Progress of Theoretical Physics, Vol. 43, No. 1, January 1970

Chiral Symmetry in the Unified Fermion Theory. I

——$SU(2)\times SU(2)$——

Kichiro HIIDA

Institute for Nuclear Study, Tokyo University, Tanashi, Tokyo

(Received July 21, 1969)

A chiral $SU(2)\times SU(2)$ transformation is introduced in a unified fermion theory in which fundamental particles have spin $\frac{1}{2}$ and all other particles are composed of them. The properties of transformation and of an invariant Lagrangian density under transformation are investigated in detail.

§ 1. Introduction and summary

A number of elementary particles were found up to the present day, and it is usually believed that a greater number of particles will be found in the future. In view of these circumstances, it would be of interest to consider a unified fermion theory in which fundamental particles have the spin $\frac{1}{2}$ and all other particles are composed of them. In this paper we shall introduce to the unified theory a chiral $SU(2)\times SU(2)$ transformation of the form as simple and analogous as possible to that in the usual theory in which the pion field and nucleon field are independent ones.

The pion field ϕ^a has a special role in the chiral $SU(2)\times SU(2)$ transformation[1] in the usual theory. Let X^a and T^a be the operators by which the chiral transformation is induced. They satisfy the relations

$$[X^a, X^b]=i\varepsilon^{abl}T^l,\quad [T^a, X^b]=i\varepsilon^{abl}X^l,\quad (a=1,2,3)\quad [T^a, T^b]=i\varepsilon^{abl}T^l, \tag{1·1}$$

where ε^{abl} is a totally anti-symmetric tensor ($\varepsilon^{123}=1$). The transformation laws of the nucleon field N are given by

$$[X^a, N]=v(\boldsymbol{\phi}^2)\,\varepsilon^{akl}t^k\phi^l N, \tag{1·2}$$

$$[T^a, N]=-t^a N, \tag{1·3}$$

where $v(\boldsymbol{\phi}^2)$ is a real scalar function of $\boldsymbol{\phi}^2$ and $t^a=\tau^a/2$. The transformation laws (1·2) and (1·3) are linear with respect to N but non-linear to ϕ^a.

Now we want to find a chiral $SU(2)\times SU(2)$ transformation in the unified theory. Let ψ be the fundamental Dirac field. Of the pseudoscalar composed of ψ, the simplest one is $P^a=i\lambda\bar{\psi}\gamma_5 t^a\psi$, where λ is a constant with the dimension of (mass)^3. One of the chiral transformation law for ψ which is analogous to Eq. (1·2) is given by

$$[X^a, \psi]=[A\varepsilon^{akl}t^k P^l + B\gamma_5 t^a]\psi\,, \tag{1·4}$$

where A and B are as yet arbitrary real scalar functions of ψ. The other transformation law is given by

$$[T^a, \psi]=-t^a\psi\,. \tag{1·5}$$

In the next section it will be shown that the conditions imposed on A and B by the relations (1·1) are

$$[X^a, A]=0\,, \tag{1·6}$$

$$B=S^0 A\pm\{1+[\boldsymbol{P}^2+(S^0)^2]A^2\}^{1/2}, \tag{1·7}$$

where S^0 is defined by $S^0=\frac{1}{2}\bar{\psi}\psi$. That is, A is an arbitrary chiral-invariant (real scalar) function, and B has two solutions, say B^+ and B^-, which are known (real scalar) functions when A is given. The chiral-invariant function A may vanish, but B cannot. Thus the mass term S^0 violates the chiral invariance:

$$[X^a, S^0]\neq 0\,. \tag{1·8}$$

The transformation law

$$[X^a, \psi]=[A\varepsilon^{akl}t^k P^l + B\gamma_5 t^a + C\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})P^a]\psi \tag{1·9}$$

will be also investigated, where C is a real scalar function of ψ. Then it will be shown that when C is given both A and B are known functions but they cannot be chiral invariant. In view of the simplicity of the theory, we prefer the transformation law (1·4) to (1·9).

In the usual theory, the nucleon field N and the pion field ϕ^a are independent fields and the chiral-invariant Lagrangian density consists of many chiral-invariant terms. For example, the parts which describe π-N scattering in S-wave and P-wave states belong to different terms in the Lagrangian density. Contrary to the usual theory, we assumd that our Lagrangian density consists of only one term

$$-\bar{\psi}\gamma_\alpha D_\alpha\psi\,, \tag{1·10}$$

which is the chiral-invariant extension of the kinetic energy part of the free Lagrangian density for ψ. We may add the mass term $(-2M_0 S^0/\lambda)$ to the Lagrangian density (1·10), where M_0 is the mechanical mass of the fundamental particles. As was shown by Eq. (1·8), the mass term violates the invariance under the chiral transformations (1·4). If we consider the spontaneous breakdown of chiral invariance, it is not necessary to introduce the mass term. We shall not introduce the mass term in this paper.

Section 3 is devoted to finding an explicit form of $(D_\alpha\psi)$. We consider the case where $(D_\alpha\psi)$ contains thirteen scalar functions. Equations for these functions are derived in the Appendix. It will be shown that six of thirteen scalar functions are arbitrary chiral-invariant functions and the remaining seven scalar functions must satisfy seventeen equations. When A is given, these seventeen equations have four sets of solutions, two of which are functions of B^+ and the remaining two are functions of B^-. When A vanishes, the transformation (1·4) reduces to the usual γ_5-transformation, and the seven scalar functions mentioned above also vanish. Then one of the simplest form of $(-\bar{\psi}\gamma_\alpha D_\alpha\psi)$ reduces to one of the equations of motion for ψ which was discussed by Heisenberg.[2)]

A simple example for A is

$$A=\frac{2}{[1-\boldsymbol{P}^2-(S^0)^2]},$$

which is evidently chiral-invariant. Section 4 is devoted to discussing phenomenologically π-N scattering at low energies by using this example. For this purpose we assume that the Lagrangian density (1·10) is a phenomenological one as is usually assumed, ψ denotes the nucleon field N and P^a is proportional to ϕ^a in an asymptotic region:

$$\boldsymbol{P}=[\Lambda+0(\boldsymbol{\phi}^2)]\boldsymbol{\phi},$$

where Λ is a constant. Then it will be shown that one of the two sets of solutions with B^- for the seven scalar functions gives

$$-\bar{\psi}\gamma_\alpha D_\alpha\psi+\bar{N}\gamma_\alpha\partial_\alpha N\simeq-\frac{2i\Lambda^2}{1+\Lambda^2\boldsymbol{\phi}^2}\bar{N}\gamma_\alpha\boldsymbol{t}N\cdot(\boldsymbol{\phi}\times\partial_\alpha\boldsymbol{\phi})+\frac{2i\Lambda}{1+\Lambda^2\boldsymbol{\phi}^2}\bar{N}\gamma_\alpha\gamma_5\boldsymbol{t}N\cdot(\partial_\alpha\boldsymbol{\phi}),$$

where only the parts which relate with π-N scattering at low energies were retained on the right-hand side. The first and second terms on the right-hand side are related with the S-wave and P-wave amplitudes for π-N scattering, respectively. Let $(g_A/g_V)_0$ denote the unrenormalized value of the ratio of the axial-vector coupling constant to the vector one in weak interactions. If $(g_A/g_V)_0=1$ is assumed, Weinberg's Lagrangian density[3)] coincides exactly with our phenomenological one written above, as far as π-N scattering is concerned.

At first sight the transformation law (1·4) seems to be non-linear with respect ψ except for the case $A=0$. An interesting problem is to study whether or not the transformation law can be reduced to that of the usual linear γ_5-transformation by a suitable redefinition of ψ. It will be shown in § 5 that the transformation law reduces to the usual one by a unitary transformation in the form $q=U\psi$, and as a result it will also be shown that the desirable terms related with π-N scattering mentioned above are absorbed into the term $(-\bar{q}\gamma_\alpha\partial_\alpha q)$. This absorption is due to the fact that the law (1·4) reduces to the usual form.

