CHILDREN'S UNDERSTANDINGS OF ALGEBRA 30 YEARS ON: WHAT HAS CHANGED?

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In this paper, we outline the design and method of the research project Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS). Phase 1 consists of a large-scale survey of attainment in algebra and multiplicative reasoning, using test items developed during the 1970s for the Concepts in Secondary Mathematics and Science (CSMS) study (Hart, 1981). This will enable a comparison of the current attainment of students aged 11-14 with that of 30 years ago. Phase 2 consists of a collaborative research study with 8 teachers extending the investigation to classroom / group settings and examining how formative assessment can be used to improve attainment. Although the focus of this paper is on reporting the research design, some early analysis of data from the initial survey data from 2008 (n = 2400) is reported.

INTRODUCTION

Over the past 30 years, there has been a great deal of work directed at, first, understanding children's difficulties in mathematics and, second, examining ways of tackling these difficulties. Yet, there is no clear evidence that that this work has had a significant effect in terms of improving either attainment or engagement in mathematics. Indeed, children continue to have considerable difficulties with algebra and multiplicative reasoning in particular (e.g., Brown, Brown & Bibby, 2008; Wiliam et al., 1999). In this paper, we describe the project Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS), a research study designed to address these problems.

ICCAMS is a 4-year research project involving a research team from King's College London and Durham University together with eight teacher-researchers from four schools. The project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England followed by a collaborative research study with the teacher-researchers extending the investigation to classroom / group settings and examining how formative assessment can be used to improve attainment and attitudes. Although the project is in its early stages, we report some initial tentative results later in this paper. These initial results compare children's current understandings with a similar survey, the Concepts in Secondary Mathematics and Science (CSMS) study (Hart, 1981), which was conducted 30 years ago. When completed, the full results will enable us to examine what gains, if any, have been made over the intervening period. The Phase 2 findings will extend the results to children's understandings in group and classroom settings.

BACKGROUND

Mathematics education in the UK[21] is facing a crisis; insufficient students are choosing to continue studying mathematics post-16, whilst university teachers and others point to falling standards in the subject (CBI, 2006; Smith, 2004). There is considerable research in the UK addressing reasons for non-participation in mathematics - students stop studying mathematics because they experience it as difficult, abstract, boring and irrelevant (e.g., Osborne et al., 1997). The most recent findings relating to 16 year-olds (Brown, Brown & Bibby, 2008) suggest that students' attainment and attitudes are strongly inter-related. A major factor is that even relatively successful students perceive that they have failed at the subject and lack confidence in their ability to cope with it at more advanced levels, especially in comparison to the perceived 'clever core' of fellow-students. When pressed about the reasons for their feelings of failure, students suggest that they do not understand parts of what they have been taught; this commonly relates to algebra and to aspects of multiplicative reasoning (e.g. percentages, and ratio) and its applications (e.g. in trigonometry). Students' negative attitudes commonly relate to the predominance of routine and formal work on algebra and multiplicative reasoning. Performance in these topics has been shown to be particularly weak in England relative to other countries (e.g. Mullis et al., 2004). Yet algebra and multiplicative reasoning are both essential for further study in mathematics, in science & engineering (as well as health and medicine, economics, etc.) and for mathematical literacy in the workplace and elsewhere (e.g., CBI, 2006).

The original CSMS study was conducted 30 years ago. The study made a very significant empirical and theoretical contribution to the documentation of children's understandings and misconceptions in school mathematics (e.g., Booth, 1984; Hart, 1981). In the intervening period, there have been various large-scale national initiatives directed at improving mathematics teaching and raising attainment: e.g., the National Curriculum, National Testing at age 7, 11 and 14, the National Numeracy Strategy and the Secondary Strategy[22]. Many of these initiatives have drawn directly on the CSMS study. During this period examination results have shown steady and substantial rises in attainment: e.g., the proportion of students achieving level 5 or above in Key Stage 3 (KS3)[23] tests has risen from 56% in 1996 to 76% in 2006 and the proportion of students achieving grade C or above at GCSE has risen from 45% in 1992 to 54% in 2006. However, independent measures of attainment suggest that that these rises may be due more to "teaching to the test" rather than to increases in genu-

[21] This crisis in mathematics education is not confined to the UK. It is also a concern in the US and elsewhere in Europe.

[22] These initiatives are particular to England. However, similar initiatives relating to testing (and accountability) and to national curricular are evident elsewhere in the world.

