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Chevalley Groups over Commutative Rings: I. Elementary Calculations

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Abstract. This is the first in a series of papers dedicated to the structure of Chevalley groups over commutative rings. The goal of this series is to systematically develop methods of calculations in Chevalley groups over rings, based on the use of their minimal modules. As an application, we give new direct proofs for normality of the elementary subgroup, description of normal subgroups and similar results due to E. Abe, G. Taddei, L. N. Vaserstein, and others, as well as some generalizations. In this first part we outline the whole project, reproduce construction of Chevalley groups and their elementary subgroups, recall familiar facts about the elementary calculations in these groups, and fix a specific choice of the structure constants.

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0. Introduction

The purpose of this series of papers is to systematically develop methods of calculations in Chevalley groups over commutative rings, based on the use of their minimal modules. Namely, we describe in *all* details two new approaches in the study of (exceptional) Chevalley groups over rings, sketched in [146] and [137]. The approaches are complementary, or, to put it plainly, antagonistic. The leading idea of one of them – what we call the *geometry of exceptional groups* – is that we can calculate with matrices for the exceptional groups as well. The leading idea of another one – the *decomposition of unipotents* – is that one can *completely* eliminate matrices from all usual calculations pertaining to the classical groups, considering only elementary matrices and isolated columns or rows instead. As is classically known, all latter calculations can be easily performed also in the exceptional groups [39, 85, 110, 112].

As an application of these methods, in the final paper of the series we give new direct proofs of the main structure theorems: normality of the elementary subgroups, classification of normal subgroups and the like. Of course, for the classical cases these results are well known (by J. S. Wilson, I. Z. Golubchik, A. A. Suslin, V. I. Kopeiko, and many others), while for the exceptional ones have been obtained by E. Abe, G. Taddei and L. N. Vaserstein [5, 6, 7, 125, 134] fairly recently. Our method of proof is more straightforward and elementary and completely eliminates distinction between exceptional and classical cases. Moreover, even for classical groups, proofs based on it, are often much easier than the known ones.

In the remaining part of the introduction we informally explain, what it is all about. Let Φ be a reduced irreducible system of roots, $G(\Phi,\ )$ be a simply connected Chevalley–Demazure group scheme of type Φ, $T(\Phi,\ )$ be a split maximal torus in it. If R is a commutative ring, the group of points $G(\Phi, R)$ is called the (simply connected) Chevalley group of type Φ over R. To each root $\alpha \in \Phi$ there correspond elementary (with respect to T) root unipotents $x_\alpha(\xi)$, $\xi \in R$. All the elementary unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, generate a group $E(\Phi, R)$, which is called the elementary subgroup of $G(\Phi, R)$.

0.1. ELEMENTARY CALCULATIONS

The most commonly known kind of calculations in Chevalley groups are the so-called elementary calculations. They are based on the fact that relations among the elementary unipotents $x_\alpha(\xi)$ – the *Steinberg relations* – are very well understood [39, 41, 111, 112]. As soon as one is in a position to relate G and E, one can go amazingly far with the help of elementary calculations (look at [113] for some striking examples). Sometimes even the knowledge that E is normal in G is enough.

E. Abe and K. Suzuki [1, 9] described all normal subgroups of elementary Chevalley groups over *arbitrary* commutative rings (actually in [9] an additional finiteness condition was imposed, but as noticed in [137], the general case immediately follows from the case considered in [9]). This is done *exclusively* in terms of elementary calculations. Now as soon as one knows that E is normal in G, one can easily describe subgroups in G normalized by E, see [4, 5, 134].

The elementary calculations are especially powerful when a Chevalley group coincides with its elementary Chevalley group, $G = E$. This is the case, for example, when the ground ring is a field, or, more generally, a semi-local ring [1, 9, 85, 108], and also for some other important cases, like, say, Euclidean rings [112], Hasse domains [27, 85], or polynomial rings with coefficients in a field [3, 4, 49, 81, 119, 121, 148]. This is not true in general, however, and the difference between G and E is measured by $K_1(\Phi, R) = G(\Phi, R)/E(\Phi, R)$.

For a field, elementary calculations are particularly efficient since in this case there is a *canonical form* of elements: Bruhat decomposition. A similar role is played for semilocal rings by the Gauss decomposition or the like. However, for general rings, the groups of finite degree do not in general admit such decompositions. This is obvious from the fact that, even when $G = E$, it is by no means true that G has *finite width* with respect to the elementary generators $x_\alpha(\xi)$. Of course, the classical groups of *infinite* degree admit a remarkable generalization of Bruhat and Gauss decompositions due to Sharpe [102]. In fact part of it survives at the *stable level* (when the rank of the group is large with respect to the dimension of the ring) in the form of Dennis–Vaserstein decomposition [69, 93, 108, 110, 157–159], which pushes all difficulties to the Levi factors of maximal parabolic subgroups. However, all these things are of little use when one looks at a *given* group over a *general* ring.

0.2. STABLE CALCULATIONS

Elementary calculations give no insight whatsoever in the interrelations of G and E. Since it has been known for some time that a simply-connected Chevalley group and the corresponding elementary Chevalley group over a field coincide, there was some way to relate the Chevalley group and its elementary subgroup. Such a technique is provided by what M. R. Stein [110] has christened the *Chevalley–Matsumoto decomposition* theorem (compare [85], Theorem 4.3 and further) which in turn is a further development of the method of '*grosse cellule*' [43]. This theorem asserts that an element g of a Chevalley group G can be written as the product of an element of a Levi factor of a proper parabolic subgroup and two factors from the unipotent radicals of this parabolic subgroup and its opposite, *if* the diagonal matrix entry $g_{\omega,\omega} \in R$ of g corresponding to the highest weight vector e^ω in a certain rational representation π: $G \to \mathrm{GL}(V)$ of G is *invertible*.

Again for fields (or such rings, as, say, semilocal ones), only occasionally the element $g_{\omega,\omega}$ is *not* invertible and it is very easy to guess what to do when it is not – this is precisely how one proves that $G = E$ in these cases. For more general rings, H. Matsumoto [85] and M. R. Stein [110] developed a general technique to make this element invertible, based on the use of various 'stability conditions', similar to the *stable rank* of H. Bass [16, 17, 23, 24, 56, 129, 130, 131]. Actually, these conditions guarantee that an appropriate *dimension* of the ring R is small with respect to the rank of the group, so that we still have enough freedom to act *essentially* as we did in the cases above.

With this end they developed what we call 'stable calculations', which make use of one column or row of the matrix of $\pi(g)$, $g \in G$, in a base of weight vectors. First of all, they restrict themselves to the representations π, for which the Weyl group $W(\Phi)$ acts transitively on the set of nonzero weights. Such representations are called *basic* (see [42, 85]) and every Chevalley group has at least one basic representation (in fact, there are precisely $|\mathrm{Cent}(G(\Phi, \mathbb{C}))|$ such representations, all of them, but, one, minuscule). Clearly, all nonzero weights of such a representation have multiplicity one. Thus, the choice of a base of weight vectors in the corresponding representation space V is essentially canonical. One can normalize a base of weight vectors in such a way, that the action of the root unipotents $x_\alpha(\xi)$ is described by very nice polynomial matrices in ξ with integral coefficients ('Matsumoto's lemma', see [85, 110]).

Additional simplification is provided by consistent use of the *weight diagrams* which allow us to visualize the computations. As graphs they are essentially the Hasse diagrams of the posets of weights with respect to the induced (weak) Bruhat order. This means that the nodes of the diagram correspond to the weights of the representation and that two nodes are joined by an edge if their difference is a fundamental root. Moreover, the edge joining two weights λ and μ is labeled by i if $\mu - \lambda = \alpha_i$ is the ith fundamental root. When the zero weight has multiplicity $\leqslant 1$, the weight diagrams are *precisely* the Hasse diagrams and the weak order coincides with the strong one. More generally, if the multiplicity of the zero weight equals m, it gives rise to m distinct 'zero-weight' nodes of the weight diagram and there are more complicated rules for the edges, taking into account the discrepancy of the weak and the strong orders.

Now one may conceive a vector $v \in V$ as such a weight diagram which has an element of R attached to every node. A standard weight vector e^λ has 1 in the λth node and zeros elsewhere, an arbitrary vector v has its λth coordinate v^λ with respect to this weight base as the label at the λth node. The above-mentioned Matsumoto lemma translates into a very simple rule describing what happens with such a vector v under the action of $x_\alpha(\xi)$. If π is minuscule and the root $\alpha = \alpha_i$ is fundamental, then $x_\alpha(\xi)$ adds or subtracts (*always* adds for a clever choice of the weight base) ξv^λ to v^μ along each edge labeled with i. For other roots, one has just to trace all paths in the diagram which have the same labels at their edges as the root α in its linear expansion with respect to the fundamental roots. For example, if $\alpha = 2\alpha_1 + \alpha_2$, one has to look at the paths which have the labels $1, 1, 2$, in any order (the order of the labels on such a path starting in λ together with the structure constants of the Lie algebra is responsible for the *sign* with which $x_\alpha(\xi)$ acts on v^λ). There are slightly more complicated rules in the presence of zero weight.

Now if $g \in G$, then the μth column $g_{*,\mu}$ of the corresponding matrix $\pi(g)$ consists of the coefficients in the expansion of $\pi(g)e^\mu$ with respect to e^λ. We may conceive any element $g \in G$ (which we seldom distinguish from its image $\pi(g)$ in $\mathrm{GL}(V)$) as a *matrix* $g_{\lambda,\mu}$ where λ and μ range over all the weights of the representation π (with multiplicities, of course). Then the columns above are obtained by freezing the second index in such a matrix. Such columns may be identified with the corresponding elements of V. Analogously, the rows $g_{\lambda,*}$ are obtained by freezing the first index and correspond to the vectors from the dual module V^*. As we know from the preceding paragraph, one can very efficiently perform calculations with such columns and rows.

0.3. GENERAL CALCULATIONS

For the *natural* representations of the classical groups (that is $SL(l+1,R)$, $SO(2l+1,R)$, $Sp(2l,R)$ and $SO(2l,R)$ for types A_l, B_l, C_l and D_l, respectively) it is reasonably easy to perform calculations involving the whole matrix $\pi(g)$. This is due to the fact that the dimensions of these representations are relatively small as compared with the dimensions of the groups. As a result, there are few equations among the matrix entries of the matrices $\pi(g)$, and, as is well known, these equations are quadratic. In these representations the unipotent root elements of these groups have residue 1 (i.e. they are the usual transvections) or 2 (i.e. they are products of two transvections which form mutual angle of $\pi/2$ or $2\pi/3$).

The use of matrices is especially important when everything else does not work, i.e. for rings of large dimension, which have few units. In these cases, the corresponding groups do not admit decompositions allowing efficient use of elementary calculations. First proofs for normality of elementary subgroups, standard description of normal subgroups, standard description of automorphisms, etc., for the classical groups over arbitrary commutative rings and their relatives, used matrices. Here one can mention the works of J. S. Wilson [154], I. Z. Golubchik [61, 62], A. A. Suslin [119], W. van der Kallen [80], W. C. Waterhouse [150], M. S. Tulenbaev [128], V. I. Kopeiko [81, 82], A. A. Suslin and V. I. Kopeiko [121], G. Taddei [123], I. Z. Golubchik and A. V. Mikhalev [65, 66], V. M. Petechuk [90], E. I. Zelmanov [123], Z. I. Borewicz and N. A. Vavilov [31, 138, 139, 140], and many others. Only later, did A. A. Suslin [128], L. N. Vaserstein [133, 134, 135, 136] and A. Bak [19] notice that for rings close to a commutative ring can apply indirect methods to these problems (like 'factorization and patching' or 'localization and patching') which allow us to avoid the use of matrices, substituting with dimension arguments similar to those appearing in the Quillen–Suslin solution of Serre's conjecture. Consult [20, 21, 22, 69, 141] and [144] for the history and comparison of various proofs. Until quite recently [114, 115, 141], there were no direct ways to avoid matrix calculations in considering structure problems of classical groups over rings.

For exceptional groups distinct from G_2, the situation seems to be very different (as for G_2, one may argue that it is a classical group – or, else that the groups of types D_4 and B_3 are exceptional). Namely, the degrees of their minimal faithful representations are already fairly large as compared to their dimensions. In fact, these degrees are equal to 26, 27, 56 and 248 for the groups of types F_4, E_6, E_7 and E_8, respectively. There are many equations among the matrix entries of the matrices $\pi(g)$ in these representations and some of these equations are cubic or quartic. Even for the two smallest groups, unipotent long root elements have residue 6. All this makes one rather uncomfortable with the idea of direct matrix calculations with the groups, using their minimal representations. Below, we refute all these objections as not being serious.

This psychological attitude created a strange double standard, when one usually thinks about the classical groups as isometry groups of bilinear forms and completely ignores analogous descriptions of the exceptional groups, acting on their minimal modules, in terms of cubic or quartic forms. This attitude is expressed, for example, in most textbooks on algebraic groups, where such realizations are not even mentioned ([105] is an exception). One of our wider objectives in this series of papers is to demonstrate that *the easiest and one of the most efficient ways to think about the groups of types* G_2, F_4, E_6, E_7 *and* E_8 *over rings is precisely to think of them as certain groups of* 7×7, 26×26, 27×27, 56×56 *or* 248×248 *matrices.*

First of all, the equations among the entries of the matrices $\pi(g)$ are very transparent. Some cubic or quartic invariants of the groups were already known to L. E. Dickson and E. Cartan. The systematic study of exceptional groups over fields as isometry groups of cubic or quartic forms was initiated in the early fifties by C. Chevalley and H. Freudenthal and continued by T. Springer, J. Tits, F. Veldkamp, N. Jacobson, G. B. Seligman, J. G. M. Mars, R. B. Brown, S. J. Haris, and others. The whole subject got a new life in the works of M. Aschbacher, A. Cohen, B. Cooperstein [11, 12, 13, 44], and others, related to the maximal subgroup classification project. Quite remarkably, most of these invariants are *characteristic-free*. Thus, for example, the group of type E_6 over an arbitrary commutative ring R is the isometry group of a certain cubic form with coefficients ± 1 on a free R-module of rank 27 and other exceptional groups admit similar uniform descriptions.

