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Chiral Symmetry in the Unified Fermion Theory. III

——*Quantization*——

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Quantization method is presented for the fundamental Dirac field when the Lagrangian densities are invariant under non-linear chiral $U(1)\times U(1)$ transformations. The Lagrangian densities are derived from a geometrical viewpoint that the space-time possesses the torsion caused by the field itself. The equal-time anti-commutators for the field are not c-numbers but functions of the field. The chiral symmetry may break down by the quantum effects. This quantization method is applicable to more general cases.

§ 1. Introduction

In the unified fermion theory in which all elementary particles and resonances have to be composed of the fundamental Dirac field ψ, the interaction parts of the Lagrangian densities invariant under chiral $SU(n)\times SU(n)$ or $U(n)\times U(n)$ transformations are, as was shown in previous papers,[1] at least of quartic form with respect to the field ψ and include in general the space-time derivatives of the field. The purpose of this paper is to show a method how to quantize the field described by the Lagrangian densities having the nature mentioned above.

In this paper we shall deal with the simplest cases that Lagrangian densities are invariant under $U(1)\times U(1)$ transformations. The method of quantization for these cases is also applicable to the cases of chiral $SU(n)\times SU(n)$ and $U(n)\times U(n)$ transformations.

We start with the Lagrangian density in the unquantized theory:

$$L_1 = -\bar{\psi}\gamma_\mu\partial_\mu\psi - i\lambda(\bar{\psi}\gamma_\mu\psi)\partial_\mu(\bar{\psi}\psi), \qquad (1\cdot1)$$

where λ is a constant with the dimension of $(\text{mass})^{-3}$. Hereafter we shall call this Lagrangian density as that of model I. The interaction Lagrangian density is of a quartic form in ψ and includes the derivative $\partial_\mu(\bar{\psi}\psi)$. Though the current $(\bar{\psi}\gamma_\mu\psi)$ is conserved, the interaction Lagrangian density has physical effects in the quantum theory. This point will be discussed later.

Let T and X be the generators of chiral $U(1)\times U(1)$ transformations. It is easy to show that L_1 is invariant under the linear transformation law

$$[T, \psi] = -\psi \tag{1·2}$$

and the non-linear one

$$[X, \psi] = -\{\gamma_5 - 2i\lambda(\bar{\psi}\gamma_5\psi)\}\psi\,. \tag{1·3}$$

We note that though L_1 is not invariant under the usual charge conjugation, it is invariant under the other kind of charge conjugation (cf. § 4). The other Lagrangian density which we shall refer to that of model II is

$$L_2 = -\bar{\psi}\gamma_\mu\partial_\mu\psi - \lambda(\bar{\psi}\gamma_\mu\gamma_5\psi)\partial_\mu(\bar{\psi}\gamma_5\psi)\,. \tag{1·4}$$

Again this is invariant under the transformation laws (1·2) and

$$[X, \psi] = -\frac{\gamma_5}{1+2\lambda(\bar{\psi}\psi)}\psi\,. \tag{1·5}$$

We can show that the Lagrangian densities L_1 and L_2 are interpreted as those for Dirac field in the space-time with the torsion caused by the field itself. It is a natural consequence of the unified fermion theory that the torsion originates in the existence of the fundamental Dirac field. This geometrical viewpoint will be discussed in the Appendix.

Now we want to quantize the field ψ for the Lagrangian densities L_1 and L_2. Readers might consider, along the following line of thought, that the quantization is trivial. Redefining the fields as

$$q_1 = [\exp\{i\lambda(\bar{\psi}\psi)\}]\psi \quad \text{and} \quad q_2 = [\exp\{\lambda(\bar{\psi}\gamma_5\psi)\gamma_5\}]\psi \tag{1·6}$$

instead of the field ψ for L_1 and L_2 respectively, we find that the Lagrangian densities reduce to

$$L_i = -\bar{q}_i\gamma_\mu\partial_\mu q_i\,. \qquad (i=1, 2)$$

The equal-time anti-commutation relations for q_i are given by

$$\begin{aligned}&\{q_{i\alpha}(\boldsymbol{x}, t), q^*_{i\beta}(\boldsymbol{y}, t)\} = \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}),\\ &\{q_{i\alpha}(\boldsymbol{x}, t), q_{i\beta}(\boldsymbol{y}, t)\} = 0\,.\end{aligned} \tag{1·7}$$

The transformations (1·6), however, are not unitary. We can show that the equal-time anti-commutation relations (1·7) are different from those for ψ given by the Lagrangian density L_1 or L_2.

The interaction parts of the Lagrangian densities L_1 and L_2 are of quartic form in ψ, ψ^* and their derivatives even after any partial integration. One may define the momenta π and $\pi^\dagger$ conjugate to the field ψ and ψ^*, respectively, by the usual method for L_1 and L_2. Since the Lagrangian densities are linear with respect to the time derivatives of the fields, there must be constraints among the fields and the momenta. The constraints are non-linear in ψ and ψ^*. The quantization must be carried out under the condition that the non-linear constraints are always satisfied without any contradictions. For this purpose we shall try to use the method of the modified Poisson bracket given by Dirac,[2] who developed this method to quantize boson fields only. After suitable modifications we find that it is applicable to quantize the fundamental Dirac field ψ in the case of the unified fermion theory.

When the field is properly quantized, the Euler equation must be identical to the Heisenberg equation of motion

$$\partial_0\psi_\alpha(x)=i\int d^3y[H(y),\psi_\alpha(x)], \tag{1·8}$$

where H is the Hamiltonian density. The Lagrangian densities L_1 and L_2 suggest that equal-time anti-commutators for ψ and ψ^* are not c-numbers but involve at least bilinear terms of ψ and ψ^*. In fact it will be shown that the anti-commutators in the case of model II depend on all even powers of ψ and ψ^*. This causes the difficulty that the ordering of ψ's involved in the anti-commutation relations is ambiguous especially in the case of model II. The ordering of ψ's is ambiguous also in the Euler equations. These ambiguities are due to the quartic form of the interaction Lagrangian densities. As a result, we are obliged to require of the quantization of ψ that the Heisenberg equation should coincide with the Euler equation only when the ordering of ψ's is ignored.

We need other conditions to determine uniquely the ordering of ψ's involved in the equal-time anti-commutation relations for ψ's. Since the transformation law (1·2) is linear in ψ, it is allowed to require that the law should hold exactly also in the quantized theory. This requirement is very severe and is useful to determine the unique ordering of ψ and ψ^* in the anti-commutation relations. However, the transformation law (1·3) or (1·5) is non-linear with respect to ψ and ψ^*. It is natural to require in the quantized theory that the law is reproduced at least when the ordering of ψ and ψ^* is ignored.

Under these requirements we can determine the equal-time anti-commutation relations for ψ and ψ^* uniquely. When Noether's theorem is assumed to hold in the quantized theory, we get expressions for the vector and axial-vector currents, from which consistency conditions for the generators T and X are obtained. It is shown in model I that the anti-commutation relations lead to the conservation of the vector current and further that there exists a solution of the consistency equation for T. On the other hand, we have no satisfactory solution of the consistency equation for X. We may conclude that Noether's theorem does not hold in the quantized theory for the case of non-linear transformation (1·3) and/or that the chiral symmetry is violated by quantum effects.

In § 2 we present the method for extending Dirac's modified Poisson brackets so that we can apply it to fermion fields. The quantization of field ψ for model I is performed in § 3. We discuss in § 4 the characteristic features due to the non-linearity of model I. The discussions are also given on the existence of an unusual charge conjugation. Section 5 is devoted to the presentation of the method of quantization for model II. The concluding remarks are given in § 6.

§ 2. Generalized Poisson brackets for the fermion field

As was mentioned in § 1, our system involves constraints which leads to inconsistencies, provided the ordinary Poisson brackets (P.b.'s) are postulated among canonical variables. We shall therefore rely upon Dirac's generalized canonical formalism based on his modified P.b.'s. He developed his theory for boson fields. The characteristic feature of our system is that the fundamental field ψ is a fermion field and that the interaction parts of the Lagrangian densities are of quartic forms of ψ, ψ^* and their derivatives. Accordingly, modified P.b.'s will in general contain powers of ψ and ψ^*, and it is necessary to take account of anti-commutability of ψ and ψ^* even in the unquantized theory. We shall extend Dirac's method so as to be applicable for our non-linear unified fermion theory.

We shall start with the Lagrangian formalism where the following anti-commutability of ψ and ψ^* is taken into account in the unquantized theory:

$$\{\psi_\alpha(x), \psi_\beta^*(y)\} = \{\psi_\alpha(x), \psi_\beta(y)\} = \{\psi_\alpha^*(x), \psi_\beta^*(y)\} = 0 . \tag{2·1}$$

The Lagrangian densities of our models are written in a symmetrized form

$$L = \frac{1}{4}[\partial_\mu \bar{\psi}, \gamma_\mu \psi] - \frac{1}{4}[\bar{\psi}\gamma_\mu, \partial_\mu \psi] - \frac{\lambda}{8}\{[\bar{\psi}\gamma_\mu \Lambda, \psi], \partial_\mu([\bar{\psi}\bar{\Lambda}, \psi])\}, \tag{2·2}$$

where

$$\begin{aligned} &\Lambda = 1 \text{ and } \bar{\Lambda} = i && \text{for model I,} \\ &\Lambda = \bar{\Lambda} = \gamma_5 && \text{for model II.} \end{aligned} \tag{2·2'}$$

Following Schwinger,[3] we introduce the right and left derivatives by

$$\frac{\partial^r P(q)}{\partial q_i} = (\delta P)(\delta q_i)^{-1}, \qquad \frac{\partial^l P(q)}{\partial q_i} = (\delta q_i)^{-1}(\delta P), \tag{2·3}$$

where q is a representative of canonical coordinates ψ and ψ^* and $P(q)$ denotes a polynomial in q's. For instance, if $P(q) = q_1 q_2$, we have

$$\frac{\partial^r P}{\partial q_2} = q_1 , \qquad \frac{\partial^l P}{\partial q_2} = -q_1 .$$

In general, for P_1 which is a product of n_1 variables q and P_2 a product of n_2 variables q the following formulas are derived:

$$\frac{\partial^r \{P_1 P_2\}}{\partial q_i} = (-1)^{n_2} \frac{\partial^r P_1}{\partial q_i} P_2 + P_1 \frac{\partial^r P_2}{\partial q_i} ,$$

$$\frac{\partial^l\{P_1P_2\}}{\partial q_i}=\frac{\partial^l P_1}{\partial q_i}P_2+(-1)^{n_1}P_1\frac{\partial^l P_2}{\partial q_i}. \tag{2·4}$$

The canonical momentum p_i conjugate to q_i is defined by

$$p_i\equiv\frac{\partial^r L}{\partial(\partial_0 q_i)} \tag{2·5}$$

whose explicit form is written as

$$\begin{aligned}\pi_\alpha&\equiv\frac{\partial^r L}{\partial(\partial_0\psi_\alpha)}=\frac{i}{2}\psi_\alpha^*+\frac{i}{4}\lambda\{[\psi^*\Lambda,\psi],(\bar{\psi}\bar{\Lambda})_\alpha\},\\ \pi_\alpha{}^\dagger&\equiv\frac{\partial^r L}{\partial(\partial_0\psi_\alpha{}^*)}=\frac{i}{2}\psi_\alpha-\frac{i}{4}\lambda\{[\psi^*\Lambda,\psi],(\beta\bar{\Lambda}\psi)_\alpha\}\end{aligned} \tag{2·5′}$$

for our system described by the Lagrangian density (2·2) with (2·2′). Since the time derivative of field variables is not contained in (2·5′), it is considered that Eqs. (2·5′) constitute a set of constraints among coordinates and momenta. We shall write them as

$$\begin{aligned}\theta_\alpha&\equiv\pi_\alpha-\frac{i}{2}\psi_\alpha^*-\frac{i}{4}\lambda\{[\psi^*\Lambda,\psi],(\bar{\psi}\bar{\Lambda})_\alpha\}=0,\\ \theta_\alpha{}^\dagger&\equiv\pi_\alpha{}^\dagger-\frac{i}{2}\psi_\alpha+\frac{i}{4}\lambda\{[\psi^*\Lambda,\psi],(\beta\bar{\Lambda}\psi)_\alpha\}=0.\end{aligned} \tag{2·6}$$

