

CHOICE FUNCTIONS OVER A FINITE SET: A SUMMARY

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IMA Preprint Series # 82

July 1984

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Abstract

A choice function picks some outcome (s) from every issue (subset of a fixed set A of outcomes). When is this function derived from one preference relation on A (the choice set being then made up of the best preferred outcomes within the issue), or from several preference relations (the choice set being then the Pareto optimal outcome within the issue, or the union of the best preferred outcome for each preference relation)? A complete and unified treatment of these problems is given based on three functional properties of the choice function. None of the main results is original.

1. Introduction

In social sciences, ranking available outcomes (or decisions) by means of preference relations yields the most common description of individual choices. It has long been recognized, however, that the transitivity of pairwise comparisons is a strong assumption: it is typically violated when these comparisons emerge from a collective body through most familiar voting methods. Less demanding properties, such as transitivity of the strict preference relation (quasi-transitivity) or the absence of cycles in it (acyclicity) are known to be more reasonable in the social choice perspective. (See, for example, Sen's possibility theorem and its account in Sen (1970).)

Given any one of these preference relations, one can still pick from every issue (subset of the finite set A of all conceivable outcomes) the best preferred outcomes, namely those to which no other outcome is strictly preferred. The interesting problem is the converse question: given a choice function (an abstract mapping picking from any issue some "good" outcomes) can we detect existence of an underlying preference relation from which our choice function is derived as above? If it exists we say that it rationalizes our choice function (in microeconomic theory one would call this relation the preference revealed by our choice function).

A companion problem is the rationalizability of a choice function by several transitive preference relations. The idea here is that of a single decision unit endowed with several primitive contradictory ranking systems: from a given issue the agent might pick the Pareto optimal outcomes corresponding to these preference relations; in that case we speak of Pareto rationalization. Alternatively he or she might pick those outcomes that are best for at least one of the given preference relations; then we speak of pseudorationalizability. These are two more ways for representing a choice function by means of preference relations.

All above-mentioned problems are solved with the help of three functional equations that a choice function may or may not satisfy. These three axioms, called Chernoff, Expansion, and Aizerman (we postpone discussion of this terminology until Sections 3 and 4) all express some consistency of the choice sets over various issues. Expansion for instance requires that whenever an outcome would be chosen within an issue B_1 and within issue B_2 , then it should be chosen as well if the issue is $B_1 \cup B_2$.

A first important result says that Chernoff plus Expansion characterize rationalizability by an acyclic relation (Theorem 2). Next, Chernoff plus Expansion plus Aizerman characterize Pareto rationalizability (Theorems 1 and 4). Finally, Chernoff plus Aizerman characterize pseudorationalizability (Theorem 5).

The Chernoff axiom is definitely needed in all rationalizability properties explored below: even to guarantee that a choice function contains some rationalizable choice function (a property that we call subrationalizability) we virtually need this axiom (see Lemma 5 as well as Exercise 4 in the final comments). In fact, Expansion plus Aizerman do not prove useful at all for abstract choice functions: in the different problem of choosing from a tournament, however, they are the key to characterize the uncovered set solution concept (see Moulin (1984, Theorem)).

This paper offers very few original results: only the Corollary to Theorems 1 and 4 and Lemmas 4 and 5 are, to the best of the author's knowledge. The aim here is mainly pedagogical: since these results are scattered through the literature, a short, self-contained exposition of its main results is believed to be useful for the nonspecialist.

Rationalizability by one binary relation is a familiar theme of social choice theory: hence Theorems 2, 3 and 4 are all at least ten years old, and well known. Less attention has been devoted to rationalizability by several preferences relations. Thanks to the representation of any quasi-transitive relation as some Pareto relation (Theorem 1), the problem of Pareto rationalizability is in effect a corollary

of previous results. Nevertheless, the combination of Theorems 1 and 4 into their corollary does not seem known. The pseudorationalizability problem is first stated and solved in the extensive work of Aizerman and his team. See the very rich survey of Aizerman (1983) from which we borrow the key observation that Chernoff, Expansion and Aizerman are the three primitive axioms which are enough to state virtually all the results.

