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AUTHOR	Mokros, Jan; Russell, Susan Jo
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ABSTRACT

This paper reports a study to address two questions concerning children's understanding of average: How do children construct and interpret representativeness within the context of data sets? and How do children think about the mean as a particular mathematical definition and relationship? Twenty-one students (seven each of 4th, 6th, and 8th graders) were interviewed using a series of seven open-ended problems that examined the notion of average. The four that yielded the most results were identified and included in the report. These were two "Construction Problems," an "Interpretation Problem," and a "Weighted Means Problem." Analysis of transcripts and summaries of the interviews produced five approaches that children used for constructing and describing average. The approaches were: (1) Average as Mode; (2) Average as Algorithm; (3) Average as Reasonable; (4) Average as Midpoint; and (5) Average as Mathematical Point of Balance. Interpretation of these approaches concluded that students whose strategies were dominated by Modal or Algorithmic approaches did not view average as a representative measure of the body of data as a whole. Interpretation also concluded that students whose strategies were dominated by the Reasonable, Midpoint, and Balance approaches embodied different aspects of constructing a mathematical definition of average. Further research questions resulting from the research are discussed. (Contains 11 references.) (MDH)

Children's Concepts of Average and Representativeness

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Children's Concepts of Average and Representativeness

Jan Mokros
and
Susan Jo Russell

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Preface

TERC is a nonprofit education research and development organization founded in 1965 and committed to improving science and mathematics learning and teaching. Our work includes research from both cognitive and sociocultural perspectives, creation of curriculum, technology innovation, and teacher development. Through our research we strive to increase knowledge of how students and teachers construct their understanding of science and mathematics.

Much of the thinking and questioning that informs TERC research is eventually integrated in the curricula and technologies we create and in the development work we engage in collaboratively with teachers. Traditionally, TERC staff present their research at conferences and report their studies in journals. By launching the TERC Working Papers series, we hope to expand our reach to the community of researchers and educators engaged in similar endeavors.

The TERC Working Papers series consists of completed research, both published and unpublished, and work-in-progress in the learning and teaching of science and mathematics. We are introducing the series with four papers and will add papers at regular intervals.

Introduction

One objective of statistics is to reduce large, unmanageable, and disordered collections of information to summary representations. The need to summarize data is present even among young children. For example, in the surveys conducted by primary grade students, we see movement from focusing on individual pieces of data ("I have one brother") to highlighting and summarizing the data in some manageable form ("*Most* of the class members have only one brother or sister" or "*few* students have no brothers or sisters"). As soon as there is the need to describe a set of data in a more succinct way, the notion of representativeness arises: What is typical of these data? How can we capture their range and distribution?

The word *average* often emerges during children's discussions about data. Younger children use this word in an informal way to refer to typical, usual, or middle. Older children also use the word to indicate the mean, median, or mode, terms they have learned in school. The connections that children make—or fail to make—between their own ways of describing data and the "averages" that they are learning about in mathematics class is a major subject of this paper.

While children and adults alike have underdeveloped notions of average, we know very little about the guiding conceptions and misconceptions from which children build their models of descriptive statistics. Most work on children's ideas about average has examined their understanding of the arithmetic mean (Goodchild, 1988; Pollatsek, Lima, & Well, 1981; Strauss & Bichler, 1988; Leon & Zawojewski, 1990). Our own work poses the broader question of how children make sense of the idea of representativeness in a data set, and how they weave the mathematical definitions of average into their ideas about representativeness. We believe that it is impossible to examine how a child understands average without also examining how that child describes and constructs sets of data.

Our pilot work (Russell & Mokros, 1987) indicated that students as young as fourth graders have developed powerful, situation-based ways of thinking about average. The pilot study showed that students' notions of representativeness, fairness, and typicality grow out of their everyday experiences and have a strong flavor of reasonableness and practicality. For example, in thinking about the average price of food items, students frequently differentiated usual values ("regular"-priced items; the price of items in supermarkets) from unusual values (soup on sale; bags of candy bought at a convenience store rather than at a large supermarket; any food labeled "natural"). The children's informal ideas about outliers helped them hone in on what was typical. Reasonableness in evaluating a data set appeared as an essential strand in understanding, a strand that plays a significant role in the development of more complex notions about average.

As an outgrowth of describing how children think about representativeness, we are also interested in how they develop mathematical definitions of average. A well-developed notion of representativeness should include an understanding of the *mean* and how it works. The mean is both central to statistical understanding and mathematically significant in a broader sense. Learning about the mean is one of a student's first encounters with a mathematical construction that expresses a *relationship* between particular numbers. The relationship between data, as reflected in the mean, is an abstract mathematical construction that has no specific referent in the real world. Summarizing data by finding a mean necessitates a manipulation that accounts for, but at the same time submerges, the concrete data points.

While most people know the procedure for finding the mean of a set of values, the mathematical relationship itself remains very opaque. For example, children and adults alike may puzzle over a household having a mean of 3.2 persons. What is the connection between the number of people in the households that I know about and this value of 3.2? Even if the value is not expressed as a decimal (which conjures up images of what .2 of a person looks like!), it is difficult to imagine that the mean can be a number not actually present in the data set. Why does this new number that doesn't even occur in the data tell us something important about the set as a whole? To understand the value and the power of this strange abstraction requires movement toward more abstract mathematics.

Research has documented that the concept of the mean is quite difficult to understand. In the small body of literature on the subject, researchers have found that 1) fourth through eighth grade students have a difficult time understanding the properties of the mean (Strauss & Bichler, 1988); 2) sixth grade students are generally unable to use the mean to compare two different-sized sets of data (Gal, Rothschild, & Wagner, 1989, 1990); and 3) even college students have difficulty working with familiar averaging problems that involve weighted means (Pollatsek, et al., 1981). Taken together, these studies indicate a lack of understanding of the mean and how it connotes representativeness, even after many years of formal schooling. They show that, given a context that requires more than a straightforward application of the algorithm, older students and adults do not have a fully developed concept of how the mean represents the data and how it relates to the distribution of those data.

This paper addresses two major questions about children's understanding of average. The first question deals with children's own understanding of representativeness within the context of data sets. When asked to describe a data set, how do children construct and interpret representativeness? The second question focuses on how children think about the mean as a particular mathematical definition and relationship. It deals with the underlying issue of how children develop mathematical definitions and how they connect these definitions with their informal mathematical understanding. This question, which has been considered by other researchers primarily in the context of experimental research designs, is addressed here in an open-ended, descriptive manner.