In the first place, we shall explicitly show that the above constraints are not consistent with each other if we postulate the P.b.'s

$$\begin{aligned}(q_i(x),p_j(y))&=(p_j(x),q_i(y))=\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}),\\ (q_i(x),q_j(y))&=(p_i(x),p_j(y))=0\end{aligned} \tag{2·7}$$

at $x_0=y_0$. In the following we shall deal with the P.b.'s for dynamical variables at equal-time. The above P.b.'s are the special cases of the following P.b. in which the anti-commutability (2·1) is taken into account. Let us define the P.b. for any dynamical variables $E(x)$ and $F(y)$ which are functions of q and p by

$$(E(x),F(y))=\sum_i\left\{\frac{\partial^r E(x)}{\partial q_i(x)}\frac{\partial^l F(y)}{\partial p_i(y)}-(-1)^{n_E n_F}\frac{\partial^r F(y)}{\partial q_i(y)}\frac{\partial^l E(x)}{\partial p_i(x)}\right\}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{2·8}$$

Here $n_E(n_F)$ takes the value 0 or 1 according as $E(F)$ is an even or odd function of q and p; for instance, if $E=q_iq_jp_k$, we have $n_E=1$. The P.b. (2·8) has the symmetry property

$$(E(x),F(y))=-(-1)^{n_E n_F}(F(y),E(x)). \tag{2·9}$$

This property corresponds correctly to that of the commutation or anti-commutation relations in quantum theory. As an example, from (2·8) we get

$$(\pi_\alpha,\psi_\beta\psi_\gamma)=(\pi_\alpha,\psi_\beta)\psi_\gamma-(\pi_\alpha,\psi_\gamma)\psi_\beta=-(\psi_\beta\psi_\gamma,\pi_\alpha).$$

If we use (2·7), i.e.

$$(\psi_\alpha(x), \pi_\beta(y)) = \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}), \qquad (\psi_\alpha{}^*(x), \pi_\beta{}^\dagger(y)) = \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2·7'}$$

etc., we have

$$\begin{aligned}(\theta_\alpha(x), \theta_\beta{}^\dagger(y)) &= -i\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) - \frac{i}{2}\lambda\{(\psi^*\Lambda)_\alpha(\beta\bar{\Lambda}\psi)_\beta - (\bar{\psi}\bar{\Lambda})_\alpha(\Lambda\psi)_\beta \\ &\quad - (\beta\bar{\Lambda}\psi)_\beta(\psi^*\Lambda)_\alpha + (\Lambda\psi)_\beta(\bar{\psi}\bar{\Lambda})_\alpha\}\delta(\boldsymbol{x}-\boldsymbol{y}), \\ (\theta_\alpha(x), \theta_\beta(y)) &= -\frac{i}{2}\lambda\{(\psi^*\Lambda)_\beta(\bar{\psi}\bar{\Lambda})_\alpha - (\bar{\psi}\bar{\Lambda})_\alpha(\psi^*\Lambda)_\beta \\ &\quad - (\bar{\psi}\bar{\Lambda})_\beta(\psi^*\Lambda)_\alpha + (\psi^*\Lambda)_\alpha(\bar{\psi}\bar{\Lambda})_\beta\}\delta(\boldsymbol{x}-\boldsymbol{y}).\end{aligned} \tag{2·10}$$ *)

The right-hand sides of (2·10) do not vanish. Thus we cannot put $\theta = \theta^\dagger = 0$ freely in dynamical equations.

In order that the constraints (2·6) are always satisfied, we shall modify the P.b. (2·8) following Dirac.[2] We now write θ_α and $\theta_\alpha{}^\dagger$ in a single $\Theta_n (n=1, 2, \cdots, 8)$ according to

$$\Theta_n = \begin{cases} \theta_n & \text{for } n=1, \cdots, 4, \\ \theta^\dagger_{n-4} & \text{for } n=5, \cdots, 8. \end{cases} \tag{2·11}$$

The modified P.b. is defined by

$$\begin{aligned}(E(x), F(y))^* \equiv (E(x), F(y)) - \iint d^3z_1 d^3z_2 (E(x), \Theta_m(z_1)) \\ \times C_{mn}(z_1, z_2)(\Theta_n(z_2), F(y)),\end{aligned} \tag{2·12}$$

where $C_{mn}(z, y)$ is an 8×8 matrix obtained from

$$\int d^3z (\Theta_l(x), \Theta_m(z)) C_{mn}(z, y) = \delta_{ln}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{2·13}$$

If we take

$$F(y) = \Theta_l(y)$$

in (2·12), we have

$$(E(x), \Theta_l(y))^* = 0$$

by virtue of (2·13). Since $E(x)$ is an arbitrary dynamical variable, the constraints (2·6) always hold. Thus the degree of freedom can be reduced without getting contradictions.

The equation $\Theta = 0$ enables us to write the Hamiltonian density derived from the Lagrangian density (2·2) as

*) Equations (2·10) indicate that $\theta=0$ and $\theta^\dagger=0$ are the second class constraints named by Dirac.[2]

$$H \equiv \pi_\alpha \partial_0 \psi_\alpha + \pi_\alpha^\dagger \partial_0 \psi_\alpha^* - L$$
$$= \frac{1}{4}[\bar{\psi}\boldsymbol{\gamma}, \boldsymbol{\nabla}\psi] - \frac{1}{4}[\boldsymbol{\nabla}\bar{\psi}, \boldsymbol{\gamma}\psi] + \frac{\lambda}{8}\{[\bar{\psi}\boldsymbol{\gamma}\Lambda, \psi], \boldsymbol{\nabla}([\bar{\psi}, \bar{\Lambda}\psi])\}. \quad (2\cdot 14)$$

Using this Hamiltonian density and the P.b. defined by (2·12), we can set up Hamilton's equation of motion

$$\partial_0 E(x) = \int d^3y \, (E(x), H(y))^*. \quad (2\cdot 15)$$

We can show that the above Hamilton's equation coincides with the Euler equation obtained from the Lagrangian density (2·2), when we know the explicit forms of $(\psi_\alpha(x), \psi_\beta^*(y))^*$, etc. In the following sections we shall give them for the individual models.

The quantization is carried out straightforwardly by means of

$$\{\psi_\alpha(x), \psi_\beta^*(y)\} = i(\psi_\alpha(x), \psi_\beta^*(y))^*, \text{ etc.} \quad (2\cdot 16)$$

Of course, it should be remarked that there arises a cumbersome problem of the ordering of ψ and ψ^*, since the right-hand sides of (2·16) are not constants but functions of ψ and ψ^* in our models.

§ 3. Quantization of model I

We shall start with the symmetrized form of the Lagrangian density (1·1), i.e.

$$L_1 = \frac{1}{4}[\partial_\mu \bar{\psi}, \gamma_\mu \psi] - \frac{1}{4}[\bar{\psi}\gamma_\mu, \partial_\mu \psi] - \frac{i\lambda}{8}\{[\bar{\psi}\gamma_\mu, \psi], \partial_\mu([\bar{\psi}, \psi])\}. \quad (3\cdot 1)$$

The P.b.'s for θ and $\theta^\dagger$ are expressed as

$$\begin{aligned} (\theta_\alpha(x), \theta_\beta^\dagger(y)) &= -i\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) - \lambda\Gamma_{\alpha\beta}[\psi_\alpha^*(x), \psi_\beta(x)]\delta(\boldsymbol{x}-\boldsymbol{y}), \\ (\theta_\alpha(x), \theta_\beta(y)) &= \lambda\Gamma_{\alpha\beta}[\psi_\alpha^*(x), \psi_\beta^*(x)]\delta(\boldsymbol{x}-\boldsymbol{y}), \end{aligned} \quad (3\cdot 2)$$

where

$$\Gamma_{\alpha\beta} = \tfrac{1}{2}(\beta_{\alpha\alpha} - \beta_{\beta\beta}),$$

whose matrix form is (3·3)

$$\Gamma = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}.$$

It should be noted that summations with respect to α and β are not carried out in the right-hand sides of (3·2) and (3·3). In order to find the modified P.b.'s we must know C_{mn} defined by (2·13). In model I, it is rather simple to find C_{mn}. The result is as follows:

$$C_{mn}(x,y)=\begin{cases}\lambda\Gamma_{\alpha\beta}[\psi_\alpha^*(x),\psi_\beta^*(x)]\delta(\boldsymbol{x}-\boldsymbol{y}) & \text{for } m=\alpha,\ n=\beta\,,\\ -i\delta_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{y})-\lambda\Gamma_{\alpha\beta}[\psi_\alpha^*(x),\psi_\beta(x)]\delta(\boldsymbol{x}-\boldsymbol{y}) & \text{for } m=\alpha,\ n=4+\beta\,,\\ -i\delta_{\alpha\beta\delta}(\boldsymbol{x}-\boldsymbol{y})-\lambda\Gamma_{\alpha\beta}[\psi_\alpha(x),\psi_\beta^*(x)]\delta(\boldsymbol{x}-\boldsymbol{y}) & \text{for } m=4+\alpha,\ n=\beta\,,\\ \lambda\Gamma_{\alpha\beta}[\psi_\alpha(x),\psi_\beta(x)]\delta(\boldsymbol{x}-\boldsymbol{y}) & \text{for } m=4+\alpha,\ n=4+\beta.\end{cases} \tag{3·4}$$

Using the above C_{mn}, we get the modified P.b.'s for ψ and ψ^*:

$$\begin{aligned}(\psi_\alpha(x),\psi_\beta^*(y))^*&=-i\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y})+\lambda\Gamma_{\alpha\beta}[\psi_\alpha(x),\psi_\beta^*(x)]\delta(\boldsymbol{x}-\boldsymbol{y}),\\(\psi_\alpha(x),\psi_\beta(y))^*&=-\lambda\Gamma_{\alpha\beta}[\psi_\alpha(x),\psi_\beta(x)]\delta(\boldsymbol{x}-\boldsymbol{y}).\end{aligned} \tag{3·5}$$

Though our P.b.'s (3·5) are quadratic polynomials in ψ and ψ^*, the transition to the quantum theory is performed straightforwardly with the aid of (2·16). We get the equal-time anti-commutation relations

$$\{\psi_\alpha(x),\psi_\beta^*(y)\}=\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y})+i\lambda\Gamma_{\alpha\beta}[\psi_\alpha(x),\psi_\beta^*(x)]\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{3·6}$$

$$\{\psi_\alpha(x),\psi_\beta(y)\}=-i\lambda\Gamma_{\alpha\beta}[\psi_\alpha(x),\psi_\beta(x)]\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{3·7}$$

From (3·6) and (3·7), we have

$$\begin{aligned}&[\psi_\alpha(x),[\psi_\beta^*(y),\psi_\gamma(z)]]\\&=2\delta_{\alpha\beta}\psi_\gamma(z)\delta(\boldsymbol{x}-\boldsymbol{y})+i\lambda\Gamma_{\alpha\beta}\{\psi_\gamma(z),[\psi_\alpha(x),\psi_\beta^*(y)]\}\delta(\boldsymbol{x}-\boldsymbol{y})\\&\qquad+i\lambda\Gamma_{\alpha\gamma}\{\psi_\beta^*(y),[\psi_\alpha(x),\psi_\gamma(z)]\}\delta(\boldsymbol{x}-\boldsymbol{y})\end{aligned} \tag{3·8}$$

$$\begin{aligned}&=2\delta_{\alpha\beta}\psi_\gamma(z)\delta(\boldsymbol{x}-\boldsymbol{y})+i\lambda\Gamma_{\alpha\beta}\{\psi_\alpha(x),[\psi_\beta^*(y),\psi_\gamma(z)]\}\delta(\boldsymbol{x}-\boldsymbol{y})\\&\qquad-i\lambda\Gamma_{\alpha\gamma}\{\psi_\beta^*(y),[\psi_\alpha(x),\psi_\gamma(z)]\}\delta(\boldsymbol{y}-\boldsymbol{z})\\&\qquad-\lambda^2\Gamma_{\alpha\beta}\Gamma_{\alpha\gamma}[\psi_\alpha(x),[\psi_\beta^*(y),\psi_\gamma(z)]]\delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{z}).\end{aligned} \tag{3·8′}$$

In the derivation of the latter equation (3·8′) we have used the identity (A·4) in Appendix A. Equations (3·8) and (3·8′) will be used extensively in the following discussions.