In the next paper the transformation law will be extended to that of a chiral $U(2)\times U(2)$ transformation. Then it will be shown that the latter law cannot be reduced to the usual form by any redefinition of ψ. We may obtain an invariant Lagrangian density under the chiral $U(2)\times U(2)$ transformation by using the same method adopted in § 4. Then the terms corresponding to the second and last ones in Eq. (4·9) have physical meanings. It would be important to study the properties of the laws for chiral $SU(3)\times SU(3)$ and $U(3)\times U(3)$ transformations which reduce to Eqs. (1·4) and (1·5) in the $SU(2)\times SU(2)$ limit and to study the problems related with the quantization of the Lagrangian density in which the interaction Lagrangian density includes derivative terms with respect to ψ.

§ 2. Chiral $SU(2)\times SU(2)$ transformations

First we shall consider the transformation laws (1·4) and (1·5) and obtain the restrictions imposed by the commutation relations (1·1) on the real scalar functions A and B. Since we are considering the unified theory, the chiral transformation laws of all quantities which appear in our theory are determined uniquely from the laws (1·4) and (1·5). For example, the laws for the pseudoscalar P^a

$$[X^a, P^b]=i[(\boldsymbol{P}^2A+S^0B)\delta^{ab}-AP^aP^b], \tag{2·1}$$

$$[T^a, P^b]=i\varepsilon^{abl}P^l \tag{2·2}$$

are obtained from Eqs. (1·4) and (1·5), respectively. The transformation laws for the scalar S^0 are

$$[X^a, S^0]=-iBP^a, \tag{2·3}$$

$$[T^a, S^0]=0\,. \tag{2·4}$$

To get restrictions imposed on A and B, we shall use the Jacobi identities

$$[X^a,[X^b,O]]-[X^b,[X^a,O]]=[[X^a,X^b],O], \tag{2·5}$$

$$[X^a,[T^b,O]]-[T^b,[X^a,O]]=[[X^a,T^b],O], \tag{2·6}$$

where O is any operator. Putting ψ for O and using Eqs. (1·1), (1·4), (1·5), (2·1) and (2·3), the identity (2·5) reduces to

$$\{[X^a,A]\varepsilon^{bcl}-[X^b,A]\varepsilon^{acl}\}P^l+i[(1+\boldsymbol{P}^2A^2)+2S^0AB-B^2]\varepsilon^{abc}$$
$$+\{[X^a,(B-S^0A)]\delta^{bc}-[X^b,(B-S^0A)]\delta^{ac}\}\gamma_5=0\,,$$

which gives

$$[X^a, A]=0\,, \tag{1·6}$$

$$(1+\boldsymbol{P}^2A^2)+2S^0AB-B^2=0\,, \tag{2·7}$$

$$[X^a, B]=A[X^a, S^0]. \tag{2·8}$$

The second equation (2·7) has two solutions for B;

$$B^{\pm}=S^0A\pm\{1+[\boldsymbol{P}^2+(S^0)^2]A^2\}^{1/2}, \qquad (1\cdot7)$$

where the superscripts $(+)$ and $(-)$ of B correspond respectively to the $(+)$ and $(-)$ signs in front of the second term of the right-hand side. Since $[X^a, \boldsymbol{P}^2+(S^0)^2]=0$ is obtained from Eqs. (2·1) and (2·3), it is evident that Eq. (2·8) is consistent with Eqs. (1·6) and (1·7). We can prove that no other restrictions than Eqs. (1·6) and (1·7) are obtained from the identity (2·5) for any operator O and the identity (2·6) always holds. Thus the restrictions imposed on A and B are given by Eqs. (1·6) and (1·7).

We can show that Eq. (1·7) and

$$[X^a, A]\pm iC\{1+[\boldsymbol{P}^2+(S^0)^2]A^2\}^{1/2}P^a=0 \qquad (2\cdot9)$$

are the restrictions imposed by the commutation relations (1·1) on the real scalar functions A, B and C in Eq. (1·9). If the $(+)$ or $(-)$ sign is adopted in Eq. (1·7), the same sign should also be adopted in Eq. (2·9). When one of A and C is given, Eq. (2·9) can be solved in principle. The function C may be chiral invariant but A cannot except for the case $C=0$.

It is possible to define many other transformation laws than Eqs. (1·4) and (1·9). Such an example is

$$[X^a, \psi]=\{C\varepsilon^{akl}t^kS^l+B'\gamma_5t^a\}\psi\,,$$

where B' and C are as yet arbitrary real scalar and real pseudoscalar functions of ψ respectively and S^a is defined by $S^a=\lambda\bar{\psi}t^a\psi/2$. We want to adopt a transformation law which is analogous (and also simple) as possible to Eq. (1·2). Thus we prefer the law (1·4) to the others.

§ 3. Lagrangian density

The purpose of this section is to obtain an invariant Lagrangian density under the transformation law (1·4). In the usual theory in which the nucleon field N and the pion field ϕ^a are independent fields, the Lagrangian density $(-\bar{N}\gamma_\alpha D_\alpha N)$ which is the chiral-invariant extension of the free Lagrangian density $(-\bar{N}\gamma_\alpha\partial_\alpha N)$ for the nucleon includes the part which describes the π-N scattering amplitude in the S-wave state but it does not include the one which describes the P-wave amplitude, so that the relative magnitude and the sign of both parts are arbitrary. We want to know what happens in our unified theory.

The real scalar function A is an arbitrary chiral-invariant function. For simplicity we shall assume that A is an arbitrary function of chiral-invariant scalar four-fermis, an example of which is $[\boldsymbol{P}^2+(S^0)^2]$. Let us introduce the notations:

$$\begin{aligned} &P^\alpha=i\lambda\bar{\psi}\gamma_5t^\alpha\psi\,, && S^\alpha=\lambda\bar{\psi}t^\alpha\psi\,, \\ &V_\beta{}^\alpha=i\lambda\bar{\psi}\gamma_\beta t^\alpha\psi\,, && A_\beta{}^\alpha=i\lambda\bar{\psi}\gamma_\beta\gamma_5t^\alpha\psi\,, \\ &T^\alpha_{\beta\gamma}=\lambda\bar{\psi}\sigma_{\beta\gamma}t^\alpha\psi\,, && (\alpha=0,1,2,3) \end{aligned} \qquad (3\cdot1)$$

where $\sigma_{\beta\gamma}=i[\gamma_\beta, \gamma_\gamma]/2$ and $t^0=1/2$. Then it is easy to show that chiral-invariant scalar four-fermis are

$$[\boldsymbol{P}^2+(S^0)^2],\qquad [\boldsymbol{S}^2+(P^0)^2],\qquad (V_\alpha{}^0)^2,\qquad (A_\alpha{}^0)^2,$$
$$(\boldsymbol{V}_\alpha{}^2+\boldsymbol{A}_\alpha{}^2),\qquad [(T^0_{\alpha\beta})^2-\boldsymbol{T}^2_{\alpha\beta}]. \tag{3·2}$$

However, all of them are not necessarily independent, because there remains the freedom of the Fierz transformation among them. In fact we can show that the three relations

$$(\boldsymbol{V}_\alpha{}^2+\boldsymbol{A}_\alpha{}^2)=(V_\alpha{}^0)^2+(A_\alpha{}^0)^2,$$
$$[\boldsymbol{P}^2+(S^0)^2]+[\boldsymbol{S}^2+(P^0)^2]=(V_\alpha{}^0)^2-(A_\alpha{}^0)^2, \tag{3·3}$$
$$[\boldsymbol{P}^2+(S^0)^2]-[\boldsymbol{S}^2+(P^0)^2]=\tfrac{1}{2}[(T^0_{\alpha\beta})^2-\boldsymbol{T}^2_{\alpha\beta}]$$

hold among them.[4] Thus three of the six invariants are independent of each other. In this paper the three terms

$$[\boldsymbol{P}^2+(S^0)^2],\qquad (V_\alpha{}^0)^2,\qquad (\boldsymbol{V}_\alpha{}^2+\boldsymbol{A}_\alpha{}^2) \tag{3·4}$$

will be chosen as independent invariants. There exist also chiral-invariant pseudoscalar four-fermis. An example is

$$[(\boldsymbol{P}\cdot\boldsymbol{S})-P^0S^0]. \tag{3·5}$$

We shall neglect in this paper the dependence of this kind of pseudoscalar invariants for A.