2. Choosing or Ranking?

A single agent faces a finite set A of conceivable outcomes. Exogenous constraints determine a subset $B \subset A$ of actual candidates; B is the issue. Within the issue B , he designates the subset $S(B) \subset B$ of those outcomes he chooses: $S(B)$ is the choice set at issue B . The choice set must contain at least one outcome; if it has more than one, the agent views these as equally good choices (outcomes are mutually exclusive: only one is eventually chosen by some unspecified tie-breaking device). The mapping S is called a choice function.

The simplest way to construct a choice function is by ordering the elements of A . Let R be an ordering of A , namely a complete, asymmetric and transitive binary relation on A . An ordering amounts to rank the candidates of A ; the associated choice function selects in B the unique outcome with higher rank.

If indifferences are possible, then R is a pre-ordering of A , namely a complete, transitive binary relation. Denote by P the asymmetric component of R (aPb iff no bRa) and by I its symmetric component (aIb iff aRb and bRa). Then I is an equivalence relation on A whose equivalence classes are ordered by P .

Figure 1 represents a pre-ordering R . Its asymmetric component xPy reads x is strictly preferred to y while xIy reads x and y are indifferent. Strict preference means "to be above" while indifference means "to be on the same horizontal level."

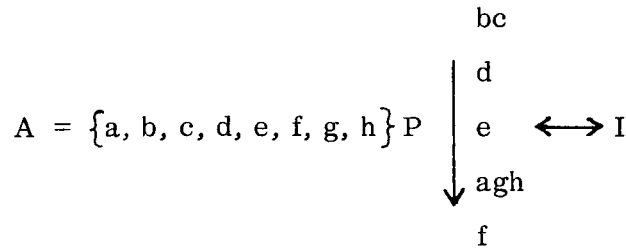


Figure 1

A preordering R yields the choice function that picks from B its maximal element(s), namely

$$S(B) = \max_B R = \{x \in B \mid \text{for no } y \in B : yPx\} \quad (1)$$

The micro-economic agents are commonly endowed with a preordering over the relevant outcome set, with the understanding that their (individual) choice over any issue B is given by (1).

In the social choice perspective, however, we use more general binary relations, meant to represent the preferences of a single agent on all pairwise comparisons of the outcomes. Think of our agent endowed with finitely many criteria $R_1, \dots, R_i, \dots, R_n$ each of them being a different preordering. When comparing two outcomes x and y , these criteria can disagree: if they don't, our agent's preferences over x, y are clear; if they do he avoids frontal opposition to all of them by declaring x and y indifferent:

$$\begin{cases} xPy \text{ iff } \{xR_i y \text{ for all } i = 1, \dots, n \text{ and } xP_i y \text{ for some } i\} \\ xIy \text{ iff } \{xI_i y \text{ for all } i = 1, \dots, n \text{ or } xP_i y \text{ for some } i \text{ and } yP_j x \\ \text{for some } j\} \end{cases} \quad (2)$$

This in turn defines a complete binary relation R that we call the Pareto relation associated with R_1, \dots, R_n . It is complete and its asymmetric component is transitive. Only its symmetric component might violate transitivity.

Say that a relation R is a quasi-ordering if it is complete and its asymmetric component is transitive. One also says that R is quasi-transitive.

Theorem 1 (Roberts, 1979). Any quasi-ordering R can be written as the Pareto relation of at most $|A|/2$ orderings if $|A| \geq 4$, and at most 2 if $|A| = 2, 3$.

When $R_1, \dots, R_i, \dots, R_n$ are orderings on A their Pareto relation is written simply as

$$\begin{cases} xPy \text{ iff } \{x \neq y \text{ and } xR_i y \text{ all } i = 1, \dots, n\} \\ xIy \text{ otherwise} \end{cases} \quad (3)$$

Sketch of the Proof: Define inductively

$$\begin{aligned} A_0 &= \emptyset \\ A_1 &= \max_A R = \{a \mid bPa \Rightarrow b \in A_0\} \\ A_{t+1} &= \{a \mid bPa \Rightarrow b \in A_t\} \end{aligned}$$

and let T be the first integer such that $A_T = A_{T+1}$. If $A_T \neq A$, pick $a \in A \setminus A_T = A \setminus A_{T+1}$. By definition of A_{T+1} , there is $b \in A \setminus A_T$ such that bPa . Repeating the argument, there is $c \in A \setminus A_T$ such that cPb . And so on. By finiteness of A , the sequence a, b, c, \dots has a cycle, which contradicts transitivity of P . We have just proved that $B_1 = A_1 \setminus A_0, \dots, B_t = A_t \setminus A_{t-1}, \dots, B_T = A_T \setminus A_{T-1}$, partition A . Moreover, if $a \in B_t, b \in B_{t'}$ and $t < t'$, then bPa is impossible.