In order to confirm the consistency of the anti-commutation relations (3·6) and (3·7), we shall first inquire whether (3·6) and (3·7) satisfy the following identity at equal time:

$$\begin{aligned}&[\psi_\alpha(x),\{\psi_\beta^*(y),\psi_\gamma(z)\}]\\&\qquad+[\psi_\beta^*(y),\{\psi_\gamma(z),\psi_\alpha(x)\}]+[\psi_\gamma(z),\{\psi_\alpha(x),\psi_\beta^*(y)\}]=0\,.\end{aligned} \tag{3·9}$$

It can be shown that (3·9) holds by using the relation

$$\begin{aligned}&[\psi_\alpha(x),\{\psi_\beta^*(y),\psi_\gamma(z)\}]\\&\qquad\qquad+[\psi_\beta^*(y),\{\psi_\gamma(z),\psi_\alpha(x)\}]+[\psi_\gamma(z),\{\psi_\alpha(x),\psi_\beta^*(y)\}]\\&=i\lambda^3\Gamma_{\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\alpha}\{[\psi_\alpha(x),[\psi_\beta^*(y),\psi_\gamma(z)]]+[\psi_\beta^*(y),[\psi_\gamma(z),\psi_\alpha(x)]]\\&\qquad+[\psi_\gamma(z),[\psi_\alpha(x),\psi_\beta^*(y)]]\}\delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z})\delta(\boldsymbol{z}-\boldsymbol{x})\end{aligned} \tag{3·10}$$

and the Jacobi identity

$$[\psi_\alpha(x), [\psi_\beta^*(y), \psi_\gamma(z)]] + [\psi_\beta^*(y), [\psi_\gamma(z), \psi_\alpha(x)]] + [\psi_\gamma(z), [\psi_\alpha(x), \psi_\beta^*(y)]] = 0\,.$$

Secondly, we shall see whether our anti-commutation relations (3·6) and (3·7) satisfy the conditions mentioned in § 1, that is, the transformation law (1·2) holds exactly, while the Heisenberg equation (1·8) coincides with the Euler equation derived from the Lagrangian density (1·1) and the transformation law (1·3) is reproduced when the ordering of ψ and ψ^* is ignored. In Appendix A, we shall give formulas to be used in the following.

The Heisenberg equation is

$$\partial_0\psi_\alpha(x) = i\int d^3y[H(y), \psi_\alpha(x)],$$

where $H(y)$ is given by (2·14) in this case. After straightforward calculations, it is written as

$$\partial_0\psi_\alpha = -i(\beta\boldsymbol{\gamma}\boldsymbol{\nabla}\psi)_\alpha + \frac{\lambda}{4}\{(\beta\boldsymbol{\gamma}\psi)_\alpha,\ \boldsymbol{\nabla}[\bar{\psi}, \psi]\} - \frac{\lambda}{4}\{\psi_\alpha,\ ([\psi^*\boldsymbol{\gamma},\ \boldsymbol{\nabla}\psi] - [\boldsymbol{\nabla}\psi^*\boldsymbol{\gamma}, \psi])\}$$
$$- \frac{i}{8}\lambda^2\{\psi_\alpha,\ \{[\psi^*\boldsymbol{\gamma}, \psi],\ \boldsymbol{\nabla}[\bar{\psi}, \psi]\}\} - \frac{i}{8}\lambda^2[[\psi^*\boldsymbol{\gamma}, \psi],\ [\psi_\alpha, \boldsymbol{\nabla}[\bar{\psi}, \psi]]]. \qquad (3\cdot11)$$

Using (3·11) and the corresponding equation for $\partial_0\psi_\alpha^*$, we get

$$\partial_0[\bar{\psi}, \psi] = -i[\psi^*\boldsymbol{\gamma}, \boldsymbol{\nabla}\psi] + i[\boldsymbol{\nabla}\psi^*\boldsymbol{\gamma}, \psi] + \frac{\lambda}{2}\{\boldsymbol{\nabla}[\bar{\psi}, \psi],\ [\psi^*\boldsymbol{\gamma}, \psi]\}. \qquad (3\cdot12)$$

From (3·11) and (3·12) we have the equation of motion

$$\partial_0\psi_\alpha = -i(\beta\boldsymbol{\gamma}\boldsymbol{\nabla}\psi)_\alpha + \frac{\lambda}{4}\{(\beta\boldsymbol{\gamma}\psi)_\alpha, \boldsymbol{\nabla}[\bar{\psi}, \psi]\} - \frac{i}{4}\lambda\{\psi_\alpha, \partial_0[\bar{\psi}, \psi]\}$$
$$- \frac{i}{8}\lambda^2[[\psi^*\boldsymbol{\gamma}, \psi],\ [\psi_\alpha, \boldsymbol{\nabla}[\bar{\psi}, \psi]]], \qquad (3\cdot13)$$

which is nothing but the Euler equation derived from the Lagrangian density L_1 except for the last term. The last term vanishes in the unquantized theory in which (2·1) is taken into account.

We want to find expressions for the generators T and X which satisfy the transformation laws (1·2) and (1·3), respectively. From (3·8) and the identity (A·4), we get

$$[\psi_\alpha(x),\ [\psi^*(y), \psi(\boldsymbol{y})]] = 2\psi_\alpha(x)\delta(\boldsymbol{x}-\boldsymbol{y})$$
$$+ i\lambda\sum_\beta \Gamma_{\alpha\beta}[\{\psi_\beta^*, \psi_\beta\}\psi_\alpha - \psi_\alpha\{\psi_\beta^*, \psi_\beta\}]\delta(\boldsymbol{x}-\boldsymbol{y}). \qquad (3\cdot14)$$

The second term on the right-hand side of (3·14) vanishes when (3·6) is used again. If we take the generator T as

$$T=\frac{1}{2}\int d^3x[\psi^*(x),\psi(x)], \tag{3·15}$$

we have (1·2), i.e.

$$[T,\psi_\alpha(x)]=-\psi_\alpha(x).$$

On the other hand, from (3·8) we have

$$\begin{aligned}(\gamma_5)_{\beta\gamma}[\psi_\alpha(x),[\psi_\beta{}^*(y),\psi_\gamma(y)]]&=2(\gamma_5\psi)_\alpha\delta(\boldsymbol{x}-\boldsymbol{y})\\&\quad-i\lambda(\beta\gamma_5)_{\beta\gamma}\{\psi_\alpha,[\psi_\beta{}^*,\psi_\gamma]\}\delta(\boldsymbol{x}-\boldsymbol{y}),\end{aligned} \tag{3·16}$$

in the derivation of which we have taken account of the following relations

$$(\gamma_5)_{\beta\gamma}\Gamma_{\alpha\beta}\Gamma_{\alpha\gamma}=0 \quad\text{and}\quad \Gamma_{\alpha\beta}-\Gamma_{\alpha\gamma}=\tfrac{1}{2}(\beta_{\gamma\gamma}-\beta_{\beta\beta}).$$

If we take

$$X^{(0)}=\tfrac{1}{2}\int d^3x[\psi^*(x)\gamma_5,\psi(x)], \tag{3·17}$$

we have (3·1), i.e.

$$[X^{(0)},\psi_\alpha]=-(\gamma_5\psi)_\alpha+\frac{i}{2}\lambda\{\psi_\alpha,[\bar{\psi}\gamma_5,\psi]\}. \tag{3·18}$$

Thus the equal-time anti-commutation relations (3·6) and (3·7) satisfy all the requirements for the quantization.

§ 4. Characteristic features due to the non-linearity of the theory

It is suggested in the quantized theory that there appear peculiar features due to the non-linear character of our Lagrangian density. In fact the Heisenberg equation of motion (3·13) is different from the equation in the unquantized theory by the term of the order of λ^2, as was shown in the preceding section. In the following it will be shown that there appears another peculiar feature related to the generator X.

To study the properties of the generators T and X, let us assume that Noether's theorem holds in the quantized non-linear fermiom theory. Since the Lagrangian density (3·1) is invariant under the transformation law (1·2), Noether's theorem leads to the following expressions for the vector current density J_μ and the generator T:

$$\begin{aligned}J_\mu=&\frac{i}{4}[[T,\bar{\psi}_\alpha],(\gamma_\mu\psi)_\alpha]-\frac{i}{4}[(\bar{\psi}\gamma_\mu)_\alpha,[T,\psi_\alpha]]\\&+\frac{\lambda}{8}\{[[T,\bar{\psi}_\alpha],\psi_\alpha]+[\bar{\psi}_\alpha,[T,\psi_\alpha]],[\bar{\psi}\gamma_\mu,\psi]\},\end{aligned} \tag{4·1}$$

$$T=\int d^3xJ_0(x). \tag{4·2}$$

From (4·1) and (4·2), we obtain the consistency condition

$$T=\frac{1}{4}\int d^3x([[T,\psi_\alpha^*],\psi_\alpha]-[\psi_\alpha^*,[T,\psi_\alpha]])$$
$$-\frac{i}{8}\lambda\int d^3x\{[[T,\bar{\psi}_\alpha],\psi_\alpha]+[\bar{\psi}_\alpha,[T,\psi_\alpha]],[\psi^*,\psi]\}. \tag{4·3}$$

If we substitute the transformation law (1·2) into the right-hand sides of (4·1) and (4·3), we get

$$J_\mu(x)=\frac{i}{2}[\bar{\psi}(x)\gamma_\mu,\psi(x)], \tag{4·4}$$

$$T=\frac{1}{2}\int d^3x[\psi^*(x),\psi(x)],$$

where the latter equation is nothing but (3·15). We have assumed that Noether's theorem holds for the vector current. Then it should be conserved:

$$\partial_\mu J_\mu(x)=0. \tag{4·5}$$

However, this conservation equation can be proved, without assuming Noether's theorem, by the direct use of the Heisenberg equation of motion (3·11) and the anti-commutation relations (3·6) and (3·7). In the result we may say that Noether's theorem is proved to be applicable even in the quantized theory as far as the transformation generated by T is concerned.

On quantizing the field ψ, we have required that the transformation law (1·2) holds exactly but the law (1·3) does only when the ordering of ψ and ψ^* is ignored. Thus we do not know the exact form of the law (1·3) in the quantized theory. Nevertheless we shall assume, to see what happens, that even in the quantized theory the Lagrangian density (3·1) is invariant under the transformation generated by X, for the Lagrangian density is invariant under (1·3) in the unquantized theory. Under this assumption we get the consistency condition for X:

$$X=\frac{1}{4}\int d^3x([[X,\psi_\alpha^*],\psi_\alpha]-[\psi_\alpha^*,[X,\psi_\alpha]])$$
$$-\frac{i}{8}\lambda\int d^3x\{[[X,\bar{\psi}_\alpha],\psi_\alpha]+[\bar{\psi}_\alpha,[X,\psi_\alpha]],[\psi^*,\psi]\}. \tag{4·6}$$

Since T is the number operator, the commutation relation

$$[X,[\psi^*,\psi]]=0 \tag{4·7}$$

should hold. Using (4·7) and (A·7) we can rewrite (4·6) as

$$X=-\frac{1}{2}\int d^3x([\psi_\alpha^*,[X,\psi_\alpha]]+\frac{i}{2}\lambda[X,[\bar{\psi},\psi]]\cdot[\psi^*,\psi]). \qquad (4\cdot8)$$

Substituting $X^{(0)}$ defined by (3·17) in place of X on the right-hand side of (4·7) and using (3·6), (3·7) and (3·18), we find

$$X=\frac{1}{2}\int d^3x([\psi^*\gamma_5,\psi]-\lambda^2\delta(\mathbf{0})^2[\psi^*\gamma_5,\psi]+\frac{\lambda^2}{2}\delta(\mathbf{0})\{\psi_\alpha^*,\{\psi_\alpha,[\psi^*\gamma_5,\psi]\}\}), \qquad (4\cdot9)^{*)}$$

which differs from $X^{(0)}$ and coincides with it only in the limit $\lambda\delta(\mathbf{0})\to0$.

We shall further examine the divergence of the axial-vector current density

$$J_\mu^{5(0)}(x)=\frac{i}{2}[\bar{\psi}(x)\gamma_\mu\gamma_5,\psi(x)] \qquad (4\cdot10)$$

corresponding to the operator $X^{(0)}$. Taking account of the Heisenberg equation of motion (3·11) and the formulas (A·8) and (A·9), we get

$$\partial_\mu J_\mu^{5(0)}=-\frac{i}{2}\lambda^2(\boldsymbol{\nabla}\delta(\boldsymbol{x}))_{x=0}\{(\bar{\psi}\boldsymbol{\gamma})_\alpha,\{\psi_\alpha,[\psi^*\gamma_5,\psi]\}\} \qquad (4\cdot11)$$

from which we have

$$[H,X^{(0)}]=-\frac{\lambda^2}{2}\iint d^3xd^3y\delta(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\nabla}_y\delta(\boldsymbol{y}-\boldsymbol{x})$$
$$\times\{(\bar{\psi}(x)\boldsymbol{\gamma})_\alpha,\{\psi_\alpha(x),[\psi_\alpha^*(x)\gamma_5,\psi(x)]\}\}. \qquad (4\cdot12)$$

The axial-vector current density $J_\mu^{5(0)}$ does not satisfy the continuity equation in the quantized theory, though it does in the unquantized theory.

