research was supported in part by the Office of Naval Research under grant number N000140810081, and by the National Science Foundation under grant number CCF-0915727.

Digital Object Identifier 10.1109/TCOMM.2011.072011.100049

II. Preliminaries

The Gaussian Q-function is defined as

$$Q(x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right), \qquad x \geq 0 \tag{1}$$

where

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt, \qquad x \geq 0 \tag{2}$$

is the complementary error function. In this letter, we analyze erfc(x), but the results can be directly applied to $Q(x)$. For the Gaussian distribution, the Chernoff bound is given by $\text{erfc}(x) \leq 2\exp\left(-x^2\right)$ [16]. Based on the results of Jacobs [19] and Hellman and Raviv [20], a factor of one-half can be applied, and the bound becomes $\text{erfc}(x) \leq \exp\left(-x^2\right)$ [21, Appendix 4B] (this bound also can be derived in other ways [17, eq. (2)][22, eq. (2.122)][23]). As given, the bound has a single-term exponential expression, and this simple expression makes it possible to analyze fading communication systems in a concise closed form [1, Table 2.2] without numerical integration. Consider the following general exponential form:

$$f(x) = \alpha\exp\left(-\beta x^2\right), \qquad x \geq 0 \tag{3}$$

where $\alpha > 0$ and $\beta > 0$ are real numbers. In the following, we derive the conditions for this to be the upper or lower bound on erfc(x).

III. Upper Bound

Theorem 1: The function $f(x)$ is an upper bound of erfc(x) if and only if $\alpha \geq 1$ and $0 < \beta \leq 1$.

Proof: We define a function $g(x)$:

$$g(x) = \alpha\exp\left(-\beta x^2\right) - \text{erfc}(x), \qquad x \geq 0 \tag{4}$$

for $\alpha > 0$ and $\beta > 0$. We will prove that $g(x) \geq 0$ if and only if $\alpha \geq 1$ and $0 < \beta \leq 1$.

i) We first show that $g(x) \geq 0$ if $\alpha \geq 1$ and $0 < \beta \leq 1$. From (4),

$$\begin{aligned} \frac{dg(x)}{dx} &= -2\alpha\beta x \exp\left(-\beta x^2\right) + \frac{2}{\sqrt{\pi}}\exp\left(-x^2\right) \\ &= \exp\left(-x^2\right) v\left(x\right), \qquad x \geq 0 \end{aligned} \tag{5}$$

where

$$v(x) = -2\alpha\beta x \exp\left(\left(1-\beta\right) x^2\right) + \frac{2}{\sqrt{\pi}}. \tag{6}$$

From (6), $v(0) = 2/\sqrt{\pi}$. In addition, if $0 < \beta \leq 1$, we have $\lim_{x\to\infty} v(x) = -\infty$ and

$$\frac{dv(x)}{dx} = -2\alpha\beta \exp\left(\left(1-\beta\right) x^2\right) \left\{2\left(1-\beta\right) x^2 + 1\right\} < 0 \tag{7}$$

Hence, $v(x)$ is zero only at a point $x_0 \in (0, \infty)$. That is, $v(x) > 0$ for $0 \leq x < x_0$, and $v(x) < 0$ for $x > x_0$. Therefore, from (5), we have $dg(x)/dx > 0$ for $0 \leq x < x_0$, and $dg(x)/dx < 0$ for $x > x_0$. If $0 < \beta \leq 1$, from (4), we also have $\lim_{x\to\infty} g(x) = 0$ If $\alpha \geq 1$, $g(0)$ satisfies $g(0) = \alpha - 1 \geq 0$ From these, it follows that $g(x) \geq 0$ if $\alpha \geq 1$ and $0 < \beta \leq 1$.

ii) We next show that $g(x) \geq 0$ does not hold if $\beta > 1$. Using integration by parts, it can be shown that [22, eq. (2.121)]

$$\text{erfc}(x) \geq \frac{1}{\sqrt{\pi}x}\left(1 - \frac{1}{2x^2}\right)\exp\left(-x^2\right), \quad x > 0. \tag{8}$$

Let ρ denote the ratio of the lower bound given by (8) to the function $f(x)$ for $x > 0$. Then, it can be shown that ρ is given by

$$\rho = \frac{1}{\alpha\sqrt{\pi}}\left(\frac{2x^2-1}{2x^3}\right)\exp\left(\left(\beta-1\right)x^2\right), \quad x > 0. \tag{9}$$