One should not be too upset that the equations have degree 3 or 4. In fact, most calculations use not the cubic or quartic invariant itself, but only its partial derivatives. In particular, equations defining the orbit of the highest-weight vector in the representations are *quadratic* (we do not discuss here further K-theoretical obstacles for an element of V to be a column of a matrix $\pi(g)$, $g \in G$). It is an utterly rare phenomenon for a simple group to be defined by equations of degree $\leqslant 2$ [28, 151]. In fact, the classical groups themselves in representations other than the natural ones are not like that. For example, the spinorial group is defined by equations of degree 4 [35], while the adjoint classical groups are usually defined by equations of degree 3.

One should also not be too upset about the sizes of the matrices either. In fact one does not usually complain that $\mathrm{GL}(248, R)$ consists of matrices of degree 248. What one does is the following: calculate with block matrices, most of whose blocks are 1's or 0's or irrelevant. One has usually to trace the destiny of very few entries or blocks to be able to attain his or her goals. The same strategy usually works for exceptional groups. In fact, a matrix of the form $\pi(g)$ has many obvious block structures corresponding to the restrictions to various subsystem subgroups. If we consider the group of type E_6 and are only interested in what happens over A_5 or D_5, we can calculate with 3×3 instead of 27×27 block matrices. If we want to go one step deeper to A_4 or D_4, we have to consider

6×6 block matrices, but this is still not *too* bad. Only occasionally – not to say never – one has to play with the whole 27×27 matrix. The same applies to all other cases.

The following obvious observation is crucial in these calculations. The columns and rows of our matrices are usually *partially* ordered and not linearly ordered. Of course, the partial order is the one described by the weight diagram. The intuition coming from the natural representation of $\mathrm{SL}(l+1, R)$ is misleading: this is one of the *very* few cases when the weight diagram is a chain. Already for the natural representation of $\mathrm{SO}(2l, R)$, it is erroneous to think that the lth column precedes the $(l+1)$-st one. This means that we can always change the numbering of columns and rows to a more convenient one (without violating the partial order, of course, if we still want our Borel subgroup to consist of upper triangular matrices). It is precisely the constant change of viewpoint, restriction to various subsystems, and comparison of the results, that constitutes the essence of matrix calculations in exceptional groups (in the classical as well, but one seldom notices this, unless he or she has ever looked at more complicated cases).

Finally, one should not be too upset about the root elements having large residues. This can be already seen in the classical cases. A usual (linear) transvection has residue 1 and is defined by a column and a row (the centre and the axis of the transvection). Root elements in the classical groups have residues 2 and should, *a priori*, depend on *two* columns and *two* rows – but they do not, because of the presence of polarity. Actually, as is well known, an Eichler–Siegel–Dickson transvection is completely defined by two columns (or, what is the same, *one* column and *one* row). The same happens for exceptional groups. We now return to our favourite example of E_6 in the 27-dimensional representation. The root element has residue 6 and so, *a priori*, should depend on *six* columns and *six* rows with some complicated dependencies among them given by the equations. Well, as is less well known, a Freudenthal transvection is again completely defined by *one* column and *one* row, the rest being most conveniently defined in terms of the *cross-product* with this column and this row. So, in this respect, the elements of root type in exceptional groups are not very much different from the usual transvections or the ESD-transvections – in fact, they are even less sophisticated than the root elements in the unitary groups!

0.4. CONTENTS OF THE SERIES

Here we briefly discuss the contents of the forthcoming papers.

In the first part, we recall necessary definitions and notation, reproduce a construction of Chevalley groups and their elementary subgroups, recall familiar facts about the elementary calculations in these groups, and fix a specific choice of structure constants.

In the second part, the minimal modules of Chevalley groups, the main tool for all that follows, are studied in detail. In particular, we construct their weight diagrams and give an explicit description of the action of root unipotents in the special bases of weight vectors (the so-called Kostant bases), which is compatible with the choice of structure constants in the first part. We also list branching rules for the restriction of these representations to subsystem subgroups.

The third part deals with the analogue of the Ree–Dieudonné identification theorem for exceptional groups. More precisely, in this part we give a new construction of multilinear forms on minimal modules, whose groups of isometries coincide with the corresponding Chevalley groups. In most cases, this was known before, but our proof of identification uses only elementary (multi)linear algebra in contrast with the previous proofs, based on deep geometrical results (like the local characterization of buildings). In this part, we explicitly list the equations on entries of matrices from a Chevalley group acting on a minimal module.

In the fourth paper, we study unipotent elements of root type in representations of Chevalley groups (including, as special cases, the usual linear transvections, the Eichler–Siegel–Dickson transvections and the Freudenthal transvections). In particular, we establish addition formulae for such elements and prove an analogue of the Whitehead–Vaserstein lemma. For fields, the class of elements of the root type coincides with the class of root unipotent elements (i.e. elements, conjugate to an elementary root unipotent), but for rings it is usually much wider. This part, together with the preceding one, sets the foundations of the geometry of exceptional groups over rings.

The fifth part contains the fundamentals of the decomposition of unipotents. We construct certain special elements of the root type (the fake root unipotents in the terminology of [141]), stabilizing a given column and prove that these elements generate the whole elementary Chevalley group as the column ranges over all columns of $\pi(g)$, $g \in G$. These results are basic for the reduction to groups of smaller rank.

In the final paper of this series, we apply the methods developed in the previous parts to get a new direct proof of the main structure theorems for the Chevalley groups over arbitrary commutative rings. We establish the standard description of subgroups, normalized by a relative elementary subgroup, the description of subnormal subgroups as well as commutator formulae for congruence-subgroups and relative elementary groups and some other results, which cannot be stated in a precise form before Part IV (see the next section for a pattern). Some of these results are new, not only for exceptional groups but even for classical cases, in some others the known estimates are improved.

Our results in this part are very different from the known ones in the following respect. Before only indirect methods of the proof of the structure theorems were known for exceptional groups. These methods were based on such things as localizations and partitions of 1 in the ground ring. Clearly, the length of such a decomposition is not bounded by any constant depending on the group. Our methods are direct and constructive, so that they result in formulae depending on the group and not on the ring. To communicate some flavour of what is *really* proven in the sixth part, we give an example. Thus, we show that in the 27-dimensional representation V of a Chevalley group G of type E_6 any root unipotent $gx_\alpha(\xi)g^{-1}$, where $\alpha \in \Phi$, $\xi \in R$, $g \in G$, is a product of exactly 27 Freudenthal transvection lying in the Weyl conjugates of the standard parabolic subgroup P_1, and express these factors in terms of g, α and ξ. Clearly, such explicit results are beyond reach of 'localization and patching'. We believe that such explicit formulae are of independent interest and, maybe, in the long term, are more important than the structure theorems themselves. An English playwright said "Theorems come and go, but a good formula remains for ever".

The proportion of new things grows at a constant pace from the first part to the fifth. The first part contains absolutely nothing new, apart from this introduction and the tables, which are not new either, only different. The second part is also not very new, but is far less known. It contains some previously unpublished recipes. The third part contains new proofs for identifications of exceptional groups with isometry groups of multilinear forms as well as a new construction of the forms and explicit description of equations. The fourth part is new for exceptional groups over rings (although mostly known for classical groups over rings and exceptional groups over fields). Finally, the fifth part is new *even* for the classical groups, only a sketch of it was published in [141, 146] (actually many details for the *classical* groups are contained in the manuscripts [114, 138], and a systematic presentation of the linear case – even over noncommutative rings – will appear in [115]).

We have included somewhat more background information in the first three parts, than what was strictly necessary for our purposes. This is motivated by several reasons. First, these three papers form an elementary and essentially self-contained introduction to the foundations of the theory of Chevalley groups over rings. We assume only the rudiments of linear algebra, Lie algebras, and representation theory. We invoke the definition of a Chevalley–Demazure group scheme, but after the identification theorem in the third part, this definition may be safely forgotten. Second, the results and methods presented here should serve as a definitive common background for our forthcoming publications, dedicated to various aspects of Chevalley groups over rings (subgroups in Chevalley groups [140, 143], Steinberg groups, unstable K-theory [93], nonsplit groups, etc.). Third, we believe that some of the methods presented here are of a wider interest, and a lot of technical details may be useful to the experts working with Chevalley groups in various fields, be it in connection with the theory of finite simple groups, the theory of algebraic groups, the theory of representations, algebraic K-theory, or arithmetic.

Our main focus in these papers are on, of course, the *exceptional* groups. However, we would like to make the following observations:

- conceptual differences between the classical and exceptional groups are negligible;

- in most problems, it is not natural to exclude exceptional groups from the consideration;
- even if one is interested solely in the classical groups, a theory also comprising exceptional groups gives a much better understanding.

If one is not convinced by these arguments, here is the ultimate argument, which he or she cannot beat:

- it is a lot of fun* to work with exceptional groups, especially with those of types E_6 and E_7.

As should be clear from the preceding text, the understanding of Chevalley groups, which we present here, comes from four major sources.**

I. 'Elementary' theory of Chevalley groups as initiated by C. Chevalley himself and developed by R. Steinberg, R. Carter, and many others in the context of fields and by E. Abe, K. Suzuki, J. Hurley, M. R. Stein, and others for rings;

II. The representation-theoretic techniques introduced to the K-theory of Chevalley groups in the works of H. Matsumoto [85] and M. R. Stein [110];

III. The 'geometry of exceptional groups' over fields as developed by H. Freudenthal, T. Springer, J. Tits, N. Jacobson, F. Veldkamp, and others; which got a new life in the works of M. Aschbacher, A. M. Cohen, B. N. Cooperstein, etc.;

IV. The 'geometry of classical groups' over rings, going back to J. Dieudonné and H. Bass, as developed by O. T. O'Meara, A. Bak, A. A. Suslin, J. S. Wilson, W. van der Kallen, I. Z. Golubchik, A. V. Mikhalev, A. J. Hahn, L. N. Vaserstein, and others.

We effectively fuse all these viewpoints in our exposition. We believe that it gives a much richer and deeper picture than the traditional approaches, based almost exclusively on elementary calculations.

1. Notation

In this section, we recall definitions associated with Chevalley groups over rings and fix notation used in the sequel.

All background information connected with Lie algebras and their representations, can be found, for instance, in [32, 33, 67, 77, 100, 147], but we recall

* We *must* admit that sometimes E_8 is *too* much of a goad thing – and working with exceptional groups, loses *some* of its beneficial effects at this stage. This might depend on the fact that we still feel much less comfortable with the geometry of an adjoint representation, than with that of a microweight one.

** Of course, there were many second and third-order influences. Thus, the calculations with one column are similar to the 'matrix problems' in the representation theory of posets [59] and to the calculations of Borel orbits in [37, 72]; our treatment of equations resembles the 'standard monomial theory' [45, 84, 101], in the construction of fake root unipotents one recognizes the 'geometry of root subgroups' [48], etc. We document all such borrowings at the appropriate places.

the notation in the text of the papers. We follow [32, 33] as far as the numbering of roots, weights, etc., is concerned.

We consider the information on algebraic groups, contained in [29, 73, 104, 105, 147] to be standard, and use it without explicit references.

The theory of affine group schemes is treated in the books [51, 52, 53, 78]. We use, in fact, only rudimentary notions of the theory, an elementary description of which can be found in [149]. An excellent overview of Hopf algebras from the point of view of their significance in the theory of linear algebraic groups can be found in [2].

As for the classical groups, we follow the notation from [10, 39, 55, 69] while the necessary rudimentary information from algebraic K-theory (definitions of lower K-functors, stability conditions, etc.) can be found in [16, 18, 23, 24, 25, 26, 27, 86, 120, 129, 130, 131, 144].

The books [39, 113] contain a systematic exposition of the theory of Chevalley groups over fields. As introductions to this theory, one can also use [29, 38, 42, 71, 74, 99]. We do not include in the bibliography, books in which *finite* Chevalley groups and their twisted analogues (i.e. the *finite groups of Lie type*) are considered from the point of view of Steinberg's theory, that is as the groups of fixed points of Frobenius endomorphisms acting on the corresponding algebraic groups.

There are no monographs on Chevalley groups over rings. We follow the papers [1, 3, 4, 5, 6, 7, 8, 9, 43, 85, 106, 107, 108, 109, 110, 124, 137, 141].

Let Φ be a reduced irreducible root system of rank l, P be a lattice, lying between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$. Then we can construct an affine group scheme $G_P(\Phi)$, over $\mathbb{Z}$ (i.e. a representable covariant functor from the category of commutative rings with 1 to the category of groups), such that for every algebraically closed field K, the value of this functor $G_P(\Phi, K)$ on K is the semisimple algebraic group of type (Φ, P) over K. The existence of such a functor was proven by C. Chevalley [43], while its uniqueness by M. Demazure [50]. We call such a scheme the *Chevalley–Demazure group scheme* of type (Φ, P), and its value $G_P(\Phi, R)$ on a commutative ring R with 1 ('the group of rational points $G_P(\Phi, R)$ with the coefficients in R') is called the *Chevalley group of type* (Φ, P) *over* R. Since the problems we treat are usually independent from the choice of the lattice P, we suppress P in the notation and speak about a Chevalley group $G(\Phi, R)$ of type Φ over R. If we want to emphasize which group we are talking about, we write $G_{\mathrm{sc}}(\Phi, R)$ for the simply connected group, when $P = P(\Phi)$, and $G_{\mathrm{ad}}(\Phi, R)$ for the adjoint group, when $P = Q(\Phi)$. *Unless specified otherwise, we assume our groups to be simply connected.* In order to fix necessary notation, we recall briefly the construction of the group scheme $G(\Phi,\ )$, certain associated objects and some important subgroups in the group $G(\Phi, R)$ of its rational points.