We shall attempt to solve the consistency equation (4·8). As was shown in (3·18), (4·9), (4·11) and (4·12), in the limit $\lambda\delta(\mathbf{0})\to0$ the operator $X^{(0)}$ is a solution of (4·8) and has all the properties as the generator of chiral transformation. The exact solution of (4·8), if it exists, should reduce to $X^{(0)}$ when the terms with $\lambda\delta(\mathbf{0})$, which come from quantum effects, are ignored. We shall adopt an iteration method to solve (4·8).

As the first step, we shall substitute $\alpha_0X^{(0)}$ in place of X on the right-hand side of (4·8) where α_0 is a c-number and reduces to unity in the limit $\lambda\delta(\mathbf{0})\to0$. Using (4·9) and (A·10), we obtain

$$X=\alpha_0\frac{1+3\varepsilon^2}{1+\varepsilon^2}X^{(0)}+\alpha_0\frac{\varepsilon^2}{1+\varepsilon^2}X^{(1)}$$
$$-\frac{1}{2}\int d^3x([\psi_\alpha^*,[X-\alpha_0X^{(0)},\psi_\alpha]]+\frac{i}{2}\lambda[X-\alpha_0X^{(0)},[\bar{\psi},\psi]]\cdot[\psi^*,\psi]),$$
$$(4\cdot13)$$

*) For the time being, we shall deal with $\delta(\mathbf{0})$ as it is a finite quantity.

where

$$\varepsilon = \lambda\delta(\mathbf{0}), \tag{4·14}$$

$$X^{(1)} = -\frac{i}{2}\lambda\int d^3x[\bar{\psi}\gamma_5, \psi]\cdot[\psi^*, \psi]. \tag{4·15}$$

Since $X^{(1)}$ appears on the right-hand side of (4·13), the generator X has a term $\alpha_1 X^{(1)}$ in addition to $\alpha_0 X^{(0)}$, α_1 being such a c-number that vanishes in the limit $\varepsilon\to 0$. When the term $\alpha_1 X^{(1)}$ is substituted into $(X-\alpha_0 X^{(0)})$ on the right-hand side of (4·13), there appear the terms $X^{(0)}$, $X^{(1)}$ and the new term:

$$X^{(2)} = \frac{\lambda^2}{2}\int d^3x[\psi^*\gamma_5, \psi][\psi^*, \psi]^2. \tag{4·16}$$

When the same procedures are repeated twice, there appear two more new terms

$$X^{(3)} = -\frac{i}{2}\lambda^3\int d^3x[\bar{\psi}\gamma_5, \psi][\psi^*, \psi]^3, \tag{4·17}$$

$$X^{(4)} = \frac{\lambda^4}{2}\int d^3x[\psi^*\gamma_5, \psi][\psi^*, \psi]^4. \tag{4·18}$$

It is not necessary to repeat the same procedures further, for the last term can be expressed as

$$X^{(4)} = 4\varepsilon^2 X^{(2)} \tag{4·19}$$

by using the anti-commutation relations (3·6) and (3·7). Thus we have shown that the generator X has the form

$$X = \sum_{n=0}^{3}\alpha_n X^{(n)}. \tag{4·20}$$

Here α_n is a function of ε with the following property:

$$\lim_{\varepsilon\to 0}\alpha_n = \begin{cases} 1 & \text{for } n=0, \\ 0 & \text{for } n=1, 2, 3. \end{cases} \tag{4·21}$$

Substituting (4·20) into both sides of (4·8), we finally have the consistency equation

$$\begin{aligned} &2\varepsilon^2(\alpha_0 - 2\alpha_1 - 2(1-\varepsilon^2)\alpha_2 - 8\varepsilon^2\alpha_3)X^{(0)} \\ &\quad + (\varepsilon^2\alpha_0 + (1+\varepsilon^2)\alpha_1 + 4\varepsilon^2\alpha_2 - 8\varepsilon^2(1-\varepsilon^2)\alpha_3)X^{(1)} \\ &\quad + (-\varepsilon^2\alpha_1 + 2\alpha_2 - 4\varepsilon^4\alpha_3)X^{(2)} + (\varepsilon^2\alpha_2 + (3-\varepsilon^2)\alpha_3)X^{(3)} = 0. \end{aligned} \tag{4·22}$$

This equation has non-trivial solutions only when

$$\varepsilon = 0$$

or (4·23)

$$\varepsilon = \pm\sqrt{3}.$$

The first case is nonsense because it leads to vanishing of the interaction ($\lambda=0$). The second case gives

$$\alpha_0=\alpha_2=0\,,\qquad \alpha_1+12\alpha_3=0\,. \tag{4·24}$$

The generator

$$X=\alpha_1\left\{X^{(1)}-\frac{X^{(3)}}{12}\right\} \tag{4·25}$$

does not satisfy the transformation law (1·3), so that the second case is inadequate. We conclude that the consistency equation (4·8) has no satisfactory solution.

The above conclusion was obtained under the assumption that Noether's theorem holds in the quantized theory. Therefore we can say that in the quantized theory Noether's theorem is not valid for such a non-linear transformation as (1·3) and/or the chiral symmetry breaks down by quantum effects.

Now we shall study the transformation property of the Lagrangian density L_1 under the charge conjugation. If a Lagrangian density has no mechanical mass term, then two kinds of charge conjugation are allowed in principle. One of them is the usual one and the other is given by

$$\begin{aligned}\boldsymbol{C}\psi\boldsymbol{C}^{-1}&=B\bar{\psi}^T,\\ \boldsymbol{C}\bar{\psi}\boldsymbol{C}^{-1}&=\psi^T B^*,\end{aligned} \tag{4·26}$$

where B is expressed as

$$B=\gamma_1\gamma_3 \tag{4·27}$$

in the usual Pauli-Dirac representation for the Dirac γ-matrices. The vector $i[\bar{\psi}\gamma_\mu, \psi]$, the scalar $[\bar{\psi}, \psi]$ and the pseudoscalar $i[\bar{\psi}\gamma_5, \psi]$ change their signs but the axial-vector $i[\bar{\psi}\gamma_\mu\gamma_5, \psi]$ does not under the unusual charge conjugation (4·26). The Lagrangian density L_1 is invariant under this unusual charge conjugation, but it is not under the usual one. Under the charge conjugation (4·26) with (4·27), we get

$$\boldsymbol{C}^2\psi\boldsymbol{C}^{-2}=-\psi\,. \tag{4·28}$$

We do not bother about this transformation property (4·28), because L_1 involves only bilinear and quartic forms in ψ and ψ^*.

In the unquantized theory, the Lagrangian density L_1 is invariant under the operation*)

$$\psi\to i\gamma_5 e^{-2i\lambda(\bar{\psi}\psi)}B\bar{\psi}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\pi}{2}\right)^n(X^{(0)},\,(X^{(0)}\cdots,\,(X^{(0)},B\bar{\psi})^*\cdots)^*)^*.$$

This operation is obtained by expressing the charge conjugation for the field q_1 (defined by (1·6)) in terms of ψ and ψ^*. When this operation is applied twice,

*) See (A·3) and (A·9) in the second paper of Ref. 1).

the field ψ returns back to itself. In the quantized theory we might have the corresponding operator

$$\psi \to (e^{i\pi X/2}C)\psi(e^{i\pi X/2}C)^{-1}.$$

As was discussed above, however, we do not know the generator X that commutes with the Hamiltonian (2·14). Therefore we use only the unusual charge conjugation.

§ 5. Quantization of model II

In this section we shall try to quantize the Lagrangian density of model II. The transformation law (1·5) includes the factor $(1+\lambda[\bar{\psi},\psi])$ in its denominator. This fact may suggest that the equal-time anti-commutation relations for ψ and ψ^* have the same denominator. If this is the case, it may be a difficult problem to find the correct ordering of ψ and ψ^* involved in the relations.

The symmetrized form of the Lagrangian density of model II is given by

$$L_2=\frac{1}{4}[\partial_\mu\bar{\psi},\gamma_\mu\psi]-\frac{1}{4}[\bar{\psi}\gamma_\mu,\partial_\mu\psi]-\frac{\lambda}{8}\{[\bar{\psi}\gamma_\mu\gamma_5,\psi],\partial_\alpha([\bar{\psi}\gamma_5,\psi])\} \tag{5·1}$$

which yields the equation of motion

$$\gamma_\mu\partial_\mu\psi-\frac{\lambda}{4}\{\gamma_\mu\gamma_5\psi,\partial_\mu[\bar{\psi}\gamma_5,\psi]\}=0. \tag{5·2}$$

The constraints equations (2·6) read

$$\begin{aligned}\theta_\alpha&\equiv\pi_\alpha-\frac{i}{2}\psi_\alpha{}^*-\frac{i}{4}\lambda\{[\psi^*\gamma_5,\psi],(\bar{\psi}\gamma_5)_\alpha\}=0,\\ \theta_\alpha{}^\dagger&\equiv\pi_\alpha{}^\dagger-\frac{i}{2}\psi_\alpha+\frac{i}{4}\lambda\{[\psi^*\gamma_5,\psi],(\beta\gamma_5\psi)_\alpha\}=0.\end{aligned} \tag{5·3}$$

In the present model it is more troublesome to find the modified P.b. than in the case of model I. Postponing the presentation of the detailed calculations to Appendix B, we here give the result only:

$$\begin{aligned}(\psi_\alpha(x),\psi_\beta{}^*(y))^*&=-i\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y})-\frac{i\lambda}{D(x)}\rho_{\alpha\beta}[\phi_\alpha(x),\phi_\beta{}^*(y)]\delta(\boldsymbol{x}-\boldsymbol{y}),\\ (\psi_\alpha(x),\psi_\beta(y))^*&=\frac{i\lambda}{D(x)}\Gamma_{\alpha\beta}[\phi_\alpha(x),\phi_\beta(y)]\delta(\boldsymbol{x}-\boldsymbol{y}),\end{aligned} \tag{5·4}$$

where

$$D(x)=1+\lambda[\bar{\psi}(x),\psi(x)], \tag{5·5}$$

$$\rho_{\alpha\beta}=\tfrac{1}{2}(\beta_{\alpha\alpha}+\beta_{\beta\beta}),\qquad \Gamma_{\alpha\beta}=\tfrac{1}{2}(\beta_{\alpha\alpha}-\beta_{\beta\beta}), \tag{5·6}$$

$$\phi_\alpha(x)=(\gamma_5\psi(x))_\alpha. \tag{5·7}$$

We can show that the modified P.b.'s (5·4) satisfy the Jacobi identities

$$(\psi_\alpha^*(x),\,(\psi_\beta(y),\,\psi_\gamma(z))^*)^* + (\psi_\beta(y),\,(\psi_\gamma(z),\,\psi_\alpha^*(x))^*)^* + (\psi_\gamma(z),\,(\psi_\alpha^*(x),\,\psi_\beta(y))^*)^* = 0, \quad \text{etc.} \tag{5·8}$$

Let us examine whether or not the Euler equation (5·2) and the transformation laws (1·2) and (1·5) are obtained from the modified P.b.'s (5·4). For this purpose, we shall use the following relation:

$$\begin{aligned}\frac{i}{2}(\psi_\alpha(x),\,[\psi_\beta^*(y),\,\psi_\gamma(z)])^* &= \delta_{\alpha\beta}\psi_\gamma(z)\delta(\boldsymbol{x}-\boldsymbol{y}) + \frac{\lambda}{D(x)}((\psi^*(y)\gamma_5)_\beta\psi_\gamma(z)(\beta\gamma_5\psi(x))_\alpha\delta(\boldsymbol{x}-\boldsymbol{y})\\ &\quad -\psi_\beta^*(y)(\gamma_5\psi(z))_\gamma(\beta\gamma_5\psi(x))_\alpha\delta(\boldsymbol{x}-\boldsymbol{z})\\ &\quad -(\bar{\psi}(y)\gamma_5)_\beta\psi_\gamma(z)(\gamma_5\psi(x))_\alpha\delta(\boldsymbol{x}-\boldsymbol{y})\\ &\quad +\psi_\beta^*(y)(\beta\gamma_5\psi(z))_\gamma(\gamma_5\psi(x))_\alpha\delta(\boldsymbol{x}-\boldsymbol{z})).\end{aligned} \tag{5·9}$$