If $(D_\alpha\psi)$ satisfies the transformation law

$$[X^a, (D_\alpha\psi)]=[A\varepsilon^{akl}t^kP^l+B\gamma_5t^a](D_\alpha\psi), \tag{3·6}$$

the Lagrangian density $(-\bar{\psi}\gamma_\alpha D_\alpha\psi)$ is invariant under the transformation law (1·4). Though the transformation law (1·4) includes γ_5, there exist two γ matrices β and γ_α between ψ^* and $(D_\alpha\psi)$ in the Lagrangian. Therefore the Lagrangian density is chiral-invariant when Eq. (3·6) holds.

As was assumed A depends on three scalar invariants written in Eq. (3·4). This means that the transformation law (1·4) depends on $\boldsymbol{P}$, S^0, $(V_\alpha{}^0)^2$ and $(\boldsymbol{V}_\alpha{}^2+\boldsymbol{A}_\alpha{}^2)$. Then the general form of $(D_\alpha\psi)$ will be given by

$$D_\alpha\psi=\partial_\alpha\psi+i[M^c(\partial_\alpha P^c)+N(\partial_\alpha S^0)+Q_\beta(\partial_\alpha V_\beta{}^0)$$
$$+R_\beta{}^c(\partial_\alpha V_\beta{}^c)+U_\beta{}^c(\partial_\alpha A_\beta{}^c)+WV_\alpha{}^0+Y^cV_\alpha{}^c+Z^cA_\alpha{}^c]\psi\,. \tag{3·7}$$

Here M^c and N are pseudoscalar and scalar functions respectively, because of the parity conservation. We shall assume that they are proportional to t^a and depend explicitly on the variable P^a alone. Then they have the form

$$M^c=D\varepsilon^{ckl}t^kP^l+E\gamma_5t^c+F\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})P^c,$$
$$N=G\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t}), \tag{3·8}$$

where D, E, F and G are real scalar functions and depend on $\boldsymbol{P}^2$, S^0, $(V_\alpha^0)^2$ and $(\boldsymbol{V}_\alpha^2+\boldsymbol{A}_\alpha^2)$. The functions Q_β, R_β^c and U_β^c would be proportional to (V_β^0), V_β^c and A_β^c respectively. We shall assume the following form for them:

$$
\begin{aligned}
Q_\beta &= [Q_1+Q_2\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})]V_\beta^0, \\
R_\beta^c &= [R_1+R_2\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})]V_\beta^c, \\
U_\beta^c &= [U_1+U_2\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})]A_\beta^c,
\end{aligned}
\tag{3·9}
$$

where Q_i, R_i and U_i $(i=1,2)$ are real scalar functions and they depend on $\boldsymbol{P}^2$, S^0, $(V_\alpha^0)^2$ and $(\boldsymbol{V}_\alpha^2+\boldsymbol{A}_\alpha^2)$. The functions W, Y^c and Z^c are scalar, scalar and pseudoscalar respectively. We shall assume the form

$$
\begin{aligned}
W &= W, \\
Y^c &= Yt^c, \\
Z^c &= Z\gamma_5 t^c,
\end{aligned}
\tag{3·10}
$$

where the real scalar functions W, Y and Z are again assumed to depend on $\boldsymbol{P}^2$, S^0, $(V_\alpha^0)^2$ and $(\boldsymbol{V}_\alpha^2+\boldsymbol{A}_\alpha^2)$.

Now our problem is, as far as it is possible, to obtain expressions for the thirteen scalar functions D, E, F, $\cdots$, Z in terms of A and B involved in the transformation laws (1·4) and (3·6). The equations for these scalar functions will be obtained in the Appendix. We shall use them here. The Lagrangian density $(-\bar{\psi}\gamma_\alpha D_\alpha\psi)$ is chiral-invariant. This fact leads directly to

$$
R_1=U_1\,, \qquad R_2=U_2\,, \qquad Y=Z\,, \tag{3·11}
$$

as will be shown by Eqs. (A·10) and (A·18). The six functions Q_1, R_1, U_1, W, Y and Z of the thirteen scalar functions are shown to be invariant under the transformation law (1·4), that is, they are arbitrary functions of $[\boldsymbol{P}^2+(S^0)^2]$, $(V_\alpha^0)^2$ and $(\boldsymbol{V}_\alpha^2+\boldsymbol{A}_\alpha^2)$:

$$
[X^a, Q_1]=0\,, \quad \text{etc.}, \tag{3·12}
$$

as will be shown by Eqs. (A·6), (A·12), (A·14), (A·16) and (A·18).

There remain for discussion the six scalar functions D, E, F, G, Q_2 and $R_2(=U_2)$. They must satisfy the following sixteen equations, as will be shown by Eqs. (A·9), (A·11), (A·13) and (A·15):

$$
\begin{aligned}
&A-BE+S^0BD=0\,, \\
&D+G-S^0F=0\,, \\
&AD+BF+2D\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{P}^2}+S^0\frac{\partial B}{\partial \boldsymbol{P}^2}\right)-2\frac{\partial A}{\partial \boldsymbol{P}^2}=0\,, \\
&AE+S^0BF+2E\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{P}^2}+S^0\frac{\partial B}{\partial \boldsymbol{P}^2}\right)-2\frac{\partial B}{\partial \boldsymbol{P}^2}=0\,,
\end{aligned}
$$

$$B(D+G)+D\left(\boldsymbol{P}^2\frac{\partial A}{\partial S^0}+S^0\frac{\partial B}{\partial S^0}\right)-\frac{\partial A}{\partial S^0}=0\,,$$

$$B(E+S^0G)+E\left(\boldsymbol{P}^2\frac{\partial A}{\partial S^0}+S^0\frac{\partial B}{\partial S^0}\right)-\frac{\partial B}{\partial S^0}=0\,,$$

$$BQ_2+2D\left(\boldsymbol{P}^2\frac{\partial A}{\partial (V_\alpha^0)^2}+S^0\frac{\partial B}{\partial (V_\alpha^0)^2}\right)-2\frac{\partial A}{\partial (V_\alpha^0)^2}=0\,,$$

$$S^0BQ_2+2E\left(\boldsymbol{P}^2\frac{\partial A}{\partial (V_\alpha^0)^2}+S^0\frac{\partial B}{\partial (V_\alpha^0)^2}\right)-2\frac{\partial B}{\partial (V_\alpha^0)^2}=0\,,$$

$$BR_2+2D\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{V}_\alpha^2}+S^0\frac{\partial B}{\partial \boldsymbol{V}_\alpha^2}\right)-2\frac{\partial A}{\partial \boldsymbol{V}_\alpha^2}=0\,, \tag{3·13}$$

$$S^0BR_2+2E\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{V}_\alpha^2}+S^0\frac{\partial B}{\partial \boldsymbol{V}_\alpha^2}\right)-2\frac{\partial B}{\partial \boldsymbol{V}_\alpha^2}=0\,,$$

$$AD+B\left(\frac{\partial D}{\partial S^0}-2S^0\frac{\partial D}{\partial \boldsymbol{P}^2}\right)=0\,,$$

$$AE-BD+B\left(\frac{\partial E}{\partial S^0}-2S^0\frac{\partial D}{\partial \boldsymbol{P}^2}\right)=0\,,$$

$$2\left(E\frac{\partial A}{\partial \boldsymbol{P}^2}+G\frac{\partial B}{\partial \boldsymbol{P}^2}\right)-2S^0F\frac{\partial B}{\partial \boldsymbol{P}^2}+B\left(\frac{\partial F}{\partial S^0}-2S^0\frac{\partial F}{\partial \boldsymbol{P}^2}\right)=0\,,$$

$$BF-\left(E\frac{\partial A}{\partial S^0}+G\frac{\partial B}{\partial S^0}\right)+S^0\frac{\partial B}{\partial S^0}-B\left(\frac{\partial G}{\partial S^0}-2S\frac{\partial G}{\partial \boldsymbol{P}^2}\right)=0\,,$$

$$2\left(E\frac{\partial A}{\partial (V_\alpha^0)^2}+G\frac{\partial B}{\partial (V_\alpha^0)^2}\right)-2S^0F\frac{\partial B}{\partial (V_\alpha^0)^2}+B\left(\frac{\partial Q_2}{\partial S^0}-2S^0\frac{\partial Q_2}{\partial \boldsymbol{P}^2}\right)=0\,,$$

$$2\left(E\frac{\partial A}{\partial \boldsymbol{V}_\alpha^2}+G\frac{\partial B}{\partial \boldsymbol{V}_\alpha^2}\right)-2S^0F\frac{\partial B}{\partial \boldsymbol{V}_\alpha^2}+B\left(\frac{\partial R_2}{\partial S^0}-2S^0\frac{\partial R_2}{\partial \boldsymbol{P}^2}\right)=0\,.$$