If $\beta > 1$, from L'Hospital's rule, we have

$$\begin{aligned} \lim_{x\to\infty} \rho = \lim_{x\to\infty} \frac{1}{\alpha\sqrt{\pi}} &\left(\frac{2\left(\beta-1\right)x^3 - \left(\beta-3\right)x}{3x^2}\right) \\ &\times \exp\left(\left(\beta-1\right)x^2\right) = \infty \end{aligned} \tag{10}$$

which shows that $f(x)$ is not the upper bound.

iii) If $\alpha < 1$, from (4), we have $g(0) = \alpha - 1 < 0$.

From i), ii), and iii), it is seen that $g(x) \geq 0$ if and only if $\alpha \geq 1$ and $0 < \beta \leq 1$.

□

Corollary 1: The function $f(x)$ is the tightest upper bound on erfc(x) if and only if $\alpha = 1$ and $\beta = 1$.

Proof: Let $f(x, \alpha, \beta)$ denote the function given by (3). Then, it can be easily shown that $f(x, \alpha, \beta)$ is strictly increasing in α for $x \geq 0$, and strictly decreasing in β for $x > 0$. Therefore, smaller α or larger β tightens the upper bound. From Theorem 1, it is seen that $f(x)$ is the tightest if and only if $\alpha = 1$ and $\beta = 1$.

□

Corollary 1 indicates that the Chernoff bound with a factor of one-half, $\text{erfc}(x) \leq \exp\left(-x^2\right)$, is the tightest upper bound on erfc(x) among any upper bounds which can be expressed in the form of (3).

IV. Lower Bound

Theorem 2: The function $f(x)$ is a lower bound of erfc(x) if

$$\beta > 1 \quad \text{and} \quad 0 < \alpha \leq \sqrt{\frac{2e}{\pi}}\frac{\sqrt{\beta-1}}{\beta}. \tag{11}$$

Proof: Suppose that (11) is satisfied. We will show that $g(x)$, given by (4), satisfies $g(x) \leq 0$. From (7), it is seen that $dv(x)/dx = 0$ only at $x = x^* \triangleq 1/\sqrt{2\left(\beta-1\right)}$ in the range of $x \geq 0$. From (6),

$$\begin{aligned} \frac{d^2v(x)}{dx^2} &= 4\alpha\beta\left(\beta-1\right)x\exp\left(\left(1-\beta\right)x^2\right) \\ &\quad \times \left\{2\left(1-\beta\right)x^2 + 3\right\} \end{aligned} \tag{12}$$

and thus,

$$\frac{d^2v(x^*)}{dx^2} = 4\sqrt{2}\alpha\beta\sqrt{\beta-1}\exp\left(-\frac{1}{2}\right) > 0. \tag{13}$$

Hence, $v(x)$ has its minimum at $x = x^*$, and from (6), $v\left(x^*\right)$ is given by

$$v\left(x^*\right) = -\sqrt{2}\alpha\beta\sqrt{\frac{1}{\beta-1}}\exp\left(-\frac{1}{2}\right) + \frac{2}{\sqrt{\pi}}. \tag{14}$$

From the supposition that α satisfies (11), we have $v(x) \geq v\left(x^*\right) \geq 0$. Hence, from (5), $g(x)$ is a non-decreasing function. We define a function $w\left(\beta\right)$:

$$w(\beta) = \sqrt{\frac{2e}{\pi}}\frac{\sqrt{\beta-1}}{\beta}, \qquad \beta > 1. \tag{15}$$

It can be shown that

$$\frac{dw(\beta)}{d\beta} = \sqrt{\frac{2e}{\pi}}\left(\frac{-\beta+2}{\beta^2\sqrt{\beta-1}}\right) \tag{16}$$

$$\frac{d^2w(\beta)}{d\beta^2} = \sqrt{\frac{e}{2\pi}}\left(\frac{3\beta^2-12\beta+8}{2\beta^3(\beta-1)^{3/2}}\right). \tag{17}$$