From (2·14), (2·15) and (5·9), we get Hamilton's equation of motion

$$\begin{aligned}i\partial_0\psi_\alpha &= (\beta\boldsymbol{\gamma}\boldsymbol{\nabla}\psi)_\alpha + \lambda(\beta\boldsymbol{\gamma}\gamma_5\psi)_\alpha\boldsymbol{\nabla}(\bar{\psi}\gamma_5\psi)\\ &\quad + \frac{\lambda}{D}(\gamma_5\psi)_\alpha((\boldsymbol{\nabla}\psi^*\boldsymbol{\gamma}\gamma_5\psi) - (\psi^*\boldsymbol{\gamma}\gamma_5\boldsymbol{\nabla}\psi)).\end{aligned} \tag{5·10}$$

Equation (5·10) and the corresponding one for $\partial_0\psi^*$ yield

$$\partial_0(\bar{\psi}\gamma_5\psi) = \frac{i}{D}((\boldsymbol{\nabla}\psi^*\boldsymbol{\gamma}\gamma_5\psi) - (\psi^*\boldsymbol{\gamma}\gamma_5\boldsymbol{\nabla}\psi)). \tag{5·11}$$

Substituting this equation into (5·10), we see that Hamilton's equation coincides with the Euler equation (5·2). With the aid of (5·9) we find

$$i(T,\psi_\alpha)^* = -\psi_\alpha, \tag{5·12}$$

$$i(X,\psi_\alpha)^* = -\frac{1}{D}(\gamma_5\psi), \tag{5·13}$$

where

$$T = \frac{1}{2}\int d^3x[\psi^*(x),\psi(x)], \tag{5·14}$$

$$X = \frac{1}{2}\int d^3x[\psi^*(x)\gamma_5,\,\psi(x)]. \tag{5·15}$$

Thus our P.b.'s (5·4) give the Euler equation (5·2) and reproduce the transformation laws (1·2) and (1·5).

The quantization may be performed by replacing the modified P.b.'s (5·4) with equal-time anti-commutation relations according to (2·16):

$$\{\psi_\alpha(x), \psi_\beta{}^*(y)\} = \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) + \frac{\lambda}{2}\rho_{\alpha\beta}\left\{[\phi_\alpha(x), \phi_\beta{}^*(x)], \frac{1}{D(x)}\right\}\delta(\boldsymbol{x}-\boldsymbol{y}),$$
$$\{\psi_\alpha(x), \psi_\beta(y)\} = -\frac{\lambda}{2}\Gamma_{\alpha\beta}\left\{[\phi_\alpha(x),\ \phi_\beta(y)], \frac{1}{D(x)}\right\}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{5·16}$$

Here the right-hand sides are so symmetrized as to be invariant under the usual charge conjugation.

Since T is the number operator, the transformation law

$$[T, \psi_\alpha] = -\psi_\alpha$$

should hold exactly in quantum theory. However, the anti-commutation relations lead to

$$[T, \psi_\alpha(x)] + \psi_\alpha(x) = \lambda^2\delta(\mathbf{0})^2\psi_\alpha(x) + O(\lambda^3). \tag{5·17}$$

Therefore our anti-commutation relations (5·16) are not rigorous. This insufficiency would be due to the fact that we have not yet known the correct position of the factor $1/D$ on the right-hand sides of (5·16). In the following, we shall show a method how to find the right position of the factor $1/D$.

We shall modify (5·16) as

$$\{\psi_\alpha(x), \psi_\beta{}^*(y)\}$$
$$= \delta_{\alpha\beta\delta}(\boldsymbol{x}-\boldsymbol{y}) + \lambda\rho_{\alpha\beta}\left(\frac{a}{2}\left\{[\phi_\alpha, \phi_\beta{}^*], \frac{1}{D}\right\} + b\left(\phi_\alpha\frac{1}{D}\phi_\beta{}^* - \phi_\beta{}^*\frac{1}{D}\phi_\alpha\right)\right)\delta(\boldsymbol{x}-\boldsymbol{y}),$$
$$\{\psi_\alpha(x), \psi_\beta(y)\}$$
$$= -\lambda\Gamma_{\alpha\beta}\left(\frac{a}{2}\left\{[\phi_\alpha, \phi_\beta], \frac{1}{D}\right\} + b\left(\phi_\alpha\frac{1}{D}\phi_\beta - \phi_\beta\frac{1}{D}\phi_\alpha\right)\right)\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{5·18}$$

where a and b are real numbers satisfying the condition

$$a + b = 1. \tag{5·19}$$

The relations (5·18) are invariant under the charge conjugation. The condition (5·19) is obtained from the requirement that the relations (5·16) and (5·18) coincide with each other except for the position of the factor $1/D$.

We shall now determine the values of the coefficients a and b so as to satisfy $[T, \psi] + \psi = O(\lambda^3)$. For this purpose, it is expedient to rewrite (5·18) by using the condition (5·19) as

$$\{\psi_\alpha(x), \psi_\beta{}^*(y)\} = \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) + \frac{\lambda}{2}\rho_{\alpha\beta}\left\{[\phi_\alpha, \phi_\beta{}^*], \frac{1}{D}\right\}\delta(\boldsymbol{x}-\boldsymbol{y})$$
$$+ \frac{\lambda}{2}b\rho_{\alpha\beta}\left(\left\{\left[\phi_\alpha, \frac{1}{D}\right], \phi_\beta{}^*\right\} - \left\{\phi_\alpha, \left[\phi_\beta{}^*, \frac{1}{D}\right]\right\}\right)\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{5·20}$$

$$\{\psi_\alpha(x), \psi_\beta(y)\} = -\frac{\lambda}{2}\Gamma_{\alpha\beta}\left\{[\phi_\alpha, \phi_\beta], \frac{1}{D}\right\}\delta(\boldsymbol{x}-\boldsymbol{y})$$

$$-\frac{\lambda}{2}b\Gamma_{\alpha\beta}\left(\left\{\left[\phi_\alpha, \frac{1}{D}\right], \phi_\beta\right\} - \left\{\phi_\alpha, \left[\phi_\beta, \frac{1}{D}\right]\right\}\right)\delta(\boldsymbol{x}-\boldsymbol{y}).$$

The last terms on the right-hand sides are already of the order of λ^2. Noting this fact, we have

$$[T, \psi_\alpha(x)] + \psi_\alpha(x) = (1-2b)\lambda^2\delta(\mathbf{0})^2 + O(\lambda^3). \tag{5·21}$$

Then

$$a = b = \tfrac{1}{2}, \tag{5·22}$$

and the relations (5·18) reduce to

$$\begin{aligned}\{\psi_\alpha(x), \psi_\beta{}^*(y)\} &= \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) \\ &\quad + \frac{\lambda}{4}\rho_{\alpha\beta}\left(\left[\left\{\phi_\alpha, \frac{1}{D}\right\}, \phi_\beta{}^*\right] + \left[\phi_\alpha, \left\{\phi_\beta{}^*, \frac{1}{D}\right\}\right]\right)\delta(\boldsymbol{x}-\boldsymbol{y}), \\ \{\psi_\alpha(x), \psi_\beta(y)\} &= -\frac{\lambda}{4}\Gamma_{\alpha\beta}\left(\left[\left\{\phi_\alpha, \frac{1}{D}\right\}, \phi_\beta\right] + \left[\phi_\alpha, \left\{\phi_\beta, \frac{1}{D}\right\}\right]\right)\delta(\boldsymbol{x}-\boldsymbol{y}).\end{aligned} \tag{5·23}$$

Since these relations satisfy the transformation law (1·2) up to the order of λ^2, it is expected that the identity (3·9) holds at least to the same order. Actually we get

$$\begin{aligned}&[\psi_\alpha(x), \{\psi_\beta{}^*(y), \psi_\gamma(z)\}] + [\psi_\beta{}^*(y), \{\psi_\gamma(z), \psi_\alpha(x)\}] + [\psi_\gamma(z), \{\psi_\alpha(x), \psi_\beta{}^*(y)\}] \\ &= -\frac{\lambda^2}{4}([\{(\beta\phi)_\alpha, (\phi^*\beta)_\beta\}, \psi_\gamma] + [\{(\phi^*\beta)_\beta, (\beta\phi)_\gamma\}, \phi_\alpha] \\ &\qquad + [\{(\beta\phi)_\gamma, (\beta\phi)_\alpha\}, \psi_\beta{}^*] + [\{\phi_\gamma, \phi_\alpha\}, \psi_\beta{}^*] \\ &\qquad - [\{\phi_\beta{}^*, \phi_\gamma\}, \psi_\alpha] - [\{\phi_\alpha, \phi_\beta{}^*\}, \psi_\gamma]) + O(\lambda^3) \\ &= O(\lambda^3).\end{aligned} \tag{5·24}$$

Let us evaluate, by using (5·23), the left-hand side of (5·21) to the higher order in λ. After straightforward but tedius calculations (cf. Appendix C), we find

$$\begin{aligned}&[T, \psi_\alpha] + \psi_\alpha \\ &= \frac{\lambda^4}{4}\delta(\mathbf{0})^2(\{[\bar{\psi}\gamma_5, \psi]\cdot[\psi^*, \psi], (\beta\gamma_5\psi)_\alpha\} - \{[\psi^*\gamma_5, \psi]\cdot[\psi^*, \psi], (\gamma_5\psi)_\alpha\}) + O(\lambda^5).\end{aligned} \tag{5·25}$$

Accordingly, we must further generalize the anti-commutation relations (5·23) so that the term of the order λ^4 in (5·25) disappears. We shall follow the same procedure as was done to remove the term of the order of λ^2 in (5·17). We set

$$\{\psi_\alpha(x), \psi_\beta{}^*(y)\}$$
$$=\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y})+\lambda\rho_{\alpha\beta}\Big(\frac{c}{4}\Big[\Big\{\phi_\alpha, \frac{1}{D}\Big\}, \phi_\beta{}^*\Big]+\frac{c}{4}\Big[\phi_\alpha, \Big\{\phi_\beta{}^*, \frac{1}{D}\Big\}\Big]$$
$$+\frac{d}{8}\Big[\Big\{\phi_\alpha, \frac{1}{D^2}\Big\}, \{\phi_\beta{}^*, D\}\Big]+\frac{d}{8}\Big[\{\phi_\alpha, D\}, \Big\{\phi_\alpha{}^*, \frac{1}{D^2}\Big\}\Big]\Big)\delta(\boldsymbol{x}-\boldsymbol{y}),$$
$$\{\psi_\alpha(x), \psi_\beta(y)\} \tag{5·26}$$
$$=-\lambda\Gamma_{\alpha\beta}\Big(\frac{c}{4}\Big[\Big\{\phi_\alpha, \frac{1}{D}\Big\}, \phi_\beta\Big]+\frac{c}{4}\Big[\phi_\alpha, \Big\{\phi_\beta, \frac{1}{D}\Big\}\Big]$$
$$+\frac{d}{8}\Big[\Big\{\phi_\alpha, \frac{1}{D^2}\Big\}, \{\phi_\beta, D\}\Big]+\frac{d}{8}\Big[\{\phi_\alpha, D\}, \Big\{\phi_\beta, \frac{1}{D^2}\Big\}\Big]\Big)\delta(\boldsymbol{x}-\boldsymbol{y})$$

with

$$c+d=1\,. \tag{5·27}$$

Taking account of (5·27), we can rewrite (5·26) as

$$\{\psi_\alpha(x), \psi_\beta{}^*(y)\}$$
$$=\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y})+\frac{\lambda}{4}\Big(\Big[\Big\{\phi_\alpha, \frac{1}{D}\Big\}, \phi_\beta{}^*\Big]+\Big[\phi_\alpha, \Big\{\phi_\beta{}^*, \frac{1}{D}\Big\}\Big]$$
$$+\frac{d}{2}\Big[[\phi_\alpha, D], \Big[\phi_\beta{}^*, \frac{1}{D^2}\Big]\Big]+\frac{d}{2}\Big[\Big[\phi_\alpha, \frac{1}{D^2}\Big], [\phi_\beta{}^*, D]\Big]$$
$$-d\Big[\frac{1}{D^2}, [D, [\phi_\alpha, \phi_\beta{}^*]]\Big]\Big)\delta(\boldsymbol{x}-\boldsymbol{y}),$$
$$\{\psi_\alpha(x), \psi_\beta(y)\} \tag{5·28}$$
$$=-\frac{\lambda}{4}\Gamma_{\alpha\beta}\Big(\Big[\Big\{\phi_\alpha, \frac{1}{D}\Big\}, \phi_\beta\Big]+\Big[\phi_\alpha, \Big\{\phi_\beta, \frac{1}{D}\Big\}\Big]+\frac{d}{2}\Big[[\phi_\alpha, D], \Big[\phi_\beta, \frac{1}{D^2}\Big]\Big]$$
$$+\frac{d}{2}\Big[\Big[\phi_\alpha, \frac{1}{D^2}\Big], [\phi_\beta, D]\Big]-d\Big[\frac{1}{D^2}, [D, [\phi_\alpha, \phi_\beta]]\Big]\Big)\delta(\boldsymbol{x}-\boldsymbol{y}).$$