As was discussed in § 2, A is an arbitrary chiral-invariant function and B has two solutions (1·7) in terms of A. For these two solutions, we get

$$\frac{\partial B}{\partial \boldsymbol{P}^2}=\frac{1}{(B-S^0A)}\left[(\boldsymbol{P}^2A+S^0B)\frac{\partial A}{\partial \boldsymbol{P}^2}+\frac{A^2}{2}\right],$$

$$\frac{\partial B}{\partial S^0}=\frac{1}{(B-S^0A)}\left[(\boldsymbol{P}^2A+S^0B)\frac{\partial A}{\partial S^0}+AB\right], \tag{3·14}$$

$$\frac{\partial B}{\partial X}=\frac{1}{(B-S^0A)}(\boldsymbol{P}^2A+S^0B)\frac{\partial A}{\partial X}\,. \qquad [X:\ (V_\alpha^0)^2 \text{ or } \boldsymbol{V}_\alpha^2]$$

Using Eq. (3·14),

$$\left(\frac{\partial}{\partial S^0}-2S^0\frac{\partial}{\partial \boldsymbol{P}^2}\right)\frac{1}{d_\pm}=-\frac{2A}{(d_\pm)^2}[(B\pm1)-S^0A] \tag{3·15}$$

is obtained, where $d_\pm$ is given by

$$d_\pm=(B\pm1)^2+\boldsymbol{P}^2A^2. \tag{3·16}$$

Now we can show for $A\neq0$ by using Eqs. (1·6), (1·7), (3·14) and (3·15) that for each solution B^+ or B^- there are two sets of solutions for the sixteen equations written in Eq. (3·13), and they are given by

$$\begin{aligned}
D_\pm&=\frac{2}{d_\pm}A^2,\\
E_\pm&=\frac{2}{d_\pm}A(B\pm1),\\
F_\pm&=\frac{4}{d_\pm}\left[(B\pm1)\frac{\partial A}{\partial\boldsymbol{P}^2}-A\frac{\partial B}{\partial\boldsymbol{P}^2}\right],\\
G_\pm&=\frac{2}{d_\pm}\left[(B\pm1)\frac{\partial A}{\partial S^0}-A\frac{\partial B}{\partial S^0}\right],\\
(Q_2)_\pm&=\frac{4}{d_\pm}\left[(B\pm1)\frac{\partial A}{\partial(V_\alpha{}^0)^2}-A\frac{\partial B}{\partial(V_\alpha{}^0)^2}\right],\\
(R_2)_\pm=(U_2)_\pm&=\frac{4}{d_\pm}\left[(B\pm1)\frac{\partial A}{\partial\boldsymbol{V}_\alpha{}^2}-A\frac{\partial B}{\partial\boldsymbol{V}_\alpha{}^2}\right],
\end{aligned} \tag{3·17}$$

where the solutions with the subscripts (+) and (−) belong to different sets of solutions. For a special case where $A=0$ and $B=\pm1$, the solutions for the sixteen equations are

$$\begin{aligned}
&D=H,\qquad E=S^0H,\qquad G=-H,\\
&F=Q_2=R_2=U_2=0,
\end{aligned} \tag{3·18}$$

where H is an arbitrary chiral-invariant function. Thus we obtained an expression for $(D_\alpha\psi)$.

Substituting Eq. (3·7) with Eqs. (3·8), (3·9) and (3·10) into $(D_\alpha\psi)$, we get the chiral-invariant Lagrangian density

$$\begin{aligned}
-\bar\psi\gamma_\alpha D_\alpha\psi=&-\bar\psi\gamma_\alpha\partial_\alpha\psi-iD\bar\psi\gamma_\alpha\boldsymbol{t}\psi\cdot(\boldsymbol{P}\times\partial_\alpha\boldsymbol{P})-iE\bar\psi\gamma_\alpha\gamma_5\boldsymbol{t}\psi\cdot(\partial_\alpha\boldsymbol{P})\\
&-iF\bar\psi\gamma_\alpha\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})\psi(\boldsymbol{P}\cdot\partial_\alpha\boldsymbol{P})-iG\bar\psi\gamma_\alpha\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})\psi(\partial_\alpha S^0)\\
&-i[Q_1\bar\psi\gamma_\alpha\psi+Q_2\bar\psi\gamma_\alpha\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})\psi](V_\beta{}^0\partial_\alpha V_\beta{}^0)\\
&-i[R_1\bar\psi\gamma_\alpha\psi+R_2\bar\psi\gamma_\alpha\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})\psi][(\boldsymbol{V}_\beta\cdot\partial_\alpha\boldsymbol{V}_\beta)+(\boldsymbol{A}_\beta\cdot\partial_\alpha\boldsymbol{A}_\beta)]\\
&-iW\bar\psi\gamma_\alpha\psi V_\alpha{}^0-iY[\bar\psi\gamma_\alpha\gamma_5\boldsymbol{t}\psi\cdot\boldsymbol{A}_\alpha],
\end{aligned} \tag{3·19}$$

where Q_1, R_1, W and Y are arbitrary chiral-invariant functions as shown by Eq. (3·12), and D, E, F, G, Q_2 and R_2 are given functions of A, as shown by Eq. (3·17).

As will be discussed in the next section, the second and third terms in

Eq. (3·19) are related formally with the S-wave and P-wave amplitudes in π-N scattering, respectively. Since these two terms are involved in the same chiral-invariant expression $(-\bar{\psi}\gamma_\alpha D_\alpha\psi)$, there is a relation between the functions D and E. This is an advantage of our unified theory. One of the simplest chiral-invariant Lagrangian density is obtained when $A=0$, $B=\pm 1$, $H=Q_1=R_1=W=0$ and Y is a constant. In this case the transformation law (1·4) reduces to the usual linear transformation, and the Lagrangian density is given by

$$-\bar{\psi}\gamma_\alpha D_\alpha\psi=-\bar{\psi}\gamma_\alpha\partial_\alpha\psi-\frac{c}{\lambda}(\boldsymbol{V}_\alpha{}^2+\boldsymbol{A}_\alpha{}^2), \tag{3·20}$$

where c is constant and $Y=c$. This is one of the Lagrangian density discussed by Heisenberg[2)] in his unified theory.

§ 4. Choice of solutions

Our theory includes an arbitrary chiral-invariant function A. When A is given, B has two solutions. Even if A and B are determined, Eqs. (3·13) for the six functions D, E, F, G, Q_2 and R_2 have two sets of solutions $D_\pm$, etc., just as given by Eq. (3·17). Thus we have as a whole four sets of solutions.