From (16), it is seen that $dw(\beta)/d\beta = 0$ only at $\beta = \beta^* \triangleq 2$ in the range of $\beta > 1$. From (17), we also have

$$\frac{d^2w(\beta^*)}{d\beta^2} = -\sqrt{\frac{e}{32\pi}} < 0. \tag{18}$$

Hence, $w(\beta)$ has its maximum at $\beta = \beta^*$, and $w\left(\beta^*\right)$ is given by

$$w\left(\beta^*\right) = \sqrt{\frac{e}{2\pi}} < 1. \tag{19}$$

From $w(\beta) \leq w(\beta^*) < 1$ and the supposition of $\alpha \leq w(\beta)$, it follows that $\alpha < 1$. Hence, from (4), we have $g(0) = \alpha - 1 < 0$. By the supposition of $\beta > 1$, we also have $\lim_{x\to\infty} g(x) = 0$. Lastly, since $g(x)$ is non-decreasing as stated below (14), it follows that $g(x) \leq 0$.

□

We examine the tightness of the lower bound achieved by the sufficient condition of Theorem 2. Note that from (3), a larger α tightens the lower bound, and thus from Theorem 2, we consider $\alpha = w(\beta)$, which is defined in (15). It is clear that $w(1) = 0$, $w(2) = \sqrt{e/2\pi}$, and from (16), $w(\beta)$ is nondecreasing for $1 \le \beta \le 2$. From this, it can be shown that the inverse of $\alpha = w(\beta)$ for $1 \le \beta \le 2$ is

$$\beta = \frac{2}{1+\sqrt{1-\frac{2\pi}{e}\alpha^2}} \qquad \text{for} \quad 0 \le \alpha \le \sqrt{\frac{e}{2\pi}}. \tag{20}$$

For comparison purposes, we also consider a single-term exponential lower bound provided by [11]. From [11, eqs. (8), (9), and (26)], for $x > 0$, the lower bound employing a series of exponential terms can be written as

$$h(x, \theta_1, \cdots, \theta_{N-1}) = \sum_{i=2}^{N} \frac{2}{\pi}(\theta_i - \theta_{i-1}) \exp\left(-\frac{x^2}{\sin^2 \theta_{i-1}}\right) \tag{21}$$

where $\theta_1, \cdots, \theta_{N-1}$ are arbitrary values satisfying $0 \le \theta_1 \le \cdots \le \theta_{N-1} \le \theta_N = \pi/2$ [11, eq. (8)]. From (21), a single exponential term lower bound (i.e., $N = 2$) is given by

$$h(x, \theta_1) = \left(1 - \frac{2}{\pi}\theta_1\right) \exp\left(-\frac{x^2}{\sin^2 \theta_1}\right), \quad x > 0 \tag{22}$$

for $0 \le \theta_1 \le \frac{\pi}{2}$. Let $\delta = 1 - 2\theta_1/\pi$ and $\lambda = 1/\sin^2 \theta_1$. Then, it can be shown that (22) is expressed as

$$h(x) = \delta \exp\left(-\lambda x^2\right), \quad x > 0 \tag{23}$$

where

$$\lambda = \sec^2 \frac{\pi}{2}\delta \quad \text{for } 0 \le \delta \le 1. \tag{24}$$

Note that from (3) and (23), smaller exponents β and λ tighten the lower bound $f(x)$ and $h(x)$ for given α and δ, respectively. In the following, we compare the magnitude of β and λ given by (20) and (24), respectively. We will show that for $0 < x \le \sqrt{e/2\pi}$,

$$\sec^2 \frac{\pi}{2}x > \frac{2}{1+\sqrt{1-\frac{2\pi}{e}x^2}}. \tag{25}$$

It can be shown that (25) is equivalent to

$$\sqrt{1 - \frac{2\pi}{e}x^2} > \cos \pi x. \tag{26}$$

i) It is obvious that for $1/2 < x \le \sqrt{e/2\pi}$, we have $\cos \pi x < 0$ and thus (26) holds.
ii) For $0 < x \le 1/2$, $\cos \pi x \ge 0$ and thus (26) is equivalent to

$$\begin{aligned} & 1 - \frac{2\pi}{e}x^2 > \cos^2 \pi x \\ \Leftrightarrow & \left(\sin \pi x - \sqrt{\frac{2\pi}{e}}x\right)\left(\sin \pi x + \sqrt{\frac{2\pi}{e}}x\right) > 0 \\ \Leftrightarrow & \sin \pi x > \sqrt{\frac{2\pi}{e}}x. \end{aligned} \tag{27}$$

