



# Chiral Gravity, Log Gravity, and Extremal CFT

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# Chiral Gravity, Log Gravity and Extremal CFT

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## Abstract

We show that the linearization of all exact solutions of classical chiral gravity around the AdS<sub>3</sub> vacuum have positive energy. Non-chiral and negative-energy solutions of the linearized equations are infrared divergent at second order, and so are removed from the spectrum. In other words, chirality is confined and the equations of motion have linearization instabilities. We prove that the only stationary, axially symmetric solutions of chiral gravity are BTZ black holes, which have positive energy. It is further shown that classical log gravity – the theory with logarithmically relaxed boundary conditions – has finite asymptotic symmetry generators but is not chiral and hence may be dual at the quantum level to a logarithmic CFT. Moreover we show that log gravity contains chiral gravity within it as a decoupled charge superselection sector. We formally evaluate the Euclidean sum over geometries of chiral gravity and show that it gives precisely the holomorphic extremal CFT partition function. The modular invariance and integrality of the expansion coefficients of this partition function are consistent with the existence of an exact quantum theory of chiral gravity. We argue that the problem of quantizing chiral gravity is the holographic dual of the problem of constructing an extremal CFT, while quantizing log gravity is dual to the problem of constructing a logarithmic extremal CFT.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>5</b>
<b>3</b>	<b>Symmetry generators</b>	<b>7</b>
<b>4</b>	<b>Classical perturbation theory</b>	<b>9</b>
4.1	Chirality confinement . . . . .	10
4.2	The CDWW modes . . . . .	13
4.3	The GKP mode . . . . .	15
4.4	Global modes . . . . .	16
<b>5</b>	<b>A Birkhoff-like theorem</b>	<b>17</b>
5.1	Stationarity and axial symmetry . . . . .	17
5.2	Boundary conditions . . . . .	19
5.3	Solutions . . . . .	20
<b>6</b>	<b>Log gravity</b>	<b>25</b>
6.1	Boundary conditions and non-chirality . . . . .	25
6.2	Decoupling the chiral gravity superselection sector . . . . .	27
<b>7</b>	<b>Quantum partition function</b>	<b>29</b>
7.1	Classical saddle points . . . . .	30
7.2	Sum over geometries . . . . .	33
7.3	Physical interpretation . . . . .	36
7.4	What could go wrong . . . . .	38
<b>8</b>	<b>Chiral gravity, log gravity, extremal CFT and log extremal CFT</b>	<b>38</b>
8.1	Nothing makes sense . . . . .	38
8.2	Chiral gravity = extremal CFT . . . . .	39
8.3	Chiral gravity $\in$ log gravity=log extremal CFT . . . . .	39
	<b>References</b>	<b>41</b>

## 1 Introduction

A consistent, non-trivial theory of pure gravity in three dimensions – classical or quantum – with a stable vacuum would undoubtedly provide invaluable insights into the many complexities of gravity in our four-dimensional world. Unfortunately, pure 3D Einstein gravity is locally trivial classically, while its quantum status remains

unclear despite decades of investigations. Recently, an exceptional and clearly non-trivial 3D theory termed “chiral gravity” was discovered [1]. This theory is a special case of topologically massive gravity [2, 3] at a particular value of the couplings, and is defined with asymptotically AdS<sub>3</sub> boundary conditions, in the sense of Fefferman-Graham-Brown-Henneaux [4, 5].<sup>1</sup> It was conjectured in [1] that at the classical level

- Chiral gravity is chiral, in the sense that the asymptotic symmetry group is generated by a single copy of the Virasoro algebra,<sup>2</sup>
- Solutions of chiral gravity have positive energy.

Some supporting evidence was given [1]. Should both conjectures turn out to be true, chiral gravity, in its quantum version, would prove an extremely interesting gedanken laboratory for the study of quantum gravity.

The chirality and positivity conjectures generated some controversy. Shortly after [1], interesting new solutions to the linearized equations which are not global energy eigenmodes and have a variety of asymptotic behaviors were discovered. These solutions are non-chiral and/or negative-energy and were argued to provide counterexamples to one or both of the classical conjectures [6, 8] – see also [10, 11, 12, 13, 14]. Subsequently the chirality conjecture was proven [15] and the claims that these modes provide counterexamples to this conjecture were revised or withdrawn [7, 9]. A proof of the chirality conjecture in a different formalism appeared in [16]. Nevertheless, claims that the proposed counterexamples disprove the positivity conjecture remain in the literature. However the modes exhibited in [6, 8] explicitly violate the chirality conjecture as well as the positivity conjecture. So if these modes are truly present in the linearization of the exact spectrum, they are fatally at odds not only with positivity but with the chirality proofs of [15, 16]. In short, the literature contains contradictory claims. For related work, see [17, 18, 19, 20, 21, 22, 23, 24, 25].

In this paper, we reconcile all these computations and hope to thereby resolve the controversy. In the process, a perturbative version of the positivity conjecture will be established to first order in the deviation around AdS<sub>3</sub>. The alleged counterexamples do not disprove positivity for exactly the same reason that they do not disprove chirality: the equations have a linearization instability. At second order in perturbation theory, explicit computation reveals that the metric perturbation develops an

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<sup>1</sup>Chiral gravity differs in this respect from log gravity which has the same action but logarithmically weaker boundary conditions.

<sup>2</sup>The quantum version of this conjecture is that physical states form representations of a single Virasoro algebra.

infrared singularity, growing logarithmically with the radius at infinity. Hence these solutions of the linearized equations are *not* the linear approximation to any exact solution of the theory with Brown-Henneaux boundary conditions. In other words, chirality is confined and chiral gravity has linearization instabilities.<sup>3</sup> This divergence was bound to appear because otherwise there would be a discrepancy between the surface integral expression for the energy (linear in the second order perturbation) and the bulk expression (quadratic in the first order perturbation). The first of these is manifestly chiral for asymptotically AdS<sub>3</sub> solutions, while the second gives a non-chiral answer. A key ingredient in reconciling herein the various computations is the discovery (independently made in [29]) of previously neglected terms in the boundary expressions for the Virasoro charges. The omission of these terms has led to some contradictory statements in the literature.

An (imperfect) analogy can be found in QCD. In the linearized approximation, the theory contains free quarks. But there is an infrared divergence in the back reaction caused by the quark and the exact finite energy spectrum contains only color singlets. A free quark is not a valid linearized approximation to any finite energy QCD state. Of course, if the boundary conditions are relaxed to allow flux tubes at infinity there are single-quark solutions. We will see below that an analogous relaxation of the boundary conditions for chiral gravity to those of log gravity allows for non-chiral excitations with finite charges.

The analogy here is imperfect in that color confinement in QCD is a difficult non-perturbative problem. In contrast, confinement in chiral gravity can be seen explicitly in second order perturbation theory. Moreover, in QCD color confinement gives one global constraint, while in chiral gravity there are an infinite number of constraints arising from the infinity of conserved (left) Virasoro charges, all of which must vanish. This is exactly what is required to eliminate an entire chiral half of the spectrum, and reconcile the chiral nature of the theory with the non-chiral “bulk” degree of freedom found in the local analyses of [6, 10, 11, 12, 13, 16, 8, 30]. Rather, we will see below these local analyses apply to log gravity.

The miraculous escape of chiral gravity from the alleged perturbative instabilities leads one to hope that there is an exact positive energy theorem for the theory. The proof of such a theorem at the non-perturbative level remains an outstanding challenge. We take one step in this direction by proving a Birkhoff-like theorem: all stationary, axially symmetric solutions of chiral gravity are BTZ black holes. The

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<sup>3</sup>Similar linearization instabilities have occurred in a number of contexts in general relativity, see e.g. [26, 27, 28].

difficulty we encounter in what would seem a straightforward exercise illustrates the complexity of the full nonlinear equations. It is interesting to note that all known solutions of chiral gravity are also solutions of the Einstein equation. This may be the case for all solutions, although we will not attempt to demonstrate this here. One might also attempt to prove a version of cosmic censorship for chiral gravity.

Armed with knowledge the perturbative spectrum, we then move on to an analysis of the quantum problem. We apply the standard methodology of Euclidean quantum gravity to compute the torus partition function as a function of the modular parameter  $\tau$ . Euclidean quantum gravity is, for a variety of reasons, a notoriously treacherous subject and the present application cannot be regarded as completely rigorous. Nevertheless the results are highly encouraging. We show that all real saddle points solve the Einstein equation, and can be classified. Moreover, at the chiral value of the coupling constants the Euclidean action is holomorphic. Following [31], we perform the sum over saddle points including all perturbative corrections, formally obtaining the exact answer for the partition function. The result is simply the “chiral part” of the extremal partition function conjectured by Witten [32] to be dual to 3D Einstein gravity. It is invariant under modular transformations and has an expansion in  $q = e^{2\pi i\tau}$  with integer coefficients, as required for a consistent quantum mechanical interpretation as a Hilbert space trace. The spectrum reproduces the entropy of the BTZ black hole, including both the Bekenstein-Hawking piece and an infinite series of corrections. Although it is not known whether a CFT exists which realizes this spectrum, the encouraging outcome of this computation might be regarded as evidence both for the existence of quantum chiral gravity as well as for the existence of such CFTs. In any case the interesting problems of understanding quantum chiral gravity and extremal CFTs are clearly closely linked.

We also consider the theory of log gravity introduced in [10, 33]. This theory has the same action as chiral gravity, but the boundary conditions are weakened to allow metric fluctuations which grow logarithmically with the proper radius. Log gravity contains a rich and interesting class of solutions [34, 13, 21, 35] which are excluded in chiral gravity. In particular, the linearization of the exact spectrum includes the non-chiral modes of [6, 8, 10], which appear in indecomposable Virasoro representations. The relaxed boundary conditions also lead to zero-norm states, violations of unitarity and violations of positivity. Interestingly, these violations resemble those found in logarithmic CFTs, suggesting that log gravity is dual to a logarithmic CFT [10]. We show here that the log gravity boundary conditions lead to finite expressions for the asymptotic symmetry generators. However, contrary to [33], the generators

are *not* chiral. This is consistent with the conjecture that log gravity is dual to a logarithmic CFT, as logarithmic conformal field theories cannot be chiral. We also show that log gravity contains within it chiral gravity as the superselection sector with vanishing left Virasoro charges. Thus although log gravity itself is not unitary, it has a potentially unitary “physical subspace”. We speculate herein that log gravity may be dual to an “extremal” logarithmic CFT whose partition function coincides with Witten’s extremal partition function.

This paper is organized as follows. Section 2 contains basic formulae and conventions. In section 3 we give the new expression for the asymptotic symmetry generators. In section 4 we work out the perturbation expansion around AdS<sub>3</sub> to second order, and show that the non-chiral negative energy solutions to the linearized equations are not the linearization of exact solutions. In section 5 we study the spectrum at the non-linear level, and prove a Birkhoff-like theorem for stationary, axially symmetric solutions. In section 6 we study log gravity, show that the asymptotic symmetry group has finite generators and discuss the problem of constructing a symplectic form as required for a canonical formulation. We show that although log gravity is non-chiral, it contains chiral gravity as a superselection sector. In section 7 we evaluate the Euclidean partition function and show that it gives the modular invariant extremal partition function. Finally section 8 concludes with a discussion of and speculations on the fascinating relation between chiral gravity, log gravity, extremal CFT and extremal logarithmic CFT.

As this work was nearing completion, the eprint [29] appeared with results which overlap with sections 3 and 6.1. All points in common are in precise agreement.