2. Chevalley Algebras

Let $L = L_{\mathbb{C}}$ be a complex semisimple Lie algebra of type Φ, $[\ ,\ ]$ be the Lie bracket on L and H be its Cartan subalgebra. Then L admits the *root decomposition* $L = H \bigoplus \sum L_\alpha$, where L_α are the *root subspaces,* i.e. one-dimensional subspaces, invariant with respect to H. For each root α denote by the same letter the linear functional on H, such that $[h, e_\alpha] = \alpha(h)e_\alpha$. The restriction of the Killing form of the algebra L to H is nondegenerate and is denoted by $(\ ,\ )$. This inner product allows us to identify H with H^* and consider α's as elements in H. However, instead of roots, it is convenient to consider *coroots* $h_\alpha = 2\alpha/(\alpha,\alpha)$. Fix an order on Φ, and let Φ^+, Φ^- and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the sets of positive, negative, and fundamental roots, respectively. A choice of nonzero elements $e_\alpha \in L_\alpha$, $\alpha \in \Phi^+$, uniquely determines elements $e_{-\alpha} \in L_{-\alpha}$, $\alpha \in \Phi^+$, such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. Then the set $\{e_\alpha, \alpha \in \Phi;\ h_\alpha, \alpha \in \Pi\}$ is a base of the Lie algebra L, called a *Weyl base* of L. The elements of such a base are multiplied as follows:

$$\begin{array}{ll}
[h_\alpha, h_\beta] = 0, & \alpha, \beta \in \Pi; \\
[h_\alpha, e_\beta] = A_{\alpha\beta} e_\beta, & \alpha \in \Pi,\ \beta \in \Phi; \\
[e_\alpha, e_{-\alpha}] = h_\alpha, & \alpha \in \Phi; \\
[e_\alpha, e_\beta] = 0, & \alpha, \beta \in \Phi,\ \alpha + \beta \notin \Phi \cup \{0\}; \\
[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}, & \alpha, \beta, \alpha + \beta \in \Phi.
\end{array} \tag{2.1}$$

Here, $A_{\alpha\beta} = 2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$ are the *Cartan integers* and $N_{\alpha\beta}$ are some complex numbers, called the *structure constants*. A direct calculation shows that

$$N_{\alpha,\beta} N_{-\alpha,-\beta} = -(p+1)^2,$$

where p is such an integer, that $-p\alpha+\beta, \dots, \beta, \dots, q\alpha+\beta$ is the α-series of roots passing through β. Chevalley has shown [42], that the elements e_α can be chosen in such a way that $N_{\alpha\beta} = \pm(p+1)$. Thus, all the structure constants in this case are integers, this fact is called a *Chevalley theorem* (see [33, 39, 74, 112] for the proof). The set of e_α, $\alpha \in \Phi$, satisfying the condition above, is called a *Chevalley system,* and a Weyl base with the integral structure constants a *Chevalley base.* An explicit choice of signs of the structure constants is a delicate matter (see Sections 14–17).

Let $L_{\mathbb{Z}}$ be the integral span of a Chevalley base. Then $L_{\mathbb{Z}}$ is a Lie algebra over $\mathbb{Z}$, which is a $\mathbb{Z}$-form of the Lie algebra L, i.e. $L = L_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. This form is called an admissible *$\mathbb{Z}$-form* or a *Chevalley order* in L. Let now R be a commutative ring. Set $L_R = L_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. In other words, L_R is the Lie algebra over R, which as an R-module is the free module with the base $e_\alpha = e_\alpha \otimes 1$, $h_\beta = h_\beta \otimes 1$, with the multiplication given by (2.1). The algebra L_R is called the *split semisimple algebra of type Φ over R* or simply the *Chevalley algebra of type Φ over R.*

Chevalley algebras over fields are the split semisimple algebras of classical type (recall that in the theory of modular Lie algebras, the exceptional algebras are also called algebras of classical type, because there are also algebras of *Cartan type*, which do not come from characteristic zero). The structure of Chevalley algebras over fields was considered, for instance, in [76, 116].

3. Adjoint Elementary Chevalley Groups

Now we are in a position to introduce the *adjoint* elementary Chevalley groups. Let us recall the corresponding construction, due to Chevalley [41] (see also [38, 39, 71, 74, 112]). Some details of this construction are essential for the treatment of the Chevalley group of type E_8.

Let first D be an arbitrary nilpotent derivation of the Lie algebra L. Then $\exp D = 1 + D + D^2 + \cdots + D^{(n)}$ (where $D^{(n+1)} = 0$ and $D^{(m)} = D^m/m!$ is the mth divided power of D) is an automorphism of the algebra L. Now we specialize D to an adjoint derivation $\operatorname{ad} e_\alpha$. It is well known that $(\operatorname{ad} e_\alpha)^4 = 0$. Since $\operatorname{ad}(\xi e_\alpha) = \xi \operatorname{ad} e_\alpha$ is also a nilpotent derivation for each $\xi \in \mathbb{C}$, we can set

$$x_\alpha(\xi) = x_\alpha^{\mathrm{ad}}(\xi) = \exp(\xi \operatorname{ad} e_\alpha).$$

Now we consider the action of the automorphisms $x_\alpha(\xi)$ on the elements of the Chevalley base. On the subalgebra $\langle e_\alpha, h_\alpha, e_{-\alpha}\rangle = \mathrm{sl}_2$, the elements $x_\alpha(\xi)$ act as follows:

$$\begin{aligned} x_\alpha(\xi)e_\alpha &= e_\alpha, \qquad x_\alpha(\xi)h_\alpha = h_\alpha - 2\xi e_\alpha, \\ x_\alpha(\xi)e_{-\alpha} &= e_{-\alpha} + \xi h_\alpha - \xi^2 e_\alpha. \end{aligned} \tag{3.1}$$

If α and β are linearly independent, then

$$\begin{aligned} x_\alpha(\xi)h_\beta &= h_\beta - A_{\beta\alpha}\xi e_\alpha, \qquad (3.2) \\ x_\alpha(\xi)e_\beta &= e_\beta + N_{\alpha\beta}\xi e_{\alpha+\beta} + M_{\alpha\beta 2}\xi^2 e_{\beta+2\alpha} + \cdots + M_{\alpha\beta q}\xi^q e_{\beta+q\alpha}, \end{aligned}$$

where

$$M_{\alpha\beta k} = \frac{1}{k!} N_{\alpha\beta} N_{\alpha,\beta+\alpha} \cdots N_{\alpha,\beta+(k-1)\alpha}. \tag{3.3}$$

Since $N_{\alpha\beta} = \pm(p+1)$, the number $M_{\alpha\beta k} = \pm C_{p+k}^k$ is an integer. Thus, the automorphism $x_\alpha(\xi) = x_\alpha^{\mathrm{ad}}(\xi)$ sends each element of the Chevalley base into a linear combination of base elements whose coefficients are linear combinations of nonnegative powers of ξ with integer coefficients. This means that we can define an automorphism $x_\alpha(\xi)$ over an arbitrary commutative ring by the same formulae (3.1)–(3.3).

The group of automorphisms of the Chevalley algebra L_R, generated by all automorphisms of the form $x_\alpha(\xi) = x_\alpha^{\mathrm{ad}}(\xi)$ is called the *elementary adjoint Chevalley group of type* Φ and is denoted by $E_{\mathrm{ad}}(\Phi, R)$. In other words,

$$E_{\mathrm{ad}}(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R\rangle.$$

Thus, by the very definition, the group $E_{\mathrm{ad}}(\Phi, R)$ is a *linear group*, being a subgroup of the group $\mathrm{GL}(L_R)$ of all automorphisms of the *free R-module* L_R. In fact, it consists of automorphisms of the *Lie algebra* L_R and in the field case it is very closely related to $\mathrm{Aut}(L_R)$ (see [112, 33] for the details).

4. Weyl Modules

The construction of adjoint Chevalley groups is based on a choice of an admissible $\mathbb{Z}$-form of the adjoint representation of Lie algebra L. In order to construct all Chevalley groups, we need to be able to choose admissible $\mathbb{Z}$-forms of an arbitrary finite-dimensional representation. Let again $L = L_{\mathbb{C}}$ be a complex semisimple Lie algebra, and π: $L \to \mathrm{GL}(V)$ be its representation in a finite-dimensional vector space V over $\mathbb{C}$. For an element $\lambda \in H^*$, denote by V^λ the corresponding *weight subspace* of the space V, regarded as H-module, i.e.

$$V^\lambda = \{v \in V \mid \pi(h)v = \lambda(h)v,\ h \in H\}.$$

We say that λ is a *weight* of the representation π if $V^\lambda \neq 0$. The dimension $m_\lambda = \mathrm{mult}(\lambda)$ of the space V^λ is called the *multiplicity* of the weight λ. Let us denote by $\bar{\Lambda}(\pi)$ the set of *weights of the representation* π, and by $\Lambda(\pi)$ the *set of weights with multiplicities*. This means that all the weights from $\bar{\Lambda}(\pi)$ are distinct, and we assign to each weight $\lambda \in \bar{\Lambda}(\pi)$ a collection of m distinct 'weights' $\lambda_1, \dots, \lambda_m \in \Lambda(\pi)$, where $m = \mathrm{mult}(\lambda)$. We denote by $\bar{\Lambda}^*(\pi)$ and by $\Lambda^*(\pi)$ the sets of nonzero weights and nonzero weights with multiplicity, respectively. Let $P = P(\pi)$ be the lattice of weights of the representation π, i.e. the subgroup in $P(\Phi)$ generated by $\bar{\Lambda}(\pi)$. Then, $V = \oplus V^\lambda$, $\lambda \in \Lambda(\pi)$. In particular, for the adjoint representation $\pi = \mathrm{ad}$, we have

$$V = L, \qquad \Lambda^*(\pi) = \Phi, \qquad \Lambda(\pi) = \Phi \cup \{0_1, \dots, 0_l\},$$

$$P = Q(\Phi), \qquad V^\alpha = L_\alpha \quad \text{for } \alpha \in \Phi \quad \text{and} \quad V^0 = H.$$

Let $\mu \in \Lambda(\pi)$ and $v^+ \in V$. The weight μ is called the *highest weight* of the representation π and the vector v^+ is called a *highest-weight vector* (or a *primitive element*) if $\pi(e_\alpha)v^+ = 0$ for all $\alpha \in \Phi^+$. Of course, this notion depends on the choice of order on the root system Φ. The representation π is irreducible if and only if V is generated as an L-module by a vector of the highest weight. The multiplicity of the highest weight of an irreducible representation is equal to 1, hence a primitive element in this case is determined uniquely up to multiplication by a nonzero scalar. It is well known that the correspondence between the finite-dimensional irreducible modules and their highest weights yields a bijection of the set of isomorphism classes of irreducible finite-dimensional L-modules onto the set $P(\Phi)_{++}$ of *dominant* integral weights (with respect to a fixed order). Recall that

$$P(\Phi)_{++} = \{\mu \in P(\Phi) \mid (\mu, \alpha) > 0,\ \alpha \in \Pi\}.$$

The famous *Chevalley–Ree theorem* asserts, that each finite-dimensional L-module V contains a $\mathbb{Z}$-lattice M, invariant with respect to all operators $\frac{1}{m!}\pi(e_\alpha)^m$, $\alpha \in \Phi$, $m \in \mathbb{Z}^+$, and each such lattice is the direct sum of its weight components $M^\lambda = M \cap V^\lambda$ (see [39, 74, 83, 96, 113]). Such a lattice $M = V_\mathbb{Z}$ is called an *admissible $\mathbb{Z}$-form* of the module V. A base v^λ, $\lambda \in \Lambda(\pi)$, is called an *admissible base* of the lattice $V_\mathbb{Z}$, if it consists of weight vectors, and is such that for all $\alpha \in \Phi$, $m \in \mathbb{Z}^+$, $\lambda \in \Lambda(\pi)$, the vector $\pi(e_\alpha^{(m)})v^\lambda$ is again an integral linear combination of the base vectors. The Chevalley–Ree theorem asserts that every finite-dimensional module has an admissible base. This theorem can be proven directly [96] by an explicit construction of admissible bases in the fundamental modules and then deducing the existence of such bases in all finite-dimensional modules by the passage to a direct sum, a tensor product, or a submodule.

Again let, R be an arbitrary commutative ring. Set $V_R = V_\mathbb{Z} \otimes_\mathbb{Z} R$. So, V_R is a free R-module with the base $v^\lambda = v^\lambda \otimes 1$, $\lambda \in \Lambda(\pi)$. It is clear that V_R is a module over the Chevalley algebra L_R. Indeed, e_α and h_α act on the first component of the product $v \otimes \xi$, $v \in V_\mathbb{Z}$, $\xi \in R$, while the scalars of R act on the second. If V is an irreducible L-module with the highest weight μ, then V_R is called the *Weyl module* of the Chevalley algebra R with the *highest weight* μ.

We make a remark concerning the field case (see [78]). It is well known that, for a field of prime characteristic, the irreducible representations are also parametrized by the highest weights. However, the Weyl module with the highest weight μ does not necessary coincide with the irreducible module with the highest weight μ. Moreover, the Weyl module is not, generally speaking, irreducible (for large dimensions they are almost always reducible). In fact, the Weyl module is indecomposable, and its unique top composition factor coincides with the irreducible module of the same highest weight. We consider minimal modules, for which such a situation occurs extremely rarely, only in characteristics 2 and 3. The fact that Weyl modules are not, generally speaking, irreducible, does not play any role in the sequel.