This in turn allows us to represent R on a Hasse diagram where each set B_t appears on a different horizontal line:

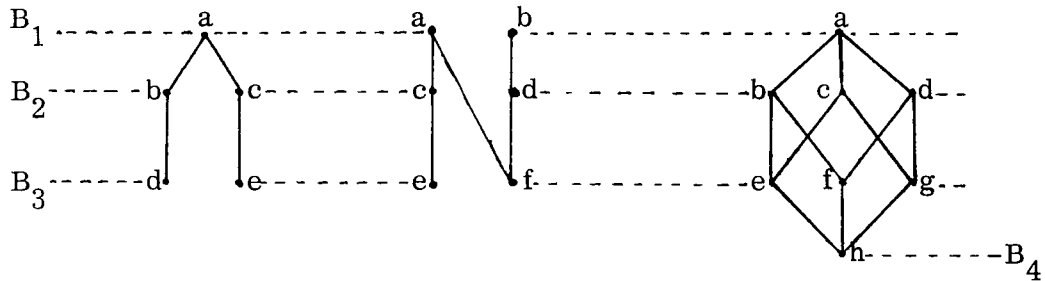


Figure 2: Three Hasse diagram. A line joining x to y downward indicates xPy . All other relations xPy are deduced by transitivity of P .

Expansion: $S(B) \cap S(B') \subset S(B \cup B')$ all B, B'

If a is a best choice against two different issues, it is still a best choice against their union; joining forces against a does not pay.

Theorem 2 (Sen, 1971): A choice function is rationalizable if and only if it satisfies Chernoff and Expansion.

Proof. Only if is straightforward. For instance, we check Expansion. Let S be rationalized by R and $a \in S(B) \cap S(B')$. This means that for no $b \in B$ nor any $b \in B'$ we have bPa . So $a \in S(B \cup B')$. Incidentally we prove more, namely: for $a \in B \cap B'$, $a \in S(B) \cap S(B')$ iff $a \in S(B \cup B')$. This property, by itself, characterizes rationalizability (see Exercise 2).

If: Let S satisfy Chernoff and Expansion. We prove (5). Take first $B \subseteq A$ and $a \in S(B)$. If $a \notin S(ab)$ for some $b \in B$, then by Chernoff $S(B) \cap \{ab\} \subset S(ab) = \{b\}$, a contradiction. Next suppose a is such that $a \in S(ab)$ for all $b \in B$. Apply successively Expansion to deduce

$$\left. \begin{array}{l} a \in S(ab) \cap S(ac) \Rightarrow a \in S(abc) \\ a \in S(ad) \end{array} \right\} \Rightarrow a \in S(abcd) \dots$$

In passing we note an equivalent formulation of Expansion:

$$\bigcap_{1 \leq k \leq K} S(B_k) \subset S\left(\bigcup_{1 \leq k \leq K} B_k\right).$$

Q. E. D.

The case of single-valued choice (deterministic) choice functions yields a simpler result. If S is single-valued, its base relation is asymmetric:

$$\text{for } a \neq b, aR_S b \Rightarrow a \in S(ab) \Rightarrow b \notin S(ab) \Rightarrow aP_S b.$$

A complete, asymmetric, acyclic binary relation is just an ordering. Hence

Corollary to Theorem 2. Let S be a single-valued choice function. Then S is rationalizable if and only if it satisfies Chernoff. In that case its base relation is an ordering.