## 2 Preliminaries

In this section we record some pertinent formulae and establish notation. Chiral gravity is a special case of topologically massive gravity (TMG) [2, 3] with a negative cosmological constant. TMG is described by the action

$$I_{TMG} = \frac{1}{16\pi G} \left[ \int d^3x \sqrt{-g} (R + 2/\ell^2) + \frac{1}{\mu} I_{CS} \right] \quad (2.1)$$

where  $I_{CS}$  is the gravitational Chern-Simons action

$$I_{CS} = \frac{1}{2} \int_{\mathcal{M}} d^3x \sqrt{-g} \varepsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma}^r \left( \partial_\mu \Gamma_{r\nu}^\sigma + \frac{2}{3} \Gamma_{\mu\tau}^\sigma \Gamma_{\nu r}^\tau \right) \quad (2.2)$$

and  $G$  has the conventional positive sign. The equation of motion in TMG is

$$E_{\mu\nu} \equiv \mathcal{G}_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \quad (2.3)$$

where we have defined

$$C_{\mu\nu} \equiv \epsilon^{\alpha\beta} {}_{(\mu} \mathcal{G}_{\nu)\beta;\alpha}, \quad \mathcal{G}_{\mu\nu} \equiv G_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu}. \quad (2.4)$$

These equations have the vacuum solution

$$\begin{aligned} ds^2 &= \ell^2 (-\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2) \\ &= \frac{\ell^2}{4} (-2\cosh 2\rho d\tau^+ d\tau^- - d\tau^{+2} - d\tau^{-2} + 4d\rho^2), \\ &\quad \tau^\pm = \tau \pm \phi. \end{aligned} \quad (2.5)$$

Chiral gravity [1] is defined by taking  $\mu\ell \rightarrow 1$  while keeping the standard Brown-Henneaux [5] asymptotically AdS<sub>3</sub> boundary conditions. These require that fluctuations  $h_{\mu\nu}$  of the metric about (2.5) fall off at the boundary according to

$$\begin{pmatrix} h_{++} = \mathcal{O}(1) & h_{+-} = \mathcal{O}(1) & h_{+\rho} = \mathcal{O}(e^{-2\rho}) \\ h_{-+} = h_{+-} & h_{--} = \mathcal{O}(1) & h_{-\rho} = \mathcal{O}(e^{-2\rho}) \\ h_{\rho+} = h_{+\rho} & h_{\rho-} = h_{-\rho} & h_{\rho\rho} = \mathcal{O}(e^{-2\rho}) \end{pmatrix} \quad (2.6)$$

The consistency of these boundary conditions for all values of  $\mu$  was demonstrated in [36]. The most general diffeomorphism which preserves (2.6) is of the form

$$\begin{aligned} \zeta &= \zeta^+ \partial_+ + \zeta^- \partial_- + \zeta^\rho \partial_\rho \\ &= [\epsilon^+(\tau^+) + 2e^{-2\rho} \partial_-^2 \epsilon^-(\tau^-) + \mathcal{O}(e^{-4\rho})] \partial_+ \\ &\quad + [\epsilon^-(\tau^-) + 2e^{-2\rho} \partial_+^2 \epsilon^+(\tau^+) + \mathcal{O}(e^{-4\rho})] \partial_- \\ &\quad - \frac{1}{2} [\partial_+ \epsilon^+(\tau^+) + \partial_- \epsilon^-(\tau^-) + \mathcal{O}(e^{-2\rho})] \partial_\rho. \end{aligned} \quad (2.7)$$

These are parameterized by a left moving function  $\epsilon^-(\tau^-)$  and a right moving function  $\epsilon^+(\tau^+)$ . We denote diffeomorphisms depending only on  $\epsilon^-$  by  $\xi^L$  and those depending only on  $\epsilon^+$  as  $\xi^R$ . The subleading terms all correspond to trivial diffeomorphisms; their generators have no surface term and hence vanish when the constraints are imposed. The asymptotic symmetry group (ASG) is defined as the general boundary-condition-preserving diffeomorphism (2.7) modulo the trivial diffeomorphisms. For



generic  $\mu$  the ASG is generated by two copies of the Virasoro algebra, which may be taken to be

$$\xi_n^L = \xi(\epsilon^- = e^{in(\tau-\phi)}, \epsilon^+ = 0) \quad \xi_n^R = \xi(\epsilon^- = 0, \epsilon^+ = e^{in(\tau+\phi)}). \quad (2.8)$$

These of course have a global  $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$  subgroup which generates the AdS<sub>3</sub> isometries. At the chiral point  $\mu\ell = 1$  the left moving generators parameterized by  $\epsilon^-(x^-)$  also become trivial [15]. Hence there is an enhancement of the trivial symmetry group and the ASG is generated by a single chiral Virasoro algebra.

### 3 Symmetry generators

In this section we present a refined expression for the symmetry generators which corrects expressions appearing in some of the literature.<sup>4</sup> The corrections are relevant only when the Brown-Henneaux boundary conditions are violated. This corrected expression is essential for demonstrating the general equality of the bulk and boundary expressions for the energy, as well as for the discussion of log gravity in section 6.

Our expression follows from the covariant formalism [37, 38], which is based on [39] and has been developed in great detail for a wide variety of applications in [40].<sup>5</sup> Let  $E_{\mu\nu}^{(1)}(h)$  denote the linearization of the equation of motion (2.3) about AdS<sub>3</sub> metric  $\bar{g}$  with respect to a small perturbation  $h$  near the boundary. One may then define the one-form

$$\mathcal{K}(\xi, h) \equiv \xi^\mu E_{\mu\nu}^{(1)}(h) dx^\nu \quad (3.1)$$

It is shown in [42] that when  $\xi$  is a Killing vector<sup>6</sup>

$$\mathcal{K}(\xi, h) = *d*\mathcal{F}(\xi, h). \quad (3.2)$$

Here  $\mathcal{F}$  is a two form “superpotential,” which is written out explicitly in [44]. It was further shown that the conserved charges associated to the ASG are then given by

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<sup>4</sup>The expressions herein were independently found using a different formalism in [29].

<sup>5</sup>Some recent discussions of TMG have used the Brown-York formalism [41], which was initially developed for diffeomorphisms which – unlike those in (3.5) – do not have components normal to the boundary. For Brown-Henneaux boundary conditions this subtlety turns out to be irrelevant. It is, however, relevant when violations of the Brown-Henneaux boundary conditions are considered. While the Brown-York formalism could likely be generalized to this case, the covariant formalism is more highly developed and hence more convenient.

<sup>6</sup>This formalism was discussed for general backgrounds in [43], and further generalized in to the case where  $\xi$  is not an asymptotic Killing vector [44]. In this case an additional term appears on the left hand side of (3.2).

the boundary integral

$$Q_{boundary}(\xi) = -\frac{1}{16\pi G} \oint_{\partial\Sigma} *F. \quad (3.3)$$

Here  $\partial\Sigma$  is the boundary of a spacelike surface  $\Sigma$ . Integrating by parts gives the bulk expression

$$Q_{bulk}(\xi) = -\frac{1}{16\pi G} \int_{\Sigma} *K = Q_{boundary}(\xi) \quad (3.4)$$

In this formula  $K$  can be taken to be any smooth extension of the boundary one-form (3.1) into the interior. In the coordinates (2.5), we shall see in the next section that

$$Q_{boundary}(\xi) = \frac{1}{32\pi\ell G} \oint_{\partial\Sigma} d\phi \left[ \epsilon^- \left( -2\partial_\rho^2 h_{--} + 4\partial_\rho h_{--} + 2\partial_\rho h_{-+} - 4h_{-+} + \frac{e^{2\rho}}{4} h_{\rho\rho} \right) + \epsilon^+ \left( 8h_{++} - 8\partial_\rho h_{++} + 2\partial_\rho^2 h_{++} + 2\partial_\rho h_{-+} - 4h_{-+} + \frac{e^{2\rho}}{4} h_{\rho\rho} \right) \right] \quad (3.5)$$

In the above expression (3.5), we have only assumed that  $h$  falls off fast enough for  $Q$  to be finite, but have not used the Brown-Henneaux boundary condition (2.6). Asymptotically, the  $\rho\rho$  component of the linearized equation of motion gives

$$2\partial_\rho h_{-+} - 4h_{-+} + \frac{e^{2\rho}}{4} h_{\rho\rho} = 0 \quad (3.6)$$

Condition (3.6) is an asymptotic constraint, as it involves only the fields and not their time derivatives and hence weakly vanishes in the Dirac bracket formalism. See [45] for a similar discussion in Gödel spacetime. Using the stricter boundary conditions (2.6), and imposing the asymptotic constraints, the expression becomes simply

$$Q_{boundary}(\xi) = \frac{1}{4\pi\ell G} \oint_{\partial\Sigma} d\phi \epsilon^+ h_{++}. \quad (3.7)$$

These charges can be decomposed into left and right charges  $Q^L$  and  $Q^R$  generating left and right diffeomorphisms  $\xi^L(\epsilon^-)$  and  $\xi^R(\epsilon^+)$ . Note that  $\epsilon^-$  does not appear in (3.7), so for Brown-Henneaux boundary conditions the  $\xi^L(\epsilon^-)$  are trivial and the left charges vanish. This implies that the theory is chiral [15]:

$$Q^L \equiv Q(\xi^L) = 0. \quad (3.8)$$

Hence the name chiral gravity.

In the following we will study violations of the asymptotic boundary conditions where the extra terms in  $Q_{boundary}$  will contribute. In particular, we will encounter

situations in which the  $\partial_\rho h_{--}$  term above does not vanish and  $Q^L \neq 0$ . In this case the left moving charges can be written in a simple gauge invariant form

$$Q_{boundary}^L = \frac{1}{8\pi G} \oint_{\partial\Sigma} \xi^{L\mu} (\mathcal{G}_{\mu\nu}^{(1)} - \frac{g_{\mu\nu}}{2} \mathcal{G}^{(1)}) dx^\nu, \quad (3.9)$$

We see that the left charges are nonzero only if the curvature perturbation does not vanish on the boundary.

## 4 Classical perturbation theory

In this section we will work out the weak field perturbation expansion of the equations of motion to second order. We start by expanding the metric around the AdS<sub>3</sub> background as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + \dots \quad (4.1)$$

The expansion parameter here is the magnitude of the first order fluctuation  $h^{(1)}$ . Inserting this into the full equation of motion

$$\mathcal{G}_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \quad (4.2)$$

and expanding to first order in the perturbation we see that  $h^{(1)}$  must satisfy

$$E_{\mu\nu}^{(1)}(h^{(1)}) \equiv \mathcal{G}_{\mu\nu}^{(1)}(h^{(1)}) + \frac{1}{\mu} C_{\mu\nu}^{(1)}(h^{(1)}) = 0 \quad (4.3)$$

In this and the following equations indices are raised and lowered using the background metric. The second order perturbation  $h^{(2)}$  is found by expanding (4.2) to second order

$$E_{\mu\nu}^{(1)}(h^{(2)}) = -E_{\mu\nu}^{(2)}(h^{(1)}) \quad (4.4)$$

Explicit computation gives the left hand side of (4.4)

$$E_{\mu\nu}^{(1)} = \mathcal{G}_{\mu\nu}^{(1)} + \frac{1}{2\mu} (\epsilon_\mu^{\alpha\beta} \nabla_\alpha \mathcal{G}_{\nu\beta}^{(1)} + \epsilon_\nu^{\alpha\beta} \nabla_\alpha \mathcal{G}_{\mu\beta}^{(1)}) \quad (4.5)$$

$$\mathcal{G}_{\mu\nu}^{(1)} = R_{\mu\nu}^{(1)} + \frac{2}{\ell^2} h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R^{(1)} + \frac{2}{\ell^2} h) \quad (4.6)$$

where

$$R_{\mu\nu}^{(1)} = \frac{1}{2} (-\nabla^2 h_{\mu\nu} - \nabla_\mu \nabla_\nu h + \nabla^\lambda \nabla_\nu h_{\mu\lambda} + \nabla^\lambda \nabla_\mu h_{\nu\lambda}) \quad (4.7)$$

$$\Gamma_{\mu\nu}^{(1)\lambda} = \frac{1}{2} \left[ \nabla_\mu h_\nu^\lambda + \nabla_\nu h_\mu^\lambda - \nabla^\lambda h_{\mu\nu} \right]. \quad (4.8)$$

The right hand side of (4.4) is

$$E_{\mu\nu}^{(2)} = \mathcal{G}_{\mu\nu}^{(2)} + \frac{1}{2\mu} [(\epsilon_\mu^{\alpha\beta} \nabla_\alpha \mathcal{G}_{\beta\nu}^{(2)} + h_{\mu\lambda} \epsilon^{\lambda\alpha\beta} \nabla_\alpha \mathcal{G}_{\beta\nu}^{(1)} - \frac{h}{2} \epsilon_\mu^{\alpha\beta} \nabla_\alpha \mathcal{G}_{\beta\nu}^{(1)} - \epsilon_\mu^{\alpha\beta} \Gamma_{\nu\alpha}^{(1)\lambda} \mathcal{G}_{\beta\lambda}^{(1)}) + (\mu \leftrightarrow \nu)] \quad (4.9)$$

$$\mathcal{G}_{\mu\nu}^{(2)} = R_{\mu\nu}^{(2)} - \frac{g_{\mu\nu}}{2} (R^{(2)} - h^{\lambda\sigma} R_{\lambda\sigma}^{(1)} + h^{\lambda\alpha} h_\alpha^\sigma R_{\lambda\sigma}) - \frac{h_{\mu\nu}}{2} (R^{(1)} + \frac{2}{\ell^2} h) \quad (4.10)$$

$$R_{\mu\nu}^{(2)} = \nabla_\lambda \Gamma_{\mu\nu}^{(2)\lambda} - \nabla_\nu \Gamma_{\mu\lambda}^{(2)\lambda} + \Gamma_{\lambda\sigma}^{(1)\lambda} \Gamma_{\mu\nu}^{(1)\sigma} - \Gamma_{\nu\sigma}^{(1)\lambda} \Gamma_{\mu\lambda}^{(1)\sigma} \quad (4.11)$$

$$\Gamma_{\mu\nu}^{(2)\lambda} = -\frac{h^{\lambda\sigma}}{2} (\nabla_\nu h_{\sigma\mu} + \nabla_\mu h_{\sigma\nu} - \nabla_\sigma h_{\mu\nu}). \quad (4.12)$$

The one-form  $\mathcal{K}$  in (3.1) may now be constructed to second order from  $E^{(1)}(h^{(2)})$  and shown to be the divergence of a two-form  $\mathcal{F}(h^{(2)})$ . The resulting boundary expression for the charges

$$Q_{boundary}(\xi) = -\frac{1}{16\pi G} \oint_{\partial\Sigma} * \mathcal{F}(h^{(2)}) \quad (4.13)$$

yields the expression quoted in (3.5). The bulk expression is then obtained by integrating by parts. When  $\xi$  is a background Killing vector it is straightforward to write this bulk charge explicitly

$$Q_{bulk}(\xi) = \frac{1}{16\pi G} \int_\Sigma * (\xi^\mu E_{\mu\nu}^{(2)}(h^{(1)}) dx^\nu). \quad (4.14)$$

For general  $\xi$  one can write a similar but somewhat more complicated expression.