Method

Twenty-one students (seven each of 4th, 6th, and 8th graders) were interviewed, using a series of open-ended problems that examined the notion of average.[1] Clinical interviews provided a means of examining, extending, and probing students' ideas. This methodology provided an opportunity to observe how school learning interacted with students' informal notions about average. The interview problems addressed the following specific questions:

- Given a data set and a problem that requires characterizing data, how do children organize and describe these data? When do they perceive the need to use a measure of representativeness?
- Given an average that describes a set of data, how do children imagine and reconstruct a distribution of data that goes with this average?
- What do children understand about how changes in the data set influence the average?
- How do children use various measures of center in different situations? How flexible are the children ?
- How do children make sense of weighted means problems?

From these questions, we designed a series of seven interview problems. The four problems that yielded the most interesting results and that were analyzed in greatest depth are described briefly below. Two of these problems involve construction of possible data sets, given an average (Construction Problems); another involves interpretation of data (Interpretation Problem); and the fourth is a Weighted Means Problem.

Construction Problems

Students were asked to demonstrate what they knew about average by constructing several data sets that could reflect a particular average. Working from the average to the data is a far more difficult task than the usual school problems that involve calculating an average from a given set of data. Construction problems are very revealing of students' understanding of the relationship between data and average. This kind of task is also what statistically literate readers of data do when they see a mean or median; that is, they think about the different distributions that could be represented by this indicator. While a memorized algorithm yields an appropriate

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[1]Several teachers were also interviewed. Results from the teacher study are reported in Russell & Mokros, 1990.

solution for the traditional averaging problem, the algorithm is almost useless for construction problems. As one eighth grader said, "I know how to get an average but I don't know how to get the numbers to go into an average, from an average. I've absolutely no idea." Just as counting backwards is more difficult for young children than counting forwards, averaging problems that involve working backwards from the average to the data are more difficult than calculating an average from given data.

The two construction tasks discussed here are the Potato Chips Problem and the Allowance Construction Problem. In the first problem, the task was to put price stickers on pictures of nine bags of potato chips so the "typical or usual or average" price of the chips would be $1.38. The students were also asked to make price stickers without using $1.38 (the average value itself) in the data set. In the Allowance Construction Problem, students used tiles and a large piece of graph paper to construct a data set of allowances for a group of students whose average allowance was $1.50. While doing this, students had to take into account several pieces of data that were placed on the graph by the interviewer. Thus, the students' task was to create a large distribution of data, including several given values. The pre-existing data demanded flexibility: Students had to accommodate their strategies to take into account the given data.

Interpretation Problem

The Allowance Interpretation Problem posed the task of examining and summarizing data presented in a skewed, bimodal bar graph of students' allowances (see Figure 1). This problem was aimed at understanding how students think about the relationship between a given data set and an indicator of center. Students were asked to imagine that their parents would give them an allowance that was "average" for the class whose data were represented in the bar graph. The students' task was to indicate the typical allowance as well as the highest amount that could be argued for, given the data in the graph. Students were encouraged to use the data to make a convincing argument to their parents.

Weighted Means Problem

The Elevator Problem (adapted from Pollatsek et al., 1981) was intended to reveal children's understanding of a more complex averaging problem. First, students decided if 10 people could safely ride in an elevator that had a weight limit of 1500 pounds, given that the average weight of the six men is 180 pounds and the average weight of the four women is 125 pounds. Students were then asked to find the average weight of the 10 people.

In the other three problems, mentioned briefly below, students investigated data sets concerning the number of candies in bags of M&Ms and people's heights.