It should be noted that the terms with the coefficient d are of the order of λ^3. When d vanishes, these relations coincide with (5·23). From (5·28), we get (cf. Appendix C)

$$[T, \psi_\alpha]+\psi_\alpha=\frac{1-4d}{4}\lambda^4\delta(\mathbf{0})^2(\{[\bar{\psi}\gamma_5, \psi]\cdot[\psi^*, \psi], (\beta\gamma_5\psi)_\alpha\}$$
$$-\{[\psi^*\gamma_5, \psi]\cdot[\psi^*, \psi], (\gamma_5\psi)_\alpha\})+O(\lambda^5). \tag{5·29}$$

Taking

$$d=\tfrac{1}{4}\,, \tag{5·30}$$

we see

$$[T, \psi_\alpha] + \psi_\alpha = O(\lambda^5). \tag{5·31}$$

We are thus able to eliminate the term of the order of λ^4 in $[T, \psi]$.

Similarly, if terms such as

$$\left[\left\{\phi_\alpha, \frac{1}{D^3}\right\}, \{\phi_\beta{}^*, D^2\}\right] + \left[\{\phi_\alpha, D^2\}, \left\{\phi_\beta{}^*, \frac{1}{D^3}\right\}\right] \tag{5·32}$$

are added to the right-hand sides of (5·26), the term $O(\lambda^5)$ will be removed from (5·31). Then the resultant anti-commutation relations will be different from (5·28) with (5·30) by the terms of the order of λ^4. Continuing the same procedures, we may finally obtain correct anti-commutation relations which satisfy $[T, \psi] = -\psi$ exactly.

§ 6. Concluding remarks

We have shown the method how to quantize the fundamental Dirac field ψ for the special cases of model I and II. The method is applicable to more general cases as far as interaction Lagrangian densities contain the first order space-time derivatives of ψ and ψ^*. Therefore the method can be applied, as it stands, to the quantization of the field ψ, when the Lagrangian density is invariant under non-linear chiral $SU(n) \times SU(n)$ or $U(n) \times U(n)$ transformations discussed in Ref. 1).

The discussions given in § 4 suggest that chiral symmetry breaks down by quantum effects. The same kind of breaking may happen also in model II. If it is the case, the higher order corrections will give rise to the non-vanishing mass of the fundamental particle in model II. In model I, however, the mass will not be generated by the corrections because the mass term $[\bar{\psi}, \psi]$ is not invariant under the unusual charge conjugation mentioned in § 4.

There are many problems left to be studied: the definition of the vacuum, the method to calculate the S matrix, C.P.T theorem and so on.

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Appendix A

——Mathematical formulas for model I——

The following identities are used in the text:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0\,, \tag{A·1}$$

$$[A, \{B, C\}] + [B, \{C, A\}] + [C, \{A, B\}] = 0\,, \tag{A·2}$$

$$\{A,\ \{B, C\}\} - \{B,\ \{C, A\}\} + [C,\ [A, B]] = 0\,, \tag{A·3}$$

$$\{A,\ [B, C]\} - [B,\ \{C, A\}] - \{C,\ [A, B]\} = 0\,. \tag{A·4}$$

From the commutation relations (3·6), (3·7) and (3·14), we get

$$\{\psi^*(x)\gamma_5, \psi(y)\} = i\lambda[\bar{\psi}\gamma_5, \psi]\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{A·5}$$

$$[\psi_\alpha(x),\ [\psi^*(y), \psi(y)]] = 2\psi_\alpha(x)\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{A·6}$$

$$[[\psi^*(x)B, \psi(x)],\ [\psi^*(y), \psi(y)]] = 0\,, \tag{A·7}$$

where B is an arbitrary 4×4 matrix. For any matrix A satisfying $\{\beta, A\}=0$, we have (cf. (3·16))

$$[\psi_\alpha(x),\ [\psi^*(y)A, \psi(y)]] = 2(A\psi)_\alpha\delta(\boldsymbol{x}-\boldsymbol{y}) - i\lambda\{\psi_\alpha,\ [\bar{\psi}A, \psi]\}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{A·8}$$

$$\begin{aligned}
&[[\psi^*(x)B, \psi(x)],\ [\psi^*(y)A, \psi(y)]]\\
&\qquad = 2[\psi^*[B, A], \psi]\delta(\boldsymbol{x}-\boldsymbol{y})\\
&\qquad\qquad + \frac{\lambda^2}{2}\delta(\mathbf{0})\ \{[\psi^*[\beta, B], \psi],\ [\bar{\psi}A, \psi]\}\delta(\boldsymbol{x}-\boldsymbol{y}).
\end{aligned} \tag{A·9}$$

Using the above formulas, we get the relations

$$\begin{aligned}
&\{\psi_\alpha(x),\ \{\psi_\alpha(y),\ [\psi^*(y)\gamma_5, \psi(y)]\}\}\\
&\quad = \{\psi_\alpha(x),\ \{\psi_\alpha^*(y),\ [\psi^*(y)\gamma_5, \psi(y)]\}\}\\
&\quad = \frac{2(3+\lambda^2\delta(\mathbf{0})^2)}{1+\lambda^2\delta(\mathbf{0})^2}[\psi^*\gamma_5, \psi]\delta(\boldsymbol{x}-\boldsymbol{y}) + \frac{2i\lambda}{1+\lambda^2\delta(\mathbf{0})^2}[\bar{\psi}\gamma_5, \psi]\cdot[\psi^*, \psi]\delta(\boldsymbol{x}-\boldsymbol{y})
\end{aligned} \tag{A·10}$$

etc.

Appendix B

——modified P.b. for model II——

In the model II, the matrix $(\Theta_m(x), \Theta_n(y))$ is given by

$$(\Theta_m(x), \Theta_n(y)) = i\delta(\boldsymbol{x}-\boldsymbol{y}) \times$$

$$\begin{pmatrix}
0 & 0 & \lambda[\phi_1^*, \phi_3^*] & \lambda[\phi_1^*, \phi_4^*] & -1-\lambda[\phi_1^*, \phi_1] & -\lambda[\phi_1^*, \phi_2] & 0 & 0 \\
0 & 0 & \lambda[\phi_2^*, \phi_3^*] & \lambda[\phi_2^*, \phi_4^*] & -\lambda[\phi_2^*, \phi_1] & -1-\lambda[\phi_2^*, \phi_2] & 0 & 0 \\
-\lambda[\phi_3^*, \phi_1^*] & -\lambda[\phi_3^*, \phi_2^*] & 0 & 0 & 0 & 0 & -1+\lambda[\phi_3^*, \phi_3] & \lambda[\phi_3^*, \phi_4] \\
-\lambda[\phi_4^*, \phi_1^*] & -\lambda[\phi_4^*, \phi_2^*] & 0 & 0 & 0 & 0 & \lambda[\phi_4^*, \phi_3] & -1+\lambda[\phi_4^*, \phi_4] \\
-1+\lambda[\phi_1, \phi_1^*] & \lambda[\phi_1, \phi_2^*] & 0 & 0 & 0 & 0 & -\lambda[\phi_1, \phi_3] & -\lambda[\phi_1, \phi_4] \\
\lambda[\phi_2, \phi_1^*] & -1+\lambda[\phi_2, \phi_2^*] & 0 & 0 & 0 & 0 & -\lambda[\phi_2, \phi_3] & -\lambda[\phi_2, \phi_4] \\
0 & 0 & -1-\lambda[\phi_3, \phi_3^*] & -\lambda[\phi_3, \phi_4^*] & \lambda[\phi_3, \phi_1] & \lambda[\phi_3, \phi_2] & 0 & 0 \\
0 & 0 & -\lambda[\phi_4, \phi_3^*] & -1-\lambda[\phi_4, \phi_4^*] & \lambda[\phi_4, \phi_1] & \lambda[\phi_4, \phi_2] & 0 & 0
\end{pmatrix}, \tag{B·1}$$

where

$$\phi = \gamma_5 \psi \quad \text{and} \quad \phi^* = \psi^* \gamma_5 .$$

To evaluate the matrix C_{mn} defined by (2·13), which is the inverse to (Θ_m, Θ_n), it is necessary to calculate the determinant Δ of (Θ_m, Θ_n).

In the calculations the anti-commutability of ψ and ψ^*, i.e.

$$\{\psi_\alpha, \psi_\beta{}^*\} = \{\psi_\alpha, \psi_\beta\} = \{\psi_\alpha{}^*, \psi_\beta{}^*\} = 0\,,$$

plays an important role. For example,

$$\psi_\alpha\psi_\beta\psi_\gamma\psi_\delta\psi_\varepsilon = 0\,.$$

The determinant Δ is calculated as

$$\begin{aligned}\Delta = \{&1+2\lambda(|\phi_1|^2+|\phi_2|^2-|\phi_3|^2-|\phi_4|^2)\\ &+8\lambda^2(|\phi_1|^2|\phi_2|^2-|\phi_1|^2|\phi_3|^2-|\phi_1|^2|\phi_4|^2-|\phi_2|^2|\phi_3|^2-|\phi_2|^2|\phi_4|^2+|\phi_3|^2|\phi_4|^2)\\ &+48\lambda^3(-|\phi_1|^2|\phi_2|^2\phi_3|^2-|\phi_1|^2|\phi_2|^2|\phi_4|^2+|\phi_1|^2|\phi_3|^2|\phi_4|^2+|\phi_2|^2|\phi_3|^2|\phi_4|^2)\\ &+384\lambda^4|\phi_1|^2|\phi_2|^2|\phi_3|^2|\phi_4|^2\}^2\\ =\{&1+2\lambda(|\phi_1|^2+|\phi_2|^2-|\phi_3|^2-|\phi_4|^2)+4\lambda^2(|\phi_1|^2+|\phi_2|^2-|\phi_3|^2-|\phi_4|^2)^2\\ &+8\lambda^3(|\phi_1|^2+|\phi_2|^2-|\phi_3|^2-|\phi_4|^2)^3+16\lambda^4(|\phi_1|^2+|\phi_2|^2-|\phi_3|^2-|\phi_4|^2)^4\}^2\\ =\{&1-2\lambda(\bar{\psi}\psi)+4\lambda^2(\bar{\psi}\psi)^2-8\lambda^3(\bar{\psi}\psi)^3+16\lambda^4(\bar{\psi}\psi)^4\}^2\\ =\{&1+2\lambda(\bar{\psi}\psi)\}^{-2},\end{aligned} \tag{B·2}$$

where $|\phi_i|^2$ denotes $\phi_i{}^*\phi_i$. In a similar way the component C_{15}, for instance, is calculated as

$$\begin{aligned}iC_{15} = -\{&1+2\lambda(|\phi_2|^2-|\phi_3|^2-|\phi_4|^2)\\ &+8\lambda^2(-|\phi_2|^2|\phi_3|^2-|\phi_2|^2|\phi_4|^2+|\phi_3|^2|\phi_4|^2)\\ &+48\lambda^3|\phi_2|^2|\phi_3|^2|\phi_4|^2\}(1+2\lambda\bar{\psi}\psi)\\ =-\{&1-2\lambda(\bar{\psi}\psi)+4\lambda^2(\bar{\psi}\psi)^2-8\lambda^3(\bar{\psi}\psi)^3+16\lambda^4(\bar{\psi}\psi)^4\}(1+2\lambda\bar{\psi}\psi)\\ &+2\lambda|\phi_1|^2\{1-4\lambda(\bar{\psi}\psi)+12\lambda^2(\bar{\psi}\psi)^2-32\lambda^3(\bar{\psi}\psi)^3\}(1+2\lambda\bar{\psi}\psi)\\ =-1&-\frac{\lambda[\phi_1, \phi_1{}^*]}{D},\end{aligned} \tag{B·3}$$