In this section we want to find some reasons to prefer a set of solutions to the others. For this purpose we shall consider an example for A:

$$A=\frac{2}{[1-\boldsymbol{P}^2-(S^0)^2]}, \tag{4·1}$$

which gives

$$B^+=\frac{(1+S^0)^2+\boldsymbol{P}^2}{[1-\boldsymbol{P}^2-(S^0)^2]}, \tag{4·2}$$

$$B^-=-\frac{(1-S^0)^2+\boldsymbol{P}^2}{[1-\boldsymbol{P}^2-(S^0)^2]} \tag{4·3}$$

for B. First we shall use B^+. Then Eq. (3·17) gives the set of the solutions

$$D_+{}^+=-G_+{}^+=\frac{2}{[(1+S^0)^2+\boldsymbol{P}^2]},\qquad E_+{}^+=\frac{2(1+S^0)}{[(1+S^0)^2+\boldsymbol{P}^2]},$$
$$F_+{}^+=(Q_2)_+{}^+=(R_2)_+{}^+=0 \tag{4·4}$$

and the set

$$D_-{}^+=\frac{2}{\{[\boldsymbol{P}^2+(S^0)^2+S^0]^2+\boldsymbol{P}^2\}},\qquad E_-{}^+=\frac{2[\boldsymbol{P}^2+(S^0)^2+S^0]}{\{[\boldsymbol{P}^2+(S^0)^2+S^0]^2+\boldsymbol{P}^2\}},$$
$$F_-{}^+=-\frac{4}{\{[\boldsymbol{P}^2+(S^0)^2+S^0]^2+\boldsymbol{P}^2\}},\qquad G_-{}^+=-\frac{2(1+2S^0)}{\{[\boldsymbol{P}^2+(S^0)^2+S^0]^2+\boldsymbol{P}^2\}},$$
$$(Q_2)_-{}^+=(R_2)_-{}^+=0\,. \tag{4·5}$$

When B^- is used, on the other hand, we get the set

$$D_-^- = -G_-^- = \frac{2}{[(1-S^0)^2+\boldsymbol{P}^2]}, \qquad E_-^- = -\frac{2(1-S^0)}{[(1-S^0)^2+\boldsymbol{P}^2]},$$
$$F_-^- = (Q_2)_-^- = (R_2)_-^- = 0 \tag{4·6}$$

and the set

$$D_+^- = \frac{2}{\{[\boldsymbol{P}^2+(S^0)^2-S^0]^2+\boldsymbol{P}^2\}}, \quad E_+^- = -\frac{[\boldsymbol{P}^2+(S^0)^2-S^0]}{\{[\boldsymbol{P}^2+(S^0)^2-S^0]^2+\boldsymbol{P}^2\}},$$
$$F_+^- = \frac{4}{\{[\boldsymbol{P}^2+(S^0)^2-S^0]^2+\boldsymbol{P}^2\}}, \quad G_+^- = -\frac{2(1-2S^0)}{\{[\boldsymbol{P}^2+(S^0)^2-S^0]^2+\boldsymbol{P}^2\}}, \tag{4·7}$$
$$(Q_2)_+^- = (R_2)_+^- = 0\,.$$

In Eqs. (4·4), (4·5), (4·6) and (4·7), the superscript of the scalar functions D, etc., denotes that of $B^\pm$.

Now we apply our Lagrangian density (3·19) to the phenomenological analysis on π-N scattering at low energies on the assumption that ψ denotes the nucleon field. Since we have the Lagrangian density, we may in principle solve bound-state problems, that is, whether or not nucleon and anti-nucleon may be in 0^-, 0^+, 1^-, 1^+ ⋯ bound states and what mass values they have. These are beyond the purpose of this paper. At this stage we assume that P^a is proportional to the pion field ϕ^a at asymptotic region:

$$\boldsymbol{P} = \Lambda\boldsymbol{\phi}\,, \tag{4·8}$$

where Λ is a constant. Further we assume that there is no 0^+ bound state at low mass region.

When the solutions (4·6) are used, the effective Lagrangian for π-N scattering at low energies is given by

$$-\bar{N}\gamma_\alpha D_\alpha N \simeq -\bar{N}\gamma_\alpha\partial_\alpha N - \frac{2i\Lambda^2}{1+\Lambda^2\boldsymbol{\phi}^2}\bar{N}\gamma_\alpha \boldsymbol{t}N\cdot(\boldsymbol{\phi}\times\partial_\alpha\boldsymbol{\phi}) + \frac{2i\Lambda}{1+\Lambda^2\boldsymbol{\phi}^2}\bar{N}\gamma_\alpha\gamma_5\boldsymbol{t}N\cdot(\partial_\alpha\boldsymbol{\phi}), \tag{4·9}$$

where the second and last terms are related with S-wave and P-wave scatterings respectively. The last term gives

$$\Lambda = \frac{G_0}{2M}, \tag{4·10}$$

where G_0 and M are the unrenormalized π-N coupling constant and the observed nucleon mass respectively. When N and ϕ^a are treated as independent fields, as was discussed by Weinberg,[1] the Lagrangian density (4·9) gives

$$\left(\frac{g_A}{g_V}\right)_0 = 1\,, \tag{4·11}$$

where the left-hand side denotes the unrenormalized value of the ratio of the axial vector coupling constant to the vector coupling constant for weak interactions. When Eq. (4·11) is satisfied, Weinberg's Lagrangian density[3)] perfectly coincides with Eq. (4·9).

Under the same assumption used to get Eq. (4·9), the solutions (4·4) give the Lagrangian density

$$-\bar{N}\gamma_\alpha D_\alpha N \simeq -\bar{N}\gamma_\alpha \partial_\alpha N - \frac{2i\Lambda^2}{1+\Lambda^2\boldsymbol{\phi}^2}\bar{N}\gamma_\alpha \boldsymbol{t} N\cdot(\boldsymbol{\phi}\times\partial_\alpha\boldsymbol{\phi}) - \frac{2i\Lambda}{1+\Lambda^2\boldsymbol{\phi}^2}\bar{N}\gamma_\alpha\gamma_5 \boldsymbol{t} N\cdot(\partial_\alpha\boldsymbol{\phi}), \tag{4·12}$$

which leads to

$$\left(\frac{g_A}{g_V}\right)_0 = -1 \tag{4·13}$$

by the same arguments used to obtain Eq. (4·11). The solutions $D_+{}^+$, etc., and $D_-{}^-$, etc., include a constant term in their denominators but the remaining ones $D_-{}^+$, etc., and $D_+{}^-$, etc., do not. The former sets of the solutions have a better property than that of the latter ones. Thus we prefer the set of the solutions (4·6) to the others.

It should be noted that Eqs. (4·11) and (4·13) were obtained under the assumption that N and ϕ^a are independent fields. In our theory, however, they are not independent fields. Thus it is necessary to show that Eqs. (4·11) and (4·13) also hold even if only ψ is treated as an independent field. The vector and axial-vector currents defined by Noether's theorem are

$$\begin{aligned} \boldsymbol{J}_\alpha &= 2i\left\{[\boldsymbol{T},\bar{\psi}]\frac{\partial L}{\partial(\partial_\alpha\bar{\psi})} + \frac{\partial L}{\partial(\partial_\alpha\psi)}[\boldsymbol{T},\psi]\right\}, \\ \boldsymbol{J}_{5\alpha} &= 2i\left\{[\boldsymbol{X},\bar{\psi}]\frac{\partial L}{\partial(\partial_\alpha\bar{\psi})} + \frac{\partial L}{\partial(\partial_\alpha\psi)}[\boldsymbol{X},\psi]\right\}. \end{aligned} \tag{4·14}$$

We shall as usual identify the vector and axial-vector currents of weak interactions with $J_\alpha{}^a$ and $J_{5\alpha}^a$ respectively. Using the general expression (3·19), Eq. (4·14) leads to the currents

$$\boldsymbol{J}_\alpha = i(1-\boldsymbol{P}^2 D)\bar{\psi}\gamma_\alpha\boldsymbol{\tau}\psi + \frac{2}{\lambda}D(\boldsymbol{V}_\alpha\cdot\boldsymbol{P})\boldsymbol{P} + \frac{2}{\lambda}E(\boldsymbol{A}_\alpha\times\boldsymbol{P}), \tag{4·15}$$

$$\begin{aligned} \boldsymbol{J}_{5\alpha} = &-i[B - S^0BE - \boldsymbol{P}^2AE]\bar{\psi}\gamma_\alpha\gamma_5\boldsymbol{\tau}\psi \\ &-\frac{2}{\lambda}[AE + BG - S^0BF](\boldsymbol{A}_\alpha\cdot\boldsymbol{P})\boldsymbol{P} \\ &-\frac{2}{\lambda}[A - S^0BD - \boldsymbol{P}^2AD](\boldsymbol{V}_\alpha\times\boldsymbol{P}). \end{aligned} \tag{4·16}$$