5. Kostant Theorem

The current approach to the proof of Chevalley–Ree theorem is due to Kostant [83] (see also [30, 74, 78, 113, 124]) and is based on the fact that divided powers $e_\alpha^{(q)} = e_\alpha^q/q!$ generate a $\mathbb{Z}$-form $\mathrm{U}(L)_\mathbb{Z}$ of the universal enveloping algebra $\mathrm{U}(L)$ of Lie algebra L. Namely, choose an order on the set of positive roots $\Phi = \{\beta_1, \dots, \beta_n\}$. Usually, we consider only *regular orders*, i.e. orders compatible with the *height* of a root $\mathrm{ht}(\alpha) = \sum p_i$, where $\alpha = \sum p_i\alpha_i$, $\alpha_i \in \Pi$. This means that $\mathrm{ht}(\alpha) \geqslant \mathrm{ht}(\beta)$ if $\alpha \geqslant \beta$. One can assume, moreover, that our order is given by the numbering of positive roots, i.e. $\beta_i \geqslant \beta_j$ if $i \geqslant j$. For an n-tuple $S = (s_1, \dots, s_n) \in \mathbb{Z}_+^n$, one introduces elements $e^{\pm S}$ by the formula

$$e^{\pm S} = e_{\pm\beta_1}^{s_1} \cdots e_{\pm\beta_n}^{s_n}.$$

The elements e^{+S} are usually denoted simply by e^S, while e^{-S} are denoted by f^S. For $x \in L$ and $q \in \mathbb{N}$ one sets

$$\binom{x}{q} = x(x-1)\cdots(x-q+1)/q! \in \mathrm{U}(L).$$

For an l-tuple $Q = (q_1, \dots, q_l) \in \mathbb{Z}_+^l$, one sets

$$h^Q = \binom{h_{\alpha_1}}{q_1} \cdots \binom{h_{\alpha_l}}{q_l}.$$

Now the *Kostant theorem* (the *'integral Poincare–Birkhoff–Will theorem'*) asserts that the elements of the form $f^S h^Q e^T$, where $S, T \in \mathbb{Z}_+^n$, $Q \in \mathbb{Z}_+^l$, constitute a base of a $\mathbb{Z}$-form $\mathrm{U}(L)_{\mathbb{Z}}$ of the universal enveloping algebra $\mathrm{U}(L)$, see [30, 74, 78, 112]. This form is called the *Kostant form* or the *Kostant–Cartier form*.

Now it is obvious how to construct an admissible $\mathbb{Z}$-form of any representation. We pick up a highest-weight vector v^+ in V and set $V_{\mathbb{Z}} = \mathrm{U}(L)_{\mathbb{Z}} v^+$. This construction leads to a very useful admissible base, the so-called *Kostant base* which consists of linearly independent vectors of the form $f^S v^+$, where the n-tuples $S = (s_1, \dots, s_n) \in \mathbb{Z}_+^n$ are chosen by a special rule. We skip the details of this construction for arbitrary L-modules, since for minimal modules (the only ones we *really* use) all the weight subspaces corresponding to nonzero weights are one-dimensional. Therefore, the choice of a Kostant base is almost canonical and much easier than in the general case.

Note, that the Kostant $\mathbb{Z}$-form $\mathrm{U}(L)_{\mathbb{Z}}$ allows us to define one more object of paramount significance connected with a Chevalley group $G = G(\Phi, R)$, namely the corresponding *hyperalgebra* $\mathrm{U}(L)_R = \mathrm{U}(L)_{\mathbb{Z}} \otimes R$. When $R = K$ is a field of prime characteristic, not the corresponding Chevalley algebra, but precisely the hyperalgebra is responsible for the irreducible representations of the group G [43, 75, 117]. (The definition above differs from the standard one, but it was shown in [70] that they are equivalent.)

6. Elementary Chevalley Groups

Again, let v^λ, $\lambda \in \Lambda(\pi)$, be an admissible $\mathbb{Z}$-base of an L-module V. Then all operators $\pi(\xi e_\alpha) \in \mathrm{GL}(V)$, $\alpha \in \Phi$, $\xi \in \mathbb{C}$, are nilpotent and we can define the exponential of such an operator by the usual formula

$$\exp(\xi e_\alpha) = e + \xi\pi(e_\alpha) + \xi^2 \pi(e_\alpha)^{(2)} + \cdots.$$

The image of each base vector under this operator is a linear combination of the base vectors whose coefficients are linear combinations of powers of ξ with integral coefficients. This means that we can define an automorphism

$$x_\alpha(\xi) = x_\alpha^\pi(\xi) = \exp\left(\xi\pi(e_\alpha)\right) \in \mathrm{GL}(V_R)$$

of the R-module V_R by the same formula as above for any commutative ring.

The subgroup of the automorphism group of the R-module V_R, generated by all the automorphisms of the form $x_\alpha(\xi) = x_\alpha^\pi(\xi)$ is called an *elementary Chevalley group of type* Φ *over* R and is denoted by $E_\pi(\Phi, R)$. Thus, we have

$$E_\pi(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R\rangle.$$

Therefore, $E_\pi(\Phi, R) \leqslant \mathrm{GL}(V_R)$, which means that the elementary Chevalley group from the very start arises as a *linear* group. As an *abstract* group, $E = E_\pi(\Phi, R)$ depends up to isomorphism not on the π itself, but just on its lattice of weights $P = P(\Phi)$. If we want to stress this, we write $E = E_P(\Phi, R)$ instead of $E_\pi(\Phi, R)$. But according to our definition, the groups $E = E_P(\Phi, R)$ *always arise in some particular representations* $E_\pi(\Phi, R)$. The corresponding modules for $E_\pi(\Phi, R)$ are also called the *Weyl modules*.

In the next paper of the series we study these actions in some details – especially for the minimal modules – and state formulae describing the action of $x_\alpha(\xi)$ analogous to (3.1)–(3.3).

7. Chevalley Groups

Now let $G = G_\mathbb{C}$ be the connected complex semisimple Lie group with the Lie algebra $L = L_\mathbb{C}$ and the lattice of weights P. Denote by $\mathbb{C}[G]$ the *affine algebra* of G, i.e. the algebra of all regular complex-valued functions on G, regarded as a *Hopf algebra* [2, 29, 73, 104]. Denote by the same letter π, the representation of G on a finite-dimensional space $V = V_\mathbb{C}$, whose differential equals π: $L_\mathbb{C} \to \mathrm{GL}(V_\mathbb{C})$. The choice of an admissible base v^λ, $\lambda \in \Lambda(\pi)$, allows us to identify $V^\mathbb{C}$ with $\mathbb{C}^n$, $n = \dim V$, and, therefore, introduces coordinate functions $x_{\lambda,\mu}$, $\lambda, \mu \in \Lambda(\pi)$, on $\mathrm{GL}(V_\mathbb{C})$. These coordinate functions identify $\mathrm{GL}(V_\mathbb{C})$ with $\mathrm{GL}(n, \mathbb{C})$ and their restrictions to $\pi(\mathrm{G}_\mathbb{C})$ generate a subring $\mathbb{Z}[G]$ of the affine algebra $\mathbb{C}[G]$. It can be verified that this subring is in fact a Hopf subalgebra of $\mathbb{C}[G]$, see [43, 50, 53]. Thus, one can define an *affine group scheme* over $\mathbb{Z}$ by

$$R \longrightarrow G_P(\Phi, R) = \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}[G], R).$$

The image of a ring R under this functor is denoted by $G_P(\Phi, R)$ and is called a *Chevalley group of type* Φ *over* R. Up to isomorphism of algebraic groups, this group depends only on Φ and R, but not on π. At the same time, by definition, we can consider the corresponding linear groups $G_\pi(\Phi, R)$ as subgroups in $\mathrm{GL}(n, R)$, where $n = \dim \pi$.

Now let $\alpha \in \Phi$ and u be an independent variable. The homomorphism of $\mathbb{Z}[G]$ on $\mathbb{Z}[u]$, assigning to each coordinate function $x_{\lambda,\mu}$ its value on $x_\alpha(u)$, induces a homomorphism

$$G_a(R) = \mathrm{Hom}(\mathbb{Z}[u], R) \longrightarrow G(\Phi, R) = \mathrm{Hom}(\mathbb{Z}[G], R)$$

of the additive group $R^+ = G_a(R)$ of the ring R to the Chevalley group $G(\Phi, R)$. The image of this homomorphism is the root subgroup $X_\alpha = \{x_\alpha(\xi),\ \xi \in R\}$. Hence, the elementary Chevalley group $E_\pi(\Phi, R)$ is contained in the Chevalley group $G_\pi(\Phi, R)$. The interrelations between these two groups constitute a major problem in the theory of Chevalley groups over rings. Whereas for an elementary Chevalley group, there is a very nice system of generators $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, and the relations among these generators are fairly well understood, nothing like this is available for the Chevalley group itself.

Let first K be an algebraically closed field. Then always $G_\pi(\Phi, R) = E_\pi(\Phi, R)$ It is easy to see that this equality is not true in general, even for the case of a field. However, if G is *simply-connected*, then for an arbitrary field one has $G_{\mathrm{sc}}(\Phi, R) = E_{\mathrm{sc}}(\Phi, R)$. This is proven by the method of '*grosse cellule*', which is a special case of the Chevalley–Matsumoto decomposition [8, 30, 43, 53, 85, 110]. In fact the equality $G_{\mathrm{sc}}(\Phi, R) = E_{\mathrm{sc}}(\Phi, R)$ holds, even for the case when R is a semilocal ring [1, 9, 85, 108]. Recall that a commutative ring is said to be *semilocal* if it has only finitely many maximal ideals. There are also some further cases when the groups G_{sc} and E_{sc} are known to coincide, say, Euclidean rings [112], Dedekind rings of arithmetic type [27, 85], and polynomial rings with coefficients in a field or a principal ideal ring [3, 4, 49, 68, 81, 119, 121, 148]. In the case of fields or semilocal rings, the distinction between the above groups is easily bridged, even for nonsimply-connected groups by adding certain semisimple generators (see Section 10).

It is well known that the group of elementary matrices $\mathrm{E}(2, R) = E_{\mathrm{sc}}(\mathrm{A}_1, R)$ is *not necessarily normal* in the special linear group $\mathrm{SL}(2, R) = G_{\mathrm{sc}}(\mathrm{A}_1, R)$ (see [47, 122, 118]), but it turns out that if Φ is an irreducible root system of rank $l \geqslant 2$, then $E(\Phi, R)$ is always *normal* in $G(\Phi, R)$ (see [82, 119, 121] for the classical groups and [124, 125] for the Chevalley groups). In fact, a new direct proof of this statement is one of the goals of the present series of papers.

Thus, for these cases one can define the quotient group

$$\mathrm{K}_1(\Phi, R) = G_{\mathrm{sc}}(\Phi, R)/E_{\mathrm{sc}}(\Phi, R),$$

which is the famous K_1-*functor of type* Φ *over* R (see [3, 108, 109, 110]). Algebraic K-theory shows that this functor is, generally speaking, nontrivial, that is why for a general ring the group $E_{\mathrm{sc}}(\Phi, R)$ can be strictly smaller than $G_{\mathrm{sc}}(\Phi, R)$.

8. Identifications

In fact, equations, defining Chevalley groups, can be explicitly listed. After that, one can identify these groups with the groups, preserving certain systems of tensors. In particular, the *Ree–Dieudonné theorem establishes that appropriate Chevalley groups of classical series coincide with split classical groups* (i.e. the groups of isometries of bilinear forms of the maximal Witt index).

1. $\Phi = A_n$. Here the group $G_{\mathrm{sc}}(\mathrm{A}_n, R)$ coincides with the *special linear group* of degree $n+1$ over R, while the group $G_{\mathrm{ad}}(\mathrm{A}_n, R)$ coincides with the corresponding *projective linear group* $\mathrm{PGL}(n+1, R)$. In turn, $E_{\mathrm{sc}}(\Phi, R) = \mathrm{E}(n+1, R)$ is the elementary group, i.e. the group generated by all *elementary transvections* $x_{ij}(\xi) = e + \xi e_{ij}$, $\xi \in R$, $i \neq j$ (as usual, e is the identity matrix and e_{ij} is a standard matrix unit).

The quotient group

$$\mathrm{K}_1(\mathrm{A}_n, R) = \mathrm{SK}_1(n+1, R) = \mathrm{SL}(n+1, R)\mathrm{E}(n+1, R)$$

is the usual (linear) *nonstable* K_1*-functor* [23, 24, 27, 69, 86].

2. $\Phi = \mathrm{B}_n$. Following [39], we number the indices from 1 to $2n+1$ as follows: $1, \dots, n, 0, -n, \dots, -1$. Let Q be the quadratic form, defined on the free R-module $V = R^m$ of rank $m = 2n+1$, by the formula

$$Q(x_1, \dots, x_{-1}) = x_0^2 + x_1 x_{-1} + \cdots + x_n x_{-n}.$$

In other words, Q is a form of the maximal Witt index, which makes V to a *split orthogonal space* (*'Artin space'*) of dimension $2l+1$.

Then $G_{\mathrm{sc}}(\Phi, R)$ is precisely the *spinorial group* $\mathrm{Spin}(n, R)$, associated with the form Q, while $G_{\mathrm{ad}}(\Phi, R)$ coincides with the corresponding *special orthogonal group* $\mathrm{SO}(n, R) = \mathrm{PSO}(n, R)$. At the same time, the elementary Chevalley groups $E_{\mathrm{sc}}(\Phi, R)$ and $E_{\mathrm{ad}}(\Phi, R)$ coincide with $\mathrm{Epin}(n, R)$ and $\mathrm{EO}(n, R)$, respectively (for the definitions of these groups see [23, 25, 17, 69, 121]). Let us note that, while $\mathrm{Spin}(n, K) = \mathrm{Epin}(K)$ for an arbitrary field K, the group $\mathrm{EO}(n, K)$ does not necessarily coincide with $SO(n, R)$. In fact, it coincides with the *kernel of spinorial norm*, which is, generally speaking, a proper subgroup of $\mathrm{SO}(n, K)$ and (apart from very small dimensions) equals the commutator subgroup $\Omega(n, K)$ of the latter group. Thus, the functor

$$\mathrm{K}_1(\mathrm{B}_n, R) = \mathrm{Spin}(2n+1, R)/\mathrm{Epin}(2n+1, R)$$

defined above is trivial for fields and differs from the Bass orthogonal K_1-functor

$$\mathrm{KO}_1(n, R) = \mathrm{SO}(n, R)/\mathrm{EO}(n, R)$$

(see [26, 17, 69, 110, 130]).