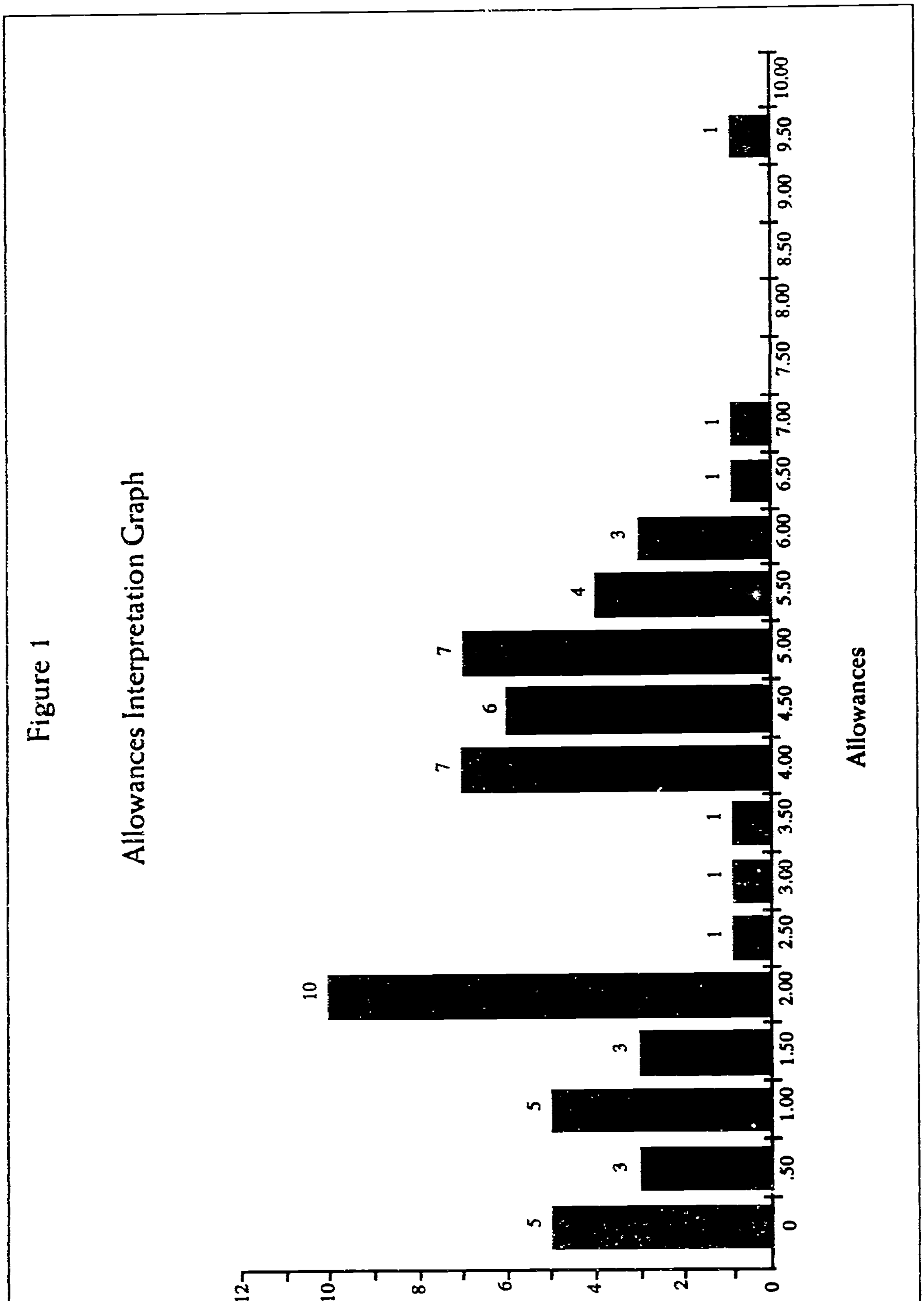


Figure 1

Allowances Interpretation Graph

11

12

Procedure and Analysis

The 45-minute individual interview sessions were conducted by one of three researchers and were videotaped and audiotaped. During the interview, students were given visual props (e.g., graphs, pictures depicting the situation in the problem) and manipulatives appropriate to each problem. A calculator was readily available, although the students were encouraged to give estimates, to reflect on what made sense to them, and to avoid becoming bogged down in calculations. The construction problems necessitated actual placement of data (e.g., price stickers or tiles) on a display.

While the initial problems were presented in a relatively standard manner for each student, follow-up probes were individualized. Throughout the interview, students were asked to think aloud. They were encouraged to develop their own ways of addressing the problems and to explain why the methods they used worked or did not work. Frequently, after students had tried the problem once, they were asked to explore a second way of doing it.

Transcripts and summaries of the interviews were examined in conjunction with the videotapes. Research team members then began identifying, discussing, and classifying strategies that the students used for each type of problem. These classifications were reviewed and revised as the analysis progressed. Each tape was then examined holistically in an attempt to classify the predominant approach used by each student.

Our analysis began with the intention of classifying the students into groups based on their predominant approaches to problem solving. We obtained initial inter-rater reliability figures on their predominant approaches, with 100% agreement on the fourth graders, 86% agreement on the eighth graders, and only 57% agreement on the sixth graders. Determining predominant approach was a difficult task, because all students demonstrated a mixture of approaches—sometimes even on the same problem. The lower reliability figure among the sixth graders may be attributable to the fact that they were learning about averages in school and were more likely than younger or older students to combine school-learned approaches with their own informal ones. Subsequent discussion and revision of the categories enabled the researchers to agree on the predominant approaches of 20 of the 21 students. In the one instance in which agreement could not be reached, the student in question was excluded from the analysis.

As the analysis progressed, we became less interested in classifying and more interested in describing the general approaches that children used for constructing and describing average. The five approaches exhibited among the 21 students are robust, easy to identify, and from what we have observed, seem to capture the whole picture. In several years of working with a wide range of children and adults (ranging from seven-year-olds to statisticians), we have effectively used these approaches to describe all of the solutions to open-ended averaging problems that we observed. We are confident that the approaches identified—while not the only useful ones for examining children's thinking about average—meet Guba and Lincoln's (1981) criteria for internal consistency, external plausibility, and inclusiveness.

While we can order the five approaches described below in terms of level of sophistication, we do not claim that they constitute a developmental sequence. All our research to date has been cross-sectional. To provide a general sense of the developmental sophistication of each approach, the results section indicates the ones that typified the various grade levels.

Results

The five approaches taken by the students can be grouped into 1) approaches that do not recognize the notion of representativeness (e.g., average as mode; average as an algorithmic procedure); and 2) approaches that embody an idea of representativeness (e.g., average as what is reasonable; average as midpoint; average as a mathematical point of balance). Table 1 summarizes their characteristics.

Approaches in Which Average is Not Viewed as Representative

Average as Mode

Five students (three 4th graders, one 6th grader, and one 8th grader) showed a predominantly modal approach to summarizing data. These students found it relatively easy to interpret and construct an "average" value, which was consistently seen as the value that occurs with the greatest frequency. With this mode in mind, building a distribution was simple for these students. However, when they were not allowed to use the typical value as part of their distributions, real difficulties were encountered. The students were not very flexible in their problem-solving approaches. Their alternative strategies for solving a problem were quite similar to their initial strategies, often involving only a minor adjustment to the frequency of data placed on the mode. While some could recite the algorithm for finding the mean, they rarely attempted to use it. For example, sixth grader Molly reported that "The only average I know is like when we add up our grades and then see how many grades you got, and that's how I get the average. I don't know how to do it with M&Ms."

However, students in this group did draw upon experience to determine the characteristics of the distributions they developed. When undertaking construction problems, they did not place all the data on the mode. Their choices about where the data should go first were dictated by placement on the mode, but also included other reasonable values and sometimes took into account their overall sense of what

Table 1

Characteristics of the Five Approaches

Average as Mode. Students with this predominant approach—

- consistently use mode to construct a distribution or interpret an existing one;
- lack flexibility in choosing strategies;
- are unable to build a distribution when not allowed to use the given average as a data point;
- use the algorithm for finding the mean infrequently or incorrectly;
- view the mode only as "the most," not as representative of the data set as a whole;
- frequently use egocentric reasoning in their solutions.

Average as Algorithm. Students with this predominant approach—

- view finding an average as carrying out the school-learned procedure for finding the arithmetic mean;
- often exhibit a variety of useless and circular strategies that confuse total, average, and data;
- have limited strategies for determining the reasonableness of their solutions.

Average as Reasonable. Students with this predominant approach—

- view an average as a tool for making sense of the data;
- choose an average that is representative of the data, both from a mathematical perspective and from a common-sense perspective;
- use their real-life experiences to judge if an average is reasonable;
- may use the algorithm for finding the mean; if so, the result of the calculation is scrutinized for reasonableness;
- believe that the mean of a particular data set is not one precise mathematical value, but an approximation which can have one of several values.

Average as Midpoint. Students with this predominant approach—

- view an average as a tool for making sense of the data;
- choose an average that is representative of the data, both from a mathematical perspective and from a common-sense perspective;
- look for a "middle" to represent a set of data; this middle is alternately defined as the median, the middle of the X axis, or the middle of the range;
- use symmetry when constructing a data distribution around the average. They show great fluency in constructing a data set when symmetry is allowed, but have significant trouble constructing or interpreting non-symmetrical distributions;
- use the mean fluently as a way to "check" answers. They seem to believe the mean and middle are basically equivalent measures.