Proof. By single-valuedness, Chernoff is rewritten as:

$$B \subset B' \quad \text{and} \quad S(B') \in B \implies S(B) = S(B') \quad (6)$$

Similarly, Expansion is now

$$S(B) = S(B') = \{a\} \implies S(B \cup B') = \{a\} \quad (7)$$

Finally, check that (6) implies (7).

Q. E. D.

3. Rationalization by Transitive and Quasi-transitive Relations

The following axiom strengthens Chernoff's condition.

Nash's Independence of
Irrelevant Alternatives : $\{B \subset B', S(B') \cap B \neq \emptyset \implies S(B') \cap B = S(B)\}$ all B, B'
(NIIA)

When an issue shrinks and some of the original choice set survives, then the new choice set is made of these survivors only. Contrary to Chernoff's condition, NIIA does not allow new outcomes to enter the choice set as the issue shrinks.

Lemma 3. The NIIA axiom is equivalently formulated as the following one.

Weak Axiom of
Revealed Preferences : $\{a \in S(B), b \in B \setminus S(B)\} \implies \text{No } \{a \in B', b \in S(B')\}$ all a, b, B, B'
(WARP)

WARP says that if outcome a was rejected once when b was chosen, then whenever a and b are both in the issue, a will never be chosen.

Proof. Suppose S satisfies NIIA and assume $a \in S(B)$, $b \in B \setminus S(B)$. Then $\{ab\} \subset B$ and $S(B) \cap \{ab\} \neq \emptyset$ so by NIIA $S(ab) = S(B) \cap \{ab\} = \{a\}$. Take now another issue, B' , such that $a \in B'$, $b \in S(B')$. By NIIA again $S(ab) = S(B') \cap \{ab\}$ so that $S(ab)$ contains b , a contradiction. Conversely, suppose S satisfies WARP and assume $B \subset B'$, $S(B') \cap B \neq \emptyset$. Denote a^* an element in $C = S(B') \cap B$. Suppose $a \in S(B)$, $a \notin C$. Then $a \notin S(B')$ so we have $\{a^* \in S(B'), a \in B' \setminus S(B')\}$ and yet $\{a^* \in B, a \in S(B)\}$, contradicting WARP. This proves $S(B) \subset C$. Conversely, suppose $a \in C$, $a \notin S(B)$

and pick some $b \in S(B)$. Then we have $\{b \in S(B), a \in B \setminus S(B)\}$ and yet $\{b \in B', a \in S(B')\}$, contradicting WARP again.

Q. E. D.

Theorem 3 (Arrow, 1959). A choice function satisfies Nash's Independence of Irrelevant Alternatives if and only if it is rationalized by a preordering.

Proof. First, NIIA implies Expansion. For any B, B' , $S(B \cup B')$ intersects B and/or B' . Say that $S(B \cup B') \cap B \neq \emptyset$, then by NIIA, $S(B) = S(B \cup B') \cap B$, hence $S(B) \subset S(B \cup B')$ so that $S(B) \cap S(B') \subset S(B \cup B')$. The case $S(B \cup B') \cap B' \neq \emptyset$ is similar.

Let S be a choice function satisfying NIIA. Since S satisfies Chernoff and Expansion, it is rationalized by its base relation R_S (Theorem 2). We only have to prove that R_S is transitive. Suppose it is not. We can choose a, b, c such that:

$$a \in S(ab) \ , \quad b \in S(bc) \quad \{c\} = S(ac) \ .$$

Consider $S(abc)$. It does not contain a , otherwise by Chernoff, $a \in S(abc) \cap \{ac\} \subset S(ac)$. It does not contain b , otherwise by NIIA,

$$S(abc) \cap \{ab\} \neq \emptyset \Rightarrow S(ab) = S(abc) \cap \{ab\} \not\supset a \ .$$

It does not contain c either, otherwise by NIIA,

$$S(abc) \cap \{bc\} \neq \emptyset \Rightarrow S(bc) = S(abc) \cap \{bc\} \not\supset b \ .$$

This proves R_S is intransitive.

Conversely, we let the reader check that the choice function associated with a preordering satisfies NIIA.

Q. E. D.

Our next axiom is again a contraction property, relating the choice set within a particular issue to the choice set of smaller issues.

Aizerman: $\{S(B') \subset B \subset B'\} \Rightarrow \{S(B) \subset S(B')\}$ all B, B' .