3. $\Phi = \mathrm{C}_n$. We number the indices from 1 to $2n$ as follows $1, \dots, n, -n, \dots, -1$ and introduce on the free module $V = R^m$, $m = 2n$, a symplectic form B by

$$B(x, y) = (x_1 y_{-1} - x_{-1} y_1) + \cdots + (x_n y_{-n} - x_{-n} y_n),$$

where

$$x = (x_1, \dots, x_{-1}), \qquad y = (y_1, \dots, y_{-1}) \in V.$$

Then $G_{\mathrm{sc}}(\Phi, R)$ is isomorphic to the *symplectic group* $\mathrm{Sp}(2n, R)$ associated with this form, while the group $G_{\mathrm{ad}}(\Phi, R)$ is isomorphic to the corresponding projective group $\mathrm{PGSp}(2n, R)$ (the factor group of the group $\mathrm{GSp}(2n, R)$ of symplectic similarities modulo the centre). The group $E_{\mathrm{sc}}(\Phi, R)$ is the corresponding elementary group $\mathrm{Ep}(2n, R)$, and

$$\mathrm{K}_1(\mathrm{C}_n, R) = \mathrm{KSp}_1(2n, R) = \mathrm{Sp}(2n, R)/\mathrm{Ep}(2n, R),$$

is the usual *nonstable symplectic* K_1*-functor* [27, 69].

4. $\Phi = \mathrm{D}_n$. Let us keep the same numbering of indices as for the case of $\Phi = \mathrm{C}_n$ and introduce a quadratic form Q on the free R-module $V = R^m$, $m = 2n$, by

$$Q(x_1, \dots, x_n) = x_1 x_{-1} + \cdots + x_n x_{-n}.$$

This form has the maximal Witt index and makes V to an *Artin space*. The group $G_{\mathrm{sc}}(\Phi, R)$ is again the *spinorial group*, associated with the form Q, while $G_{\mathrm{ad}}(\Phi, R) = \mathrm{PGO}(n, R)$ is the projective orthogonal group (the factor group of the group $\mathrm{GO}(2n, R)$ of orthogonal similarities modulo the centre).

For one of the proper intermediate lattices P, lying between $Q(\Phi)$ and $P(\Phi)$ (if n is odd, this lattice is unique), we have $G_P(\Phi, R) = \mathrm{SO}(2n, R)$. In this case, the situation is similar to $\Phi = \mathrm{B}_n$ and the functor

$$\mathrm{K}_1(\mathrm{D}_n, R) = \mathrm{Spin}(2n, R)/\mathrm{Epin}(2n, R)$$

is again different from the Bass functor $\mathrm{KO}_1(2n, R)$.

The identifications for the classical groups go back to the works of R. Ree [94] and J. Dieudonné [54] (where the case of an odd ortogonal group over a nonperfect field of characteristic 2 was treated). For a modern exposition, see J.-Y. Hée [71].

For the exceptional cases, explicit identifications with groups, preserving systems of tensors, are far less commonly known, although there are quite a number of works dedicated to this topic. We describe these identifications very briefly, referring to [11, 12, 13, 44, 105, 141] for further information and references. The whole subject will be discussed in detail in a forthcoming paper by the first-named author. That is why we do not go into technical details (preservation of a three-linear form and the corresponding cubic form, partial polarizations, etc.), and we restrict ourselves to the case when K is a field of characteristic distinct from 2 and 3. In fact, most of the invariants constructed below are characteristic free and thus define the Chevalley groups over an arbitrary commutative ring.

5. $\Phi = \mathrm{G}_2$. The group of type G_2 was introduced by L. E. Dickson in 1905 and Ree [94] verified that Dickson's group actually coincides with the Chevalley group of type G_2. In fact, the group of type G_2 is, in many respects, close to the classical groups and can be considered in this context.

Let $V = K^7$ be a seven-dimensional vector space with the coordinates numbered as follows: $1, 2, 3, 0, -3, -2, -1$. We consider the following pair of forms on V: the same quadratic form, as for the case of B_3

$$Q(x) = 2x_0^2 + x_1x_{-1} + x_2x_{-2} + x_3x_{-3},$$

where

$$x = (x_1, x_2, x_3, x_0, x_{-3}, x_{-2}, x_{-1}) \in V,$$

and the alternating three linear form F, defined by the monomials

$$x_0x_1x_{-1} + x_0x_2x_{-2} + x_0x_3x_{-3} + x_1x_2x_3 + x_{-1}x_{-2}x_{-3}.$$

This should be understood in the following sense. To get the form $F(x, y, z)$, one should alternate each monomial to get six summands. For example, the first monomial gives the following summands

$$x_0y_1z_{-1} + y_0z_1x_{-1} + z_0x_1y_{-1} - y_0x_1z_{-1} - z_0y_1x_{-1} - x_0z_1y_{-1}.$$

The group $\mathrm{Isom}(Q, F, K)$ consists of such elements g of $\mathrm{GL}(V) = \mathrm{GL}(7, K)$, that $Q(gx) = Q(x)$ and $F(gx) = F(x)$ for every $x \in V$. It was proven in [95] that $G(\mathrm{G}_2, K) = \mathrm{Isom}(Q, F, K)$.

6. $\Phi = \mathrm{E}_6$. Let $\mathrm{M}(3, K)$ be the full matrix ring of degree 3 over K, and $V = \{(x, y, z) \mid x, y, z \in \mathrm{M}(3, K)\}$ be the 27-dimensional vector space over K. Define a cubic form F on V by the following formula:

$$F((x, y, z)) = \det(x) + \det(y) + \det(z) - \mathrm{tr}(xyz).$$

Then $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ can be identified with the group $\mathrm{Isom}(F, K)$, consisting of all transformations $g \in \mathrm{GL}(V) = \mathrm{GL}(27, K)$, such that $F(g(x, y, z)) = F((x, y, z))$.

This remarkable construction of simply connected Chevalley groups of type E_6 is due to Freudenthal. A similar (but slightly more complicated) realization of this group was discovered by Dickson as early as 1905. Usually, it is more convenient not to consider the cubic form, but the trilinear form, associated with the above form.

7. $\Phi = \mathrm{F}_4$. The group of type F_4 is obtained by restricting the group of type E_6 constructed above to certain hyperplanes in the 27-dimensional space V. (Actually, almost *any* hyperplane will work, the variety of hyperplanes which do not has positive codimension. For the exceptional hyperplanes, one gets groups of type B_4 or – falling still deeper – D_4.)

8. $\Phi = \mathrm{E}_7$. The first explicit construction of the form in this case is again due to Freudenthal and may be described as follows. Let V be the vector space

$$V = \Big\{(x, y) \mid x, y \in \mathrm{M}(8, K),\ x^t = -x,\ y^t = -y\Big\}$$

consisting of pairs of 8×8 alternating matrices with entries in the field K (its dimension over K equals 56). Define a symplectic inner product h on the space V by the formula

$$h((x_1, x_2), (y_1, y_2)) = 1/2(\mathrm{tr}(y_1 x_2) - \mathrm{tr}(x_1 y_2))$$

and a *quartic* form F by

$$F((x, y)) = \mathrm{pf}(x) + \mathrm{pf}(y) - 1/4\mathrm{tr}((xy)^2) + 1/16\mathrm{tr}(xy)^2,$$

where $\mathrm{pf}(x)$ denotes the *Pfaffian* of an alternating matrix x. Polarize F to a symmetric four-linear form f on V. Then the isometry group of the pair of forms h and f coincides with $G_{\mathrm{sc}}(\mathrm{E}_7, K)$. In other words, $G_{\mathrm{sc}}(\mathrm{E}_7, K)$ consists of those symplectic (with respect to h) matrices which preserve the form f or, what is the same (recall that $\mathrm{char}(K) \neq 2, 3$), the form F, in the usual sense

$$G_{\mathrm{sc}}(\mathrm{E}_7, K) = \Big\{g \in \mathrm{Sp}(56, K) \mid f(g(x, y)) = f((x, y))\Big\}.$$

The group we get in this way is the simply connected Chevalley group $G_{\mathrm{sc}}(\mathrm{E}_7, R)$.

9. $\Phi = \mathrm{E}_8$. Let $V = L$ be the Chevalley algebra of type E_8 over K. It has dimension 248. Recall that $V = L$ bears the Lie bracket $[\ ,\ ]$: $V \times V \to V$ as well as the Killing form $(\ ,\ )$: $V \times V \to K$. Then $G_{\mathrm{sc}}(\mathrm{E}_8, K)$ can be identified with the isometry group of the trilinear form $f(x, y, z) = ([x, y], z)$, where $x, y, z \in V$.

9. Chevalley Commutator Formula

Let us recall some properties of the *elementary root unipotent elements* $x_\alpha(\xi)$, which do not depend on a representation. It is clear that

$$x_\alpha(\xi) x_\alpha(\eta) = x_\alpha(\xi + \eta), \tag{9.1}$$

for every $\xi \in R$ and $\eta \in R$ and, thus, for a fixed $\alpha \in \Phi$, the map x_α: $\xi \to x_\alpha(\xi)$ is a homomorphism of the additive group R^+ of the ring R to the one-parameter subgroup $X_\alpha\{x_\alpha(\xi) \mid \xi \in R\}$. This subgroup is called the *elementary unipotent root subgroup* corresponding to α. In fact, x_α is an isomorphism of R^+ on X_α. When it does not lead to a confusion, we omit epithets 'elementary' and 'unipotent' and speak about *root elements* and *root subgroups*. For two elements $x, y \in G$, we denote by $[x, y]$ their commutator $xyx^{-1}y^{-1}$. Now let $\alpha, \beta \in \Phi$, $\alpha + \beta \neq 0$, $\xi, \eta \in R$. Then the *Chevalley commutator formula* asserts, that

$$[x_\alpha(\xi), x_\beta(\eta)] = \prod x_{i\alpha + j\beta}(N_{\alpha\beta ij} \xi^i \eta^j), \tag{9.2}$$

where the product on the right-hand side is taken over all roots of the form $i\alpha + j\beta \in \Phi$, $i, j \in \mathbb{N}$, in any given order. The constants $N_{\alpha\beta ij}$ do not depend on ξ and η (though they may depend on the order of roots). The numbers $N_{\alpha\beta ij}$ are called the *structure constants of the Chevalley group*, and we will see that all of them are integers [14, 15, 38, 39, 42, 76, 94, 95, 113], etc.

First of all, $N_{\alpha\beta 11} = N_{\alpha\beta}$, where $N_{\alpha\beta}$ are the structure constants of the Lie algebra L in the Chevalley base, i.e. $[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$ and, therefore, they are equal to $\pm 1, \pm 2, \pm 3$. Thus,

$$[x_\alpha(\xi), x_\alpha(\eta)] = 1, \tag{9.3}$$

if $\alpha + \beta \notin \Phi$, and

$$[x_\alpha(\xi), x_\alpha(\eta)] = x_{\alpha+\beta}(N_{\alpha\beta}\xi\eta), \tag{9.4}$$

if $\alpha + \beta$ is the only linear combination of the roots α and β with natural coefficients, belonging to Φ.

In the case when all roots of Φ have the same length (the '*simply laced*' case), only these two possibilities occur. Thus, in this case, the coefficients in the Chevalley formula are completely determined by the structure constants in the corresponding Lie algebra (and, therefore, can be found in the tables [36, 60, 87, 88, 142], see the details below). For the general case, the coefficients $N_{\alpha\beta ij}$ are expressed via the constants $M_{\alpha\beta i}$, mentioned in Section 2. More precisely, if we order the roots $i\alpha + j\beta$ according to increasing $i + j$, the following formulae hold (see [38, 39]):

$$N_{\alpha\beta i1} = M_{\alpha\beta i}, \qquad N_{\alpha\beta 1j} = -M_{\beta\alpha j},$$

$$N_{\alpha\beta 23} = 1/3 M_{\alpha+\beta,\beta,2}, \qquad N_{\alpha\beta 32} = 2/3 M_{\alpha+\beta,\alpha,2}.$$

Recall that the constants $M_{\alpha\beta k}$ were defined as follows:

$$M_{\alpha\beta k} = 1/k! \prod_{i=1}^{k} N_{\alpha,\beta+(i-1)\alpha} = \pm C_k^{p+k}.$$

A different formula for calculation of $N_{\alpha\beta ij}$ is contained in [53, 76]. Suppose the roots $i\alpha + j\beta$ are arranged in *any* order. Then

$$\begin{aligned}
N_{\alpha\beta 32} &= 2M_{\alpha\beta 3}N_{\beta,3\alpha+\beta}, && \text{if } \alpha+\beta < 2\alpha+\beta,\\
N_{\alpha\beta 32} &= -M_{\alpha\beta 3}N_{\beta,3\alpha+\beta}, && \text{if } \alpha+\beta > 2\alpha+\beta,\\
N_{\alpha\beta 23} &= -2M_{\beta\alpha 3}N_{\alpha,\alpha+3\beta}, && \text{if } \alpha+2\beta < \alpha+\beta,\\
N_{\alpha\beta 23} &= M_{\beta\alpha 3}N_{\alpha,\alpha+3\beta}, && \text{if } \alpha+2\beta > \alpha+\beta.
\end{aligned}$$

Yet another choice of signs in the Chevalley commutator formula is reproduced in [103].

Now, an inspection of the root systems of types A_2, B_2, G_2 shows that $N_{\alpha\beta ij}$ may only take the values $\pm 1, \pm 2, \pm 3$. Clearly G_2 is the only (irreducible) system Φ, in which the number of linear combinations of two roots $\alpha, \beta \in \Phi$ of the form

TABLE I. Positive roots of F_4.