At this point we have not assumed Brown-Henneaux boundary conditions. We note that it is crucial that the  $\partial_\rho h_{--}$  terms in (3.5) are included; these terms are omitted in some discussions in the literature. Without them the bulk and boundary expressions (4.14) and (4.13) would not be equal.

## 4.1 Chirality confinement

We now turn to a discussion of solutions of the linearized equations and their second-order back reaction. One may consider a basis of eigenmodes of  $\xi_0^L = \partial_-$  and  $\xi_0^R = \partial_+$ , or equivalently energy and angular momentum. Such eigenmodes were constructed in [1], where it was shown that all the (non-gauge) modes obeying the boundary conditions (2.6) have vanishing left charges and are in the  $(h_L, h_R) = (0, 2)$  highest

weight representation of  $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ . These are the right-moving boundary gravitons and can be constructed from non-trivial  $\xi_{-2}^R$  and  $\xi_{-1}^R$  diffeomorphisms on the  $AdS_3$  background. There are also weight  $(2, 0)$  left-moving excitations, but these can be eliminated by trivial  $\xi_{-2}^L$  and  $\xi_{-1}^L$  diffeomorphisms. This is in contrast with the situation for generic  $\mu$  where there are three types of eigenmodes in highest weight representations: chiral left and right boundary gravitons as well as massive gravitons transforming in a non-chiral highest weight  $\frac{1}{2}(3 + \mu\ell, -1 + \mu\ell)$  representation. As  $\mu\ell \rightarrow 1$ , the weight of the massive graviton approaches  $(2, 0)$  and its wave function degenerates with that of the left-moving boundary graviton. Consequently it can also be eliminated by a trivial diffeomorphism. Hence the disappearance of the massive and left moving representations at the chiral point is a direct result of the enhancement of the group of trivial symmetries.

However, there is no guarantee that all solutions of the linearized equations obeying the boundary conditions (2.6) have an expansion in terms of  $(\xi_0^L, \xi_0^R)$  eigenmodes, or fall into highest weight representations. Interestingly, modes without such an expansion do exist. Examples were explicitly constructed in [8] (building on results of [10]) and will be denoted  $h_{GKP}^{(1)}$ .  $h_{GKP}^{(1)}$  cannot be Fourier expanded as eigenmodes of  $\partial_\tau$  because it grows linearly in  $\tau$ . Moreover the GKP modes are non-chiral: the quadratic bulk expressions for left and right moving charges are non-zero

$$E_L \equiv Q_{bulk}^L(\xi_0^L, h_{GKP}^{(1)}) = -\frac{\ell}{12G}, \quad E_R \equiv Q_{bulk}^R(\xi_0^R, h_{GKP}^{(1)}) = -\frac{\ell}{24G}. \quad (4.15)$$

On the other hand, we can also compute the charges from the boundary expression. This involves first solving for the second order perturbation  $h_{GKP}^{(2)}$  and then evaluating the boundary integral. Since the bulk and boundary expressions are equal we must have

$$E_L = Q_{boundary}^L(\xi_0^L, h_{GKP}^{(2)}) = -\frac{\ell}{12G}, \quad (4.16)$$

where we have imposed the condition (3.6). This cannot be nonzero if  $h_{GKP}^{(2)}$  obeys the boundary condition (2.6). We conclude that  $h_{GKP}^{(2)}$  violates the boundary condition, and there is no exact solution to chiral gravity already at second order with the prescribed boundary condition. Explicit expressions for  $h_{GKP}^{(1)}$  and  $h_{GKP}^{(2)}$  are given below.  $h_{GKP}^{(2)}$  grows linearly at infinity so that  $\partial_\rho h_{GKP}^{(2)}$  gives a nonvanishing contribution to the boundary expression for the left charge.

This resolves the apparent contradiction between the vanishing of  $E_L$  and the existence of boundary-condition-obeying solutions of the linearized equations with

nonzero  $E_L$ . The latter are obstructed at second order and are not the linearization of boundary-condition-obeying solutions of the exact equations.

In the introduction we made an analogy between non-chiral solutions of linearized chiral gravity and free quark solutions of linearized QCD: neither are approximate finite-energy solutions of the exact theory. An alternate, purely classical, analogy can be found in Maxwell electromagnetism coupled to a charged scalar in 1+1 dimensions. At linear order there are scalar field configurations of order  $\epsilon$  with finite charge and finite energy. However these disappear from the finite energy spectrum at quadratic order: there is an electric field of order  $\epsilon^2$  which carries infinite energy due to an infrared divergence. So there are no finite energy excitations with nonzero charge: charge is confined. Here we are finding in analogy that non-chiral excitations are confined. In the Maxwell case, there is only one conserved quantity – the electric charge – which must vanish. This implies that the linearized solutions must obey a one parameter constraint in order to approximate exact solutions to the theory. In chiral gravity there are infinitely many conserved charges  $Q^L$  which must vanish. This leads to infinitely many constraints, and the elimination of an entire (left) chiral sector of the theory.

We have shown that the linearization of all finite excitations of chiral gravity must be chiral in the sense that the quadratic bulk expression for  $E_L$  (as well as the other left charges) must vanish. This is irrelevant to the energy eigenmodes which are in any case chiral, but it eliminates the nonchiral mode  $h_{GKP}^{(1)}$  which, from (4.15), has  $E_L = -\frac{\ell}{12G}$ . In principle there could be additional modes which are chiral but still have negative energy  $E = E_L + E_R = E_R$ . This would ruin perturbative stability. This seems highly unlikely since all linear chiral modes are associated with asymptotic symmetries, and we know already that the ASG is generated by exactly one copy of the Virasoro algebra. This Virasoro algebra is already accounted for by the (0, 2) mode.

For the sake of completeness, in sections (4.2) and (4.3) we will compute explicitly the second order perturbation resulting from the various alleged counterexamples to the chiral gravity conjecture. We will see the infrared divergence described above and conclude that the linearization of the exact spectrum consists only of the right-moving boundary gravitons.

We note that it is in principle straightforward to find *all* solutions of the linearized constraint equations in global coordinates, rather than just the energy eigenmodes described above. However, the analogous computation has already been solved in Poincaré coordinates [7]. So we will work primarily in Poincaré coordinates. We will

then show in section (4.4) that on global AdS<sub>3</sub> all linearized solutions which are non-singular at second order must be chiral and obey the linearized Einstein equations.

## 4.2 The CDWW modes

Carlip, Deser, Waldron and Wise (CDWW) have described all solutions of the linearized equations of motion which are smooth in Poincaré coordinates [7]. These include many nonchiral modes. We will first show that all of these nonchiral modes are singular at second order on the boundary of the Poincaré patch. All modes which are nonsingular at second order are chiral and obey the linearized Einstein equation.

We use Poincaré coordinates<sup>7</sup>

$$ds^2 = \frac{-dt^+ dt^- + dz^2}{z^2} \quad (4.17)$$

and light-front gauge

$$h_{--}^{(1)} = h_{+-}^{(1)} = h_{-z}^{(1)} = 0. \quad (4.18)$$

Following [7] we may integrate out  $\partial_- h_{+z}^{(1)}$  and  $\partial_-^2 h_{++}^{(1)}$  in the quadratic action. The equation of motion for  $h_{zz}^{(1)}$  becomes

$$\partial_+ \partial_- h_{zz}^{(1)} = \frac{1}{4z^2} [z^2 \partial_z^2 + 3z \partial_z + (-\mu^2 + 4\mu - 3)] h_{zz}^{(1)} \quad (4.19)$$

The general solution of (4.19) is a real linear combination of the modes

$$h_{\omega^+, \omega^-}^{(1)} = \sqrt{\frac{\omega}{4\pi E}} \frac{1}{z} e^{-i(\omega^+ t^+ + \omega^- t^-)} J_{|2-\mu|}(2\omega z) \quad (4.20)$$

$$h_{\omega^+, \omega^-}^{(1)*} = \sqrt{\frac{\omega}{4\pi E}} \frac{1}{z} e^{i(\omega^+ t^+ + \omega^- t^-)} J_{|2-\mu|}(2\omega z) \quad (4.21)$$

where

$$\omega^2 = \omega^+ \omega^-, \quad E = \frac{\omega^+ + \omega^-}{2}, \quad k = \frac{\omega^+ - \omega^-}{2}$$

A general solution can be written as a wave packet

$$h_{zz}^{(1)} = \int d\omega dk [a(\omega^+, \omega^-) h_{\omega^+, \omega^-}^{(1)} + a^*(\omega^+, \omega^-) h_{\omega^+, \omega^-}^{(1)*}] \quad (4.22)$$

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<sup>7</sup>The orientation here is  $\epsilon_{t\phi\rho} = \sqrt{-g}$ , or equivalently,  $\epsilon_{+-z} = \sqrt{-g}$ .

The other components of the linear fluctuation are obtained from (4.22) by

$$\partial_- h_{+z}^{(1)} = \frac{1}{2} \left[ \partial_z + \frac{-\mu + 2}{z} \right] h_{zz}^{(1)} \quad (4.23)$$

$$\partial_-^2 h_{++}^{(1)} = \frac{1}{2} \left[ 2\partial_+ \partial_- - \frac{\mu}{z} \partial_z + \frac{\mu^2 - 3\mu}{z^2} \right] h_{zz}^{(1)} \quad (4.24)$$

The left moving charges  $Q^L$  can now be computed from the bulk quadratic expression (4.9). They are in general nonzero. For example

$$E_L = -\frac{1}{128\pi G} \int dz dx \left[ z^3 \left( (\partial_z h_{zz}^{(1)})^2 + 4(\partial_- h_{zz}^{(1)})^2 \right) + \frac{1}{2} \partial_z \left( z^2 (9 + z\partial_z) (h_{zz}^{(1)})^2 \right) \right] \quad (4.25)$$

$$= -\frac{1}{128\pi G} \int dz dx \left[ 4z^3 \left( -\partial_+ \partial_- h_{zz}^{(1)} h_{zz}^{(1)} + (\partial_- h_{zz}^{(1)})^2 \right) + \frac{1}{2} \partial_z \left( z^2 (9 + z\partial_z) (h_{zz}^{(1)})^2 \right) \right] \quad (4.26)$$

We have discarded here total derivatives of  $t^-$  which vanish upon integration over  $x$ . This expression is a total derivative plus a negative semi-definite term.

$$E_L = -\frac{1}{32\pi G} \int dz dx z^3 \left[ -\partial_+ \partial_- h_{zz}^{(1)} h_{zz}^{(1)} + (\partial_- h_{zz}^{(1)})^2 \right] \quad (4.27)$$

$$= -\frac{1}{64\pi G} \int d\omega dk \omega^- |a(\omega^+, \omega^-)|^2 \quad (4.28)$$

This vanishes if and only the mode has support in the region

$$w^- = 0, \quad (4.29)$$

In this case  $h^{(1)}$  obeys the linearized Einstein equation

$$\mathcal{G}_{\mu\nu}^{(1)}(h^{(1)}) = 0. \quad (4.30)$$

In order to make this completely explicit, we will now compute the second order perturbation of the CDWW modes. We will compute the curvature rather than the metric, as divergences in the latter can be coordinate artifacts. The  $--$  component of (4.4) is

$$\partial_z \mathcal{G}_{--}^{(1)}(h^{(2)}) - \partial_- \mathcal{G}_{z-}^{(1)}(h^{(2)}) \quad (4.31)$$



$$= \frac{1}{2}z^3 \left[ \partial_- h_{zz}^{(1)} \partial_- h_{zz}^{(1)} + \partial_- \left( 2z \partial_- (h_{+z}^{(1)} \partial_- h_{zz}^{(1)}) - 2z h_{zz}^{(1)} \partial_-^2 h_{+z}^{(1)} - \partial_- h_{zz}^{(1)} (z \partial_z + \frac{5}{2}) h_{zz}^{(1)} \right) \right].$$