Deleting from a given issue some outcomes outside the choice set cannot make new outcomes chosen. Although this axiom has been in the literature for a while (for example, see Fishburn (1975)), its prominent role was recognized only recently (Aizerman and Malishevski, 1981). Together with the Chernoff and Expansion axioms, this axiom characterizes all choice functions that are quasi-transitively rationalizable, and hence Pareto rationalizable.

Theorem 4 (Schwartz, 1976). A choice function satisfies {Chernoff, Expansion and Aizerman} if and only if it is rationalized by a quasi-transitive relation.

Corollary to Theorems 1 and 4. A choice function S satisfies {Chernoff, Expansion and Aizerman} if and only if it is Pareto rationalizable: there exist at most $|A|/2$ orderings R_1, \dots, R_n such that for all issue B , $S(B)$ is the set of Pareto optimal outcomes w.r.t. R_1, \dots, R_n on B .

Proof. Only if: By Theorem 2, it is enough to prove, if S is rationalized by its base relation R_S and S satisfies Aizerman, then R_S is quasi-transitive. In other words we want:

$$\left[S(ab) = \{a\}, S(bc) = \{b\} \right] \Rightarrow \left[S(ac) = \{a\} \right] . \quad (8)$$

Since S is rationalizable, $S(ab) = \{a\}$ and $S(bc) = \{b\}$ imply $S(abc) = \{a\}$.

Next by Aizerman,

$$S(abc) \subset \{ac\} \subset \{abc\} \Rightarrow S(ac) \subset S(abc) = \{a\} ,$$

proving (8).

Conversely, let S be rationalized by the quasi-transitive relation R ; we prove it satisfies Aizerman. Fix an issue B' and pick $a \in B' \setminus S(B') = B' \setminus \max_{B'} R$. By definition of the maximal elements there is $a_1 \in B'$ such that $a_1 P a$. We claim that b can be taken in $S(B')$. To prove this construct inductively a sequence $a = a_0, a_1, \dots, a_t$, where $a_t \in B'$ and $a_t P a_{t-1}$. By transitivity of P , the sequence cannot cycle, so by finiteness of A it must stop. When it stops we have reached a maximal element of R on B' , namely $a_t \in S(B')$. By transitivity of P , we conclude $a_t P a$ and the claim is proved.

We take now some issue B such that $S(B') \subset B \subset B'$. By the above argument, any outcome $a \in B \setminus S(B')$ is such that bPa for some $b \in S(B')$. Therefore a is outside $\max_B R = S(B)$. This proves

$$B \setminus S(B') \subset B \setminus S(B) \Rightarrow S(B) \subset S(B')$$

Q. E. D.

The three axioms Chernoff, Expansion, and Aizerman are of the same vein, yet they are not logically related when A contains at least 3 outcomes: one can find a choice function satisfying any subset of those three and no more. (See Aizerman and Malishevski (1981).) These 8 examples will be easily constructed in a set A of size 3 by the patient reader. Notice, however, that for single-valued choice functions, Chernoff and Aizerman coincide and imply Expansion.

4. Pseudo-rationalization

Say that S is subrationalizable if it contains a rationalizable choice function:

$$\text{for some acyclic relation } R: \max_B R \subset S(B), \text{ all } B. \quad (9)$$

Then relation R alone allows to compute at least one acceptable choice per issue. In some contexts this might be enough for all practical purposes. This property was introduced by Deb (1983) under the name of weak rationalizability. Before analyzing this condition we formulate it in a seemingly stronger form.

Lemma 4. A choice function S is subrationalizable if and only if it contains a single-valued rationalizable choice function, that is to say,

$$\text{for some orderings } R: \max_B R \in S(B), \text{ all } B. \quad (10)$$

Proof. Let R be an acyclic relation. We prove existence of an ordering R_1 such that

$$\left\{ \max_B R_1 \in \max_B R, \text{ all } B \right\} \iff \left\{ aPb \Rightarrow aR_1b, \text{ all } a, b \right\},$$

thus (9) implies (10), as is to be proved.