1000	0100	0010	0001	1100	0110
0011	1110	0120	0111	1120	1111
0121	1220	1121	0122	1221	1122
1231	1222	1232	1242	1342	2342

TABLE II. The $N_{\alpha\beta}$-matrix for F_4.

	1	0	0	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	1	1	1	1	1	2
	0	1	0	0	1	1	0	1	1	1	1	1	1	2	1	1	2	1	2	2	2	2	3	3
	0	0	1	0	0	1	1	1	2	1	2	1	2	2	2	2	2	2	3	2	3	4	4	4
	0	0	0	1	0	0	1	0	0	1	0	1	1	0	1	2	1	2	1	2	2	2	2	2
1000	0	1	0	0	0	1	0	0	1	1	0	0	1	0	0	1	0	0	0	0	0	0	1	0
0100	−1	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0
0010	0	−1	0	1	−1	2	0	2	0	1	0	1	0	0	0	0	1	0	0	1	2	0	0	0
0001	0	0	−1	0	0	−1	0	−1	−1	0	−1	0	2	−1	2	0	2	0	1	0	0	0	0	0
1100	0	0	1	0	0	0	1	0	−1	0	0	0	−1	0	0	−1	0	0	0	0	0	1	0	0
0110	−1	0	−2	1	0	0	−1	2	0	0	0	1	0	0	−1	0	0	−1	0	0	2	0	0	0
0011	0	−1	0	0	−1	1	0	1	0	−2	0	−2	0	−1	0	0	1	0	2	0	0	0	0	0
1110	0	0	−2	1	0	−2	−1	0	0	−1	0	0	1	0	0	1	0	0	0	0	2	0	0	0
0120	−1	0	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0	−1	0	−1	0	0	0	0
0111	−1	0	−1	0	0	0	2	1	0	0	1	−2	0	0	−1	0	0	0	2	0	0	0	0	0
1120	0	−1	0	1	0	0	0	0	0	−1	0	0	0	0	0	1	0	0	0	−1	0	0	0	0
1111	0	0	−1	0	0	−1	2	0	−1	2	0	0	1	0	0	0	0	0	2	0	0	0	0	0
0121	−1	0	0	−2	1	0	0	−1	0	0	0	−1	0	0	−2	0	−2	0	0	0	0	0	0	0
1220	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0
1121	0	−1	0	−2	0	1	0	0	0	1	0	0	2	0	0	0	−2	0	0	0	0	0	0	0
0122	−1	0	0	0	1	0	0	−1	0	0	−1	0	0	−1	0	0	0	0	0	0	0	0	0	0
1221	0	0	−1	−2	0	0	−1	0	0	0	0	0	2	0	2	0	0	0	0	0	0	0	0	0
1122	0	−1	0	0	0	1	0	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0
1231	0	0	0	−1	0	0	−2	0	0	−2	0	−2	0	0	0	0	0	0	0	0	0	0	0	0
1222	0	0	−1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1232	0	0	−2	0	0	−2	0	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1242	0	−1	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1342	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2342	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$i\alpha + j\beta \in \Phi$, $i, j \in \mathbb{Z}_+$, can be larger than two. Hence, for all systems except G_2, the two first formulae suffice. More precisely, if two roots α and β generate the system B_2 with α being long and β being short, then

$$[x_\alpha(\xi), x_\beta(\eta)] = x_{\alpha+\beta}(\pm\xi\eta)x_{\alpha+2\beta}(\pm\xi\eta^2), \tag{9.5}$$

TABLE III. The $N_{\alpha\beta 21}$-matrix for F_4.

	1	0	1	0	1	1	0	1	1	1	1	2
	0	1	1	1	1	2	1	1	2	2	3	3
	0	0	0	2	2	2	2	2	2	4	4	4
	0	0	0	0	0	0	2	2	2	2	2	2
0010	0	−1	−1	0	0	0	0	0	1	0	0	0
0001	0	0	0	−1	−1	−1	0	0	0	0	0	0
0110	−1	0	0	0	0	0	0	−1	0	0	0	0
0011	0	1	1	0	0	−1	0	0	0	0	0	0
1110	0	0	0	0	0	0	1	0	0	0	0	0
0111	1	0	0	0	1	0	0	0	0	0	0	0
1111	0	0	0	−1	0	0	0	0	0	0	0	0
0121	1	0	−1	0	0	0	0	0	0	0	0	0
1121	0	1	0	0	0	0	0	0	0	0	0	0
1221	0	0	0	0	0	0	0	0	0	0	0	0
1231	0	0	0	0	0	0	0	0	0	0	0	0
1232	0	0	0	0	0	0	0	0	0	0	0	0

TABLE IV. Chevalley commutator formula for G_2.

$$[x_{10}(\xi), x_{01}(\zeta)] = x_{11}(\xi\zeta)x_{21}(\xi^2\zeta)x_{31}(\xi^3\zeta)x_{32}(2\xi^3\zeta^2)$$
$$[x_{10}(\xi), x_{11}(\zeta)] = x_{21}(2\xi\zeta)x_{31}(3\xi^2\zeta)x_{32}(-3\xi\zeta^2)$$
$$[x_{10}(\xi), x_{21}(\zeta)] = x_{31}(3\xi\zeta)$$
$$[x_{01}(\xi), x_{31}(\zeta)] = x_{32}(\xi\zeta)$$
$$[x_{11}(\xi), x_{21}(\zeta)] = x_{32}(-3\xi\zeta)$$

where the signs are specified by the above formulae.

For the classical cases, the structure constants are nothing mysterious: use the same Chevalley order as in [33]. For the cases E_l, one has only to list the structure constants for the corresponding Lie algebra. This is done, for example, in [60, 142] (see references there for a broader picture). Actually, it is *the* choice of the structure constants which we will always use (see Section 15). This leaves us with the analysis of the cases F_4 and G_2. These cases are relatively small and the corresponding structure constants were calculated by several authors on various occasions.

A possible choice of the coefficients $N_{\alpha\beta ij}$ for the system F_4 is given in Tables II and III (Table I lists Dynkin forms of positive roots of F_4 in the height lexicographic order). Namely, Table II lists the structure constants $N_{\alpha\beta} = N_{\alpha\beta 11}$ of the corresponding Lie algebra (see Section 15). Table III lists only the constants $N_{\alpha\beta 21}$, but, as we know, $N_{\alpha\beta 12} = -N_{\beta\alpha 21}$.

For the group of type G_2, the Chevalley commutator formula is somewhat more complicated. Namely, the two following cases are not covered by the formulae (9.3)–(9.4) above. If two short roots α and β form the angle of $2\pi/3$, then

$$[x_\alpha(\xi), x_\beta(\eta)] = x_{\alpha+\beta}(\pm 2\xi\eta) x_{2\alpha+\beta}(\pm 3\xi^2\eta) x_{\alpha+2\beta}(\pm 3\xi\eta^2). \tag{9.6}$$

If a long root α and a short root β form the angle of $5\pi/6$, then

$$[x_\alpha(\xi), x_\beta(\eta)] = x_{\alpha+\beta}(\pm\xi\eta) x_{\alpha+2\beta}(\pm\xi\eta^2) x_{\alpha+3\beta}(\pm\xi\eta^2) x_{2\alpha+3\beta}(\pm\xi^2\eta^3). \tag{9.7}$$

These relations are studied in detail in [112], Section 10. A particular choice of signs in these formulae is listed in Table IV.

10. Split Maximal Torus

In this section, we explain how the distinction between a Chevalley group and its elementary subgroup is bridged for fields. Let $T = T_P(\Phi, \ )$ be a split maximal torus of a Chevalley–Demazure group scheme $G = G_P(\Phi, \ )$. If R is a commutative ring, the corresponding group of points $T = T_P(\Phi, R)$ is called a *split maximal torus* of the Chevalley group $G = G_P(\Phi, R)$. It is well known that

$$T = T_P(\Phi, R) = \mathrm{Hom}(\mathbb{Z}[T], R) \cong \mathrm{Hom}(P, R^*),$$

where $\mathbb{Z}[T] = \mathbb{Z}[\lambda_1, \lambda_1^{-1}, \dots, \lambda_l, \lambda_l^{-1}]$ is the algebra of Laurent polynomials for some $\mathbb{Z}$-base $\lambda_1, \dots, \lambda_l$ of the lattice P.

Let, as usual, v^λ, $\lambda \in \Lambda(\pi)$, be an admissible base of the Weyl module $V = V_R$ of the Chevalley group $G = G_P(\Phi, R)$, constructed starting from a finite-dimensional representation π of the Lie algebra $L = L_{\mathbb{C}}$. We arbitrarily number the weights $\lambda \in \Lambda(\pi)$ and set $v_i = v^{\lambda_i}$. Thus, $v_1, \dots, v_n$ is a base V, consisting of weight vectors corresponding to the weights $\lambda_1, \dots, \lambda_n$. Let us denote by $X(P, R)$ the set of R-characters of the weight lattice $P = P(\pi)$, i.e. the group $\mathrm{Hom}(P, R^*)$, and by $X'(P, R)$ the subgroup in $X(P, R)$ consisting of those R-characters, which can be extended to $P(\Phi)$.

If the group G is simply connected, that is $P = P(\Phi)$, then $X'(P, R) = X(P, R)$ but, in general, not every character of P can be extended to the whole weight lattice $P(\Phi)$. For a $\chi \in X(P, R)$, denote by $h^\pi(\chi)$ the endomorphism of the module V, which is defined in the base $v_1, \dots, v_n$ by the diagonal matrix $\mathrm{diag}(\chi(\lambda_1), \dots, \chi(\lambda_n))$. In other words, $h^\pi(\chi)v^\lambda = \chi(\lambda)v^\lambda$ for all $\lambda \in \lambda(\pi)$. Then

$$T = T_P(\Phi, R) = \{h^\pi(\chi),\ \chi \in X(P, R)\}.$$

Let

$$H = H_P(\Phi, R) = \{h^\pi(\chi),\ \chi \in X'(P, R)\}.$$

It can be shown that

$$H_P(\Phi, R) = T_P(\Phi, R) \cap \mathrm{E}_P(\Phi, R).$$

Thus, $H_P(\Phi, R) \leqslant T_P(\Phi, R)$ and if $P = P(\Phi)$, then $H_P(\Phi, R) = T_P(\Phi, R)$. The group

$$G_P^0(\Phi, R) = \mathrm{E}_P(\Phi, R) T_P(\Phi, R)$$

is contained in the Chevalley group $G_P(\Phi, R)$ and it is well known that these groups coincide in the case when $R = K$ is a field or, more generally, a semilocal ring [1, 9, 85]:

$$G_P(\Phi, K) = G_P^0(\Phi, K).$$

Impressed by the fact that $E_{\mathrm{ad}}(\Phi, K)$ is usually simple as an abstract group, many authors call the latter group the 'Chevalley group'. In this context, it is interesting to notice that in the original paper [41] Chevalley studied the groups $G_{\mathrm{ad}}^0(\Phi, K)$ rather than $E_{\mathrm{ad}}(\Phi, K)$.

The elements $h(\chi)$ are related to the elementary generators $x_\beta(\xi)$ by the formula

$$h(\chi) x_\beta(\xi) h(\chi)^{-1} = x_\beta(\chi(\beta)\xi).$$

Many more details about the elements $h(\chi)$ as well as an exhaustive bibliography may be found in [145].

11. Regularly Embedded Subgroups

Now we recall definition of some important subgroups in Chevalley group $G = G(\Phi, R)$. First, set

$$U = U(\Phi, R) = \langle x_\alpha(\xi),\ \xi \in R,\ \alpha \in \Phi^+ \rangle,$$

$$U^- = U^-(\Phi, R) = \langle x_\alpha(\xi),\ \xi \in R,\ \alpha \in \Phi^- \rangle.$$

The subscript P is redundant in the notation as these groups do not depend on representation π up to isomorphism. In the field case, these groups are *maximal unipotent subgroups* in G. For an arbitrary commutative ring R and an arbitrary ordering of the set Φ^+ of positive roots, we have $U = \prod X_\alpha$, $U^- = \prod X_{-\alpha}$, where $\alpha \in \Phi^+$ in the given order. More precisely, each element $u \in U$ can be uniquely expressed in the form $\prod x_\alpha(u_\alpha)$, $\alpha \in \Phi^+$, where the coefficients $u_\alpha \in R$ depend only on u and the chosen order of roots. If $u = \prod x_\alpha(u'_\alpha)$, $\alpha \in \Phi^+$, where the product is taken in another order, then it follows from the Chevalley commutator formula that $u'_\alpha - u_\alpha$ is a linear combination of the products of the form $u_{\beta_1} \cdot \ldots \cdot u_{\beta_s}$, where $\beta_i \in \Phi^+$ are such that $\beta_1 + \cdots + \beta_s = \alpha$. In particular, $u'_\alpha = u_\alpha$ for all simple roots α.