We define a function $p(x)$:

$$p(x) = \sin \pi x - \sqrt{\frac{2\pi}{e}}x. \tag{28}$$

From (28), we have

$$p(0) = 0 \quad \text{and} \quad p\left(\frac{1}{2}\right) = 1 - \sqrt{\frac{\pi}{2e}} > 0. \tag{29}$$

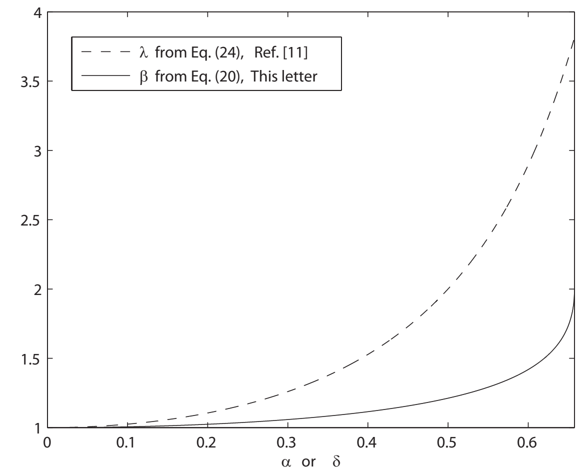

Fig. 1. Exponents β and λ versus α or δ, in the functions of $\alpha \exp\left(-\beta x^2\right)$ of this letter and $\delta \exp\left(-\lambda x^2\right)$ of [11].

For $0 < x \le 1/2$, $p(x)$ is strictly concave since

$$\frac{d^2 p(x)}{dx^2} = -\pi^2 \sin \pi x < 0. \tag{30}$$

From (29) and (30), it is seen that for $0 < x \le 1/2$, $p(x) > 0$ is satisfied, and thus (26) holds for $0 < x \le 1/2$. As a result, from i) and ii), (26) is satisfied for $0 < x \le \sqrt{e/2\pi}$. We have proved the single exponential lower bound achieved by Theorem 2 is tighter than that provided by [11]. In Fig. 1, β and λ are plotted versus α or δ using (20) and (24).

V. Numerical Evaluation

In this section, we numerically evaluate how tightly a single exponential function, given by (3), bounds erfc(x). For the lower bound, from Theorem 2, we consider

$$f(x, \beta) = \sqrt{\frac{2e}{\pi}} \frac{\sqrt{\beta - 1}}{\beta} \exp\left(-\beta x^2\right), \quad x > 0 \tag{31}$$

for $\beta \ge 1$. We also consider the lower bound in [11], $h(x, \theta_1, \cdots, \theta_{N-1})$, which is given by (21). To compare the tightness of $f(x, \beta)$ and $h(x, \theta_1, \cdots, \theta_{N-1})$, we use the accuracy metric given in [11, eq. (13)], where the optimal parameters β and $[\theta_1, \cdots, \theta_{N-1}]$ of (31) and (21) are chosen as

$$\begin{aligned} \beta_{opt} &= \arg\min_{\beta} \frac{1}{R_2 - R_1} \int_{R_1}^{R_2} \frac{|f(x, \beta) - \text{erfc}(x)|}{\text{erfc}(x)} dx, \\ [\theta_1, \cdots, \theta_{N-1}]_{opt} &= \arg\min_{\theta_1, \cdots, \theta_{N-1}} \frac{1}{R_2 - R_1} \\ &\quad \times \int_{R_1}^{R_2} \frac{|h(x, \theta_1, \cdots, \theta_{N-1}) - \text{erfc}(x)|}{\text{erfc}(x)} dx. \end{aligned} \tag{32}$$

From (32), the optimal parameters are determined to minimize the integral of the relative error in the range of R_1 to R_2 [11] (see [8, eq. (22)] for the definition of relative error). Fig. 2 depicts the numerical evaluation of the lower bounds given by (21) and (31) with the optimal parameters obtained from (32), where $20 \log_{10} R_1 = 0$ (dB) and $20 \log_{10} R_2 = 12$ (dB) is