A boundary condition for this differential equation is obtained by noting that at the boundary point  $z = \infty$ , (2.6) implies  $\mathcal{G}_{--}^{(1)}(h^{(2)}; \infty, t^+, t^-) = 0$ . If  $h^{(1)}$  is one of the  $\omega_+, \omega_-$  eigenmodes, equation (4.31) decomposes into three equations which depend on  $t^\pm$  as either  $e^{\pm 2i(\omega_+ t^+ + \omega_- t^-)}$  or are constant in  $t^\pm$ . Consider the constant piece, for which  $\mathcal{G}_{z-}^{(1)}(h^{(2)})$  vanishes by symmetry. We may then solve for the constant part of  $\mathcal{G}_{--}^{(1)}(h^{(2)})$ :

$$\mathcal{G}_{--}^{(1)}(h^{(2)}; z, t^+, t^-) = -\frac{1}{2}\omega_-^2 \int_{-\infty}^z dz' z'^3 h_{\omega^+, \omega^-}^{(1)*} h_{\omega^+, \omega^-}^{(1)} \quad (4.32)$$

which is strictly negative unless  $\omega^- = 0$ . Thus it is impossible for  $\mathcal{G}_{--}^{(1)}(h^{(2)})$  to vanish everywhere on the boundary  $z = 0$  and  $z = \infty$  unless  $\partial_- h^{(1)}$  itself vanishes everywhere. This leaves only the chiral  $\omega^- = 0$  modes which solve the linearized Einstein equation. We see explicitly that the linearized modes or, since the right hand side is always negative, any superposition thereof must obey the linearized Einstein equation. Looking at the Fourier modes of  $\mathcal{G}_{--}^{(1)}(h^{(2)})$  gives more constraints leading again to  $\omega^- = 0$ .

We note that the above expressions for the curvature at second order, and hence the conclusion that the boundary conditions are violated, follows directly from the perturbative expansion of the equations of motion. Thus although our discussion was motivated by charge conservation, our conclusions ultimately do not rely on any particular expressions for or properties of the charges.

### 4.3 The GKP mode

A interesting nonchiral solution of the linearized equations was constructed by Giribet, Kleban and Porrati (GKP) in [9]. This mode is not an  $(\xi_0^L, \xi_0^R)$  eigenmode but nevertheless obeys the Brown-Henneaux boundary conditions (2.6). It may be written as

$$h_{\mu\nu}^{(1)GKP} = \mathcal{L}_{-1}^R(y(\tau, \rho) \mathcal{L}_{-2}^L \bar{g}_{\mu\nu}) + \mathcal{L}_\xi \bar{g}_{\mu\nu} \quad (4.33)$$

where  $\mathcal{L}_n^{L,R}$  is the Lie derivative with respect to  $\xi_n^{L,R}$  and

$$y(\tau, \rho) = -i\tau - \ln(\cosh \rho) \quad (4.34)$$

$$\xi = -\frac{iy(\tau, \rho) \sinh(\rho)}{2\ell^2 \cosh^5(\rho)} e^{-i(\tau^+ + 2\tau^-)} \xi_0^R. \quad (4.35)$$

The conserved charges are

$$E_{GKP}^L = -\frac{\ell}{12G} \quad (4.36)$$

$$E_{GKP}^R = -\frac{\ell}{24G} \quad (4.37)$$

We may now solve asymptotically for  $h^{(2)GKP}$  using (4.4), which reduces to

$$E^L = \frac{1}{16\pi\ell G} \oint d\phi (2\partial_\rho h_{--}^{(2)} - \partial_\rho^2 h_{--}^{(2)}) = E_{GKP}^L \quad (4.38)$$

$$E^R = \frac{1}{16\pi\ell G} \oint d\phi (4h_{++}^{(2)} - 4\partial_\rho h_{++}^{(2)} + \partial_\rho^2 h_{++}^{(2)}) = E_{GKP}^R \quad (4.39)$$

The  $\phi$  independent solution is

$$h_{--}^{(2)} = 4G\ell E^L \rho + \dots \quad (4.40)$$

$$h_{++}^{(2)} = 2G\ell E^R + \dots \quad (4.41)$$

where  $\dots$  denotes terms which are subleading in  $\rho$ . In particular, for the GKP modes, we have

$$h_{--}^{(2)GKP} = -\frac{\ell^2 \rho}{3} + \dots \quad (4.42)$$

$$h_{++}^{(2)GKP} = -\frac{\ell^2}{12} + \dots \quad (4.43)$$

From (4.42) we see that the Brown-Henneaux boundary conditions (2.6) are violated. We conclude that  $h^{(1)GKP}$  is not the linearization of an exact solution to the equations of motion.

#### 4.4 Global modes

We can now argue that *all* solutions to the linearized equations of motion that obey Brown-Henneaux boundary conditions at second order must be solutions of the linearized Einstein equations. In particular, we are left only with the right-moving boundary gravitons.

To prove this, one could study the linearized equations of motion in global rather than Poincaré coordinates. However this can be avoided by noting that every mode which is smooth and asymptotically AdS<sub>3</sub> in global coordinates is smooth on the

Poincare patch and hence has an expansion in CDWW modes of section (4.2).<sup>8</sup> We have seen that of these modes only the ones with vanishing Einstein tensor obey Brown-Henneaux boundary conditions at second order. As the right hand side of (4.32) is negative definite, we cannot cancel this divergence for any linear superpositions of modes. Hence all global modes must obey the linearized Einstein equation.

## 5 A Birkhoff-like theorem

We have seen that any solution of chiral gravity is, at the linearized level, locally AdS<sub>3</sub>. This might lead one to suspect that all solutions of chiral gravity are locally AdS<sub>3</sub> at the full non-linear level. In this section we will see that this is indeed the case for a particularly simple class of solutions: those which are stationary and axially symmetric. For this class of solutions the full non-linear equations of motion, although still surprisingly complicated, are reasonably tractable. We will conclude that, once we impose Brown-Henneaux boundary conditions, the only solutions are the BTZ black holes.

A similar result was obtained for general values of  $\mu$  by [46, 47], who made the somewhat stronger assumption of a hypersurface orthogonal Killing vector field.

### 5.1 Stationarity and axial symmetry

We start by studying the equations of motion of TMG for stationary, axially symmetric spacetimes, following the approach of [48].

A three dimensional spacetime with two commuting  $U(1)$  isometries may, through judicious choice of coordinates, be written in the form

$$ds^2 = -X^+(\sigma)d\tau^2 + X^-(\sigma)d\phi^2 + 2Y(\sigma)d\tau d\phi + \frac{d\sigma^2}{X^+(\sigma)X^-(\sigma) + Y(\sigma)^2} \quad (5.1)$$

The two  $U(1)$  isometries are the generated by Killing vectors  $\partial_\tau$  and  $\partial_\phi$ . We are interested in axially symmetric solutions, so we will take the angular direction to be periodic  $\phi \sim \phi + 2\pi$ . We have chosen the coefficient of  $d\sigma^2$  for future convenience.

The geometry of the solution is encoded in the three functions  $X^\pm(\sigma), Y(\sigma)$ , which we will package into a three dimensional vector  $\mathbf{X}$  with components  $X^i, i = 0, 1, 2$

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<sup>8</sup>Of course, the converse is not true: modes which are well behaved on the Poincare patch may not be well-behaved globally.

given by

$$X^0 = \frac{X^+(\sigma) - X^-(\sigma)}{2}, \quad X^1 = \frac{X^+(\sigma) + X^-(\sigma)}{2}, \quad X^2 = Y \quad (5.2)$$

The dynamics of stationary, axial metrics in three dimensions may be thought of as the dynamics of a particle with position  $\mathbf{X}(\sigma)$  moving in the auxiliary space  $\mathbb{R}^{2,1}$  parameterized by  $\mathbf{X}$ .

For the ansatz (5.1), the equations of motion of TMG are

$$-2\mu\mathbf{X}'' = 2\mathbf{X} \times \mathbf{X}''' + 3\mathbf{X}' \times \mathbf{X}'' \quad (5.3)$$

$$4 = \mathbf{X}'^2 - \frac{2}{\mu}\mathbf{X} \cdot (\mathbf{X}' \times \mathbf{X}'') \quad (5.4)$$

We have set  $\ell = 1$  for convenience. Here  $'$  denotes  $\partial_\sigma$  and we have defined the Lorentz invariant dot product and cross product<sup>9</sup>

$$\mathbf{A} \cdot \mathbf{B} \equiv \eta_{ij}A^iB^j, \quad \text{and} \quad (\mathbf{A} \times \mathbf{B})^i \equiv \eta^{il}\epsilon_{ljk}A^jB^k, \quad \epsilon_{012} = 1 \quad (5.5)$$

In order to understand these equations, it is helpful to note that for our ansatz (5.1) the action of TMG is

$$I = \frac{1}{16\pi G} \int d\sigma \frac{1}{2} \left( \mathbf{X}'^2 - \frac{1}{\mu}\mathbf{X} \cdot (\mathbf{X}' \times \mathbf{X}'') \right) \quad (5.6)$$

This is the action of a Lorentz invariant particle mechanics in  $\mathbb{R}^{2,1}$ . Equation (5.3) is found by varying this reduced action with respect to  $\mathbf{X}$ . Equation (5.4) represents an additional constraint which arises due to gauge-fixing. In fact, the right hand side of (5.4) is just the conserved Hamiltonian of the reduced action (5.6).

The equation of motion (5.3) can easily be integrated once. To see this, note that the action (5.6) is invariant under Lorentz transformations in  $\mathbb{R}^{2,1}$ . Hence there is a conserved angular momentum  $\mathbf{J}$  associated to these Lorentz transformations, which we can compute using the Noether procedure (taking into account the higher derivative terms)

$$\mathbf{J} = \frac{1}{16\pi G} \left( \mathbf{X} \times \mathbf{X}' - \frac{1}{2\mu}[\mathbf{X}' \times (\mathbf{X} \times \mathbf{X}') - 2\mathbf{X} \times (\mathbf{X} \times \mathbf{X}'')] \right) \quad (5.7)$$

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<sup>9</sup>In Lorentzian signature some of the usual cross product identities must be altered (e.g.  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{A} \cdot \mathbf{C})$  differs by a sign from the usual case).

One can check explicitly that (5.7) is the the first integral of (5.3). The dynamics of the system is given by the set of second order differential equations (5.7) and (5.4). The auxiliary angular momentum  $\mathbf{J}$  should not be confused with the physical angular momentum of the spacetime, although we shall see that they are closely related. With the help of (5.3) and (5.4) we can write (5.7) as a second order differential equation for  $\mathbf{X}''$ :

$$2\mathbf{X}^2\mathbf{X}'' = 32\pi G\mu\mathbf{J} - 2\mu\mathbf{X} \times \mathbf{X}' + \mathbf{X}'(\mathbf{X} \cdot \mathbf{X}') + \mathbf{X}(6 - \frac{5}{2}\mathbf{X}'^2) \quad (5.8)$$

This equation can then be integrated to give a solution  $\mathbf{X}(\sigma)$  to the equations of motion. For future reference, we note that with the help of the equations of motion we can write the Hamiltonian constraint as

$$(\mathbf{X}'^2 - 4) = \frac{2}{\mu}\mathbf{X} \cdot (\mathbf{X}' \times \mathbf{X}'') = -\frac{4}{3}\mathbf{X} \cdot \mathbf{X}'' \quad (5.9)$$

## 5.2 Boundary conditions

We will now consider axially symmetric, stationary solutions of TMG obeying Brown-Henneaux boundary conditions.

We start by noting that global metric on  $\text{AdS}_3$  can be written as

$$ds^2 = -(2\sigma + 1)d\tau^2 + 2\sigma d\phi^2 + \frac{d\sigma^2}{2\sigma(2\sigma + 1)} \quad (5.10)$$

which is of the form (5.1) with

$$\mathbf{X}_{AdS} = 2\sigma(0, 1, 0) + \frac{1}{2}(1, 1, 0) \quad (5.11)$$

Here  $\sigma$  is related to the usual global radial coordinate on  $\text{AdS}_3$  by  $\sigma = \frac{1}{2}\sinh^2 \rho$ . The asymptotic boundary is at  $\sigma \rightarrow \infty$ . Likewise, the BTZ black hole can be written in the form (5.1) with

$$\mathbf{X}_{BTZ} = 2\sigma(0, 1, 0) - 4GM(1, 1, 0) + 4GJ(0, 0, 1) \quad (5.12)$$

Here  $M$  and  $J$  are the ADM mass and angular momentum of the BTZ black hole in Einstein gravity. As we are working in units with  $\ell = 1$ , empty  $\text{AdS}_3$  has energy  $M = -1/8G$ .