Recall that a subset $S \subseteq \Phi$ is called *closed* if for any $\alpha, \beta \in S$ such that $\alpha + \beta \in \Phi$, one has $\alpha + \beta \in S$. With a closed subset S, one can associate the subgroup

$$E(S) = E(S, R) = \langle x_\alpha(\xi),\ \xi \in R,\ \alpha \in S \rangle.$$

Each closed set S may be presented as a disjoint union of two parts, *reductive* (alias *symmetric*) $S^r = \{\alpha \in S,\ -\alpha \in S\}$ and *special* (alias *unipotent*) $S^u = \{\alpha \in S, -\alpha \notin S\}$. It is clear that the special part of a closed set is an *ideal* in it (i.e. if $\alpha \in S$, $\beta \in S^u$, $\alpha + \beta \in \Phi$ then $\alpha + \beta \in S^u$) and, therefore, it follows from the Chevalley commutator formula that $E(S)$ may be presented as a semidirect product of its *unipotent radical* $E(S^u)$ (which is a normal subgroup in $E(S)$) and a reductive subgroup $E(S^r)$ (a *Levi subgroup* of $E(S)$):

$$E(S) = E(S^u) \leftthreetimes E(S^r).$$

In the sequel, we use subgroups which stand in the same relation to $E(S)$ as a Chevalley group does to its elementary subgroup. In order to construct them, we must return to the situation considered in Section 7. Let $R = \mathbb{C}$. We set

$$G = G(S, \mathbb{C}) = E(S, \mathbb{C}) \quad \text{and} \quad G^0 = G^0(S, \mathbb{C}) = G(S, \mathbb{C})T(\Phi, \mathbb{C}).$$

These are algebraic subgroups of the Chevalley group $G = G(\Phi, \mathbb{C})$. Let $X(S)$ denote one of the groups G or G^0. Restrictions of the coordinates $x_{\lambda,\mu}$, $\lambda, \mu \in \Lambda(\pi)$ (with respect to an admissible base v^λ), $\lambda \in \Lambda(\pi)$ to the group $X(S)$, generate an affine $\mathbb{Z}$-algebra $\mathbb{Z}[X(S)]$. Now, the natural projection $\mathbb{Z}[G] \to \mathbb{Z}[X(S)]$ induces the homomorphism

$$\mathrm{Hom}(\mathbb{Z}[X(S), R]) \longrightarrow \mathrm{Hom}(\mathbb{Z}[G], R) = G(\Phi, R).$$

We denote the group on the left-hand side by

$$G(S, R) = G(S) \quad \text{or} \quad G^0(S, R) = G^0(S),$$

respectively. These groups admit Levi decompositions

$$G(S) = E(S^n) \leftthreetimes G(S^r) \quad \text{and} \quad G^0(S) = E(S^u) \leftthreetimes G^0(S^r)$$

with the same unipotent radical as $E(S)$. Clearly, S^r is a root system and the semisimple group $G(S^r) = G(S^r, R)$ coincides with the corresponding Chevalley group. The reductive group $G^0(S^r) = G^0(S^r, R)$ is the product of $G(S^r, R)$ and $T(\Phi, R)$. It not only depends on S, but also on the ambient group $G(\Phi, R)$.

The following two special cases of the above construction are of particular interest.

First, if $\Delta \subseteq \Phi$ is a *root subsystem*, then there is an embedding of Chevalley groups $G(\Delta, R) \subseteq G(\Phi, R)$, taking roots to roots. Following Dynkin, we used to call such embeddings '*regular*'. At the same time, the expression 'subsystem subgroups' coined by Liebeck and Seitz seems to be more suggestive.

Second, let Q be a *parabolic* set of roots, i.e. a closed set of roots such that for any root $\alpha \in \Phi$ either $\alpha \in Q$ or $-\alpha \in Q$. The group $G^0(Q) = G^0(Q, R)$ and its conjugates are called 'parabolic' subgroups. The proofs of many results are based on a reduction to the groups of smaller rank using parabolic subgroups. Note that in the case when the ring R is not a field, the subgroups $G^0(Q)$, $\Phi^+ \subseteq Q \subseteq \Phi$, by far do not exhaust all the subgroups containing the *standard Borel subgroup*

$$B = B(\Phi, R) = U(\Phi, R)T(\Phi, R).$$

See [137, 152, 153] for the description of other such subgroups and further references.

12. Normalizer of Maximal Torus

Let us apply the construction from the previous section to the special case of $\Delta = \{\pm\alpha\}$. We get an embedding φ_α of the group $G(\Delta, R)$, which is isomorphic to $\mathrm{SL}(2, R)$ or $\mathrm{PGL}(2, R)$, into the Chevalley group $G(\Phi, R)$. The image of this embedding will be denoted by $G_\alpha = G_\alpha(R)$. In the case when $R = K$ is a field, G_α is equal to $\langle X_\alpha, X_{-\alpha}\rangle$. This embedding can be normalized in such a way that:

$$\varphi_\alpha\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} = x_\alpha(\xi), \qquad \varphi_\alpha\begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} = x_{-\alpha}(\xi).$$

For $\varepsilon \in R^*$ we set

$$\varphi_\alpha\begin{pmatrix} 0 & \varepsilon \\ \varepsilon^{-1} & 0 \end{pmatrix} = w_\alpha(\varepsilon), \qquad \varphi_\alpha\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} = h_\alpha(\varepsilon).$$

In fact, these elements can be easily expressed in terms of x_α and $x_{-\alpha}$, namely

$$w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon), \tag{12.1}$$

$$h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1}. \tag{12.2}$$

The elements $h_\alpha(\varepsilon)$ and their conjugates are called *semisimple root elements.* Sometimes, in order to be specific, we speak about *long* or *short* root semisimple elements depending on whether α is long or short. The elements h_α can be easily expressed in terms of Section 10. Namely, let us fix an $\varepsilon \in R^*$ and consider the R-character $\chi_{\alpha,\varepsilon}$ of the lattice $P(\Phi)$, defined by the formula $\chi_{\alpha,\varepsilon}(\lambda) = \varepsilon^{\langle\lambda,\alpha\rangle}$. Then $h_\alpha(\varepsilon) \in H(\Phi, R)$, and

$$H(\Phi, R) = \langle h_\alpha(\varepsilon),\ \varepsilon \in R^*,\ \alpha \in \Phi\rangle.$$

Moreover, in fact, we can take only $\alpha \in \Pi$.

Now we introduce some important subgroups. Let

$$N_0(\Phi,R)=\langle w_\alpha(\varepsilon),\ \varepsilon\in R^*,\ \alpha\in\Phi\rangle.$$

Denote by $N(\Phi,R)$ the product $N_0(\Phi,R)T(\Phi,R)$. Then it is well known that $H(\Phi,R)$, and $T(\Phi,R)$ are normal subgroups of $N(\Phi,R)$ and

$$N(\Phi,R)/T(\Phi,R)\cong N_0(\Phi,R)/H(\Phi,R)\cong W,$$

where $W=W(\Phi,R)$ is the Weyl group of the root system Φ. These isomorphisms can be defined in such a way that a root reflection $w_\alpha\in W$ maps to the classes $w_\alpha(\varepsilon)T$ or $w_\alpha(\varepsilon)H$, respectively. The group $N(\Phi,R)$ is the *normalizer of the maximal torus* in the sense of the theory of algebraic groups. When $R=K$ is a field, the group $N(\Phi,K)$ almost always coincides with the normalizer of $T(\Phi,K)$ in the abstract sense. The possible exceptions are the field $\mathbb{F}_2$ and for some systems $\mathbb{F}_3$. In an appropriate base of weight vectors, the elements from $T(\Phi,R)$ are represented by diagonal matrices, while those from $N(\Phi,R)$ by monomial matrices.

The key role in various problems concerning Chevalley groups is played by the so-called *extended Weyl group* $\widetilde{W}=\widetilde{W}(\Phi)$ of the roots system Φ, which is isomorphic to the group $N(\Phi,\mathbb{Z})$ (see [53, 127]). This group is called also the *Tits–Demazure group* (see [89]). In the case, when $2\in R^*$, this group coincides with the group $\widetilde{W}=\langle w_\alpha(1),\ \alpha\in\Phi\rangle$. It is known that the group $\widetilde{W}$ is an extension of the usual Weyl group W by the elementary abelian group of order 2^l, thus $|\widetilde{W}|=2^l|W|$.

13. Steinberg Relations

Apart from Relations (9.1), (9.2), (12.1), (12.2) from Sections 9 and 12 elements $x_\alpha(\xi)$, $w_\alpha(\varepsilon)$ and $h_\alpha(\varepsilon)$ satisfy the following relations, which are also called *Steinberg relations*

$$\begin{aligned}
w_\alpha(\varepsilon)x_\beta(\xi)w_\alpha(\varepsilon)^{-1} &= x_{w_\alpha\beta}(\eta_{\alpha\beta}\varepsilon^{-\langle\beta,\alpha\rangle}\xi), &(13.1)\\
w_\alpha(\varepsilon)w_\beta(\omega)w_\alpha(\varepsilon)^{-1} &= w_{w_\alpha\beta}(\eta_{\alpha\beta}\varepsilon^{-\langle\beta,\alpha\rangle}\omega), &(13.2)\\
w_\alpha(\varepsilon)h_\beta(\omega)w_\alpha(\varepsilon)^{-1} &= h_{w_\alpha\beta}(\omega), &(13.3)\\
h_\alpha(\varepsilon)x_\beta(\xi)h_\alpha(\varepsilon)^{-1} &= x_\beta(\varepsilon^{\langle\beta,\alpha\rangle}\omega), &(13.4)\\
h_\alpha(\varepsilon)w_\beta(\omega)h_\alpha(\varepsilon)^{-1} &= w_\beta(\varepsilon^{\langle\beta,\alpha\rangle}\omega), &(13.5)
\end{aligned}$$

for all $\varepsilon,\omega\in R^*$, $\xi\in R$, $\alpha,\beta\in\Phi$. The numbers $\eta_{\alpha\beta}$ are constants equal to ±1 and depending only on α,β (and, of course, on the choice of the structure constants of Chevalley algebra). It is known [39, 94, 113], that $\eta_{\alpha\beta}$ satisfy the following properties:

$$\eta_{\alpha\beta}=\eta_{\alpha,-\beta},\qquad \eta_{\alpha\alpha}=\eta_{\alpha,-\alpha}=-1,\qquad \eta_{\alpha\beta}\eta_{\alpha w_\alpha(\beta)}=(-1)^{A_{\alpha\beta}},\quad (13.6)$$

and

$$\begin{aligned}\eta_{\alpha\beta} &= 1, \quad \alpha \pm \beta \neq 0,\ \alpha \pm \beta \notin \Phi,\\ \eta_{\alpha\beta} &= \eta_{\beta\alpha} = -1, \quad \langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle = -1,\\ \eta_{\alpha\beta} &= -1, \quad \langle\alpha,\beta\rangle = 0,\ \alpha \pm \beta \in \Phi.\end{aligned} \tag{13.7}$$

There are further useful relations among the element $w_\alpha(\varepsilon)$ and $h_\alpha(\varepsilon)$. Thus,

$$w_\alpha(\varepsilon)w_\alpha(\omega) = h_\alpha(-\varepsilon\omega^{-1}). \tag{13.8}$$

In particular,

$$w_\alpha(\varepsilon)^2 = h_\alpha(-1), \qquad w_\alpha(\varepsilon)^{-1} = w_\alpha(-\varepsilon).$$

Finally, $w_\alpha(\varepsilon) = w_{-\alpha}(-\varepsilon)$. Any two elements $h_\alpha(\varepsilon)$ and $h_\beta(\omega)$ commute and the symbol $h_\alpha(\varepsilon)$ is multiplicative on ε, i.e.

$$h_\alpha(\varepsilon)h_\alpha(\omega) = h_\alpha(\varepsilon\omega). \tag{13.9}$$

Besides, $h_\alpha(\varepsilon)^{-1} = h_{-\alpha}(\varepsilon)$.

Let us show, how the numbers $\eta_{\alpha\beta}$ are expressed in terms of structure constants. If the roots α and β are proportional, then $\eta_{\alpha\beta} = -1$, and if they are orthogonal, then $\eta_{\alpha\beta} = 1$ (see (13.6), (13.7) above). Suppose the roots α and β have the same length and $\Phi \neq \mathrm{G}_2$.

If $\alpha, \beta, \alpha+\beta \in \Phi$, then a straightforward calculation, using the definition of elements $w_\alpha(\varepsilon)$ and the Chevalley commutator formula, shows that

$$w_\alpha(\varepsilon)x_\beta(\xi)w_\alpha(\varepsilon)^{-1} = x_{\alpha+\beta}(N_{\alpha\beta}\varepsilon\xi)$$

(we used the equality $N_{\alpha\beta} = N_{-\alpha,\alpha+\beta}$). Thus

$$\eta_{\alpha\beta} = N_{\alpha\beta}, \quad \alpha, \beta, \alpha+\beta \in \Phi.$$

Similarly, if $\alpha, \beta, \beta-\alpha \in \Phi$, then

$$w_\alpha(\varepsilon)x_\beta(\xi)w_\alpha(\varepsilon)^{-1} = x_{\beta-\alpha}(-N_{-\alpha,\beta}\varepsilon^{-1}\xi)$$

and

$$\eta_{\alpha\beta} = -N_{\alpha\beta} = N_{\alpha,-\beta} = N_{\beta-\alpha,\alpha}.$$

This completely determines all the constants $\eta_{\alpha\beta}$ in the case when $|\alpha| = |\beta|$, and the system Φ is distinct from G_2.

If the roots α and β have different length, α is a long root, β is a short one, then a similar calculation using the Chevalley commutator formula shows that

$$\eta_{\alpha\beta} = N_{\alpha\beta}, \quad \alpha+\beta \in \Phi,$$

$$\eta_{\alpha,-\beta} = N_{\alpha\beta}, \quad \beta-\alpha \in \Phi.$$

Thus, it remains to consider the case when β is long and α is short. Then

$$\eta_{\alpha\beta} = N_{\alpha\beta q1}, \quad \alpha+\beta \in \Phi,$$

$$\eta_{\alpha,-\beta} = (-1)^p N_{\alpha\beta p1}, \quad \beta-\alpha \in \Phi,$$

where p and q have the same sense as before, i.e. they correspond to the α-series of roots passing through β.

There are similar formulae expressing $\eta_{\alpha\beta}$ for the case when α, β are short roots, generating a root system of type G_2.