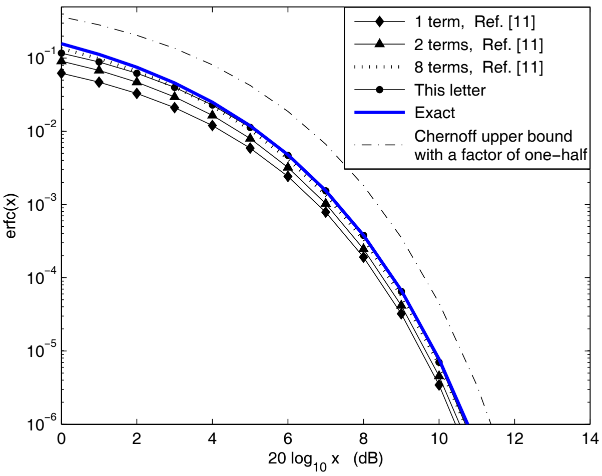

Fig. 2. Lower bound of this letter using single exponential term, $f(x, \beta)$, given by (31), and lower bound of [11] using the series of exponential terms, $h(x, \theta_1, \cdots, \theta_{N-1})$, given by (21) for pure Gaussian error function. The Chernoff upper bound with a factor of one-half is also plotted.

TABLE I
LOWER BOUND OF THIS LETTER USING SINGLE EXPONENTIAL TERM: $f(x, \beta)$, GIVEN BY (31), FOR PURE GAUSSIAN ERROR FUNCTION, AND $f_{STC}(\gamma_s, \beta)$, GIVEN BY (35), FOR AVERAGE PEP OF STCS IN FADING CHANNELS

	β_{opt}	Integral of the relative error in eq. (32) or (37)
(a) Pure Gaussian error function $f(x, \beta)$	1.080	0.1116
(b) Average PEP of STCs in fading channels $f_{STC}(\gamma_s, \beta)$	1.240	0.1027

TABLE II
LOWER BOUND OF [11] USING THE SERIES OF EXPONENTIAL TERMS: $h(x, \theta_1, \cdots, \theta_{N-1})$, GIVEN BY (21), FOR PURE GAUSSIAN ERROR FUNCTION

Number of terms	$[\theta_1, \cdots, \theta_{N-1}]_{opt}$	Integral of the relative error in eq. (32)
1	[1.280]	0.5465
2	[1.125, 1.360]	0.3791
4	[0.955, 1.155, 1.300, 1.425]	0.2348
6	[0.865, 1.035, 1.160, 1.265, 1.360, 1.455]	0.1702
7	[0.835, 0.995, 1.115, 1.215, 1.300, 1.380, 1.465]	0.1497
8	[0.805, 0.960, 1.075, 1.170, 1.250, 1.325, 1.395, 1.470]	0.1336

used. The integral of the relative error in (32) (i.e., the average relative error) with the optimal parameters is also listed in Table I (a) and Table II. From Fig. 2, Table I (a) and Table II, it is seen that the single exponential lower bound of this letter outperforms that of [11], as proved in Section IV. Further, it is observed that the tightness of the lower bound of this letter is comparable to that of the lower bound in [11] employing eight exponential terms.

As an example of an application, we next consider the average pairwise error probability (PEP) of space-time codes (STCs) in the same scenario as in [11, Sec. VII. A] and [24, Sec. IV. B], where a four-state quadrature phase shift keying (QPSK), two transmit and one receive antennas, and an independent Rayleigh fading channel are assumed. The average PEP, $P(\mathbf{X} \rightarrow \hat{\mathbf{X}})$, is given by [11, eq. (36)] as $P(\mathbf{X} \rightarrow \hat{\mathbf{X}}) = \frac{1}{2} E\left[\mathrm{erfc}\left(\sqrt{\rho}\right)\right]$, where ρ is the instantaneous signal-to-noise ratio (SNR) per symbol. Hence, from (21), the lower bound on $P(\mathbf{X} \rightarrow \hat{\mathbf{X}})$ provided by [11] is given by

$$P(\mathbf{X} \rightarrow \hat{\mathbf{X}}) \geq \frac{1}{2} \sum_{i=2}^{N} E\left[h\left(\sqrt{\rho}, \theta_1, \cdots, \theta_{N-1}\right)\right] = \sum_{i=2}^{N} \frac{1}{\pi}(\theta_i - \theta_{i-1}) \Phi\left(-\frac{1}{\sin^2 \theta_{i-1}}\right) \quad (33)$$