Table 1 (continued)

Average as Mathematical Point of Balance. Students with this predominant approach—

- view an average as a tool for making sense of the data;
- look for a point of balance to represent the data;
- take into account the *values* of all the data points;
- use the mean with a beginning understanding of the quantitative relationships among data, total, and average; students are able to work from a given average to data, from a given average to total, from a given total to data.
- break problems into smaller parts and find "sub-means" as a way to solve more difficult averaging problems.

the distribution should look like. Consider the following explanation of potato chip pricing, from a fourth grader:

> OK, first, not all chips are the same, as you just told me, but the lowest chips I ever saw was $1.30 myself, so since the typical price is $1.38, I just put most of them at $1.38, just to make it typical, and highered the prices on a couple of them, just to make it realistic.

At first glance, it appears that children who use modal strategies have simply developed a very literal meaning for the word "typical." Could it be a matter of semantics? It is certainly appropriate to equate typical with the mode. Any specific instance of the word "typical" may have drawn a typical=modal analogy, but the interview involved many different phrasings and requests for alternative descriptions and interpretations. Despite our probing for additional, deeper understandings of "typical," "usual," "average," or "representative," the students who used predominantly modal strategies could not be prompted to construct other meanings of average.

Students who relied on the mode did not see the average as being representative of the data set as a whole. They understood that the mode was "the highest one" on a bar graph, and therefore the most typical. However, they saw little else other than the mode. On the Allowances Interpretation Problem, for example, these students were almost always drawn to the mode of $2.00, but they did not see the cluster of data at higher values being more (or as) representative. Even when there was a strong motivation to see the higher numbers as more representative (e.g., they would help one argue for a higher allowance), they could not make an argument based on representativeness. According to these students, $2.00 was the only number that mattered—at least mathematically—in the distribution.

Our attempts to get these students to justify their choices mathematically, using anything other than the mode, sometimes resulted in surprising, non-mathematical turns to the discussion. In the following dialogue, the interviewer asks a sixth grader whether it is possible to use the graph shown in Figure 1 to convince parents that the typical allowance might be higher than $2.00 (the modal amount):

Interviewer: So what would you tell them you thought was fair for you to get based on this?

Student: $4.00.

Interviewer: And why would you say that?

Student: Because even though $2.00 is the typical, I don't know, I spend a lot of money sometimes... I think I could get $4.00.

Interviewer: So even if I was your mother and I said, well, but you know, most kids get $2.00.

Student: But I'm not most kids.

Interviewer: OK, so you'd say you deserve something better.

Student: Right.

Interviewer: Is there any way you could argue for anything higher than $4.00?

Student: The highest would be $5.00, $5.50. $5.50 would be the highest.

Interviewer: OK, and how could you make an argument for $5.50?

Student: I'd butter her up.

So much for arguing from the data! The students who were predominantly modal often exhibited egocentric arguments, as well as arguments based purely on the mode.

One student, who used mostly modal strategies, meshed her informal knowledge of typical with her school-learned knowledge about the algorithm for the mean. On the Potato Chip Problem, fourth grader Dina divided $1.38 (the average price) by 9, resulting in a value close to 15 cents. She then priced all the bags at 15 cents. She was convinced that the "typical" value was 15 cents, because it was both the mode *and* the result of the misapplied algorithm she learned at school. When asked whether the typical price for a bag of potato chips would come out to $1.38 using her scheme, she replied, "Yeah, that's close enough." This student's synthesis of her informal understanding of mode with her school-based understanding of the algorithm has resulted in a serious loss of meaning.

Average as an Algorithmic Procedure

The three students in this group (two 6th graders and an 8th grader) relied almost exclusively on the algorithm for finding the mean, and none used it effectively. Several additional students, including fourth grader Dina discussed above, made ineffective use of the algorithm on one or more problems. It is clear that these students either did not have an idea of average as representative, or chose not to use this idea when they had an algorithm they could apply. They understood that to calculate the mean, one needs to find a total and divide by some number. But in the process of finding the mean, these students seemed to forget how the procedure related to the data at hand. They did not use the data as a standard against which to judge their solutions to problems. Three students, all girls, were placed in this category. The two sixth graders misapplied the algorithm to a range of problems. The eighth grader used the algorithm primarily on the construction problems.

The students using the Algorithmic approach clearly knew something about the procedure for finding a mean and were eager to move directly to it when confronted with a problem. They over-trusted the algorithm, which often meant they had to give up what they knew about reality. An eighth grader provides a most interesting example: On the Weighted Means Problem, Emily teetered back and forth between seeing average as a reasonable approximation and seeing it as something that made sense only computationally. She understood the point of the problem (i.e., finding an average weight), and proceeded to do it by getting the correct total (1580 pounds) and then dividing by the number of groups (2), rather than by the number of individuals (10). She registered shock and disbelief at the resulting 790 pounds, then recalculated a second and third time. At that point, she knew the value was absurd, but rather than rejecting it she slumped back in her chair and said, "I can live with that."

Student: I guess it's logical.

Interviewer: Logical?

Student: Logical in that it's done the right way.

In watching this episode on videotape, the viewer can almost see Emily giving up what she knows about the world in order to apply a procedure that resulted in very unreasonable (and unrepresentative) results. But at least she registered dismay at unreasonable values. Most other students who used the algorithm found it quite easy to "live with" the results.

Two students in this group (as well as one in the Modal group above) exhibited a particular "meaningless algorithm" which they applied to the Construction Problem. It involved constructing unusual pairs of sub-means for the Construction Problems. For example, on the Potato Chip Problem, where the task was to construct prices that yielded an average value of $1.38, one student chose pairs of numbers where the "cents had to add to .38" (e.g., $1.08 and $1.30). Her pairs consistently totaled $2.38, which she thought yielded an average of $1.38. Another student chose pairs that *differed* by 38 cents. On the Allowances Construction Problem, this student chose data points that differed by $1.50 (the average allowance). Yet another strategy was to find a pair of numbers and *divide* by a number that appears in the average (such as dividing by 38 cents.)

Another meaningless algorithm involved the circular use of average and total, in the ill-conceived hope of arriving at the data. For this reason, one of the sixth graders in this group was stymied by the Potato Chip Problem. Although she was given the average and asked to find the data, her initial conceptualization of the task was to try to find the average. (Interchangeable use of average, total, and the individual data points was often seen in this group.) She used a guess-and-check approach to this problem, wherein she chose a reasonable value (e.g., 1.45), multiplied it by 9, then divided it by 9 in hopes of getting the mean. She did not notice that these operations got her back to her original number. After four successive attempts to get closer to 1.38, she finally noticed the circularity and chose 1.38, multiplied by 9, and divided by 9. "There" she said proudly, "It works!" When the interviewer asked how she could use this information to put prices on the chip bags, she replied that she knew how to get an average but could not reverse the process.