Let us now consider an arbitrary metric obeying Brown-Henneaux boundary conditions. Comparing with the AdS metric (5.10) one can check that a metric of the

form (5.1) obeys Brown-Henneaux boundary conditions if

$$\mathbf{X} = 2\sigma(0, 1, 0) + \mathcal{O}(1), \quad \text{as } \sigma \rightarrow \infty \quad (5.13)$$

In analogy with (5.12) we will write this boundary condition as

$$\mathbf{X} = 2(\sigma - \sigma_0)(0, 1, 0) - 4GM(1, 1, 0) + 4GJ(0, 0, 1) + \dots \quad (5.14)$$

where  $\sigma_0$ ,  $M$  and  $J$  are constants,  $G$  is Newton's constant and  $\dots$  denotes terms which vanish as  $\sigma \rightarrow \infty$ . By comparing with (5.12), we see that  $M$  and  $J$  are the usual ADM mass and angular momentum of the spacetime as measured at asymptotic infinity in Einstein gravity. The parameter  $\sigma_0$  is just a shift in the radial coordinate and does not have a coordinate-independent meaning.

Let us now consider solutions to the equations of motion of TMG with the boundary conditions (5.14). As the angular momentum  $\mathbf{J}$  is a constant of motion we can compute it at  $\sigma \rightarrow \infty$ . Plugging (5.14) into (5.7) we find

$$2\pi \mathbf{J} = (J, 0, -M) + \frac{1}{\mu}(-M, 0, J) \quad (5.15)$$

The auxiliary angular momentum  $\mathbf{J}$  is just a rewriting of the usual mass and angular momentum of the solution. We note that

$$(2\pi\mathbf{J})^2 = \left(1 - \frac{1}{\mu^2}\right)(M^2 - J^2) \quad (5.16)$$

For values of  $\mu > 1$ , we see that the angular momentum vector is spacelike for solutions obeying the cosmic censorship bound  $M > J$ . For extremally rotating solutions  $\mathbf{J}$  is null,  $\mathbf{J}^2 = 0$ .

Finally, we turn to the case of chiral gravity ( $\mu = 1$ ) with Brown-Henneaux boundary conditions. In this case the angular momentum vector is always null:

$$2\pi\mathbf{J} = (J - M)(1, 0, 1), \quad \mathbf{J}^2 = 0 \quad (5.17)$$

This property will turn out to be very useful.

### 5.3 Solutions

We will now specialize to chiral gravity ( $\mu = 1$ ) and study axially symmetric, stationary solutions obeying Brown-Henneaux boundary conditions. We will demonstrate

that if the spacetime has a single asymptotic boundary obeying Brown-Henneaux boundary conditions, then the spacetime must be locally AdS<sub>3</sub>. We will also assume that  $\mathbf{X}(\rho)$  is an analytic function of  $\rho$ .

At  $\sigma \rightarrow \infty$ , the vector  $\mathbf{X}$  is spacelike. As long as  $\mathbf{X}$  remains spacelike, we can continue to smoothly evolve our metric into the interior using (5.8). In fact,  $\mathbf{X}$  must become null – with  $\mathbf{X}^2 = 0$  – for some finite value of  $\sigma$ . To see this, consider what would happen if  $\mathbf{X}^2$  remained strictly positive for all values of  $\sigma$ . In this case the evolution equation (5.8) would allow us to evolve  $\mathbf{X}$  all the way to  $\sigma \rightarrow -\infty$ . The region  $\sigma \rightarrow -\infty$  then represents an additional asymptotic boundary. To prove this, note that in order for  $\mathbf{J}$  to remain finite at  $\sigma \rightarrow -\infty$ ,  $\mathbf{X}^2$  must either remain finite or diverge no more quickly than  $\sigma^2$ . The line element

$$ds^2 \sim \frac{d\sigma^2}{\mathbf{X}^2} + \dots \quad (5.18)$$

then implies that points with  $\sigma \rightarrow -\infty$  lie an infinite proper distance from points with finite  $\sigma$ . As we are assuming Brown-Henneaux boundary conditions with a *single* asymptotic boundary, we must not allow this additional boundary at  $\sigma \rightarrow -\infty$ . We conclude that there must be finite value of  $\sigma$  where  $\mathbf{X}$  becomes null, i.e.  $\mathbf{X}^2 = 0$ .

We will now proceed to study the equation of motion near the point where  $\mathbf{X}^2 = 0$ . First, let us shift our coordinate  $\sigma$  so that this point occurs at  $\sigma = 0$ . We will assume that the metric is analytic at this point, so is equal to its Taylor expansion

$$\mathbf{X} = \sum_{n \geq 0} \frac{1}{n!} \sigma^n \mathbf{X}_n \quad (5.19)$$

The coefficients  $\mathbf{X}_n$  are finite and given by derivatives of  $\mathbf{X}$  at  $\sigma = 0$ , with  $\mathbf{X}_0^2 = 0$ . In fact, one can check that the point  $\sigma = 0$  is either a horizon or an origin of polar coordinates, depending on the relative values of  $\mathbf{X}_0$  and  $\mathbf{X}_1$ .

We now turn to the equations of motion. By plugging (5.19) into (5.8) and expanding order by order in powers of  $\sigma$  we obtain a set of recursion relations which determine the  $\mathbf{X}_n$  in terms of  $\mathbf{X}_0$  and  $\mathbf{X}_1$ . We will now proceed to show that these recursion relations imply that all the terms with  $n \geq 2$  in the Taylor expansion (5.19) vanish. This will imply that our Taylor expansion converges, hence the solution can be smoothly matched on to the metric at infinity. Indeed, comparing with equation (5.12) we see that our solution

$$\mathbf{X} = \mathbf{X}_0 + \sigma \mathbf{X}_1 \quad (5.20)$$

is simply the BTZ black hole.<sup>10</sup> This allows us to conclude that our solution is locally AdS<sub>3</sub>.

In order to demonstrate this, let us now expand equation (5.8) order by order in  $\sigma$ . The order  $\sigma^0$  term just fixes the angular momentum vector

$$32\pi G\mathbf{J} = 2\mathbf{X}_0 \times \mathbf{X}_1 - \mathbf{X}_1(\mathbf{X}_0 \cdot \mathbf{X}_1) - \mathbf{X}_0(6 - \frac{5}{2}\mathbf{X}_1^2) \quad (5.21)$$

In chiral gravity,  $\mathbf{J}$  must be null, so that

$$(32\pi G\mathbf{J})^2 = -4(\mathbf{X}_0 \cdot \mathbf{X}_1)^2(\mathbf{X}_1^2 - 4) = 0 \quad (5.22)$$

implying that either  $\mathbf{X}_1^2 - 4$  or  $\mathbf{X}_0 \cdot \mathbf{X}_1$  vanish. We will consider the following cases separately:

*Case 0:*  $\mathbf{X}_0 = 0$

In this case  $\mathbf{J} = 0$  and it is easy to prove directly that all the higher order terms vanish. In particular, (5.7) implies that

$$\mathbf{J} \cdot \mathbf{X} = \mathbf{X}^2\mathbf{X}'^2 - (\mathbf{X} \cdot \mathbf{X}')^2 = 0 \quad (5.23)$$

so that  $\mathbf{X} \times \mathbf{X}'$  is null. We also see that

$$\mathbf{J} \cdot (\mathbf{X} \times \mathbf{X}' - 2\mathbf{X}) = \mathbf{X}^2(\mathbf{X}'^2 - 4) = 0 \quad (5.24)$$

so that  $\mathbf{X}'^2 = 4$ . Thus in the region where  $\mathbf{X}^2$  is positive,  $\mathbf{X}$  and  $\mathbf{X}'$  are spacelike vectors whose cross product  $\mathbf{X} \times \mathbf{X}'$  is null. One can use this condition to show that  $\mathbf{X}$  and  $\mathbf{X}'$  obey<sup>11</sup>

$$\mathbf{X} \times \mathbf{X}' = -\sqrt{\mathbf{X}'^2}\mathbf{X} + \sqrt{\mathbf{X}^2}\mathbf{X}' \quad (5.25)$$

Plugging these identities into (5.8) we conclude that  $\mathbf{X}'' = 0$ . Thus all of the higher order terms in the Taylor expansion vanish and the solution is just the BTZ black hole.

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<sup>10</sup>In comparing (5.20) with (5.12) we must remember that in (5.20) we have shifted  $\sigma$  to put the horizon at  $\sigma = 0$ , in contrast to (5.12).

<sup>11</sup>For any two spacelike vectors  $\mathbf{A}$  and  $\mathbf{B}$  whose cross product is null one has the identity  $\mathbf{A} \times \mathbf{B} = \pm\sqrt{\mathbf{A}^2}\mathbf{B} \pm \sqrt{\mathbf{B}^2}\mathbf{A}$  where the signs depend on the relative orientations of the vectors. We have fixed the signs here by comparing to the behavior at asymptotic infinity (where  $\mathbf{X}$  approaches that of an extremally rotating BTZ black hole with  $M = J$ ).



Case 1:  $\mathbf{X}_1^2 = 4$  and  $\mathbf{X}_0 \cdot \mathbf{X}_1 \neq 0$

In this case we must work a little harder and examine the terms in the Taylor expansion (5.19) term by term. The equation for  $\mathbf{X}_2$  is found by expanding (5.8) to linear order in  $\sigma$ :

$$3\mathbf{X}_0 \cdot \mathbf{X}_1 \mathbf{X}_2 + 2\mathbf{X}_0 \times \mathbf{X}_2 - \mathbf{X}_1 \mathbf{X}_0 \cdot \mathbf{X}_2 + 5\mathbf{X}_0 \mathbf{X}_1 \cdot \mathbf{X}_2 = 0 \quad (5.26)$$

The Hamiltonian equation expanded to order  $\sigma^0$  is

$$\mathbf{X}_0 \cdot \mathbf{X}_2 = 0 \quad (5.27)$$

In fact,  $\mathbf{X}_2 = 0$  is the only solution to this equation. To see this, note that since  $\mathbf{X}_0 \cdot \mathbf{X}_1 = 0$  the vectors  $\mathbf{X}_0$ ,  $\mathbf{X}_1$  and  $\mathbf{X}_0 \times \mathbf{X}_1$  form a basis for  $\mathbb{R}^{2,1}$ . So we may expand

$$\mathbf{X}_2 = a\mathbf{X}_0 + b\mathbf{X}_1 + c\mathbf{X}_0 \times \mathbf{X}_1 \quad (5.28)$$

for some constants  $a, b, c$ . Plugging into the equations of motion we find that each of these constants must vanish, so  $\mathbf{X}_2 = 0$ .

We will now prove by induction that all of the higher order terms in the expansion (5.19) must vanish as well. Let us start by assuming that all of the quadratic and higher terms in the expansion (5.19) vanish up to a given order  $m$ . That is, let us assume that

$$\mathbf{X} = \mathbf{X}_0 + \sigma\mathbf{X}_1 + \sum_{n \geq m} \frac{1}{n!} \sigma^n \mathbf{X}_n \quad (5.29)$$

for some  $m \geq 2$ . Expanding equation (5.8) to order  $\sigma^{m-1}$  gives

$$(4m - 5)\mathbf{X}_0 \cdot \mathbf{X}_1 \mathbf{X}_m + 2\mathbf{X}_0 \times \mathbf{X}_m - \mathbf{X}_0 \cdot \mathbf{X}_m \mathbf{X}_1 + 5\mathbf{X}_1 \cdot \mathbf{X}_m \mathbf{X}_0 = 0 \quad (5.30)$$

and the Hamiltonian constraint at order  $\sigma^{m-2}$  gives

$$\mathbf{X}_0 \cdot \mathbf{X}_m = 0 \quad (5.31)$$

Expanding  $\mathbf{X}_m$  in the basis as above, one can again show that each term in the basis expansion vanishes separately. Hence  $\mathbf{X}_m$  vanishes and the inductive hypothesis (5.29) holds up to order  $m + 1$ . In the previous paragraph we proved the case  $m = 2$ , so by induction it follows that that *all*  $\mathbf{X}_n$ ,  $n \geq 2$  must vanish.