Using the expressions for $D_\pm$, $E_\pm$, $F_\pm$ and $G_\pm$ given in Eq. (3·17), the axial vector current (4·16) can be rewritten in the form

$$J_{5\alpha}=-i(1-\boldsymbol{P}^2D_+)\bar{\psi}\gamma_\alpha\gamma_5\boldsymbol{\tau}\psi-\frac{2}{\lambda}D_+(\boldsymbol{A}_\alpha\cdot\boldsymbol{P})\boldsymbol{P}-\frac{2}{\lambda}E_+(\boldsymbol{V}_\alpha\times\boldsymbol{P}) \tag{4·17}$$

or

$$J_{5\alpha}=i(1-\boldsymbol{P}^2D_-)\bar{\psi}\gamma_\alpha\gamma_5\boldsymbol{\tau}\psi+\frac{2}{\lambda}D_-(\boldsymbol{A}_\alpha\cdot\boldsymbol{P})\boldsymbol{P}+\frac{2}{\lambda}E_-(\boldsymbol{V}_\alpha\times\boldsymbol{P}), \tag{4·18}$$

showing that Eqs. (4·11) and (4·13) also hold even if only ψ is the independent field.

§ 5. Linearlization of the transformation law (1·4)

In the usual theory chiral transformations are in general non-linear ones with respect to boson fields but linear ones to fermion fields. Since only ψ is included as an independent field in our unified fermion theory, it would be expected that chiral transformations in our theory in general become non-linear ones with respect to ψ. This expectation is not necessarily true.

It will be shown in this section that the transformation law (1·4) can be reduced to the form of the usual γ_5-transformation, that is, a linear transformation law by the redefinition of ψ. However, chiral $SU(2)\times SU(2)$ transformations are special ones. In the next paper the transformation laws (1·4) and (1·5) will be extended to the form of a chiral $U(2)\times U(2)$ transformation and it will be shown that one of the transformation laws cannot be reduced to the form of the usual γ_5-transformation by any redefinition of ψ.

Let us assume that the transformation law (1·4) can be reduced to the form

$$[X^a, q]=g\gamma_5 t^a q\,, \tag{5·1}$$

where g is a real constant and q is a Dirac spinor under the proper Lorentz transformation and carries the same iso-spin index as ψ has. The general form for q is given by

$$q=\{\beta+i\delta\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})\}\psi\,, \tag{5·2}$$

where β and δ are real scalar functions. We shall take as the normalization condition

$$\bar{q}\gamma_\alpha q=\bar{\psi}\gamma_\alpha q\,. \tag{5·3}$$

Using the expression (5·2), the normalization condition leads to

$$\beta^2+\tfrac{1}{4}\boldsymbol{P}^2\delta^2=1\,. \tag{5·4}$$

The normalization condition (5·4) guarantees that the transformation from ψ to q is unitary.

Substituting the expression (5·2) into Eq. (5·1) and using the known commutation relations (1·4) and (2·1), the relation (5·1) reduces to

$$\left\{i[X^a,\beta]-\frac{1}{4}(B-g)\delta P^a\right\}-\frac{i}{2}[(B+g)\delta-2A\beta]\varepsilon^{akl}t^kP^l$$

$$+\left\{i[X^a,\delta]+\frac{1}{2}A\delta P^a\right\}\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})+\left\{(B-g)\beta-\frac{1}{2}[A\boldsymbol{P}^2+2S^0B]\delta\right\}\gamma_5 t^a=0\,. \tag{5·5}$$

The four terms involved in Eq. (5·5) are independent of each other. Thus we get

$$\delta=\frac{2A\beta}{(B+g)}\,, \tag{5·6}$$

$$(B-g)\beta-\tfrac{1}{2}[A\boldsymbol{P}^2+2S^0B]\delta=0\,, \tag{5·7}$$

$$i[X^a,\beta]=\tfrac{1}{4}(B-g)\delta P^a, \tag{5·8}$$

$$i[X^a,\delta]=-\tfrac{1}{2}A\delta P^a. \tag{5·9}$$

A set of the solutions for β and δ obtained from Eqs. (5·4) and (5·6) is

$$\beta=\frac{(B+g)}{\{(B+g)^2+A^2\boldsymbol{P}^2\}^{1/2}}\,,\qquad \delta=\frac{2A}{\{(B+g)^2+A^2\boldsymbol{P}^2\}^{1/2}}\,. \tag{5·10}$$

Using Eqs. (2·7) and (5·6), the second equation (5·7) leads to

$$g=\pm 1\,. \tag{5·11}$$

It is easy to show that the remaining two equations (5·8) and (5·9) are satisfied by the solutions (5·10). Thus it was shown that the transformation law (1·4) can be reduced to the form of the usual γ_5-transformation by the unitary transformation (5·2).

Substituting q for ψ in Eq. (3·1), we get a new set of variables P_q^a, etc. Then new variables can be expressed in terms of old variables by using Eq. (5·2). Such examples are

$$P_q^a=(1-S^0\beta\delta-\tfrac{1}{2}\boldsymbol{P}^2\delta^2)P^a,\qquad S_q^0=(1-\tfrac{1}{2}\boldsymbol{P}^2\delta^2)S^0+\boldsymbol{P}^2\beta\delta\,. \tag{5·12}$$

From Eq. (5·12) we get

$$\boldsymbol{P}_q^2+(S_q^0)^2=\boldsymbol{P}^2+(S^0)^2. \tag{5·13}$$

Likewise we get

$$V_{q,\alpha}^0=V_\alpha^0,\qquad \boldsymbol{V}_{q,\alpha}^2+\boldsymbol{A}_{q,\alpha}^2=\boldsymbol{V}_\alpha^2+\boldsymbol{A}_\alpha^2. \tag{5·14}$$

Using these new variables, the Lagrangian density (3·19) is rewritten as

$$-\bar{\psi}\gamma_\alpha D_\alpha\psi=-\bar{q}\gamma_\alpha\partial_\alpha q-\frac{1}{\lambda}V_{q,\alpha}^0\{Q_1\partial_\alpha(V_{q,\beta}^0)^2+R_1\partial_\alpha[\boldsymbol{V}_{q,\beta}^2+\boldsymbol{A}_{q,\beta}^2]\}$$

$$-\frac{2W}{\lambda}(V^0_{q,\alpha})^2-\frac{Y}{\lambda}[\boldsymbol{V}^2_{q,\alpha}+\boldsymbol{A}^2_{q,\alpha}]. \qquad (5\cdot 15)$$

In the previous section we assumed that the Lagrangian density (3·19) is an effective one and showed in a phenomenological way that it can explain π-N scattering phenomena at low energies. As was shown by Eq. (5·15), however, the parts related with π-N scattering in it are absorbed into the free Lagrangian density $(-\bar{q}\gamma_\alpha\partial_\alpha q)$ by the unitary transformation (5·2). If the transformation law (1·4) could not be reduced to the linear one (5·1), such desirable parts had physical meanings. In a subsequent paper the transformation law (1·4) will be extended to that for a chiral $U(2)\times U(2)$ transformation, and it will be shown that the latter law cannot be reduced to the usual linear one (5·1). From the discussions given in the previous section, we may infer to some extent the properties of the invariant Lagrangian density under the chiral $U(2)\times U(2)$ transformation. This is the reason why we treat in the previous section the second and last terms in Eq. (4·9) as if they cannot be absorbed into the free Lagrangian density. It would be important to study the properties of chiral $SU(3)\times SU(3)$ and $U(3)\times U(3)$ transformation laws which reduce to Eqs. (1·4) and (1·5) in the $SU(2)\times SU(2)$ limit.

Acknowledgements

The author would like to thank Professor G. Takeda, Professor Y. Yamaguchi, Dr. T. Ohta and Dr. T. Okabayashi for valuable discussions. He realized the need to study non-linear chiral transformations for ψ through discussions with Professor Y. Yamaguchi. He thinks that this paper could not appear without the discussions.

Appendix

Equations for the thirteen scalar functions D, E, etc.