The two approaches discussed above (modal and algorithmic) are characteristic of children who exhibit little understanding of representativeness. The following three groups (average as reasonable, as midpoint, and as a mathematical point of balance) describe the approaches used by children who understand representativeness and are using it to develop their own definitions of average.

Approaches Based on the Idea of Representativeness

Average as Reasonable

Two 4th graders and two 6th graders relied upon solutions that were "reasonable" in two respects. First, the values used in solving the problems were based on the students' own experience with real phenomena (e.g., prices, allowances, heights). One clear meaning of average is what these students knew to be typical in their own lives. A second characteristic of this approach is *mathematical* reasonableness. Children who use reasonable strategies, unlike those who prefer the mode, know something about average as it relates to the distribution of data. Usually, they indicate that the average is centered roughly among the rest of the data. This is not a precise middle, but, when thinking about average, it is a sense that high values must be countered by low values.

Students in the Reasonable group saw the mathematics as intertwined with real-life problems. For example, fourth grader Suzanne tended to emphasize the real-life dimension of reasonableness, but the statistical dimension was a definite part of her understanding. Here is how she discussed the distribution she made on the Allowances Construction Problem, as she attempted to construct an average value of $1.50.

> Student: [It depends] on how old they are . . . If there are some kids that were like 15 and 16 years old and there are other kids that were 10 years old . . . It depends on how rich their parents are sometimes.
>
> Interviewer: Tell me how you're thinking about this one.

Student: Well, just as they get higher, sometimes they should get lower. And you said the typical allowance is about $1.50, so some kids can get $1.50. And if it were $1.75 that would be pretty close and so would ($1.25) because that's around it . . . Parents don't like to waste their money on kids.

Interviewer: Is that why you put a lot below $1.50?

Student: If the typical (allowance) is $1.50, you're not going to really go above $5 for any kid. If I got $5, it would be good . . . And you know that when you run around with a lot of kids, most of them are like $1.50 or $1.75 or $1.25 or $1.00, something like that.

Students with predominantly Reasonable approaches seemed to make good use of the algorithm for calculating the mean. Two of the four students who used mostly reasonable strategies arrived at a correct answer for the relatively difficult Weighted Mean Problem. Helen, a fourth grader, was the youngest student in the sample who achieved an accurate answer for this problem. The algorithm for these students was used largely in service of confirming what made sense to them based on their understanding of the distributions.

However, this common-sense approach was not without problems. Students calculated a mean fairly readily but often did not believe that the mean had a precise mathematical value. Average was viewed as a reasonable approximation that could have one of several values. In order to know the exact value, one would have to know the precise values of each piece of data. It would not be possible to extract the precise value from knowledge about sub-means (e.g., the 6 men waiting for the elevator weigh an average of 180 pounds, the 4 women an average of 125 pounds). The following discussion with sixth grader Meghan reveals a misconception concerning the average as an approximation for real data that are not known. She, like other students who used the strategy, believed that the average cannot be exact because the individual data pieces are not known. Meghan arrived at an appropriate total weight for the Elevator Problem, but balked at finding an average:

Interviewer: So the total is 1580.

Student: But unless . . . because you don't know exactly how much each woman or each male . . . weighs, so one of them might weigh 100, like one woman might weigh 118, another woman might weigh 126.

Interviewer: Umhm.

Student: But *this is just the average* (emphasis added).

Interviewer: Is there any way of telling what the average weight of a person is here?

Student: Um, well, no I don't think so. I'm not sure how to do that.

Interviewer: OK. Would you have any good guesses?

Student: Well, if you like um, if you compare their height, maybe.

Interviewer: Their heights?

Student: Yeah, their heights and like how, like how they're built, and you could just take an educated guess.

Even the two students who derived the correct answers were not convinced that their answers were exact. Both believed their answers were in the right ball park (e.g., reasonable based on their knowledge of weights, and mathematically reasonable), but as Maida explained, "It's probably off because you don't know the exact weights of everybody." All one has is a mean, which students did recognize as an abstraction. What they failed to recognize is that while it is not possible to know the individual weights of the me. or women, it is possible to know the new mean because there is a precise mathematical definition of mean.

While reasonableness as an approach allows students to go a long way in their understanding of average, it does not allow them to reach complete mathematical understanding. The approach lacks precision, much in the way the everyday expressions of "on the average" or "more or less" are imprecise. The everyday and the mathematical definitions have much in common, but they are not synonymous. Mathematical thinking, unlike the vernacular use of "on the average," does involve a precise definition, one which these students have yet to fully establish.

Initially, the reasoning of students in this group looked fairly egocentric and undeveloped. But there is an important element of reasoning in this approach that is simply not apparent in the Modal or Algorithmic groups. Students in the Reasonable group appear to have a concept of average as *representative*, a concept that is not yet a well-articulated mathematical definition, but which is nonetheless a critical foundation for future development.

Average as Midpoint

Like students in the Reasonable group, the six Midpoint group students (one 4th, one 6th, and four 8th graders) were flexible and based in the reality of data distributions as well as in real-life problems. Reality was used as a standard against which they judged the reasonableness of their answers. However, these students had an additional component of a mathematical definition of average: They gravitated to middles in making their constructions and their interpretations.

While most could not formally identify the median, the students in this group had a strong sense of middle. Nearly all dealt with the Construction Problems by placing an equal number of prices above and below the given average, usually in a symmetrical manner. In some cases, this strategy appeared to be entirely visual. Eighth grader Stacy constructed her allowances distribution by simultaneously placing tiles with her right and left hands, on opposite sides of the average. When finished with her distribution, she commented, "I think that might work, because I was just looking at it, and the spaces (on the left and right sides of the mean) are equal, so they have to be equal over here to be averaged at $1.50. I think that it would work just by looking at it." There is something simple and satisfying about a symmetrical distribution. It yields an "average" that is both a mean and a median, but no computation is needed. In some cases, students in this group had accurate working definitions of the median. Eighth grader Noreen, for example, explained on one of the Construction Problems: "I'm trying to find out what the number is that has about the same number on both sides." She knew that this number was "either the mean or the median."

The question still arises about what students in this group understood about the mean itself. All knew the algorithm for finding the mean and used it as a check on their more informal methods. Some quite obviously equated their median-like strategies as a short-cut for finding the mean. Alan, an eighth grader, explained that one could find the mean through the algorithmic method or through his invented median method. He explained, "I think it would equal out to the same, but I think this [his median method] is a quicker way." And Alan was right, as long as the context was symmetrical distributions.