where

$$D = 1+\lambda[\bar{\psi}, \psi].$$

Making use of similar calculations, we get

$$C_{mn}(x,y)=-i\delta(\boldsymbol{x}-\boldsymbol{y})\frac{1}{D(x)}\times$$

$$\begin{bmatrix} 0 & 0 & \lambda[\phi_1,\phi_3] & \lambda[\phi_1,\phi_4] & -D-\lambda[\phi_1,\phi_1^*] & -\lambda[\phi_1,\phi_2^*] & 0 & 0 \\ 0 & 0 & \lambda[\phi_2,\phi_3] & \lambda[\phi_2,\phi_4] & -\lambda[\phi_2,\phi_1^*] & -D-\lambda[\phi_2,\phi_2^*] & 0 & 0 \\ -\lambda[\phi_3,\phi_1] & -\lambda[\phi_3,\phi_2] & 0 & 0 & 0 & 0 & -D+\lambda[\phi_3,\phi_3^*] & \lambda[\phi_3,\phi_4^*] \\ -\lambda[\phi_4,\phi_1] & -\lambda[\phi_4,\phi_2] & 0 & 0 & 0 & 0 & \lambda[\phi_4,\phi_3^*] & -D+\lambda[\phi_4,\phi_4^*] \\ -D+\lambda[\phi_1^*,\phi_1] & \lambda[\phi_1^*,\phi_2] & 0 & 0 & 0 & 0 & -\lambda[\phi_1^*,\phi_3^*] & -\lambda[\phi_1^*,\phi_4^*] \\ \lambda[\phi_2^*,\phi_1] & -D+\lambda[\phi_2^*,\phi_2] & 0 & 0 & 0 & 0 & -\lambda[\phi_2^*,\phi_3^*] & -\lambda[\phi_2^*,\phi_4^*] \\ 0 & 0 & -D-\lambda[\phi_3^*,\phi_3] & -\lambda[\phi_3^*,\phi_4] & \lambda[\phi_3^*,\phi_1^*] & \lambda[\phi_3^*,\phi_2^*] & 0 & 0 \\ 0 & 0 & -\lambda[\phi_4^*,\phi_3] & -D-\lambda[\phi_4^*,\phi_4] & \lambda[\phi_4^*,\phi_1^*] & \lambda[\phi_4^*,\phi_2^*] & 0 & 0 \end{bmatrix}. \tag{B·4}$$

Appendix C

From (5·23) we get

$$\{\psi_\alpha(x),\psi_\beta^*(y)\}=\left(\delta_{\alpha\beta}+\frac{\lambda}{2}[(\beta\gamma_5\psi)_\alpha,\ \bar{\psi}_\beta]-\frac{\lambda}{2}[(\gamma_5\psi)_\alpha,\ (\bar{\psi}\gamma_5)_\beta]\right)\delta(\boldsymbol{x}-\boldsymbol{y})+O(\lambda^2),\tag{C·1}$$

$$\{\psi_\alpha(x),\psi_\beta(y)\}=-\frac{\lambda}{2}([(\beta\gamma_5\psi)_\alpha,\ (\gamma_5\psi)_\beta]+[(\gamma_5\psi)_\alpha,\ (\beta\gamma_5\psi)_\beta])\delta(\boldsymbol{x}-\boldsymbol{y})+O(\lambda^2),$$

$$\begin{aligned}[\psi_\alpha(x),D(x)]&=2\lambda(\beta\psi)_\alpha\delta(\mathbf{0})+2\lambda^2\delta(\mathbf{0})(-(\beta\gamma_5\psi)_\alpha[\bar{\psi}\gamma_5,\psi]+(\gamma_5\psi)_\alpha[\psi^*\gamma_5,\psi]-2\psi_\alpha\delta(\mathbf{0}))+O(\lambda^3)\\&=2\lambda(\beta\psi)_\alpha\delta(\mathbf{0})+2\lambda^2\delta(\mathbf{0})(-[\bar{\psi}\gamma_5,\psi](\beta\gamma_5\psi)_\alpha+[\psi^*\gamma_5,\psi](\gamma_5\psi)_\alpha+2\psi_\alpha\delta(\mathbf{0}))+O(\lambda^3),\end{aligned}\tag{C·2}$$

$$\begin{aligned}[\psi_\alpha^*(x),D(x)]&=-2\lambda\bar{\psi}_\alpha\delta(\mathbf{0})+2\lambda^2\delta(\mathbf{0})((\bar{\psi}\gamma_5)_\alpha[\bar{\psi}\gamma_5,\psi]-(\psi^*\gamma_5)_\alpha[\psi^*\gamma_5,\psi]-2\psi_\alpha^*\delta(\mathbf{0}))+O(\lambda^3),\end{aligned}$$

$$\begin{aligned}\left[\psi_\alpha(x),\frac{1}{D(x)}\right]&=-2\lambda(\beta\psi)_\alpha\delta(\mathbf{0})+2\lambda^2\delta(\mathbf{0})([\bar{\psi}\gamma_5,\psi](\beta\gamma_5\psi)_\alpha-[\psi^*\gamma_5,\psi](\gamma_5\psi)_\alpha\\&\qquad+2[\bar{\psi},\psi](\beta\psi)_\alpha)+O(\lambda^3)\\&=-2\lambda(\beta\psi)_\alpha\delta(\mathbf{0})+2\lambda^2\delta(\mathbf{0})((\beta\gamma_5\psi)_\alpha[\bar{\psi}\gamma_5,\psi]-(\gamma_5\psi)_\alpha[\psi^*\gamma_5,\psi]\\&\qquad+2(\beta\psi)_\alpha[\bar{\psi},\psi])+O(\lambda^3),\end{aligned}\tag{C·3}$$

$$\left[\psi_\alpha^*(x),\frac{1}{D(x)}\right]=2\lambda\bar{\psi}_\alpha\delta(\mathbf{0})-2\lambda^2\delta(\mathbf{0})([\bar{\psi}\gamma_5,\psi](\bar{\psi}\gamma_5)_\alpha-[\psi^*\gamma_5,\psi](\psi^*\gamma_5)_\alpha+2[\bar{\psi},\psi]\bar{\psi}_\alpha)+O(\lambda^3),$$

$$\begin{aligned}\left[\psi_\alpha(x),\frac{1}{D^2(x)}\right]&=-4\lambda(\beta\psi)_\alpha\delta(\mathbf{0})+4\lambda^2\delta(\mathbf{0})((\beta\gamma_5\psi)_\alpha[\bar{\psi}\gamma_5,\psi]-(\gamma_5\psi)_\alpha[\psi^*\gamma_5,\psi]\\&\qquad+3(\beta\psi)_\alpha[\bar{\psi},\psi]-\psi_\alpha\delta(\mathbf{0}))+O(\lambda^3)\\&=-4\lambda(\beta\psi)_\alpha\delta(\mathbf{0})+4\lambda^2\delta(\mathbf{0})([\bar{\psi}\gamma_5,\psi](\beta\gamma_5\psi)_\alpha-[\psi^*\gamma_5,\psi](\gamma_5\psi)_\alpha\\&\qquad+3[\bar{\psi},\psi](\beta\psi)_\alpha+\psi_\alpha\delta(\mathbf{0}))+O(\lambda^3),\end{aligned}\tag{C·4}$$

$$\begin{aligned}\left[\psi_\alpha^*(x),\frac{1}{D^2(x)}\right]&=4\lambda\bar{\psi}_\alpha\delta(\mathbf{0})-4\lambda^2\delta(\mathbf{0})((\bar{\psi}\gamma_5)_\alpha[\bar{\psi}\gamma_5,\psi]-(\psi^*\gamma_5)_\alpha[\psi^*\gamma_5,\psi]+3\bar{\psi}_\alpha[\bar{\psi},\psi]\\&\qquad+\psi_\alpha^*\delta(\mathbf{0}))+O(\lambda^3).\end{aligned}$$

It is easily shown that the following relation holds exactly:

$$\sum_{\beta}\left(\rho_{\alpha\beta}\left\{\psi_\beta, \left[\left\{\psi_\alpha, \frac{1}{D}\right\}, \phi_\beta{}^*\right]\right\} + \Gamma_{\alpha\beta}\left\{\psi_\beta{}^*, \left[\left\{\phi_\alpha, \frac{1}{D}\right\}, \phi_\beta\right]\right\}\right) = 0\,. \qquad (\mathrm{C}\cdot 5)$$

Using (C·5) and the anti-commutation relations (5·23) we have

$$\begin{aligned} A_\alpha &\equiv [T, \psi_\alpha] + \psi_\alpha \\ &= -\frac{\lambda}{8}\left(\rho_{\alpha\beta}\left\{\psi_\beta, \left[\phi_\alpha, \left\{\phi_\beta{}^*, \frac{1}{D}\right\}\right]\right\} + \Gamma_{\alpha\beta}\left\{\psi_\beta{}^*, \left[\phi_\alpha, \left\{\phi_\beta, \frac{1}{D}\right\}\right]\right\}\right). \end{aligned} \qquad (\mathrm{C}\cdot 6)$$

The expression (C·6) can further be reduced to

$$A_\alpha = -\frac{\lambda}{8}\left(\rho_{\alpha\beta}\left\{\psi_\beta, \left[\{\phi_\alpha, \phi_\beta{}^*\}, \frac{1}{D}\right]\right\} + \Gamma_{\alpha\beta}\left\{\psi_\beta{}^*, \left[\{\phi_\alpha, \phi_\beta\}, \frac{1}{D}\right]\right\}\right) \qquad (\mathrm{C}\cdot 7)$$

by making use of the identity (A·2) and (C·5) again.

From the anti-commutation relations (5·23), we obtain

$$\begin{aligned} \{\phi_\alpha(x), \phi_\beta{}^*(x)\} &= \delta_{\alpha\beta}\delta(\mathbf{0}) - \frac{\lambda}{4}\rho_{\alpha\beta}\underline{[\psi_\alpha, \psi_\beta{}^*]}\,\delta(\mathbf{0}), \\ \{\phi_\alpha(x), \phi_\beta(y)\} &= \frac{\lambda}{4}\Gamma_{\alpha\beta}\underline{[\psi_\alpha, \psi_\beta]}\,\delta(\mathbf{0}), \end{aligned} \qquad (\mathrm{C}\cdot 8)$$

where

$$\underline{[\psi_\alpha, \psi_\beta{}^*]} = \left[\left\{\psi_\alpha, \frac{1}{D}\right\}, \psi_\beta{}^*\right] + \left[\psi_\alpha, \left\{\psi_\beta{}^*, \frac{1}{D}\right\}\right],$$

$$\underline{[\psi_\alpha, \psi_\beta]} = \left[\left\{\psi_\alpha, \frac{1}{D}\right\}, \psi_\beta\right] + \left[\psi_\alpha, \left\{\psi_\beta, \frac{1}{D}\right\}\right].$$

Substituting (C·8) into (C·7), we get

$$A_\alpha = \frac{\lambda^2}{32}\delta(\mathbf{0})\left(\rho_{\alpha\beta}\rho_{\alpha\beta}\left\{\psi_\beta, \left[\underline{[\psi_\alpha, \psi_\beta{}^*]}, \frac{1}{D}\right]\right\} - \Gamma_{\alpha\beta}\Gamma_{\alpha\beta}\left\{\psi_\beta{}^*, \left[\underline{[\psi_\alpha, \psi_\beta]}, \frac{1}{D}\right]\right\}\right) + O(\lambda^5). \qquad (\mathrm{C}\cdot 9)$$

Using (C·4) and the relations

$$\begin{aligned} \left[\underline{[\psi_\alpha, \psi_\beta{}^*]}, \frac{1}{D}\right] &= 2\left[[\psi_\alpha, \psi_\beta{}^*], \frac{1}{D^2}\right] + O(\lambda^3), \\ \left[\underline{[\psi_\alpha, \psi_\beta]}, \frac{1}{D}\right] &= 2\left[[\psi_\alpha, \psi_\beta], \frac{1}{D^2}\right] + O(\lambda^3), \end{aligned} \qquad (\mathrm{C}\cdot 10)$$

we find

$$A_\alpha = \frac{\lambda^4}{4}\delta(\mathbf{0})^2(\{[\bar{\psi}\gamma_5, \psi][\psi^*, \psi], (\beta\gamma_5\psi)_\alpha\} - \{[\psi^*\gamma_5, \psi][\psi^*, \psi], (\gamma_5\psi)_\alpha\}) + O(\lambda^5), \qquad (\mathrm{C}\cdot 11)$$

which is nothing but (5·25).