Case 2:  $\mathbf{X}_0 \neq 0$  but  $\mathbf{X}_0 \cdot \mathbf{X}_1 = 0$

This special case is a bit more complicated. We note that since  $\mathbf{X}_0$  is null and  $\mathbf{X}_0 \cdot \mathbf{X}_1 = 0$  it follows that

$$\mathbf{X}_0 \times \mathbf{X}_1 = \pm \sqrt{\mathbf{X}_1^2} \mathbf{X}_0 \quad (5.32)$$

Expanding the Hamiltonian constraint to order  $\sigma^0$  gives

$$\mathbf{X}_1^2 - 4 = 2\mathbf{X}_2 \cdot (\mathbf{X}_0 \times \mathbf{X}_1) = \pm 2\sqrt{\mathbf{X}_1^2} \mathbf{X}_2 \cdot \mathbf{X}_0 \quad (5.33)$$

Comparing to the second form of the Hamiltonian constraint

$$\mathbf{X}_1^2 - 4 = -\frac{4}{3} \mathbf{X}_2 \cdot \mathbf{X}_0 \quad (5.34)$$

we see that either  $\mathbf{X}_1^2 = 4/9$  or  $\mathbf{X}_1^2 = 4$ . We will consider these two cases separately.

If  $\mathbf{X}_1^2 = 4$  then we can show that all higher order terms in the Taylor expansion vanish. The Hamiltonian constraint is  $\mathbf{X}_2 \cdot \mathbf{X}_0 = 0$ . Expanding the equation of motion to linear order in  $\sigma$  gives equation (5.26), which in this case has the solution

$$\mathbf{X}_2 = a\mathbf{X}_0 \quad (5.35)$$

for some constant  $a$ . However, we find that the equation at order  $\sigma^3$  implies that  $a = 0$  so that  $\mathbf{X}_2 = 0$ . Likewise, equation (5.30) as a solution of the form

$$\mathbf{X}_m = a_m \mathbf{X}_0 \quad (5.36)$$

but the coefficient  $a_m$  is set to zero by the equation of motion at order  $\sigma^{m+1}$ . Proceeding in this manner we conclude that all of the  $\mathbf{X}_n$  must vanish for  $n \geq 2$ , so the solution is just the BTZ black hole.

Finally, let us consider the case where  $\mathbf{X}_1^2 = 4/9$ . In this case the Hamiltonian constraint

$$\mathbf{X}_2 \cdot \mathbf{X}_0 = \frac{8}{3} \quad (5.37)$$

implies that  $\mathbf{X}_2$  is non-zero. Aside from this small difference, the argument proceeds exactly as above. Expanding the equations of motion order by order in  $\sigma$  we discover that all of the terms in the Taylor expansion vanish except for  $\mathbf{X}_0$ ,  $\mathbf{X}_1$  and  $\mathbf{X}_2$ , which

obey

$$\mathbf{X}_0^2 = \mathbf{X}_2^2 = \mathbf{X}_0 \cdot \mathbf{X}_1 = \mathbf{X}_1 \cdot \mathbf{X}_2 = 0, \quad \mathbf{X}_1^2 = 4, \quad \mathbf{X}_0 \cdot \mathbf{X}_2 = \frac{8}{3}, \quad \mathbf{X}_0 \times \mathbf{X}_2 = \pm 4\mathbf{X}_1. \quad (5.38)$$

One may check that the resulting solution

$$\mathbf{X} = \mathbf{X}_0 + \mathbf{X}_1\rho + \frac{1}{2}\mathbf{X}_2\rho^2 \quad (5.39)$$

is warped AdS<sub>3</sub>. This solution does not obey Brown-Henneaux boundary conditions, because of the  $\mathcal{O}(\rho^2)$  behavior at asymptotic infinity.

This completes the proof.

## 6 Log gravity

In this section we consider log gravity, which differs from chiral gravity in that the boundary conditions are relaxed to allow certain types of growth linear in  $\rho$  (and logarithmic in the proper radius) at infinity. The solutions of log gravity will have energies which are unbounded below as well as unbounded above. Nevertheless, the theory is of considerable interest as it contains a novel and mathematically natural class of solutions [34, 13, 21, 35] excluded from chiral gravity. Here we will show that log gravity is consistent insofar as the expressions for the conserved charges are finite. However, the left charges are in general nonzero, so log gravity is not chiral. This result is in agreement with [29].<sup>12</sup> Moreover, we shall see that log gravity contains within it a decoupled superselection sector which is identical to chiral gravity. The relation between this chiral  $Q^L = 0$  subsector and the full spectrum of log gravity is reminiscent of the relation between the physical states of a first-quantized string (or any 2d gravity theory) and the larger Hilbert space including longitudinal modes and ghosts. Indeed, logarithmic CFTs appeared in the 2D gravity context in [49].

### 6.1 Boundary conditions and non-chirality

The starting point for the development of log gravity was the observation by Grumiller and Johansson (GJ) in that a solution to TMG at the chiral point can be obtained as [10]

$$h^{GJ} \equiv \lim_{\mu\ell \rightarrow 1} \frac{h^\mu - h^L}{\mu\ell - 1}. \quad (6.1)$$

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<sup>12</sup>However, this result is not in agreement with reference [33], which neglected a term in the generators.

Here  $h^\mu$  and  $h^L$  are the wave functions for the massive and left-moving gravitons, respectively. The mode  $h^{GJ}$  is a solution of the linearized equations of motion, but it is not an energy eigenstate and was not considered in [1]. Moreover  $h^{GJ}$  does not obey the Brown-Henneaux boundary conditions as certain components grow linearly in  $\rho$  at the boundary.

GJ then proposed that the Brown-Henneaux boundary conditions (2.6) be relaxed to allow metric fluctuations to grow linearly as  $\rho \rightarrow \infty$  [10]. The mode  $h^{GJ}$  would be included in the spectrum of such a theory. However, this proposal does not lead to a consistent theory, because for the general such asymptotic perturbation the right-moving charge  $Q^R$  is linearly divergent and hence ill-defined. A modified approach [15] is to impose chiral boundary conditions for which  $h_{--}$  is allowed to grow linearly in  $\rho$  but  $h_{++}$  or  $h_{+-}$  are not. Specifically, we take [29, 33]

$$\begin{pmatrix} h_{++} = \mathcal{O}(1) & h_{+-} = \mathcal{O}(1) & h_{+\rho} = \mathcal{O}(e^{-2\rho}) \\ & h_{--} = \mathcal{O}(\rho) & h_{-\rho} = \mathcal{O}(\rho e^{-2\rho}) \\ & & h_{\rho\rho} = \mathcal{O}(e^{-2\rho}) \end{pmatrix} \quad (6.2)$$

The allowed diffeomorphisms are

$$\xi^+ = \epsilon^+(x^+) + 2e^{-2\rho} \partial_-^2 \epsilon^-(x^-) + \dots \quad (6.3)$$

$$\xi^- = \epsilon^-(x^-) + 2e^{-2\rho} \partial_+^2 \epsilon^+(x^+) + \dots \quad (6.4)$$

$$\xi^\rho = -\frac{1}{2} (\partial_+ \epsilon^+(x^+) + \partial_- \epsilon^-(x^-)) + \dots \quad (6.5)$$

The leading terms give two sets of Virasoro generators. The subleading terms are trivial and do not appear in the charges. It is straightforward to see that both  $Q^L$  and  $Q^R$  are finite for all elements of the ASG. However, since  $\partial_\rho h_{--} \neq 0$  we find that

$$Q^L \neq 0, \quad (6.6)$$

so that log gravity is *not* chiral.

This opens up the possibility [10] that log gravity could be holographically dual to a logarithmic CFT. In fact, a logarithmic CFT can never be chiral.<sup>13</sup> Moreover it is not hard to see that the mode (6.1) lies in an indecomposable Virasoro representation (containing the left-moving highest weight representation of massless gravitons) characteristic of a logarithmic CFT [50].

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<sup>13</sup>We thank M. Gaberdiel for pointing this out.

While we have seen that the charges are finite for log gravity, more work must be done to show that they actually generate the asymptotic symmetry group, or indeed if log gravity has a canonical formulation at all. A canonical formulation requires the construction of a closed invertible symplectic form  $\Omega$ , or equivalently a Dirac bracket, on the physical phase space. The Dirac bracket is nonlocal in space and its construction involves inverting the constraints. Whether or not the constraints can be inverted depends on the boundary conditions, and so there is in general no guarantee that Dirac brackets exist for any boundary condition. Since there are physical zero norm states in log gravity, invertibility is not manifest. An elegant covariant construction of  $\Omega$  was given for general relativity in [51] in the form of an integral  $\Omega = \int d\Sigma_\alpha J^\alpha$  over a spatial slice, with  $J^\alpha = \delta\Gamma_{\nu\lambda}^\alpha \wedge [\delta g^{\nu\lambda} + \frac{1}{2}g^{\nu\lambda}\delta \ln g] - \delta\Gamma_{\nu\lambda}^\lambda [\delta g^{\alpha\nu} + \frac{1}{2}g^{\alpha\nu}\delta \ln g]$  and  $\delta$  the exterior derivative on the phase space. For TMG, there is an additional term proportional to  $\frac{1}{\mu}\epsilon^{\alpha\lambda\nu}\delta\Gamma_{\lambda\rho}^\sigma \wedge \delta\Gamma_{\nu\sigma}^\rho$ . It would be interesting to see by direct computation if this symplectic form is both finite and invertible for log gravity.

## 6.2 Decoupling the chiral gravity superselection sector

Log gravity in and of itself does not seem of so much interest because it is not unitary. Nonunitary theories of quantum gravity are generally easy to construct, and are not expected to shed much light on the presumably unitary theory which describes our four dimensional world. What makes log gravity interesting is that it contains chiral gravity, which has the possibility of being unitary, within it. The structure of this embedding is intriguing and could be useful for a full understanding of chiral gravity. In this section we explain how this embedding works.

Let  $Q_n^L$  denote the left Virasoro charges. The classical computation of the central charge is insensitive to the boundary conditions as long as the charges are well defined. Therefore the Dirac bracket algebra

$$\{Q_m^L, Q_n^L\} = i(m-n)Q_{m+n}^L, \quad (6.7)$$

has

$$c_L = 0 \quad (6.8)$$

as is the case for chiral gravity. The charges  $Q_n^L$  are conserved for all  $n$ . Therefore we can consistently truncate to the charge superselection sector of the theory with

$Q_n^L = 0$ .  $Q_n^L$  are the Fourier transforms of the linearly growing terms in  $h$

$$\partial_\rho h_{--} = 4\ell G \sum_n Q_n^L e^{in\phi}. \quad (6.9)$$

Therefore in the  $Q_n^L = 0$  superselection sector we have

$$\partial_\rho h_{--} = 0. \quad (6.10)$$

This condition reduces the log gravity boundary conditions (6.2) to the chiral gravity boundary conditions (2.6).<sup>14</sup> Therefore the  $Q_n^L = 0$  superselection sector of log gravity is precisely chiral gravity. Charge conservation guarantees that time evolution preserves the chiral boundary conditions and chiral gravity completely decouples from log gravity. Note that this result is nonperturbative.

At the classical level, this shows that solutions of chiral gravity cannot smoothly evolve into geometries with logarithmic behavior at infinity. Of course, we have not proven cosmic censorship so we cannot rule out singularities on the boundary for either log or chiral gravity.

One may phrase the issue of classical decoupling of chiral gravity in a different way in perturbation theory, where one can see the decoupling by direct computation without invoking charge conservation. If we excite two linearized modes of chiral gravity, will a log mode be excited at the next order? Do the chiral modes source the log modes? This question has already been answered to second order in our perturbative analysis of section 4. It is immediate from inspection of (4.31)-(4.32) that if we take  $h^{(1)}$  to solve the linearized Einstein equation, then at second order  $\mathcal{G}^{(2)}$  vanishes. Of course one can always add a log mode obeying the homogenous equation at second order, but as this is not required the log mode can be decoupled.

This analysis can be extended to all orders. If  $h^{(1)}$  is nontrivial and is a linearized solution of chiral gravity, it also solves the linearized Einstein equation, and is an infinitesimal nontrivial diffeomorphism. The exact all-orders corrected solution is then just the finite diffeomorphism. This obviously is a solution of chiral gravity with no log modes excited.

At the quantum level, the question is trickier. Of course we do not know whether or not either theory exists quantum mechanically. If log gravity does exist as a logarithmic conformal field theory we know it contains chiral gravity as a superselection

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<sup>14</sup> If present, terms in  $h_{-\rho}$  proportional to  $\rho e^{-2\rho}$  may then, as in [9] be eliminated by a trivial diffeomorphism along  $\xi = -2e^{-2\rho} h_{-\rho} \partial_+$ .

sector. In perturbation theory, chiral gravity has only the massless gravitons which are dual to the quantum stress tensor current algebra. The OPEs of these fields obviously close and hence decouple from log gravity. But we do not know if that superselection sector contains more than just the current algebra, or if it is local or modular invariant. Equivalently, we do not know if the black hole microstates are *exactly* chiral at the quantum level, or contain small nonchiral charges which are lost in the semiclassical limit. As we shall now see, this is more or less equivalent to the question of whether or not there are extremal CFTs with large central charge. More discussion of this point can be found in the concluding section.