[23] In England, compulsory secondary school consists of two Key Stages: KS3 (11-14 years) and KS4 (14-16 years). In 2008, and for more than a decade previously, 14 year olds took a 'high stakes' test at the end of KS3, although this assessment has been abandoned for 2009 and future KS3 assessment arrangements are currently under review. GCSE (General Certificate in Secondary Education) is the examination taken at age 16, the end of compulsory schooling. Almost all 16 year olds in England take GCSE mathematics.

ine mathematical understanding. Replication results from the science strand of the CSMS study (using a test on volume and density) suggest that students' understanding of some mathematical ideas as well as the related science concepts has declined (Shayer et al., 2007). Studies at the primary level indicate that any increases in attainment due to the introduction of the National Numeracy Strategy have been at best modest (Brown, Askew, Hodgen et al., 2003; Tymms, 2004). Results from the Leverhulme Numeracy Research Programme suggest that any increase in attainment at Year 6 is followed by a reduction in attainment at Year 7 (Hodgen & Brown, 2007) Further, Williams et al. (2007) find that, following this dip at Year 7, there is a plateau in attainment across Key Stage 3.

AN ALTERNATIVE APPROACH: FORMATIVE ASSESSMENT?

National initiatives in mathematics education in England have largely focused on specifying what mathematics should be taught (e.g., the National Curriculum), how mathematics should be taught (e.g., the Secondary Strategy) and summatively assessing what mathematics has been learnt (e.g., National Tests). However, research suggests that a much more effective approach to increasing attainment and engagement would be formative and diagnostic assessment: the tailoring of teaching to students' learning needs (Black & Wiliam, 1998). In an extensive meta-analysis study Hattie (1999) found that interventions involving feedback are more effective than any other educational intervention, with an effect size of 1.13. Further, Wiliam (2007) calculates that, for the achieved effect size, the cost of formative assessment is lower than for other comparative educational interventions. Yet, whilst there has been a great deal of activity nationally and internationally in formative assessment, there is also considerable evidence that teachers have substantial difficulties implementing these ideas (Bell, 1993). These difficulties in implementation relate to three issues. First, formative assessment has largely been described generically rather than in subject-specific terms (Watson, 2006). Second, formative assessment has been poorly described theoretically and pedagogically (Black & Wiliam, 2006). Third, teachers' ability to use formative assessment in mathematics is limited by their knowledge about key ideas, and the likely patterns of progression in student learning. Thus if teachers focus on teaching mathematical procedures they may find it difficult to see what is causing problems for students in mastering and applying these, and though aware of the importance of questioning, they may not know what questions they should ask (Hodgen, 2007).

THE NEED FOR A COLLABORATIVE APPROACH TO DISSEMINATION

Much of both the research and the implementation of initiatives in these areas of mathematics have been "done to" teachers, which may in part explain the limited influence in schools. Leach et al. (2006) found that research evidence cannot simply be presented to teachers; research findings need to be "re-worked" as teaching materials. However this process of re-working, or recontextualisation, is not straightforward

(Ruthven, 2002). We hypothesise that in order for change to occur teachers must have greater insight into the problems of student understandings and attitudes, a profound understanding of fundamental mathematics (Ma, 1999), and understanding of how available resources relate to student understandings and underlying mathematical ideas. These approaches have been tried before in e.g. diagnostic teaching experiments – also based on the CSMS research - and have proven success (Bell, 1993; Swan 2006). Existing experience of collaborative research methods (e.g., Black & Wiliam, 2003) suggests that disseminating these approaches more widely and implementing them in ordinary classrooms is more likely to be successful if these approaches have been grounded in teachers' practices.

THE RESEARCH STUDY

ICCAMS is investigating engagement and achievement by focusing on the two topics at KS3 that are central to the current mathematics curriculum: algebra, and multiplicative reasoning. These topics are also fundamental to further study in mathematics and other numerate disciplines (e.g., science, engineering, economics[24], etc.) The study will focus on KS3, because this is where students first meet algebra and more abstract multiplicative reasoning, and where attitudes begin to deteriorate (Mullis et al., 2004). There is also evidence of a plateau in student achievement at KS3 (Williams et al., 2007).