where $\Phi(s) \triangleq E\left[\exp\left(s\rho\right)\right]$ is the moment-generating function associated with the random variable ρ [11, eq. (33)]. In this scenario, $\Phi(s)$ is given by $\Phi(s) = (1 - s\gamma_s)^{-2}$ [11, eq. (35)] , where $\gamma_s = E_s/N_0$ is the SNR per symbol. Hence, from (33), the lower bound of $P(\mathbf{X} \rightarrow \hat{\mathbf{X}})$ provided by [11], $h_{STC}(\gamma_s, \theta_1, \cdots, \theta_{N-1})$, is given by

$$P(\mathbf{X} \rightarrow \hat{\mathbf{X}}) \geq \sum_{i=2}^{N} \frac{1}{\pi}(\theta_i - \theta_{i-1}) \left(1 + \frac{1}{\sin^2 \theta_{i-1}} \gamma_s\right)^{-2} \triangleq h_{STC}(\gamma_s, \theta_1, \cdots, \theta_{N-1}). \quad (34)$$

In the same way as in (33) and (34), from (31), the lower bound of $P(\mathbf{X} \rightarrow \hat{\mathbf{X}})$ provided by this letter, $f_{STC}(\gamma_s, \beta)$, is derived as

$$P(\mathbf{X} \rightarrow \hat{\mathbf{X}}) \geq \frac{1}{2} E\left[f(\sqrt{\rho}, \beta)\right] = \sqrt{\frac{e}{2\pi}} \frac{\sqrt{\beta - 1}}{\beta} \left(1 + \beta\gamma_s\right)^{-2} \triangleq f_{STC}(\gamma_s, \beta). \quad (35)$$

The exact expression for the average PEP is given by [24, eq. (16)] as:

$$P(\mathbf{X} \rightarrow \hat{\mathbf{X}}) = \frac{1}{2}\left\{1 - \sqrt{\frac{\gamma_s}{1 + \gamma_s}} \sum_{k=0}^{1} \binom{2k}{k} \left(\frac{1}{4(1 + \gamma_s)}\right)^k\right\}. \quad (36)$$

Similar to (32), the optimal parameters is obtained as (37) on the next page. Fig. 3 depicts the numerical evaluation of the lower bounds given by (34) and (35) with the optimal parameters obtained from (37), where $10 \log_{10} R_1 = -4$ (dB) and $10 \log_{10} R_2 = 20$ (dB) is used (note that $10 \log_{10}(\cdot)$ is used for decibel scale). The integral of the relative error in (37) with the optimal parameters is also listed in Table I (b) and Table III. From Fig. 3, Table I (b) and Table III, it is also seen that the single exponential lower bound of this letter outperforms that of [11]. Further, the tightness of the lower bound of this letter is comparable to that of the lower bound in [11] employing eight exponential terms.

In the computation of the pure Gaussian error function, from Tables I and II, the optimal exponents for the single-term exponential lower bound of this letter and [11] are given by $\beta_{opt} = 1.080$ and $\lambda_{opt} = 1/\sin^2\left([\theta_1]_{opt}\right) = 1.090$, where the first expression for λ_{opt} follows from the one below (22), and the second expression follows from $[\theta_1]_{opt} = 1.280$ in Table II. In the computation of the average PEP of STCs in fading channels, from Tables I and III, the optimal exponents are given by $\beta_{opt} = 1.240$ and $\lambda_{opt} = 1/\sin^2\left([\theta_1]_{opt}\right) = 1.345$. Note that even for fading channels, the optimal exponents are not very much larger than 1 due to the fact that the error

$$\beta_{opt} = \arg\min_{\beta} \frac{1}{R_2 - R_1} \int_{R_1}^{R_2} \frac{\left| f_{STC}(\gamma_s, \beta) - P(\mathbf{X} \to \hat{\mathbf{X}}) \right|}{P(\mathbf{X} \to \hat{\mathbf{X}})} d\gamma_s$$

$$[\theta_1, \cdots, \theta_{N-1}]_{opt} = \arg\min_{\theta_1, \cdots, \theta_{N-1}} \frac{1}{R_2 - R_1} \int_{R_1}^{R_2} \frac{\left| h_{STC}(\gamma_s, \theta_1, \cdots, \theta_{N-1}) - P(\mathbf{X} \to \hat{\mathbf{X}}) \right|}{P(\mathbf{X} \to \hat{\mathbf{X}})} d\gamma_s \quad (37)$$