## 7 Quantum partition function

At this point we have seen that with Brown-Henneaux boundary conditions the linearized spectrum of perturbations of chiral gravity around an  $\text{AdS}_3$  background includes only right moving boundary gravitons. We will now use this observation to compute the partition function of the quantum theory assuming applicability of the standard Euclidean methodology. Quite nontrivially, we will find that the resulting partition function has a consistent quantum mechanical interpretation. This can be regarded as evidence that the quantum theory actually exists.

We wish to compute the torus partition function, which is defined as the generating function

$$Z(\tau) = \text{Tr } q^{L_0} \bar{q}^{\bar{L}_0}, \quad q = e^{2\pi i \tau} \quad (7.1)$$

encoding the spectrum of the theory.  $Z(\tau)$  may be thought of as a canonical ensemble partition function at fixed temperature  $\beta^{-1}$  and angular potential  $i\theta$  given by  $\tau = \frac{1}{2\pi}(\theta + i\beta)$ . We will not attempt to perform the trace in (7.1) directly; this would require an understanding of the Hilbert space of chiral gravity at the non-perturbative level. We instead adopt an indirect approach, following [31, 52, 53, 54].

The standard strategy for computing canonical ensemble partitions in quantum field theory is to go to Euclidean signature. In this case  $Z(\tau)$ , originally formulated in terms of the Hilbert space of the Lorentzian theory, takes the form of a path integral

$$Z(\tau) \sim \int Dg e^{-kI[g]} \quad (7.2)$$

The dependence of this path integral on  $\tau$  enters through the boundary conditions imposed on the metric  $g$ ; the boundary is taken to be a two torus  $T^2$  with conformal structure parameter  $\tau$ . For clarity, we have extracted from equation (7.2) an explicit

factor of the dimensionless coupling constant of the theory,  $k = \ell/16G$ . In terms of the central charge of the dual boundary theory,  $k = c/24$ .

At large  $k$  the dominant contribution to the path integral is given by the saddle point approximation

$$Z(\tau) = \sum_{g_c} e^{-kI[g_c] + I^{(1)}[g_c] + \frac{1}{k}I^{(2)}[g_c] + \dots} \quad (7.3)$$

Here the sum is over classical solutions  $g_c$  to the equations of motion of the theory.  $I[g_c]$  denotes the corresponding classical action. The subleading terms of the form  $k^{1-n}I^{(n)}[g_c]$  represent quantum corrections to the effective action at  $n^{\text{th}}$  order in perturbation theory.

We will take equation (7.3) to be our working definition of the path integral of chiral gravity (7.2) and assume the equivalence of (7.2) with (7.1). In quantum mechanics, this equivalence can be rigorously established. In quantum field theory in general it cannot be proven, but has worked well in many situations. In quantum gravity, the Euclidean approach is less well-founded because, among other problems, the action is unbounded below. Nevertheless, straightforward applications in quantum gravity have tended to yield sensible answers. We will simply assume that this is the case for the path integral of chiral gravity. At the end of this paper we will discuss various ways in which this assumption might fail.

## 7.1 Classical saddle points

Our first task is to determine which classical saddle points contribute to the partition function (7.3). These saddles  $g_c$  are solutions to the classical equations of motion with  $T^2$  conformal boundary. In Euclidean signature, the bulk action of chiral gravity is

$$I_{TMG} = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left( R + \frac{2}{\ell^2} \right) + i\ell \int d^3x \sqrt{g} \varepsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma}^r \left( \frac{1}{2} \partial_\mu \Gamma_{r\nu}^\sigma + \frac{1}{3} \Gamma_{\mu\tau}^\sigma \Gamma_{\nu r}^\tau \right) \quad (7.4)$$

The factor of  $i$  appearing in front of the final term is the usual one that appears for Chern-Simons theories in Euclidean signature. To see that it must be there, note that the Chern-Simons Lagrangian is a pseudo-scalar rather than a scalar. Hence in Lorentzian signature the Chern-Simons term is odd under time reversal  $t \rightarrow -t$ . Rewriting in terms of the Euclidean time variable  $t_E = it$  we see that the Chern-Simons action is pure imaginary in Euclidean signature. The corresponding Euclidean



equations of motion are

$$\mathcal{G}_{\mu\nu} + i\ell C_{\mu\nu} = 0 \tag{7.5}$$

where  $\mathcal{G}_{\mu\nu}$  and  $C_{\mu\nu}$  are defined as in (2.4). One can verify directly that this is just the Lorentzian equation of motion (2.3) written in terms of a Euclidean time coordinate  $t_E = it$ .

The classical saddle points are smooth, real<sup>15</sup> Euclidean metrics which solve (7.5). For these metrics both  $\mathcal{G}_{\mu\nu}$  and  $C_{\mu\nu}$  are real, so must vanish separately. Thus these saddle points obey the equations of motion of Einstein gravity with a negative cosmological constant

$$\mathcal{G}_{\mu\nu} = 0 \tag{7.6}$$

The fact that Euclidean saddle points must be locally Einstein is in contrast with the quite difficult problem of solving the equations of motion Lorentzian signature. This dramatic simplification will allow us to compute the partition function exactly. One might interpret this simplification either as evidence that the Euclidean formulation does not correctly capture the complexity of the Lorentzian theory, or as evidence that the Lorentzian theory has a hidden simplicity. Indeed it is possible that all Lorentzian solutions of chiral gravity are locally Einstein.

Solutions of the equation of motion (7.6) are locally isometric to three dimensional hyperbolic space  $\mathbb{H}_3$  with Ricci curvature  $R = -6/\ell^2$ . So we just need to classify locally hyperbolic three manifolds with  $T^2$  boundary. Any locally hyperbolic three manifold is a quotient of  $\mathbb{H}_3$  by a discrete subgroup of its isometry group  $SL(2, \mathbb{C})$ . In fact, it is straightforward to show (see e.g. [31]) that any such smooth geometry with a  $T^2$  conformal boundary must be of the form  $H^3/\mathbb{Z}$ . We will not review this classification in detail, but simply summarize the salient points.

We will take the boundary  $T^2$  to be parameterized by a complex coordinate  $z$ , in terms of which the periodicity conditions are

$$z \sim z + 1 \sim z + \tau \tag{7.7}$$

This complex coordinate is related to the usual time and angular coordinates of global  $\text{AdS}_3$  by  $z = \frac{1}{2\pi}(\phi + it)$ .<sup>16</sup> To find a Euclidean geometry whose boundary has this

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<sup>15</sup>One might wonder whether complex saddle points should be considered. In Euclidean quantum field theory, one is instructed to include complex saddle points when, for example, momenta are held fixed at the boundary. As we are fixing the boundary metric here there is no obvious reason to include complex saddle points.

<sup>16</sup>In equation (2.5) we used  $\tau$  and  $\phi$  to denote the time and angular coordinates of global  $\text{AdS}_3$ . Here we use  $t$  and  $\phi$  to avoid confusion with the conformal structure parameter  $\tau$ .

conformal structure, write  $\mathbb{H}_3$  in planar coordinates as

$$\frac{ds^2}{\ell^2} = \frac{dw\bar{w} + dy^2}{y^2} \quad (7.8)$$

The conformal boundary is at  $y = 0$ , along with the point  $y = \infty$ . To obtain  $\mathbb{H}_3/\mathbb{Z}$  we will quotient by the identification

$$w \sim qw, \quad q = e^{2\pi i\tau} \quad (7.9)$$

If we identify  $w = e^{2\pi iz}$ , then the identifications (7.7) follow. We will call the quotient  $\mathbb{H}_3/\mathbb{Z}$  constructed in this way  $M_{0,1}$ .

Now, the geometry described above is not the only locally hyperbolic manifold with the desired boundary behavior. To see this, note that the geometry (7.8) does not treat the two topologically non-trivial cycles of the boundary  $T^2$  in a democratic manner. In particular, the  $\phi$  (real  $z$ ) cycle of the boundary torus is contractible in the interior of the geometry (7.8), while the  $t$  cycle is not. In fact, for every choice of cycle in the boundary  $T^2$  one can find a quotient dimensional manifold  $\mathbb{H}_3/\mathbb{Z}$  which makes this cycle contractible. A topologically nontrivial cycle  $ct + d\phi$  in  $T^2$  is labeled by a pair of relatively prime integers  $(c, d)$ . The associated quotient  $\mathbb{H}_3/\mathbb{Z}$  will be denoted  $M_{c,d}$ . These geometries were dubbed the “ $SL(2, \mathbb{Z})$  family of black holes” by [55].

To describe these manifolds, consider the group of modular transformations

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (7.10)$$

which act as conformal transformations of the boundary  $T^2$ . The cycles  $t$  and  $\phi$  transform as a vector  $\begin{pmatrix} t \\ \phi \end{pmatrix}$  under  $SL(2, \mathbb{Z})$ , so the element  $\gamma$  takes  $\phi \rightarrow c\phi + dt$ . Under these transformations the conformal structure of the boundary  $T^2$  is invariant, and  $\tau$  transforms in the usual way

$$\tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d} \quad (7.11)$$

These conformal transformations of the boundary  $T^2$  extend to isometries in the interior. These isometries are easiest to write down by combining the bulk coordinates  $(w, y)$  into a single quaternionic coordinate  $h = w + jy$ . The modular transformation

acts as

$$h \rightarrow \gamma h = (ah + b)(ch + d)^{-1} \quad (7.12)$$

Applying this isometry to the geometry  $M_{0,1}$  described in (7.8) gives a geometry  $M_{c,d}$  in which the cycle  $c\phi + dt$  is contractible. This geometry will represent a saddle point contribution to the partition function. Moreover, it is possible to demonstrate the  $M_{c,d}$  so constructed are in fact the only smooth real saddle point contributions to the partition sum.

We should emphasize that the pair of relatively prime integers  $(c, d)$  determines the geometry  $M_{c,d}$  uniquely. Note that  $(c, d)$  does not determine  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  uniquely, as  $a$  and  $b$  are determined only up to an overall shift  $(a, b) \rightarrow (a + nc, b + nd)$  for some  $n \in \mathbb{Z}$ . However, one can check that the geometry  $M_{c,d}$  is in fact independent of  $n$  up to a diffeomorphism which leaves the boundary invariant. Thus the geometries  $M_{c,d}$  are in one to one correspondence with elements of the coset  $SL(2, \mathbb{Z})/\mathbb{Z}$ .

We conclude that the partition function takes the form

$$Z(\tau) = \sum_{(c,d)} Z_{c,d}(\tau) \quad (7.13)$$

where  $Z_{c,d}(\tau)$  denotes the contribution from the saddle  $M_{c,d}$ . Since the geometries are related by modular transformations we may write this as

$$Z(\tau) = \sum_{(c,d)} Z_{0,1}(\gamma\tau) \quad (7.14)$$

where  $\gamma$  is given by (7.10). The sum over  $(c, d)$  may be thought of as a sum over the coset  $SL(2, \mathbb{Z})/\mathbb{Z}$ . Such sums are known as Poincaré series and first appeared in the context of three dimensional gravity in [52].

## 7.2 Sum over geometries

We now wish to compute the perturbative partition function  $Z_{0,1}(\tau)$  around the saddle point geometry  $M_{0,1}$  given in (7.8). The computation of the classical piece, including the gravitational Chern-Simons term, was given in [56]. This computation is rather subtle as the appropriate boundary terms must be included in the action. The result is

$$e^{-I[M_{0,1}]} = q^{-k} \quad (7.15)$$

We note that this answer is complex, since the original Euclidean action (7.4) was complex. In particular, equation (7.15) is holomorphic in  $\tau$ . It is crucial that we are at the chiral point  $\mu\ell = 1$ , otherwise the action would not be holomorphic in  $\tau$ .

In order to determine the perturbative corrections to this saddle point action, we will follow the argument of [31]. The geometry  $M_{0,1}$  is simply the Euclidean geometry found by imposing the identifications (7.7) on the global  $t$  and  $\phi$  coordinates of  $\text{AdS}_3$ . It is therefore the usual Euclidean geometry associated with the canonical ensemble partition function computed in a fixed Anti-de Sitter background. The partition function  $Z_{0,1}$  therefore has the interpretation in Lorentzian signature as

$$Z_{0,1} = \text{Tr}_{\mathcal{H}} q^{L_0} \bar{q}^{\bar{L}_0} \quad (7.16)$$

where the trace is over the Hilbert space  $\mathcal{H}$  of quantum fluctuations around a fixed Anti-de Sitter background. The classical contribution (7.15) may be interpreted as the contribution to this trace from a ground state  $|0\rangle$  of conformal dimension  $L_0|0\rangle = -k|0\rangle$ . This ground state is just empty Anti-de Sitter space in the absence of any excitations.