An equation for the thirteen scalar functions D, E, etc., defined in Eqs. (3·8), (3·9) and (3·10) is obtained by substituting Eq. (3·7) with Eqs. (3·8), (3·9) and (3·10) into Eq. (3·6) and by using Eq. (1·4). The equation consists of eight parts which are proportional to $(\partial_\alpha P^c)$, $(\partial_\alpha S^0)$, $(\partial_\alpha V_\beta{}^0)$, $(\partial_\alpha V_\beta{}^c)$, $(\partial_\alpha A_\beta{}^c)$, $V_\alpha{}^0$, $V_\alpha{}^c$ and $A_\alpha{}^c$. Since these eight terms are independent, we get eight equations. They are

$$AD(\boldsymbol{P}\cdot\boldsymbol{t})\varepsilon^{acl}P^l$$
$$-[A-BE+(A\boldsymbol{P}^2+S^0B)D]\varepsilon^{acl}t^l+\left[2AD+B\left(\frac{\partial D}{\partial S^0}-2S^0\frac{\partial D}{\partial \boldsymbol{P}^2}\right)\right]P^a\varepsilon^{ckl}t^kP^l$$
$$-\left[BF+2D\left(A+\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{P}^2}+S^0\frac{\partial B}{\partial \boldsymbol{P}^2}\right)-2\frac{\partial A}{\partial \boldsymbol{P}^2}\right]P^c\varepsilon^{akl}t^kP^l$$

$$+\Big\{B[D+G-S^0F](\boldsymbol{P}\cdot\boldsymbol{t})\delta^{ac}$$

$$+\Big[2\Big(E\frac{\partial A}{\partial \boldsymbol{P}^2}+G\frac{\partial B}{\partial \boldsymbol{P}^2}\Big)+B\Big(\frac{\partial F}{\partial S^0}-2S^0\frac{\partial F}{\partial \boldsymbol{P}^2}\Big)-2S^0F\frac{\partial B}{\partial \boldsymbol{P}^2}\Big](\boldsymbol{P}\cdot\boldsymbol{t})P^aP^c$$

$$+\Big[AE-BD+B\Big(\frac{\partial E}{\partial S^0}-2S^0\frac{\partial E}{\partial \boldsymbol{P}^2}\Big)\Big]P^at^c$$

$$-\Big[S^0BF+E\Big(A+2\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{P}^2}+2S^0\frac{\partial B}{\partial \boldsymbol{P}^2}\Big)-2\frac{\partial B}{\partial \boldsymbol{P}^2}\Big]t^aP^c\Big\}\gamma_5$$

$$+2\frac{\partial B}{\partial \boldsymbol{P}^2}[(R_1-U_1)+(R_2-U_2)\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})]P^c\varepsilon^{akl}V_\alpha{}^kA_\alpha{}^l=0\,, \tag{A·1}$$

$$\Big[BG-\frac{\partial A}{\partial S^0}+D\Big(B+\boldsymbol{P}^2\frac{\partial A}{\partial S^0}+S^0\frac{\partial B}{\partial S^0}\Big)\Big]\varepsilon^{akl}t^kP^l$$

$$+\Big\{-\Big[B\Big(\frac{\partial G}{\partial S^0}-2S^0\frac{\partial G}{\partial \boldsymbol{P}^2}\Big)+\Big(E\frac{\partial A}{\partial S^0}+G\frac{\partial B}{\partial S^0}\Big)-F\Big(B+S^0\frac{\partial B}{\partial S^0}\Big)\Big](\boldsymbol{P}\cdot\boldsymbol{t})P^a$$

$$+\Big[S^0BG-\frac{\partial B}{\partial S^0}+E\Big(B+\boldsymbol{P}^2\frac{\partial A}{\partial S^0}+S^0\frac{\partial B}{\partial S^0}\Big)\Big]t^a\Big\}\gamma_5$$

$$-\frac{\partial B}{\partial S^0}[(R_1-U_1)+(R_2-U_2)\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})]\varepsilon^{akl}V_\alpha{}^kA_\alpha{}^l=0\,, \tag{A·2}$$

$$B\Big[\frac{\partial Q_1}{\partial S^0}-2S^0\frac{\partial Q_1}{\partial \boldsymbol{P}^2}\Big]P^a$$

$$-\Big[BQ_2-2\frac{\partial A}{\partial (V_\alpha{}^0)^2}+2D\Big(\boldsymbol{P}^2\frac{\partial A}{\partial (V_\alpha{}^0)^2}+S^0\frac{\partial B}{\partial (V_\alpha{}^0)^2}\Big)\Big]\varepsilon^{akl}t^kP^l$$

$$+\Big\{-\Big[S^0BQ_2-2\frac{\partial B}{\partial (V_\alpha{}^0)^2}+2E\Big(\boldsymbol{P}^2\frac{\partial A}{\partial (V_\alpha{}^0)^2}+S^0\frac{\partial B}{\partial (V_\alpha{}^0)^2}\Big)\Big]t^a$$

$$+\Big[2\Big(E\frac{\partial A}{\partial (V_\alpha{}^0)^2}+G\frac{\partial B}{\partial (V_\alpha{}^0)^2}\Big)$$

$$-2S^0F\frac{\partial B}{\partial (V_\alpha{}^0)^2}+B\Big(\frac{\partial Q_2}{\partial S^0}-2S^0\frac{\partial Q_2}{\partial \boldsymbol{P}^2}\Big)\Big](\boldsymbol{P}\cdot\boldsymbol{t})P^a\Big\}\gamma_5$$

$$+2\frac{\partial B}{\partial (V_\alpha{}^0)^2}[(R_1-U_1)+(R_2-U_2)\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})]\varepsilon^{akl}V_\beta{}^kA_\beta{}^l=0\,, \tag{A·3}$$

$$\Big\langle B\Big[\frac{\partial R_1}{\partial S^0}-2S^0\frac{\partial R_1}{\partial \boldsymbol{P}^2}\Big]P^a$$

$$-\Big[BR_2-2\frac{\partial A}{\partial \boldsymbol{V}_\beta{}^2}+2D\Big(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{V}_\beta{}^2}+S^0\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}\Big)\Big]\varepsilon^{akl}t^kP^l$$

$$+\Big\{-\Big[S^0BR_2-2\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}+2E\Big(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{V}_\beta{}^2}+S^0\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}\Big)\Big]t^a$$

$$+\left[2\left(E\frac{\partial A}{\partial \boldsymbol{V}_\beta{}^2}+G\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}\right)-2S^0F\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}+B\left(\frac{\partial R_2}{\partial S^0}-2S^0\frac{\partial R_2}{\partial \boldsymbol{P}^2}\right)\right](\boldsymbol{P}\cdot\boldsymbol{t})\boldsymbol{P}^a\Big\}\gamma_5$$

$$+2\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}[(R_1-U_1)+(R_2-U_2)\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})]\varepsilon^{akl}V_\gamma{}^kA_\gamma{}^l\Big\rangle V_\alpha{}^c$$

$$+B[(R_1-U_1)+(R_2-U_2)\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})]\varepsilon^{acl}A_\alpha{}^l=0\,, \tag{A·4}$$

$$\Big\langle B\left[\frac{\partial U_1}{\partial S^0}-2S^0\frac{\partial U_1}{\partial \boldsymbol{P}^2}\right]P^a$$

$$-\left[BU_2-2\frac{\partial A}{\partial \boldsymbol{V}_\beta{}^2}+2D\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{V}_\beta{}^2}+S^0\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}\right)\right]\varepsilon^{akl}t^kP^l$$

$$+\Big\{-\left[S^0BU_2-2\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}+2E\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{V}_\beta{}^2}+S^0\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}\right)\right]t^a$$

$$+\left[2\left(E\frac{\partial A}{\partial \boldsymbol{V}_\beta{}^2}+G\frac{\partial A}{\partial \boldsymbol{V}_\beta{}^2}\right)-2S^0F\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}+B\left(\frac{\partial U_2}{\partial S^0}-2S^0\frac{\partial U_2}{\partial \boldsymbol{P}^2}\right)\right](\boldsymbol{P}\cdot\boldsymbol{t})P^a\Big\}\gamma_5$$

$$+2\frac{\partial B}{\partial \boldsymbol{V}_\beta{}^2}[(R_1-U_1)+(R_2-U_2)\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})]\varepsilon^{akl}V_\gamma{}^kA_\gamma{}^l\Big\rangle A_\alpha{}^c$$

$$+B[(R_1-U_1)+(R_2-U_2)\gamma_5(\boldsymbol{P}\cdot\boldsymbol{t})]\varepsilon^{acl}V_\alpha{}^l=0\,, \tag{A·5}$$

$$[X^a, W]=0\,, \tag{A·6}$$

$$\left(\frac{\partial Y}{\partial S^0}-2S^0\frac{\partial Y}{\partial \boldsymbol{P}^2}\right)P^at^c+(Y-Z)\gamma_5\varepsilon^{acl}t^l=0\,, \tag{A·7}$$

$$\left(\frac{\partial Z}{\partial S^0}-2S^0\frac{\partial Z}{\partial \boldsymbol{P}^2}\right)\gamma_5P^at^c+(Z-Y)\varepsilon^{acl}t^l=0\,. \tag{A·8}$$