Define P^* to be the transitive closure of P , namely,

aP^*b iff {for some K there is a sequence:

$$a = c_0, c_1, \dots, c_K = b : c_0 P c_1, \dots, c_t P c_{t+1}, \dots, c_{K-1} P c_K \}$$

Then P^* is the assymmetric component of a quasi-transitive relation R^* (defined by aR^*b iff $\text{No}\{bP^*a\}$). By Theorem 1, R^* can be written as the Pareto relation associated with some orderings R_1, \dots, R_n . In particular, aP^*b implies aR_1b (see (3)).

Q. E. D.

Subrationalizability is a weaker property than Chernoff:

Lemma 5. If S satisfies the Chernoff axiom it is subrationalizable. More precisely, an ordering $R = (a_1, a_2, \dots, a_g)$ of A (where a_1 is ranked first and a_g is ranked last) defines a selection of S (satisfying (10)) if and only if we have

$$a_1 \in S(A); a_2 \in S(A \setminus a_1); \dots; a_j \in S(A \setminus \{a_1 \dots a_{j-1}\}); \dots \quad j = 1, 2, \dots, g \quad (11)$$

Proof. Only if: Let $R = \{a_1, a_2, \dots, a_g\}$ be an ordering satisfying (11). Taking $B = A$ yields $a_1 \in S(A)$; next $B = A \setminus a_1$ yields $a_2 \in S(A \setminus a_1)$ and so on.

If: Suppose Chernoff and construct an ordering $R = (a_1, \dots, a_g)$ according to (11). We must prove (10). Fix B and denote $\max_B R = a_j$. Since $B \subset A \setminus \{a_1, \dots, a_{j-1}\}$ we can apply Chernoff:

$$S(A \setminus \{a_1, \dots, a_{j-1}\}) \cap B \subset S(B) \quad .$$

By construction a_j is in $S(A \setminus \{a_1, \dots, a_{j-1}\})$ and in B . Hence $a_j \in S(B)$ as was to be proven.

One easily constructs a subrationalizable choice function which does not satisfy Chernoff. Pick an ordering $R = \{a_1, \dots, a_g\}$ and choose S such that:

$$\left\{ \begin{array}{l} S(B) = \max_B R \quad \text{for all } B \subsetneq A \\ \{a_1\} \subsetneq S(A) \subsetneq A \end{array} \right.$$

Chernoff applied to $B = S(A) \not\subset A$ would imply

$$S(A) \cap B \subset S(B) \Rightarrow B \subset S(B) \Rightarrow B = S(B) ,$$

a contradiction since B is not a singleton.

Exercise 5 offers a characterization of subrationalizability based on algorithm (11) (Deb, 1983).

Given a subrationalizable choice function S , there are typically several orderings satisfying (11). Do these orderings provide enough information to describe S entirely? Specifically, is S the union of its single-valued rationalizable selections?

We shall say that S is pseudo-rationalized by the orderings R_1, \dots, R_n if S is written as

$$S(B) = \bigcup_{1 \leq i \leq n} \max_B R_i , \quad \text{all } B . \quad (12)$$

A moment's reflexion will convince the reader that S is pseudo-rationalizable if and only if S is the union of all choice functions $B \rightarrow \max_B R$, where ordering R satisfies (10).

To any set of orderings (R_1, \dots, R_n) we have already associated the Pareto relation $\text{Par}(R_1, \dots, R_n)$ (defined by (3)) and the choice function \bar{S} that retains the Pareto undominated outcomes in any issue

$$\bar{S}(B) = \max_B \text{Par}(R_1, \dots, R_n), \quad \text{all } B . \quad (13)$$

Clearly S (defined by (12)) is contained in \bar{S} . Also the base relation R_S of S is just $\text{Par}(R_1, \dots, R_n)$. Hence a pseudo-rationalizable choice function S is contained in the choice function rationalized by its base relation, $S(B) \subset \max_B R_S$, all B . But equality does not hold in general, which means that not all pseudo-rationalizable choice functions are rationalizable. Here is an example:

$$A = \{a, b, c\} \quad R_1 = (a, b, c) \quad R_2 = (c, b, a) .$$

Applying (12) we get $S(abc) = \{ac\}$, yet $S(ab) = \{ab\}$ and $S(bc) = \{bc\}$ thus Expansion is violated: $S(ab) \cap S(bc) \not\subset S(abc)$.