At the linearized level, as shown above, the Hilbert space  $\mathcal{H}$  includes only right moving boundary gravitons. The linearized metrics of these boundary gravitons are obtained by acting with a right moving Virasoro generator on the empty  $\text{AdS}_3$  vacuum state. The generator  $L_{-1}$  annihilates the vacuum, as  $L_{-1}$  is an isometry of  $\text{AdS}_3$ . The other  $L_{-n}$ , with  $n \geq 2$  act as creation operators, and describe non-trivial boundary graviton states. In the CFT language, such a boundary graviton is thought of as a state of the form

$$L_{-i_1} \dots L_{-i_n} |0\rangle, \quad i_n \geq 2 \quad (7.17)$$

The resulting trace over these states is easy to compute. It is a character of the Virasoro algebra

$$Z_{1,0} = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1 - q^n} \quad (7.18)$$

which is closely related to the Dedekind eta function.

It is illustrative to compare this formula to equation (7.3). The trace over Virasoro descendants can be interpreted as the one loop contribution to the free energy; this is to be expected, as the boundary gravitons are solutions to the linearized equations of motion. It would be interesting to derive this result directly by computing an appropriate one loop determinant, as in [57].

We now ask to what extent the formula (7.18) may be altered by higher order corrections in powers of the inverse coupling  $k^{-1}$ , i.e. by the terms  $I^{(n)}[g_c]$  for  $n \geq 2$  in (7.3). We first note that the dimensions of the states appearing in the representation (7.17) are completely fixed by the Virasoro algebra. Once the dimension of the vacuum state is known, the result (7.18) is the only answer consistent with the existence of a Virasoro algebra. This implies that equation (7.18) is one-loop exact, in the sense that the energy levels of the known states can not be altered in perturbation theory. The only possible modification of this formula at higher orders in perturbation theory in  $1/k$  is a shift in the dimension of the vacuum state. This shift is interpreted as a renormalization of the cosmological constant. It may be absorbed by a shift of the bare coupling constant order by order in perturbation theory.

One might wonder whether there are additional states which are not present at linear order which might contribute to the sum. We do not claim to have a complete understanding of the Lorentzian spectrum at the non-linear level, and so can not rule out this possibility. If such states do exist, they are not solutions of the Einstein equation and hence do not appear in the Euclidean formulation followed here. So, if the Euclidean methodology assumed here is correct, it implies that either no such additional states/corrections appear or they cancel among themselves. In this case equation (7.18) includes the contributions to the saddle point action to all orders in the perturbation expansion in  $1/k$ .

Putting together the results of the previous analyses, we conclude that the partition function of chiral gravity takes the form

$$Z(\tau) = \sum_{c,d} Z_{0,1}(\gamma\tau), \quad Z_{0,1}(\tau) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \quad (7.19)$$

This sum is naively divergent, but has a well defined regularization (analogous to zeta function regularization) which is consistent with modular invariance. We will not review the details of this regularization, which has been discussed elsewhere [31, 52, 53, 54, 58], but simply state the result.

To start, we expand  $Z_{0,1}(\tau)$  in powers of  $q$

$$Z_{0,1}(\tau) = \sum_{\Delta=-k}^{\infty} a(\Delta)q^{\Delta}, \quad a(\Delta) = p(\Delta+k) - p(\Delta+k-1) \quad (7.20)$$

where  $p(N)$  is the number of partitions of the integer  $N$ . Then the regularization of

the sum (7.19) is

$$Z(\tau) = \sum_{\Delta'=-k}^0 a(\Delta') T_{-\Delta'} J \quad (7.21)$$

where  $T_n J$  denotes the action of the  $n^{\text{th}}$  Hecke operator on the modular function  $J(\tau)$ . From a practical point of view,  $T_n J$  may be defined as the unique holomorphic, modular invariant function on the upper half plane which has a pole of order  $n$  at  $\tau = i\infty$ . In particular, it is the unique  $SL(2, \mathbb{Z})$  invariant function whose Taylor expansion in integer powers of  $q$  is

$$T_n J(\tau) = q^{-n} + \mathcal{O}(q) \quad (7.22)$$

The coefficients in this Taylor expansion are positive integers which are straightforward to compute; we refer the reader to [59] for a more detailed discussion of these Hecke operators and their properties.

### 7.3 Physical interpretation

The above analysis implies that, with the assumptions noted above, the partition function of chiral gravity takes the form

$$Z(\tau) = \sum_{\Delta=-k}^{\infty} N(\Delta) q^{\Delta} \quad (7.23)$$

where the  $N(\Delta)$  are positive integers. These positive integers may be computed for any desired value of  $k$ , as described in detail in [32]. In fact, the partition function (7.23) is precisely the holomorphic part of the partition function conjectured to be dual to pure gravity in [32]. This is not a coincidence, as chiral gravity apparently is a theory with all the properties shown in [32] to lead uniquely to (7.23). As this partition function contains as few states as possible consistent with modular invariance, it is referred to as the extremal CFT partition function.

This partition function is exactly of the form that one expects for a consistent quantum theory; it is a discrete sum over a positive spectrum, with positive integer coefficients. We contrast the present situation with that of pure Einstein gravity [31], where the corresponding computation did not yield a consistent quantum mechanical partition function unless complexified geometries were included in the sum. The inclusion of the gravitational Chern-Simons term has resolved this apparent inconsistency.

The partition function (7.23) has several additional interesting properties. First, we note that the partition function makes sense only when  $k$  is an integer. Thus the cosmological constant and the Chern-Simons coefficient are quantized in Planck units. Moreover, the spectrum of dimensions is quantized,  $\Delta \in \mathbb{Z}$ . Thus the masses and angular momenta of all states in the theory – including black holes – are quantized as well.

These two rather remarkable statements are consequences of the fact that the theory is chiral. To see this, note that in a chiral theory the partition function  $Z(\tau)$  must depend holomorphically on  $\tau$ . The complex structure of the boundary  $T^2$  is modular invariant, so we may think of  $Z(\tau)$  as a holomorphic function on the modular domain  $\mathbb{H}_2/SL(2, \mathbb{Z})$ . Including the point at  $\tau = i\infty$  this modular domain may be thought of as a Riemann surface of genus zero, which is mapped analytically to the usual Riemann sphere  $\mathbb{C} \cup \{\infty\}$  by the  $j$ -invariant  $J(\tau)$  (see e.g. [59]). Since the partition function  $Z(\tau)$  is meromorphic, it is therefore a rational function of  $J(\tau)$ . Moreover, if we assume that the canonical ensemble partition sum is convergent,  $Z(\tau)$  must be holomorphic at all points, except possibly at  $\tau = i\infty$ . Thus  $Z(\tau)$  is a polynomial in the  $J$ -invariant

$$Z = \sum_{n \geq 0} a_n J(\tau)^n, \quad J(\tau) = q^{-1} + 744 + 196884q + \dots \quad (7.24)$$

for some real coefficients  $a_n$ . It follows that both the coupling constant and the spectrum of dimensions are quantized.

We should note, however, that this argument does not imply that the coefficients  $N(\Delta)$  appearing in the expansion are positive integers. This fact was crucial for a consistent quantum mechanical interpretation of the partition function.

These coefficients  $N(\Delta)$  for large  $\Delta$  can be interpreted as the exact degeneracies of quantum black holes in chiral gravity. One can demonstrate that these coefficients reproduce precisely the black hole entropy, including an infinite series of corrections. This is done by reorganizing the modular sum (7.19) into a Rademacher expansion [31, 52, 58]. The computation proceeds exactly as in [31], so we refer the reader there for details.

Finally, we emphasize that it is not at all clear that conformal field theories with the spectrum described above exist. No examples have been constructed with  $k > 1$ . Indeed, a potential objection to the existence of these theories at large  $k$  was noted in [60, 61].<sup>17</sup> Although the results of this paper do not imply the existence of such

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<sup>17</sup>See also [62] for a discussion of these objections and [63] for a related discussion in the context

extremal CFTs, they certainly fit harmoniously with their conjectured existence.

## 7.4 What could go wrong

We have argued that (7.23) follows from a conservative set of assumptions. Nevertheless, our argument is not watertight. We now list some possible reasons why (7.23) might not actually be the quantum chiral gravity partition function.

- The Euclidean approach is invalid because the path integral is unbounded.
- Other complex saddle points are encountered in the analytic continuation from Lorentzian to Euclidean signature and must be included.
- There are nonperturbative Lorentzian classical solutions other than black holes which correspond to additional primaries in the CFT and non-perturbative corrections to the Euclidean saddle point approximation.
- Non-smooth saddle points must be included.

These various possibilities are not mutually exclusive.

# 8 Chiral gravity, log gravity, extremal CFT and log extremal CFT

We have presented several results pertaining to chiral gravity, log gravity, extremal CFT and their interconnections. Much remains to be understood. In this concluding section we will draw lessons from what we learned and freely speculate on various possible outcomes. There are many possibilities – we will limit ourselves below to the most pessimistic and the two most optimistic outcomes.

## 8.1 Nothing makes sense

The least interesting possibility, which cannot be excluded, is that none of the theories under discussion are physically sensible. It might turn out that at the classical level chiral gravity has negative energy solutions, non-perturbative instabilities and/or generically develops naked singularities. In this case the quantum theory is unlikely to be well defined. If chiral gravity is not classically sensible, log gravity – which contains chiral gravity – is not likely to be well defined either. Extremal CFTs

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of supersymmetric theories.



with large central charges may simply not exist. The main conraindicator to this possibility is that we have so far discovered a rich and cohesive mathematical structure with no apparent internal inconsistencies.

## 8.2 Chiral gravity = extremal CFT

An obvious and interesting possibility is that chiral gravity is fully consistent and unitary, and has the modular invariant partition function proposed in [32]. In this case chiral gravity is holographically dual to a local extremal CFT . There seems to be no room here for non-Einstein Lorentzian solutions of chiral gravity because there are no corresponding primaries in an extremal CFT. An important indicator in favor of this scenario is that the torus partition function, formally computed using Euclidean methods, gives the extremal CFT partition function on the nose. In this case the genus  $g$  partition function of the extremal CFT would be simply the chiral-gravity weighted sum over geometries with genus  $g$  boundary. Conversely, if the extremal CFTs are constructed, we are finished: we can simply declare them, in the spirit of [32], to be the quantization of chiral gravity.

Of course extremal CFTs have not been constructed for  $k > 1$ . Indeed, arguments against the existence of extremal CFTs at large  $k$  were described in [60, 61], although no proof was given. An optimist might view the failure of these valiant efforts to produce an actual non-existence proof as indication that extremal CFTs do exist as highly exceptional mathematical objects. A pessimist, on other other hand, might take the fact that no extremal CFTs have been constructed for  $k > 1$  as evidence that they do not exist. Further investigation is clearly needed.

## 8.3 Chiral gravity $\in$ log gravity=log extremal CFT

A third interesting possibility is the following.<sup>18</sup> Assume that quantum log gravity exists and has a well-defined Hilbert space, and that there is a holographically dual CFT which is logarithmic and not chiral. Of course, this is not of so much interest in and of itself, as there is no shortage of non-unitary quantum theories of gravity. However, chiral gravity then also necessarily exists as the superselection sector in which all left charges vanish. This superselection sector could still itself have undesirable properties. In particular there is no a priori guarantee that it is modular invariant. Since a modular transformation is a large diffeomorphism in Euclidean space, this is certainly a desirable property. Modular invariance should be violated if

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<sup>18</sup>We are grateful to V. Schomerus for discussions on this point.

the chiral states are in some sense incomplete. For example, consider the truncation of a generic non-chiral CFT to the purely right-chiral sector. Generically, the only chiral operators are the descendants of the identity created with the right moving stress-tensor. The partition function is simply a Virasoro character and is not modular invariant. In the previous section it was argued that in the context of chiral gravity the primaries associated to black holes complete, in the manner described in [32], this character to a modular invariant partition function. However, it is possible that no such completion exists. It might be that chiral gravity is in some sense a physical, unitary subsector of log gravity, but its dual does not obey all the axioms of a local CFT. Interestingly, in [49] it was found that some 2D gravity theories coupled to matter are logarithmic CFTs.

At first the compelling observation that the Euclidean computation of the chiral gravity partition function gives the extremal CFT partition function would seem to be evidence against this possibility. One would expect that any extra states present in log gravity would spoil this nice result. However, as log gravity is not unitary, the extra contributions to the partition function can vanish or cancel. Indeed it is a common occurrence in logarithmic CFTs for the torus partition function to contain no contributions from the logarithmic partners. We see hints of this here: as  $c_L = 0$ , the left-moving gravitons of log gravity have zero norm and hence do not contribute. This suggests the at-present-imprecise notion of a “log extremal CFT”: a logarithmic CFT whose partition function is precisely the known extremal partition function. Perhaps previous attempts to construct extremal CFTs have failed precisely because the theory was assumed to be unitary rather than logarithmic. Clearly there is much to be understood and many interesting avenues to pursue.

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