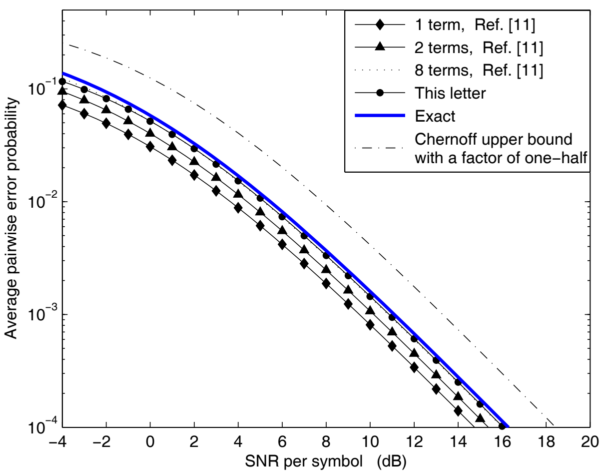

Fig. 3. Lower bound of this letter using single exponential term, $f_{STC}(\gamma_s, \beta)$, given by (35), and lower bound of [11] using the series of exponential terms, $h_{STC}(\gamma_s, \theta_1, \cdots, \theta_{N-1})$, given by (34) for average PEP of STCs in fading channels. The Chernoff upper bound with a factor of one-half is also plotted.

TABLE III
LOWER BOUND OF [11] USING THE SERIES OF EXPONENTIAL TERMS: $h_{STC}(\gamma_s, \theta_1, \cdots, \theta_{N-1})$, GIVEN BY (34), FOR AVERAGE PEP OF STCS IN FADING CHANNELS

Number of terms	$[\theta_1, \cdots, \theta_{N-1}]_{opt}$	Integral of the relative error in eq. (37)
1	[1.040]	0.4877
2	[0.840, 1.185]	0.3245
4	[0.655, 0.885, 1.090, 1.300]	0.1954
6	[0.560, 0.745, 0.900, 1.045, 1.195, 1.360]	0.1402
7	[0.530, 0.700, 0.840, 0.970, 1.100, 1.230, 1.375]	0.1229
8	[0.500, 0.655, 0.785, 0.905, 1.020, 1.135, 1.255, 1.390]	0.1094

function, erfc(x), roughly decays exponentially with x^2 (i.e., erfc$(x) \approx \exp\left(-x^2\right)/\sqrt{\pi}x$ for $x >> 1$).

The Chernoff upper bound with a factor of one-half, which is proved to be the tightest in Section III, is also plotted in Figs. 2 and 3. Compared to the lower bound, this single exponential upper bound is not tight. The advantage of employing the Chernoff upper bound, which is widely used in the literature, is focused on the simplicity of the expression. For example, the average block error probability in fading channels typically involves $E\left[Q^N(x)\right]$ with a high order of N [10], and the use of single-term exponential bound on $Q(x)$ instead of multiple-terms leads to a concise closed-form even for large N.

VI. CONCLUSION

One of the principal reasons for using the bounds on $Q(x)$ is to obtain a simple form that facilitates analysis of communication systems. In this letter, we studied single-term exponential-type bounds on $Q(x)$. This type of bound is analytically the simplest such that the performance metrics such as average bit, symbol, or block error probabilities can be expressed in a concise closed form for a wide variety of fading channel models. We derived the conditions for such functions to be upper or lower bounds on $Q(x)$. We proved that there exists no tighter single-term exponential upper bound beyond the Chernoff bound employing a factor of one-half. Regarding the lower bound, we proved that the single exponential lower bound achieved in this letter outperforms that provided by [11]. In fact, numerical results showed that the tightness of the single-term lower bound of this letter is comparable to that of the lower bound in [11] employing eight exponential terms.

REFERENCES

[1] M. K. Simon and M.-S. Alouini, *Digital Communication over Fading Channels*, 2nd edition. Wiley, 2005.

[2] Z. W. Birnbaum (1942), "An inequality for Mills' ratio," *Ann. of Math. Statist.*, vol. 13, pp. 245–246.