The characterization of pseudorationalizable choice functions by functional properties is, again, very simple.

Theorem 5 (Aizerman and Malishevski, 1981). A choice function is pseudo-rationalizable if and only if it satisfies Chernoff and Aizerman.

Proof. Only if: Observe that the Chernoff and Aizerman properties are both preserved by taking the union of choice functions.

Before proving the if statement a remark is in order. If a choice function S satisfies $S^2 = S$ then Aizerman's axiom is equivalent to:

$$S(B') \subset B \subset B' \implies S(B) = S(B') . \quad (14)$$

Namely, if B, B' satisfy the premises of (14) we deduce from Aizerman:

$S(B) \subset S(B') \subset B$. Apply Aizerman again to $\tilde{B}' = B$ and $\tilde{B} = S(B')$: this gives $S^2(B') \subset S(B)$, i. e., $S(B') \subset S(B)$, which proves the claim. Now to the proof of the if statement.

Suppose S satisfies Chernoff and Aizerman. Chernoff implies $S^2 = S$ hence S satisfies (14).

There are finitely many orderings R satisfying (10). We must prove that the union of their associated (single-valued) choice functions is no less than S .

Take an arbitrary $B \subset A$ and $a \in S(B)$; we must construct an ordering R satisfying

(10) and moreover, $\max R = a$. By Lemma 5 this is equivalent to constructing a

sequence (a_1, a_2, \dots, a_g^B) satisfying (11) and moreover,

$a = a_j \implies \{a_1, \dots, a_{j-1}\} \subset A \setminus B$ (in such a way that all outcome in B are ranked no higher than a).

$$\left\{ \begin{array}{l} \text{If } S(A) \subset B \text{ then (by (14)) } S(A) = S(B) \text{ so we can pick } a_1 = a \text{ and we} \\ \text{are home} \\ \text{If } S(A) \not\subset B, \text{ pick } a_1 \in S(A) \setminus B \text{ so that } B \subset A \setminus a_1 \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{If } S(A \setminus a_1) \subset B \text{ then (by (14) again) } S(A \setminus a_1) = S(B), \text{ so choose } a_2 = a \\ \text{and we are done} \\ \text{If } S(A \setminus a_1) \not\subset B, \text{ pick } a_2 \in S(A \setminus a_1) \setminus B \text{ so that } B \subset A \setminus \{a_1, a_2\} \end{array} \right.$$

and so on. If a_1, \dots, a_j does not contain a then $B \subset A \setminus \{a_1, \dots, a_j\}$ and a_{j+1} is selected as follows:

$$\left\{ \begin{array}{l} \text{If } S(A \setminus \{a_1, \dots, a_j\}) \subset B \text{ then (by (14)), } S(A \setminus \{a_1, \dots, a_j\}) = S(B), \text{ so} \\ \text{choose } a_{j+1} = a \\ \text{If } S(A \setminus \{a_1, \dots, a_j\}) \not\subset B, \text{ pick } a_{j+1} \in S(A \setminus \{a_1, \dots, a_j\}) \setminus B . \end{array} \right.$$

Q. E. D.

Compare Theorem 4 and 5: pseudo-rationalizability (equivalent to Chernoff and Aizerman) is a strictly weaker property than Pareto (or quasi-transitive) rationalizability (equivalent to Chernoff + Expansion + Aizerman). In other words, representing a choice function in the form (12) requires less rationality (less consistency of the choice set across various issues) than representing it in the Pareto form (13). In particular, any quasi-transitive choice function can be expressed as (13) for some orderings R_1, \dots, R_n or as (12) for some other orderings R'_1, \dots, R'_n (typically more numerous). The upper bound of the minimal number of orderings necessary to such representations is not known.

Last but not least, the conjunction of the Chernoff and Aizerman axioms is captured by a unique, handsome axiom due to Plott (1973). It allows us to decompose the choice problem over a (large) issue into choices over smaller issues: we can cut any piece out of the original issue, and replace it by its own choice set. Hence the original choice problem can be converted into a quite arbitrary path of smaller problems.