14. Properties of the Structure Constants

First of all we recall some well-known properties of the structure constants

$$\begin{aligned} N_{\alpha\beta} &= 0, \quad \alpha,\beta\in\Phi,\ \alpha+\beta\notin\Phi,\ \alpha+\beta\neq 0, && (14.1)\\ N_{\alpha\beta} &= -N_{\beta\alpha}, && (14.2)\\ N_{\alpha,-\beta} &= N_{\beta,-\alpha}, && (14.3)\\ N_{\alpha,\beta} &= \pm(p+1), && (14.4)\\ N_{\alpha\beta}N_{-\alpha,-\beta} &= -(p+1)^2. && (14.5)\end{aligned}$$

The proof of the following two formulae uses the Jacobi identity. If $\alpha+\beta+\gamma = 0$, then

$$N_{\alpha\beta}/(\gamma,\gamma) = N_{\beta\gamma}/(\alpha,\alpha) = N_{\gamma\alpha}/(\beta,\beta). \tag{14.6}$$

Let now $\alpha,\beta,\gamma,\delta$ be four roots such that $\alpha+\beta+\gamma+\delta=0$. Then

$$\begin{aligned} &N_{\alpha\beta}N_{\gamma\delta}/(\alpha+\beta,\alpha+\beta) + N_{\beta\gamma}N_{\alpha\delta}/(\beta+\gamma,\beta+\gamma)+\\ &\quad +N_{\gamma\alpha}N_{\beta\delta}/(\gamma+\alpha,\gamma+\alpha) = 0. \end{aligned} \tag{14.7}$$

We consider the matrix $N = (N_{\alpha\beta})$, $\alpha,\beta\in\Phi^+$. Formula (14.2) says that this matrix is antisymmetric. Now (14.3) (or (14.5)) expresses the matrix $(N_{\alpha\beta})$, $\alpha,\beta\in\Phi^-$ via N. The same formula expresses $(N_{\alpha\beta})$, $\alpha\in\Phi^-$, $\beta\in\Phi^+$ via $(N_{\alpha\beta})$, $\alpha\in\Phi^+$, $\beta\in\Phi^-$. It remains only to express $(N_{\alpha\beta})$, $\alpha\in\Phi^+$, $\beta\in\Phi^-$ via N. Indeed, applying (14.6) we get

$$N_{\alpha\beta} = \begin{cases} N_{\beta,-\alpha-\beta}(\alpha+\beta,\alpha+\beta)/(\alpha,\alpha) & \text{if } \alpha+\beta\in\Phi^+,\\ N_{-\alpha-\beta,\alpha}(\alpha+\beta,\alpha+\beta)/(\beta,\beta) & \text{if } \alpha+\beta\in\Phi^-. \end{cases} \tag{14.8}$$

This completely reduces the calculation of all the structure constants $N_{\alpha\beta}$ to the calculation of N.

When all the roots of Φ have the same length, the formulae above may be substantially simplified. Namely, for all roots $\alpha,\beta,\alpha+\beta\in\Phi$, one necessarily has $\alpha - \beta \notin \Phi$, so that $p = 0$ and $N_{\alpha\beta} = \pm 1$. Formulae (14.6) and (14.7) become

$$N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha}, \quad \text{if } \alpha + \beta + \gamma = 0 \tag{14.9}$$

and

$$N_{\alpha\beta}N_{\gamma\delta} + N_{\beta\gamma}N_{\alpha\delta} + N_{\gamma\alpha}N_{\beta\delta} = 0, \quad \text{if } \alpha + \beta + \gamma + \delta = 0. \tag{14.10}$$

Observe that in the last case, only two of the summands can be nonzero. Indeed, $N_{\alpha\beta} \neq 0$ and $N_{\beta\gamma} \neq 0$ implies that $\alpha + \beta, \beta + \gamma \in \Phi$. Thus, β forms angle $2\pi/3$ with both α and γ. Suppose $N_{\alpha\gamma} \neq 0$. Then the angle between α and γ also equals $2\pi/3$ and, thus, α, β, γ lie in one plane. But then $\delta = 0$, a contradiction. Thus, (14.7) is equivalent to a piece of the 2-cocycle equation:

$$N_{\beta\gamma}N_{\alpha,\beta+\gamma} = N_{\alpha+\beta,\gamma}N_{\alpha\beta}. \tag{14.11}$$

This equation is extremely important. Actually, the equation alone settles many cases where [133] refers to the explicit knowledge of signs.

15. Calculation of the Structure Constants

There is a natural inductive procedure based on the formulae from the proceeding section, which expresses all the structure constants in terms of the structure constants for certain pairs of roots, the so-called extraspecial pairs, see [126, 39]. In turn, the signs of the structure constants for the extraspecial pairs may be taken arbitrary (it is customary to take all of these signs to be '+').

Let us choose the height lexicographic ordering of positive roots which is *regular* (i.e. a root of smaller height always proceeds a root of larger height) and *lexicographic* at the roots of a given height. It is a total ordering of Φ^+ and we write $\alpha \prec \beta$ if α precedes β with respect to this ordering. By definition, this means that either $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$ or $\mathrm{ht}(\alpha) = \mathrm{ht}(\beta)$ and the integer represented by the string Dynkin form of α is *bigger* than the integer represented by the string Dynkin form of β.

Recall that a pair (α, β) of positive roots is called *special* if $\alpha + \beta \in \Phi$ and $\alpha \prec \beta$ with respect to the ordering described above. A pair (α, β) is called *extraspecial*, if it is special and for any special pair (γ, δ) such that $\alpha + \beta = \gamma + \delta$ one has $\alpha \prec \gamma$. Then the values of the structure constants $N_{\alpha\beta}$ may be taken arbitrarily at the extraspecial pairs and all the other structure constants may be uniquely determined using only properties (14.2), (14.5), (14.6) and (14.7) (see [39], pp. 58–60).

Since, for classical groups, the structure constants can be easily determined (it is sufficient to take Chevalley systems, considered in [33]), and for the group G_2 the structure constants are well known [39, 94, 113], etc., one has only to consider the cases $\mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$. Here is an inductive procedure from the paper

[60] (modulo correcting two misprints, see [134]), which works for E_6, E_7, E_8. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$. *Set* $N_{\alpha_i\beta} = 1$ *if* $\alpha_i + \beta \in \Phi^+$ *and there is no* $j < i$ *such that* $\alpha_i + \beta = \alpha_j + \bar\beta$ *for some* $\bar\beta \in \Phi^+$. If $\alpha, \beta, \alpha + \beta \in \Phi^+$, let i be minimal such that $\alpha + \beta = \alpha_i + \bar\beta$ for some $\bar\beta \in \Phi^+$. Since all the roots have the same length, either $\alpha - \alpha_i \in \Phi^+$, or $\beta - \alpha_i \in \Phi^+$, but not both. Now $N_{\alpha\beta}$ is expressed as follows

$$N_{\alpha\beta} = \begin{cases} -N_{\alpha_i,\beta-\alpha_i} N_{\beta-\alpha_i,\alpha} & \text{if } \beta - \alpha_i \in \Phi^+, \\ N_{\alpha_i,\alpha-\alpha_i} N_{\alpha-\alpha_i,\beta} & \text{if } \alpha - \alpha_i \in \Phi^+, \end{cases}$$

and, since the height of $\alpha - \alpha_i$ or $\beta - \alpha_i$ is strictly smaller than that of α or β, respectively, we get an algorithm to calculate the matrix $N = (N_{\alpha\beta})$, $\alpha, \beta \in \Phi^+$.

In order to get the matrix $N_{\alpha\beta}$, $\alpha, \beta \in \Phi$, the following relations can be used

$$N_{\alpha,-\beta} = \begin{cases} N_{\alpha-\beta,\beta}(\alpha - \beta, \alpha - \beta)/(\alpha, \alpha) & \text{if } \alpha - \beta \in \Phi^+, \\ N_{\beta-\alpha,\beta}(\beta - \alpha, \beta - \alpha)/(\beta, \beta) & \text{if } \beta - \alpha \in \Phi^+. \end{cases}$$

Especially important for the subsequent calculations is the matrix $N = (N_{\alpha\beta})$ of type $\Phi = E_8$. This is connected with the fact that the minimal representations of the groups $G(E_6, R)$ and $G(E_7, R)$ appear in the restrictions of the adjoint representation for the group of type $\Phi = E_8$. Therefore, for these types the signs of actions of the elementary root unipotent elements $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, on a weight base v^λ of a minimal module may be determined from the $N_{\alpha\beta}$-matrix of type E_8.

16. Frenkel–Kac Cocycle

Recall that a root system Φ is called *simply laced* if $\langle \alpha, \beta \rangle = 0, \pm 1$ for any two linearly independent roots $\alpha, \beta \in \Phi$. For an irreducible root system, this is equivalent to saying that all the roots of Φ have the same length. For a simply laced system, always $N_{\alpha\beta} = 0, \pm 1$, so the only problem is to determine the *signs* of the structure constants.

For the simply laced root systems there is a much more elegant and efficient way to calculate the structure constants, which is due to I. Frenkel and V. Kac [57] (see also [58, 79, 98, 104]). (Note that, in some sense, a similar algorithm for groups of type G_2 and F_4 is given in [97].)

Let $Q(\Phi)$ be the *root lattice* of the root system Φ, i.e. the $\mathbb{Z}$-lattice spanned by Φ endowed with the inner product induced by the inner product in V. Any fundamental root system $\pi = \{\alpha_1, \dots, \alpha_l\}$ forms a $\mathbb{Z}$-base of $Q(\Phi)$. In particular, the lattice $Q(\Phi)$ is *even*, i.e. $(x, x) \in 2\mathbb{Z}$ for all $x \in Q(\Phi)$.

Now denote by f any bilinear $\mathbb{Z}$-valued form on $Q(\Phi)$ such that

$$(x, y) \equiv f(x, y) + f(y, x) \pmod 2$$

and, moreover,

$$\tfrac{1}{2}(x,x) \equiv f(x,x) \pmod 2$$

for all $x, y \in Q(\Phi)$.

It is obvious that such a form exists. In fact, take Π as a basis of $Q[\Phi]$ and define $f(\alpha_i, \alpha_j)$ by the following formulae:

$$f(\alpha_i,\alpha_j) = \begin{cases} 0, & 1 \leqslant i < j \leqslant l, \\ \tfrac{1}{2}(\alpha_i,\alpha_i), & 1 \leqslant i \leqslant l, \\ (\alpha_i,\alpha_i), & 1 \leqslant j < i \leqslant l. \end{cases}$$

Of course, usually the inner product is normalized so that $(\alpha_i, \alpha_i) = 2$ and, thus, $f(\alpha_i, \alpha_i) = 1$.

Let $\varepsilon(\alpha)$ be a function on $Q(\Phi)$ such that $\varepsilon(x) = 0$ if $x \notin \Phi$ and $\varepsilon(\alpha)\varepsilon(-\alpha) = -1$ for all $\alpha \in \Phi$. Now for all linearly independent $\alpha, \beta \in \Phi$, we may set

$$N_{\alpha\beta} = \varepsilon(\alpha)\varepsilon(\beta)\varepsilon(\alpha+\beta)(-1)^{f(\alpha,\beta)}.$$

It is easy to show that these numbers actually form a system of structure constants for a Lie algebra of type Φ.

For example, one may set $\varepsilon(x)$ to be the sign of a vector $x \in Q(\Phi)$ with respect to the fixed fundamental system Π:

$$\varepsilon(x) = \begin{cases} +1, & x \in \Phi^+, \\ -1, & x \in \Phi^-, \\ 0, & x \notin \Phi. \end{cases}$$

Unfortunately, this natural sign does not lead to the choice of the structure constants we want. That is why one has to modify the choice of signs (see [142]) or of the form f (see [46]). We refer to [142] for further details.

17. Hall Polynomials of Ringel

In [97] C. M. Ringel proposed a beautiful generalization of the Kac–Frenkel construction which works for all root systems Φ, not just for the simply laced ones. Below, we reproduce his construction of the structure constants.

Let $\alpha, \beta, \gamma \in \Phi^+$. Then one can define certain polynomials $\phi^\gamma_{\alpha\beta}$ (‘*Hall polynomials*’). These polynomials have the property that $\phi^\gamma_{\alpha\beta} = 0$ if $\gamma \neq \alpha + \beta$ and if $\gamma = \alpha + \beta$, then exactly one of two polynomials $\phi^\gamma_{\alpha\beta}$, $\phi^\gamma_{\beta\alpha}$ is nonzero.

Now the structure constants may be expressed via the evaluations of the Hall polynomials at 1. More precisely, one has $N_{\alpha\beta} = \phi^\gamma_{\alpha\beta}(1) - \phi^\gamma_{\beta\alpha}(1)$.

This formula may be restated as a convenient recipe to express the structure constants explicitly, without any reference to the Hall polynomials. Namely, define a (nonsymmetric) bilinear form f on the root lattice $Q(\Phi)$ as follows. Let $\operatorname{diag}(f_1, \ldots, f_l)$ be the minimal symmetrization of the Cartan matrix

TABLE V.

$f(\alpha,\beta)$	A_l, D_l, E_l	B_l, C_l, F_4	G_2
0	+	+	+
1	−	+	+
2	+	−	+
3	−	−	−
4	+	+	−
5	−	+	−

of Π. This means that $f_1, \dots, f_l$ are relatively prime natural numbers such that $A_{ij} f_j = A_{ji} f_i$ for all i, j, where, as usual, $A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$. Then the form f is defined by

$$f(\alpha_i, \alpha_j) = \begin{cases} f_i & \text{for } i = j, \\ A_{ij} f_i & \text{for } i < j, \\ 0 & \text{for } i > j. \end{cases}$$

Now assume that $\alpha, \beta, \alpha + \beta \in \Phi^+$. Then precisely one of the numbers $f(\alpha, \beta)$, $f(\beta, \alpha)$ is negative. Suppose $f(\beta, \alpha) < 0$. Then the sign of $N_{\beta\alpha}$ is determined by the value of $f(\alpha, \beta)$ according to Table V.

Of course, in fact the form f does not take value 5 for $\Phi = B_l$, C_l or F_4 and values 4 and 5 for $\Phi = G_2$.

Tables of the structure constants of types E_6, E_7 and E_8 may be found in [60, 142]. In Tables I–IV, we have reproduced tables of the structure constants for F_4 and G_2, corresponding to the choice of signs described in Section 15; the signs of $N_{\alpha\beta}$ for all extraspecial pairs are taken to be $+$.

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