[3] Y. Komatu (1955), "Elementary inequalities for Mills' ratio," *Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs.*, vol. 4, pp. 69–70.

[4] H. O. Pollak (1956), "A remark on 'Elementary inequalities for Mills' ratio'," by Y. Komatu, *Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs.*, vol. 4, pp. 110.

[5] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. Dover Publications, 1965.

[6] P. O. Borjesson and C. E. Sundberg, "Simple approximations of the error function $Q(x)$ for communications applications," *IEEE Trans. Commun.*, vol. 27, no. 3, pp. 639–643, Mar. 1979.

[7] N. C. Beaulieu, "A simple series for personal computer computation of the error function $Q(.)$," *IEEE Trans. Commun.*, vol. 37, no. 9, pp. 989–991, Sep. 1989.

[8] C. Tellambura and A. Annamalai, "Efficient computation of erfc(x) for large arguments," *IEEE Trans. Commun.*, vol. 48, no. 4, pp. 529–532, Apr. 2000.

[9] G. K. Karagiannidis and A. S. Lioumpas, "An improved approximation for the Gaussian Q-function," *IEEE Commun. Lett.*, vol. 11, no. 8, pp. 644–646, Aug. 2007.

[10] Y. Isukapalli and B. D. Rao, "An analytically tractable approximation for the Gaussian Q-function," *IEEE Commun. Lett.*, vol. 12, no. 9, pp. 669–671, Sep. 2008.

[11] M. Chiani, D. Dardari, and M. K. Simon, "New exponential bounds and approximations for the computation of error probability in fading channels," *IEEE Trans. Wireless Commun.*, vol. 2, no. 4, pp. 840–845, July 2003.

[12] C. Ling, K. H. Li, and A. C. Kot, "Performance of space-time codes: Gallager bounds and weight enumeration," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3592–3610, Aug. 2008.

[13] K. T. Phan and C. Tellambura, "Receive antenna selection based on union-bound minimization using convex optimization," *IEEE Signal Process. Lett.*, vol. 14, no. 9, pp. 609–612, Sep. 2007.

[14] F. Heliot, M. Ghavami, and M. R. Nakhai, "An accurate closed-form approximation of the average probability of error over a log-normal fading channel," *IEEE Trans. Wireless Commun.*, vol. 7, no. 5, pp. 1495–1500, May 2008.

[15] H. A. Suraweera, D. S. Michalopoulos, and G. K. Karagiannidis, "Performance of distributed diversity systems with a single amplify-and-forward relay," *IEEE Trans. Veh. Technol.*, vol. 58, no. 5, pp. 2603–2608, June 2009.

[16] J. G. Proakis, *Digital Communication*, 3rd edition. McGraw-Hill, 1995.

[17] M. K. Simon, "A new twist on the Marcum Q-function and its applications," *IEEE Commun. Lett.*, vol. 2, no. 2, pp. 39–41, Feb. 1998.

[18] M. K. Simon and M. Alouini, "Exponential-type bounds on the generalized Marcum Q function with application to error probability analysis over fading channels," *IEEE Trans. Commun.*, vol. 48, no. 3, pp. 359–366, Mar. 2000.

[19] I. M. Jacobs, "Probability-of-error bounds for binary transmission on the slowly fading Rician channel," *IEEE Trans. Inf. Theory*, vol. 12, no. 4, pp. 431–441, Oct. 1966.

[20] M. E. Hellman and J. Raviv, "Probability of error, equivocation, and the Chernoff bound," *IEEE Trans. Inf. Theory*, vol. 16, no. 4, pp. 368–372, July 1970.

[21] M. K. Simon, J. K. Omura, R. A. Scholtz, and B. K. Levitt, *Spread Spectrum Communications*, vol. I. Computer Science Press, 1985.

[22] J. M. Wozencraft and I. M. Jacobs, *Principles of Communication Engineering*, 1st edition. Wiley, 1965.

[23] M. K. Simon and D. Divsalar, "Some new twists to problems involving the Gaussian probability integral," *IEEE Trans. Commun.*, vol. 46, no. 3, pp. 200–210, Feb. 1998.

[24] M. K. Simon, "Evaluation of average bit error probability for space-time coding based on a simpler exact evaluation of pairwise error probability," *J. Commun. Netw.*, vol. 3, no. 3, pp. 257–264, Aug. 2001.