In order to push the reasoning of these students, we occasionally readjusted their symmetrical distributions by moving a few of their data points, and then asked them to make the average still come out at the same value. How would these students adjust their distributions? We found that it was not difficult for them to make the equivalent move on the other side of the distribution—as long as it was possible to do so (i.e., the compensation for a higher allowance could be made without going below zero on the other side). In only one case did a student make a non-symmetrical adjustment (e.g., moved two tiles down one-half the distance that one tile was moved up). Symmetry had a strong draw for these children.

The other way in which we tried to push students beyond symmetrical reasoning was to ask them to do Construction Problems without using the average price originally provided. Students in the Midpoint group were often stumped by this specification. Three of the six students said it couldn't be done; two made the closest possible adjustment in one number (e.g., moving a price at $1.38 to $1.37) and said the outcome would be "close enough." Only one student, eighth grader Noreen, came close to breaking the symmetry barrier. Noreen, who had constructed equally-spaced pairs of prices around the average potato chip price of $1.38, now had to deal with the ninth number—which she had placed at $1.38. She made two adjustments: she moved $1.38 to $1.39, and then moved $1.34 to $1.35. Unfortunately, her final adjustment was in the wrong direction. It is possible—but unclear from her explanation—that she was beginning to consider a more flexible view of deviations from the mean, but simply compensated in the wrong direction.

One obvious shortcoming of most students in this group was their inability to deal in an effective mathematical manner with nonsymmetrical distributions. They used the algorithm for finding the mean; had sound, reasonable ideas about the relation of average to the data; and were very good at finding "middles." Moreover, their strategies were efficient and elegant in dealing with symmetrical distributions. What these students did not have was a fully developed quantitative sense of the relationship between the data and the average.

Moving Toward Mean as a Mathematical Point of Balance

One of the strongest notions that emerges, especially among older students, is related to the idea that the average embodies some kind of "balance." Students often used the word *balance* explicitly and, as they constructed data sets that might be represented by a given average, they often explained that "a higher value must be balanced by a lower one." It is at the point of trying to figure out what kind of balance the average is and to relate this idea to a variety of data sets that we begin to see students in the middle grades engage in a mental struggle to develop a model of the way the data balance around the average. The notion of balance was definitely easier to construct in a symmetrical distribution, as exhibited by the several students in the Midpoint group.

Discovering the nature of the balance represented by the mean (that is, that the total of the deviations of the data on one side of the mean are equivalent to those on the other side) is a much more difficult task than simple symmetrical balancing. Yet because this balance is a critical mathematical characteristic of the mean, it is important to examine the steps that students take toward it. None of the students in our sample fully understood the way in which the data on either side of the mean must balance. However, one 6th grader and one 8th grader attempted a balancing act that differed substantially from other strategies. This balancing act, which we have subsequently seen in a number of adults (including many teachers and colleagues) involves a combination of using sound intuitive balancing strategies with a misapplied algorithm for finding the mean. We label this phenomenon "balancing totals," and it is illustrated in the following example:

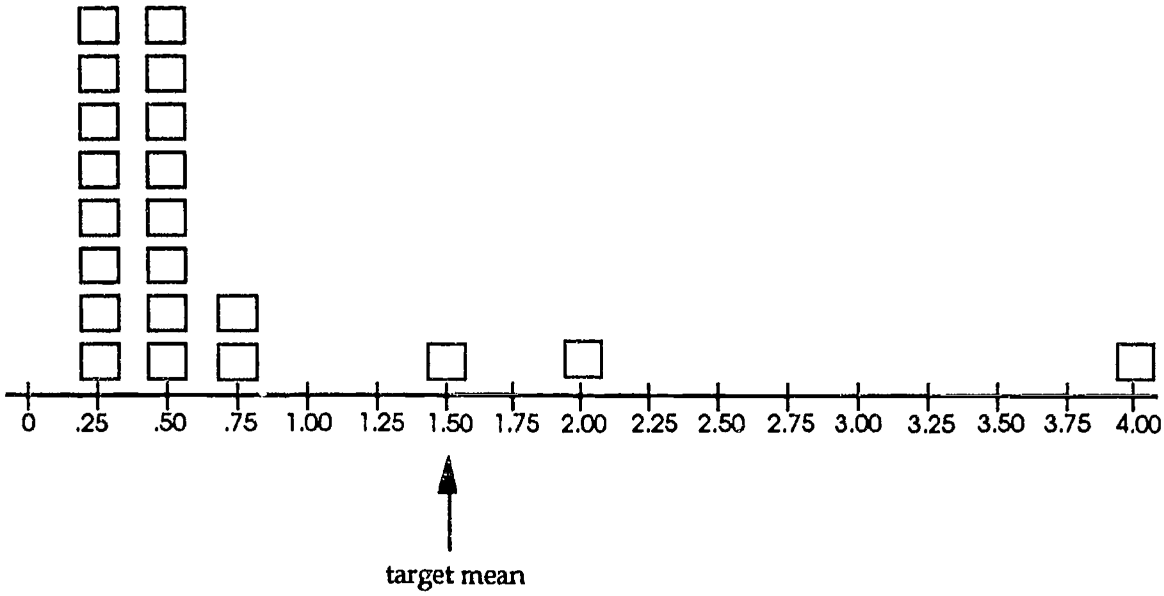
Eighth grader Billy began the Allowance Construction Problem by putting two tiles on $.75 and one on $1.50; then eight tiles on $.25 and one tile on $2.00; then 8 tiles on $.50 and one on $4.00 (see Figure 2).

In each case the *sum* of the data points below the mean is equivalent to those above the mean. If one were actually to calculate the mean, it would be nowhere near $1.50—it's actually about 72 cents in Billy's distribution. But the totals are equal on either side of $1.50; hence, they "balance."

Sixth grader Phil, like Billy, moved toward a mathematical balancing strategy that involved balancing totals. But Phil began by making a very accurate symmetrical distribution that is balanced around a mean and median at $1.50. Then he noticed something fishy:

> Well, on either side of $1.50 there are about the same amount of pieces. . . . Wait a second . . . but umm . . . I think I made a mistake. But there should be more numbers on this side [points to lower side of distribution]. This [points to the upper side of the distribution] is a lot more money than these [lower side].

Figure 2
Billy's Allowance Construction



Phil came to the same conclusion Billy reached: that the sum of the values on one side of the mean must be the same as the sum on the other side.