Finally, we shall evaluate the contributions of the terms with the coefficient d in (5·28) to $[T, \psi_\alpha]$. The contributions consist of two parts B_α and C_α:

$$B_\alpha = -\frac{\lambda}{16} d \left(\rho_{\alpha\beta} \left\{ \psi_\beta, \left(\left[[\phi_\alpha, D], \left[\phi_\beta^*, \frac{1}{D^2} \right] \right] + \left[\left[\phi_\alpha, \frac{1}{D^2} \right], [\phi_\beta^*, D] \right] \right) \right\} \right.$$

$$\left. + \Gamma_{\alpha\beta} \left\{ \psi_\beta^*, \left(\left[[\phi_\alpha, D], \left[\phi_\beta, \frac{1}{D^2} \right] \right] + \left[\left[\phi_\alpha, \frac{1}{D^2} \right], [\phi_\beta, D] \right] \right) \right\} \right), \tag{C·12}$$

$$C_\alpha = \frac{\lambda}{8} d \left(\rho_{\alpha\beta} \left\{ \psi_\beta, \left[\frac{1}{D^2}, [D, [\phi_\alpha, \phi_\beta^*]] \right] \right\} + \Gamma_{\alpha\beta} \left\{ \psi_\beta^*, \left[\frac{1}{D^2}, [D, [\phi_\alpha, \phi_\beta]] \right] \right\} \right). \tag{C·13}$$

By the aid of (C·2) and (C·4), we obtain

$$B_\alpha = d\lambda^4 \delta(\mathbf{0})^2 (\{[\psi^* \gamma_5, \psi] \cdot [\psi^*, \psi], (\gamma_5 \psi)_\alpha\} - \{[\bar{\psi}\gamma_5, \psi] \cdot [\psi^*, \psi], (\beta\gamma_5\psi)_\alpha\}) + O(\lambda^5). \tag{C·14}$$

Similarly, we have

$$C_\alpha = O(\lambda^5). \tag{C·15}$$

Combining (C·9) and (C·14), we get (5·29).

Appendix D

——*Geometrical derivation of Lagrangians*——

We shall show that our Lagrangian densities (1·1) and (1·4) can be derived from a geometrical viewpoint when our space-time possesses a torsion. In this appendix we shall distinguish a covariant vector from a contravariant one. The lowering and raising of indices are performed by the metric tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$ respectively.

In the first we briefly sketch such a space-time that is called the neutral space by Finkelstein.[4] The space-time is characterized by an asymmetric connection

$$L^\mu{}_{\alpha\beta} \neq L^\mu{}_{\beta\alpha}. \tag{D·1}$$

We can associate a given vector v^μ at x^β with another vector $v^\mu + \delta v^\mu$ at a neighbouring point $x^\beta + \delta x^\beta$, according to either of the two equations

$$\delta v^\mu = -L^\mu{}_{\alpha\beta} v^\alpha \delta x^\beta, \tag{D·2+}$$

$$\delta v^\mu = -L^\mu{}_{\alpha\beta} v^\beta \delta x^\alpha. \tag{D·2−}$$

The symmetrical part of $L^\mu{}_{\alpha\beta}$, i.e.

$$\Gamma^\mu{}_{\alpha\beta} \equiv \tfrac{1}{2}(L^\mu{}_{\alpha\beta} + L^\mu{}_{\beta\alpha}) \tag{D·3}$$

defines (0) connection. The anti-symmetrical part of $L^\mu{}_{\alpha\beta}$, i.e.

$$\Omega^\mu{}_{\alpha\beta} \equiv \tfrac{1}{2}(L^\mu{}_{\alpha\beta} - L^\mu{}_{\beta\alpha}) \tag{D·4}$$

is called the torsion tensor.

Our space-time has the following structures.

(i) Equations (D·2±) are not integrable and the curvature tensors do not vanish:

$$L^{\mu}{}_{\alpha\beta\gamma}(\pm)\equiv\partial_{\gamma}L^{\mu}{}_{\alpha\beta}(\pm)-\partial_{\beta}L^{\mu}{}_{\alpha\gamma}(\pm)+L^{\sigma}{}_{\alpha\beta}(\pm)L^{\mu}{}_{\sigma\gamma}(\pm)-L^{\sigma}{}_{\alpha\gamma}(\pm)L^{\mu}{}_{\sigma\beta}(\pm)$$
$$\neq 0\,, \tag{D·5}$$

where

$$L^{\mu}{}_{\alpha\beta}(+)=L^{\mu}{}_{\alpha\beta}\,,\ L^{\mu}{}_{\alpha\beta}(-)=L^{\mu}{}_{\beta\alpha}(+).$$

(ii) When some equation holds for (+) connection, it holds also for (−) connection in our geometry.

(iii) We postulate that

$$L^{\mu}{}_{\mu\beta\gamma}(+)=L^{\mu}{}_{\mu\beta\gamma}(-), \tag{D·6a}$$

$$L^{\mu}{}_{\alpha\beta\mu}(+)=L^{\mu}{}_{\alpha\beta\mu}(-), \tag{D·6b}$$

$$\Omega^{\sigma}{}_{\alpha\beta}\Omega^{\mu}{}_{\sigma\gamma}+\Omega^{\sigma}{}_{\beta\gamma}\Omega^{\mu}{}_{\sigma\alpha}+\Omega^{\sigma}{}_{\gamma\alpha}\Omega^{\mu}{}_{\sigma\beta}=0. \tag{D·7}$$

From (D·5) we get

$$L^{\mu}{}_{\alpha\beta\gamma}(+)-L^{\mu}{}_{\alpha\beta\gamma}(-)=2(\Omega^{\mu}{}_{\alpha\beta|\gamma}-\Omega^{\mu}{}_{\alpha\gamma|\beta}), \tag{D·8}$$

where

$$\Omega^{\mu}{}_{\alpha\beta|\gamma}=\partial_{\gamma}\Omega^{\mu}{}_{\alpha\beta}+\Omega^{\sigma}{}_{\alpha\beta}\Gamma^{\mu}{}_{\sigma\gamma}-\Omega^{\mu}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\alpha\gamma}-\Omega^{\mu}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\gamma}\,.$$

Using (D·6) and (D·8), we have

$$\Omega_{\alpha|\beta}=\Omega_{\beta|\alpha}\,, \tag{D·9a}$$

$$\Omega^{\mu}{}_{\alpha\beta|\mu}=-\Omega^{\mu}{}_{\mu\alpha|\beta}\,, \tag{D·9b}$$

where

$$\Omega_{\alpha}=\Omega^{\mu}{}_{\mu\alpha}\,. \tag{D·10}$$

Owing to the relation (D·9a) the "torsion vector" Ω_{α} is expressed as

$$\Omega_{\alpha}=\partial_{\alpha}\Omega\,. \tag{D·11}$$

Because of the constraints (D·7), eight components are independent among the twenty-four components of $\Omega^{\alpha}{}_{\beta\gamma}$. Therefore, we can choose the "torsion pseudo-vector density" φ^{μ},

$$\varphi^{\mu}=\frac{1}{3!}\varepsilon^{\mu\alpha\beta\gamma}\Omega_{\alpha\beta\gamma},\qquad(\varepsilon^{0123}=-\varepsilon_{0123}=1) \tag{D·12}$$

as irreducible components in addition to Ω_{α}.

We can consider the following two cases:

(I) $\Omega_\alpha \neq 0$; $\varphi^\alpha = 0$.

(II) $\Omega_\alpha = 0$, $\Omega_{\alpha\beta\gamma} + \Omega_{\beta\alpha\gamma} = 0$; $\varphi^\alpha \neq 0$.

In the latter case, from (D·9b) we have

$$\Omega^\mu{}_{\alpha\beta|\mu} = 0, \tag{D·13}$$

which means $(g^{\mu\nu}\varepsilon_{\nu\alpha\beta\gamma}\varphi^\gamma)_{|\mu} = 0$, i.e.

$$\varphi_{\alpha|\beta} = \varphi_{\beta|\alpha}\,. \tag{D·14}$$

Here we have postulated that $g_{\mu\nu|\alpha} = 0$, that is, $\Gamma^\alpha{}_{\mu\nu}$ is equal to the Christoffel symbol. If we now put the coordinate condition

$$\partial_\mu(\sqrt{-g}) = 0$$

or restrict the metric tensor $g^{\alpha\beta}$ to the flat one, we can express φ_α as

$$\varphi_\alpha = \partial_\alpha \varphi\,. \tag{D·15}$$

We shall consider the field equation for a spin 1/2 particle in our space time. Even in our space time where $L^\mu{}_{\alpha\beta\gamma}(\pm) \neq 0$, we can introduce the tetrad field $\lambda^\mu(i)$ by

$$\lambda^\alpha(i)_{|\beta} + \sum_j O_\beta(i,j)\,\lambda^\alpha(j) = 0\,, \tag{D·16}$$

where $O_\beta(i,j) = -O_\beta(j,i)$. In terms of the constant Dirac matrix $\gamma(i)$, the generalized matrix γ^μ is expressed as

$$\gamma^\mu = \sum_i \lambda^\mu(i)\,\gamma(i)\,, \tag{D·17}$$

which satisfies $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. From (D·16) and (D·17) we have

$$\delta\gamma^\mu = -(\Gamma^\mu{}_{\alpha\beta} + \bar{O}^\mu{}_{\alpha\beta})\,\gamma^\alpha \delta x^\beta, \tag{D·18}$$

where

$$\bar{O}^\mu{}_{\alpha\beta} = \sum_{s,t} \lambda_\alpha(s)\,O_\beta(s,t)\,\lambda^\mu(t)\,. \tag{D·19}$$

Let ψ_+ and ψ_- be the fields characterized by $(+)$ and $(-)$ connections respectively. We want to determine the transformation law for $\psi_\pm$ so that the following equation holds:

$$\delta(\bar{\psi}_\pm \gamma^\mu \psi_\pm) = -L^\mu{}_{\sigma\alpha}(\pm)\,(\bar{\psi}_\pm \gamma^\sigma \psi_\pm)\,\delta x^\alpha. \tag{D·20}$$

As a solution of (D·20) we obtain

$$\delta\psi_\pm = -S_\pm \psi_\pm, \qquad \delta\bar{\psi}_\pm = \bar{\psi}_\pm S_\pm, \tag{D·21}$$

where

$$S_\pm = \tfrac{1}{4}(\bar{O}_{\alpha\beta\mu} \mp \Omega_{\alpha\beta\mu})\,\gamma^\alpha\gamma^\beta \delta x^\mu. \tag{D·22}$$

Thus we get the covariant spinor equation

$$\gamma^\mu\{\partial_\mu + \tfrac{1}{4}(\bar{O}_{\alpha\beta\mu} \mp \Omega_{\alpha\beta\mu})\,\gamma^\alpha\gamma^\beta\}\,\psi_\pm = 0\,. \tag{D·23}$$

In case (I), we get

$$\Omega_{\alpha\beta\mu}\gamma^{\alpha}\gamma^{\beta}=\Omega_{\mu}=\partial_{\mu}\Omega\,, \tag{D·24}$$

while in case (II)

$$\begin{aligned}\gamma^{\mu}\Omega_{\alpha\beta\mu}\gamma^{\alpha}\gamma^{\beta}&=\varepsilon_{\alpha\beta\mu\lambda}\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\varphi^{\lambda}\\&=-6\gamma_{\lambda}\gamma_{5}\varphi^{\lambda}\\&=-6\gamma^{\lambda}\gamma_{5}\partial_{\lambda}\varphi\,,\end{aligned} \tag{D·25}$$

where we used $\gamma_{\lambda}\gamma^{5}=(1/3!)\varepsilon_{\lambda\alpha\beta\mu}\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}$. Therefore, if we set

$$\begin{aligned}\Omega&=4i\lambda(\bar{\psi}\psi) &&\text{for (I)},\\ \varphi&=\tfrac{2}{3}\lambda(\bar{\psi}\gamma_{5}\psi) &&\text{for (II)},\end{aligned} \tag{D·26}$$

we are able to get our Lagrangian densities (1·1) and (1·4) respectively.

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