Phase 1: The large-scale survey of algebra and multiplicative reasoning 11-14

In Phase 1, we are conducting a large-scale survey of attainment in algebra and multiplicative reasoning and attitude to mathematics, involving both cross-sectional and longitudinal elements. This will use test items first developed during the 1970s as part of the CSMS study (Hart, 1981). Based on a representative sample of schools and students in England, the survey will provide a comprehensive and detailed analysis of current student attainment in algebra and multiplicative reasoning. It will provide up-to-date information on student understandings of basic ideas in the areas of algebra and multiplicative reasoning enabling us to plot where changes have occurred since the original study. It will extend the CSMS study by linking understanding of concepts and student progression to student attitudes, to teaching, and to demographic factors. Analysis is being conducted using a variety of techniques, extending those used in the original CSMS study with Rasch and other techniques.

The full survey will consist of both cross-sectional (n=6000) and longitudinal (n=600) samples identified using the MidYIS database (Tymms & Coe, 2003). Three original CSMS tests (Ratio, Algebra, Decimals) will be administered with some additional items relating to fractions (drawn from the CSMS Fractions test) and spread-

[24] ICCAMS is funded by the Economic and Social Research Council in the UK as part of a wider initiative aimed at identifying ways to participation in Science, Technology, Engineering and Mathematics disciplines.

sheet items. Piloting indicated that only minor updating of language and contexts was required.

The test items range from very basic to sophisticated, allowing broad stages of attainment in each topic to be reported, but also each item, or linked group of items, is diagnostic in order to inform teachers about one aspect of student understanding.

Phase 2: The collaborative research study investigating formative assessment

In Phase 2, we are conducting a collaborative research study with teachers, which will indicate how they can best use a formative assessment focus within these curriculum areas to improve student confidence and competence, and thus participation, engagement and attainment. In this phase, we adopt a design research methodology (Cobb et al., 2003). Central to our approach will be the analysis of children's difficulties from both teaching and research perspectives.

Initially teachers will be supported in interpreting and acting upon the survey results of their students; later they will use classroom-based formative assessment based on the frameworks for learning provided by the tests, and assessment for learning approaches. They will also draw on research-informed approaches to the teaching of these curriculum areas. This study will, first, examine how teachers can make use of existing resources and initiatives to respond to students' learning needs, and, second, develop and evaluate an intervention designed to enable a wider group of teachers with much less support to do this. In the final year of the study, the approach will be implemented and evaluated with a further group of teachers and classes.

The Phase 1 findings will provide up-to-date information on student understandings of basic ideas in the areas of algebra and multiplicative reasoning to inform the teachers and teacher-researchers in Phase 2 both about their own students and about where they lie relative to the general population.

A central question for Phase 2 is how the generic approach of formative assessment can be adapted to the particular needs of mathematics teaching and learning. This will be done in several ways. First, the diagnostic results for individual students assessed against the learning and progression framework developed by CSMS will guide teachers in planning appropriate work for students and in further formative assessment. The CSMS tests were carefully designed over the 5-year project starting with diagnostic interviews in order to focus on student progression in understanding of key concepts such as variable and rational number. (See below for a fuller description of the Algebra test.) Second, we will identify and link existing teaching resources into the developmental and diagnostic learning structure provided by CSMS, building on and extending our existing work in this area which is underpinned by a combination of Piagetian and Vygotskian theories (Adhami, et al., 1995; Brown, 1992). There is extensive research evidence relating to the teaching and learning of both algebra and multiplicative reasoning that can inform this intervention (e.g. Bednarz et al., 1996; Sutherland et al., 2000; Ainley et al., 2006), but these research findings and resources have only made a limited impact on teaching practices in classrooms. The solution lies not in designing yet another resource for the teaching of algebra and multiplicative reasoning, but in supporting the judicious use and interpretation of existing resources by teachers (Askew, 1996). Third, we will develop our existing work in this area (Hodgen & Wiliam, 2006).

THE WORK TO DATE AND EARLY ANALYSIS

In June 2008, tests were administered to a sample of around 3000 students in each of Years 7, 8 and 9[25]. Approximately 2000 of these students took the Algebra test. The full cross-sectional sample will be completed in Summer 2009 when a further subsample of around 2000 students will be tested. We report here on the early analysis of this data. We note that these early results should be treated with caution. In particular, the current sample of students appears to be slightly higher attaining than the general population in England. This early analysis suggests that student attainment in algebra at age 14 is broadly similar to that of 30 years ago, although the patterns across the attainment range and in earlier years are more complex.

Students' understandings of letters

We now focus on just four linked items due to space constraints: 9a-d, illustrated in Figure 1. These items have been chosen to give a flavour of the test.