Neither Phil nor Billy saw the flaw in the balancing act they performed. (In fact, most teachers did not see the flaw; see Russell and Mokros, 1990.) The balancing totals approach may be modeled on an underlying notion of balance represented by a pan balance or by an equation: Whatever is added to one side must be added to the other. Viewing the mean as the fulcrum of a pan balance leads students to pay attention to the *value* of each data point only, rather than to the *distance* between the mean and the piece of data.

The students who used this approach were in fact constructing an underlying mathematical model for the average which has a great deal of power. It embodies a more precise (but not precise enough) meaning for the lower-balanced-by-higher principle that they want to apply. Whereas students who focused on the mode or the middle of the data set tended to disregard the frequency of the values, the students who balance totals are attempting to find a point of balance that takes into account *all* the data values in relation to each other.

It is noteworthy that both Phil and Billy could successfully use the procedure for finding the mean and also had a beginning understanding of how the quantitative relationships worked. Their understanding of the data-total-mean relationship was most visible in the Construction Problems. On the Potato Chips Problem, for example, one boy figured out the total price for all bags of chips (9 bags X $1.38 = $12.42), then constructed prices that would add up to $12.42. The other child found the total, then subtracted 9 prices from $12.42 until he ended up at zero. These were the only two children in the sample who understood how to use the data-total-mean relationship in the context of Construction Problems.

The other indicator of understanding the data-total-mean relationship was exhibited in the boys' use of "sub-means," particularly on the Allowance Construction Problem. The sub-mean strategy, of which we also saw glimmers in other children and more frequently among teachers, is as follows: Think of the problem as a series of much smaller averaging tasks. If one keeps finding combinations of numbers whose mean is $1.50, then the result will be an overall mean at $1.50. This translates into placing two tiles so the total value is $3.00, three tiles totaling $4.50, or 4 tiles totaling $6.00. The use of this strategy does demand that one make use of the relationship between data, total, and mean.

What are the limitations of those students who successfully and flexibly negotiated the algorithm? While generalizations are not possible given the small sample, it is interesting that the most successful algorithm-user had one striking difficulty: Billy, who spent a great deal of time laboring over his sub-mean combinations, was only successful on the construction problem when he could construct *all* of the data himself. He could not deal with the few data points we provided on the Allowance Construction Problem. In fact, he asked if he could take them off. When these data points were later reintroduced, Billy had no strategies for even beginning the problem. Billy's algorithmic approach, particularly when he used a series of sub-means, was quite complex and demanded good organizational and tracking strategies. It is possible that he simply could not handle the information load involved in incorporating the additional data points.

The two students (along with a few teachers) who were beginning to understand the mean as a mathematical point of balance were actually constructing a definition of mean as they worked on the problems during the interview. The definition was in development and included contradictions as people struggled with the different kinds of balance that might be represented by the mean. The most striking contradiction was seen in the students who both balanced totals and balanced in a symmetrical way using distance from the mean as a criterion. These procedures obviously did not yield the same results! In future research, it would be fruitful to work with children and adults who believe that balancing totals works. Why does it work (or not work), and how can it be reconciled with the relationships one understands among the data, total, and mean?

Discussion

We identified five approaches used by fourth, sixth, and eighth graders in solving averaging problems. These five approaches illustrate ways in which students are (or are not) developing useful, general definitions for the statistical concept of average. Much in the way that students develop a general definition for rectangle or triangle through encounters with describing and sorting many kinds of polygons, students construct their definition of average through encounters with many examples of data sets. However, it makes no sense to construct a definition for something that is not yet seen as an entity. As long as a rectangle is simply a collection of lines and corners, it cannot be named and defined; as long as 10 is only the thing you get to at the end of 1-2-3-4-5-6-7-8-9-10, it is not yet a unit with its own particular characteristics and relationship to other units. Similarly, until a data set can be thought of as a unit, not simply as a series of values, it cannot be described and summarized as something that is more than the sum of its parts. An average is a measure of the center of the data, a value that represents aspects of the data set as a whole. This notion of representativeness is the foundation of any meaningful work with averages. Students whose strategies were dominated by Modal or Algorithmic approaches had not developed—or at least were not using—this foundation. They did not view average as a representative measure.

Students in the Modal group—mostly the youngest students in the study—were not yet treating a set of data like an entity that might be described and summarized. A "representative" value had no meaning because the data set was, for them, no more than the values themselves. Our classroom work with third and fourth graders confirms this clinical work: through third grade, most students "describe" their data by enumerating observations about individual values in the data set or, sometimes, moving up one level of aggregation to describe frequencies ("vanilla had the most," "there are six of us with four kids in their families"). However, they seem to have no informal idea about a representative value that could capture the center of the data. By fourth grade, students are just beginning to develop informal ideas about values that represent the data set as a whole. Therefore, it is not surprising that some of the fourth graders in this study noticed only the mode and had not enriched their idea of representativeness beyond looking at the most frequent value, even when this value did not reasonably represent the data set as a whole. It is more surprising that two of the older students had limited, modal strategies in all the problem contexts.

Students in the Algorithm group are similar to those in the Modal group in that they are not concerned with constructing a definition of average. However, the histories of these two groups may be quite different. Definitions are constructed around meaning. Students in the Modal group had not yet developed—or were just beginning to develop—the idea of a data set as a unit, which we hypothesize is prerequisite to constructing any notion of representativeness. Students in the Algorithm group were concerned only with procedures, not with meaning. For them, average means a series of steps involving addition and division; it is not a mathematical object. The students in this group were those who were so much in the grip of procedures that non-standard problems and clinical approaches could not break through to any informal understanding of average they might have—or might have had at one time.

Students in the Reasonable, Midpoint, and Balance groups are all working toward constructing a definition of average. Their strategies for solving averaging problems are varied and ingenious. Like emergent readers, they use many contextual clues for arriving at meaningful interpretations. The shape of the distribution, including its outliers, clumps, modes, and middles provides students with clues about what is representative. The data themselves, as filtered through students' everyday experience with similar phenomena, provide important contextual clues. Use of these contextual clues is not the same as simple egocentrism. In particular, the students who used the Reasonable and Midpoint approaches based their solutions on what made sense in the real world as well as what made sense in terms of characteristics of the data distributions. These students are well on their way to employing average and other summary statistics in a useful and mathematically sound manner.

Students in these groups applied some combination of the following important—albeit incomplete—informal notions to the problems presented:

- A value in the middle of the data can be used to represent a set of data.
- The average is situated in the data set in such a way that values higher than the average are countered in some way by values lower than the average; the average is a point of "balance" in the data.
- An average for a given set of data provides a reasonable sense of the values of the data.
- The shape of the data represented by a given average should reasonably represent what we know about the context from which these values are derived (heights, family size, prices, etc.).