At first we shall consider Eq. (A·1). This equation consists of ten terms. It is easy to show that nine of them are independent. Thus we obtain

$$\begin{aligned}
&A-BE+S^0BD=0\,,\\
&D+G-S^0F=0\,,\\
&AD+BF+2D\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{P}^2}+S^0\frac{\partial B}{\partial \boldsymbol{P}^2}\right)-2\frac{\partial A}{\partial \boldsymbol{P}^2}=0\,,\\
&AE+S^0BF+2E\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{P}^2}+S^0\frac{\partial B}{\partial \boldsymbol{P}^2}\right)-2\frac{\partial B}{\partial \boldsymbol{P}^2}=0\,,\\
&AD+B\left(\frac{\partial D}{\partial S^0}-2S^0\frac{\partial D}{\partial \boldsymbol{P}^2}\right)=0\,,\\
&AE-BD+B\left(\frac{\partial E}{\partial S^0}-2S^0\frac{\partial E}{\partial \boldsymbol{P}^2}\right)=0\,,\\
&2\left(E\frac{\partial A}{\partial \boldsymbol{P}^2}+G\frac{\partial B}{\partial \boldsymbol{P}^2}\right)-2S^0F\frac{\partial B}{\partial \boldsymbol{P}^2}+B\left(\frac{\partial F}{\partial S^0}-2S^0\frac{\partial F}{\partial \boldsymbol{P}^2}\right)=0
\end{aligned} \tag{A·9}$$

and

$$R_1 = U_1\,, \qquad R_2 = U_2\,. \tag{A·10}$$

Equation (A·2) consists of five terms. They are independent. Thus we get

$$\begin{aligned}
&B(D+G)+D\left(\boldsymbol{P}^2\frac{\partial A}{\partial S^0}+S^0\frac{\partial B}{\partial S^0}\right)-\frac{\partial A}{\partial S^0}=0\,,\\
&B(E+S^0G)+E\left(\boldsymbol{P}^2\frac{\partial A}{\partial S^0}+S^0\frac{\partial B}{\partial S^0}\right)-\frac{\partial B}{\partial S^0}=0\,,\\
&BF-\left(E\frac{\partial A}{\partial S^0}+G\frac{\partial B}{\partial S^0}\right)+S^0\frac{\partial B}{\partial S^0}-B\left(\frac{\partial G}{\partial S^0}-2S^0\frac{\partial G}{\partial \boldsymbol{P}^2}\right)=0
\end{aligned} \tag{A·11}$$

and Eq. (A·10).

Likewise Eqs. (A·3), (A·4) and (A·5) give

$$\frac{\partial Q_1}{\partial S^0}-2S^0\frac{\partial Q_1}{\partial \boldsymbol{P}^2}=0\,, \tag{A·12}$$

$$\begin{aligned}
&BQ_2+2D\left(\boldsymbol{P}^2\frac{\partial A}{\partial (V_\alpha{}^0)^2}+S^0\frac{\partial B}{\partial (V_\alpha{}^0)^2}\right)-2\frac{\partial A}{\partial (V_\alpha{}^0)^2}=0\,,\\
&S^0BQ_2+2E\left(\boldsymbol{P}^2\frac{\partial A}{\partial (V_\alpha{}^0)^2}+S^0\frac{\partial B}{\partial (V_\alpha{}^0)^2}\right)-2\frac{\partial B}{\partial (V_\alpha{}^0)^2}=0\,,\\
&2\left(E\frac{\partial A}{\partial (V_\alpha{}^0)^2}+G\frac{\partial B}{\partial (V_\alpha{}^0)^2}\right)-2S^0F\frac{\partial B}{\partial (V_\alpha{}^0)^2}+B\left(\frac{\partial Q_2}{\partial S^0}-2S^0\frac{\partial Q_2}{\partial \boldsymbol{P}^2}\right)=0
\end{aligned} \tag{A·13}$$

and Eq. (A·10),

$$\frac{\partial R_1}{\partial S^0}-2S^0\frac{\partial R_1}{\partial \boldsymbol{P}^2}=0\,, \tag{A·14}$$

$$\begin{aligned}
&BR_2+2D\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{V}_\alpha{}^2}+S^0\frac{\partial B}{\partial \boldsymbol{V}_\alpha{}^2}\right)-2\frac{\partial A}{\partial \boldsymbol{V}_\alpha{}^2}=0\,,\\
&S^0BR_2+2E\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{V}_\alpha{}^2}+S^0\frac{\partial B}{\partial \boldsymbol{V}_\alpha{}^2}\right)-2\frac{\partial B}{\partial \boldsymbol{V}_\alpha{}^2}=0\,,\\
&2\left(E\frac{\partial A}{\partial \boldsymbol{V}_\alpha{}^2}+G\frac{\partial B}{\partial \boldsymbol{V}_\alpha{}^2}\right)-2S^0F\frac{\partial B}{\partial \boldsymbol{V}_\alpha{}^2}+B\left(\frac{\partial R_2}{\partial S^0}-2S^0\frac{\partial R_2}{\partial \boldsymbol{P}^2}\right)=0
\end{aligned} \tag{A·15}$$

and Eq. (A·10), and

$$\frac{\partial U_1}{\partial S^0}-2S^0\frac{\partial U_1}{\partial \boldsymbol{P}^2}=0\,, \tag{A·16}$$

$$\begin{aligned}
&BU_2+2D\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{V}_\alpha{}^2}+S^0\frac{\partial B}{\partial \boldsymbol{V}_\alpha{}^2}\right)-2\frac{\partial A}{\partial \boldsymbol{V}_\alpha{}^2}=0\,,\\
&S^0BU_2+2E\left(\boldsymbol{P}^2\frac{\partial A}{\partial \boldsymbol{V}_{\ddot\alpha}{}^2}+S^0\frac{\partial B}{\partial \boldsymbol{V}_{\ddot\alpha}{}^2}\right)-2\frac{\partial B}{\partial \boldsymbol{V}_{\ddot\alpha}{}^2}=0\,,
\end{aligned} \tag{A·17}$$

$$2\left(E\frac{\partial A}{\partial \boldsymbol{V}_\alpha^2}+G\frac{\partial B}{\partial \boldsymbol{V}_\alpha^2}\right)-2S^0F\frac{\partial B}{\partial \boldsymbol{V}_\alpha^2}+B\left(\frac{\partial U_2}{\partial S^0}-2S^0\frac{\partial U_2}{\partial \boldsymbol{P}^2}\right)=0$$

and Eq. (A·10) respectively. Finally, Eqs. (A·7) and (A·8) give

$$Y=Z\,, \qquad [X^a, Y]=0\,. \tag{A·18}$$

References

1) For the references see S. Weinberg, Phys. Rev. **166** (1968), 1568.
2) W. Heisenberg, *Introduction to the Unified Field Theory of Elementary Particles* (Interscience Pub., New York, 1966). See Eq. 3(22) in this book.
3) Equations (4·9) and (4·21) in reference 1).
4) These relations are also obtained by Dr. T. Ohta independently (private communication).