The CSMS algebra test was carefully designed over the 5-year project starting with diagnostic interviews. The original test consisted of 51 items. Of these 51 items, 30 were found to perform consistently across the sample and were reported in the form of a hierarchy (Booth, 1981; Küchemann, 1981). The test items range from the basic to the sophisticated allowing broad stages of attainment to be reported, but also each item, or linked group of items, is diagnostic in order to inform teachers about one aspect of student understanding. The focus of the test was on generalised arithmetic, and in particular it looked at different ways in which pronumerals can be interpreted (Collis, 1975). Items were devised to bring out these six categories (Küchemann, 1981):

> Letter evaluated, Letter not used, Letter as object, Letter as specific unknown, Letter as generalised number, and Letter as variable.

The four items, 9a-d, were amongst the consistently performing items that formed part of the original hierarchy. Item 9a, at Level 1 in the hierarchy, and items, 9b and c, at Level 2, can be solved without having to operate on the letters as unknowns; the letters can be treated as objects (i.e., the name of the various sides of the figures). Items 9b and c additionally require the explicit use of some mathematical syntax. Item 9d, at Level 3, was designed to test whether students would readily 'accept the lack of closure' (Collis, 1972) of the expression $2n$, where the given letter, n, has to be treated as at least a specific unknown. The proportions of 14 year old students an-

[25] Key Stage 3 is made up of three academic years: Y7 (age 11-12), Y8 (age 12-13) and Y9 (age 13-14).

swering these items correctly in 1976 reflect this variation in difficulty: 94% for 9a; 68% for 9b; 64% for 9c; 38% for 9d.

The item facilities for 1976 and 2008 are presented graphically in Figure 1. This suggests that the *pattern* of progression is similar in 1976 and 2008: an initial relatively steep rise is followed by a much smaller rise subsequently. However, although the initial steep rise now appears to take place a year earlier, this initial advantage is not sustained and by age 14 students' attainment appears similar in 1976 and 2008. The results for item 9a are more of an anomaly: this relatively easy item appears to be more difficult now than in 1976.

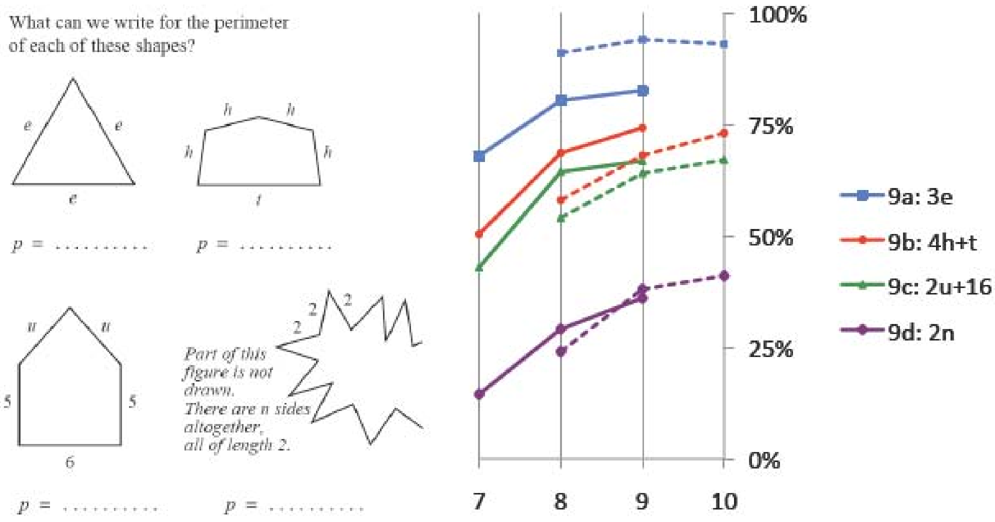

Figure 1: Items 9a-d. Facilities for items in both 2008 [continuous] and 1976 [dotted] for Year 7 to Year 10 (ages 11-14). In 2008 data were not collected for Year 10; in 1976 data were not collected for Year 7.

DISCUSSION

In comparison to 30 years ago, in England, formal algebra is taught to all students earlier. This is partly as a consequence of the introduction of a National Curriculum. The initial results of the study reported here suggest that, whilst this practice confers an initial advantage to students, this increased attainment may not be sustained. Our early analysis suggests that, by age 14, current performance in algebra is broadly similar to that of students in 1976. Moreover, it is worth noting again that the sample of students tested in 2008 is in general a relatively high attaining group. Hence, the data presented here suggest that increases in examination performance are not matched by increased conceptual understanding and, thus, add weight to the research reported earlier in this paper.

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