The three groups—Reasonable, Midpoint, and Balance—embody different aspects of constructing a mathematical definition of average. We view the act of defining a mathematical object as iterative. Through constructing and working with a variety of examples and counterexamples in different contexts, definitions are formed, challenged, and revised. In this process, definitions gain clarity and precision. For example, children begin to form a definition of rectangle or square as they hear these words used to describe shapes in their environment and as they experiment with drawing these shapes. Gradually, given sufficient experience, their definitions expand so, for example, they include a "diamond" with right angles in their definition of square and a square in their definition of rectangle. As they debate whether or not a square is a rectangle, they examine their definition of rectangle. In this process, they clarify and rethink the necessary elements for their definition. What is also happening here is that, as mathematical definitions are constructed, *the idea of mathematical definition itself* is also being constructed. Wh' ' does a mathematical definition require? How can it be constructed so it includes exactly what is intended and excludes what is not intended? When is it precise enough? What makes a definition simple and elegant?

Students in the Reasonable, Midpoint, and Balance groups are in different places in relation to the creation of a definition of average. Those in the Reasonable group have constructed an idea of representativeness but do not appear interested in developing a precise measu:e of representativeness. They have developed elements of a definition as well as an overall qualitative sense of the idea they are defining. Those in the Midpoint group are developing a more precise definition for average, but their definition of representativeness breaks down as they encounter cases to which the Midpoint approach cannot be applied (i.e., non-symmetrical distributions).

Students in the Balance group are struggling not only with the concept of average but with the idea of mathematical definition. Their idea of representativeness is embodied in their view of average as a mathematical balance, and they are attempting to define this balance in both a precise and general way. Their attempts to construct possible distributions for a given average—even when these attempts go awry—are characterized by trying to connect the average and data in a precise, mathematical relationship that can be applied in any situation and that defines the average in such a way that it represents the data set as a whole. Watching students in the Balance group is almost like seeing a glimpse of statistical history; their strong sense that a point of balance in the data set must provide an important piece of information leads them to attempt to find a way to define this balance precisely.

This research raises many questions. Primary among them is the issue of developmental and pedagogical paths in learning and teaching the concept of average. It appears clear to us, both from this study and from our classroom work, that constructing the idea of representativeness is the foundation for any work with averages and is a prerequisite to the development of specific definitions for particular averages such as the median or mean. Children construct the idea of representativeness, just as they construct any fundamental mathematical idea, through many encounters with instances of that idea, in this case, a variety of real data sets. In order for children to develop an understanding of representativeness, they collect, represent, describe, and interpret data about meaningful topics (National Council of Teachers of Mathematics, 1989) and in this process gradually broaden and deepen their uses of *typical, average, representative, balance,* and *center*. In particular, students in the middle grades who focus only on modes in their descriptions of data need continuing encounters with data and with situations that challenge them to summarize data, such as describing the "typical" sixth grader or comparing the heights of first graders and sixth graders (Corwin & Russell, 1989; Friel, Mokros, & Russell, 1992).

We believe that children wedded to the algorithm must also be pulled away from their narrow view of average as a procedure to a focus on describing and comparing data sets. It is possible that the students in the Algorithm group once had sound informal ideas about average as a representative measure, ideas that were stunted or destroyed by rote learning of the algorithm for finding the mean. It may be that these students gave up notions about finding middles or points of balance in favor of what they were "supposed" to do in school, or that these students never developed any idea of representativeness and had only the learned procedure to draw upon as a strategy. In any case, we believe this group simply presents extreme examples of the way most of the students in our study treated the algorithm for finding the mean. When interviewed about what they did in school, many students in all categories talked about the add-'em-up-and-divide algorithm, but related it to very limited contexts ("We use it for getting our spelling score," "We use it to get the average temperature") and were unable to use it meaningfully in our problems.

It appears that premature introduction of the algorithm for finding the mean may cause a short-circuit in the reasoning of some children and that, for many, it is not at all connected to their development of fundamental ideas about representativeness and balance. Because most of the younger children in our sample were developing sound intuitive notions about average, we wonder about the impact of the algorithm on these emerging concepts of representativeness. Does gaining the algorithm mean losing a meaningful concept of representativeness? If so, the trade-off is not educationally sound. Rather than sacrifice understanding, we need to work with children's intuitive notions and help them develop ways of mapping new, varied, and richer concepts onto the ones that they already possess. We envision classroom environments in which students are given many opportunities to work with data sets; to develop their own informal ideas about typicality or representativeness as they describe and compare data sets; to articulate and compare their emerging concepts of "middle" and "balance point;" to be introduced to useful averages that connect with these emerging concepts (in particular, the median), and only late in the middle grades to encounter the more explicit definition for the mean.

One of the questions that remains is how children can ultimately come to understand the mean as a mathematical point of balance in the data. Since none of the students in our study or in previous pilot work showed a thorough grasp of the mathematical relationships embodied by the mean, we had little opportunity to probe the nature of this knowledge. However, in a small intervention study with sixth graders, using Construction Problems such as those in this study, we experimented with a pedagogical strategy that emphasizes balancing the deviations around the mean and encountered some preliminary success (Mokros & Russell, 1991). Our work with adults also leads us to suspect that the arithmetic mean is a mathematical object of unappreciated complexity (belied by the "simple" algorithm for finding it) and that it should only be introduced relatively late in the middle grades, well after students have developed a strong foundation of the idea of representativeness and have a great deal of experience using the median, a measure that is not only extremely useful, but more easily connects to children's informal ideas about average.

On a final note, we have found that this research has deepened our own appreciation of the elegance and power of the arithmetic mean. Amy Shulman Weinberg, a researcher on the project, captured these feelings in an ode that she wrote while the analysis was in progress. We find it a fitting tribute to the meaning-making (pun intended!) endeavors of the students in our study.

Ode to a Mean

How elegant the mean, and how fair!
You take a bunch of numbers, even two will do.
What happens next is nothing fancy, just a little bit of addition and division
and a powerful relationship develops.
On the one hand there's the mean itself, respectful of all
whether large or small,
alone,
or one of many,
quite the diplomat and consensus builder.
And never wishy-washy, but always apparent;
you know where you stand.
On the other hand, the data.
They flank the mean, sometimes crowding, sometimes more distant
but always, always in a mysteriously measured formation.
What is their secret? How do they know where to place themselves?
But in the end they are mutable, even dispensable, and the mean stands alone,
a confirmation of all that they were.

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