Path Independence: $S(B \cup B') = S(S(B) \cup B')$ all B, B'

Here is an example of an algorithm for computing $S(B)$, that Path Independence validates: order the outcomes in B as b_1, \dots, b_g ; compute successively $S(b_1 b_2) = B_2, S(B_2 b_3) = B_3, \dots, S(B_{g-1} b_g) = B_g = S(B)$.

Also, Path Independence is equivalently formulated (see Blair, et al. (1976))

as:

$$S(B \cup B') = S(S(B) \cup S(B')) \quad \text{all } B, B' \quad (15)$$

or as

$$S\left(\bigcup_{1 \leq k \leq K} B_k\right) = S\left(\bigcup_{1 \leq k \leq K} S(B_k)\right) \quad \text{all } B_1, \dots, B_K \quad . \quad (16)$$

We prove this claim: clearly (16) implies (15). Next suppose (15). Taking $B = B'$ gives $S^2 = S$. Using (15) again yields

$$S(S(B) \cup B') = S(S^2(B) \cup S(B')) = S(S(B) \cup S(B')) ,$$

hence Path Independence. Finally, suppose Path Independence: we prove (16) for $K = 3$ and leave the obvious induction argument to the reader.

$$\begin{aligned} S(B_1 \cup B_2 \cup B_3) &= S(S(B_1) \cup (B_2 \cup B_3)) = S(B_2 \cup (S(B_1) \cup B_3)) \\ &= S(S(B_2) \cup (S(B_1) \cup B_3)) = S(B_3 \cup (S(B_1) \cup S(B_2))) \\ &= S(S(B_3) \cup (S(B_1) \cup S(B_2))) \quad . \end{aligned}$$

Strikingly enough, Path Independence is just the conjunction of two of our three basic axioms.

Lemma 6. (Aizerman and Malishevski, 1981). Plott's Path Independence is equivalent to {Chernoff and Aizerman}.

Proof. Suppose S satisfies Chernoff and Aizerman. Remember Aizerman can be expressed as (14) since $S^2 = S$.

Let us prove that S satisfies Path Independence. Pick any two issues B_1, B_2 . By Chernoff we have

$$\{S(B_1 \cup B_2) \cap B_i \subset S(B_i), i = 1, 2\} \implies \{S(B_1 \cup B_2) \subset S(B_1) \cup S(B_2)\} .$$

Apply (14) with $B' = B_1 \cup B_2$ and $B = S(B_1) \cup S(B_2)$. This yields (15), as was to be proved.

Conversely, let S satisfy Path Independence. Pick $B \subset B'$, $B \neq B'$ and apply the axiom to B' and $B' \setminus B$:

$$S(B') = S(B \cup (B' \setminus B)) = S(S(B) \cup (B' \setminus B)) .$$

Thus $S(B') \subset S(B) \cup (B' \setminus B)$ so that $S(B') \cap B \subset S(B)$, establishing Chernoff. To prove Aizerman, we suppose $S(B') \subset B \subset B'$. By Chernoff, $S(B') \subset S(B)$; we want to prove equality. Apply (15):

$$S(B') = S(B \cup B') = S(S(B) \cup S(B')) = S(S(B)) = S(B) ,$$

the last equation by $S^2 = S$, implied by Chernoff.

Q. E. D.

As a matter of conclusion, we list a few additional results that the motivated reader can see as exercises.

Exercise 1) Equivalent formulations of Chernoff's condition

Chernoff's condition is equivalent to any one of the eight following properties:

- | | | |
|--|---|-------------|
| a) $S(B \cup B') \subset S(B) \cup B'$ | } | all B, B' |
| b) $S(B \cup B') \subset S(B) \cup S(B')$ | | |
| c) $S(B \cup B') \subset S(S(B) \cup B')$ | | |
| d) $S(B \cup B') \subset S(S(B) \cup S(B'))$ | | |

e) through h) same as above, stated for all pairs B, B' of disjoint subsets of A .

Exercise 2) One more characterization of rationalizability (Schwartz, 1976)

Prove that S is rationalizable if and only if

$$S(B) \cap S(B') = S(B \cup B') \cap B \cap B' \quad \text{all } B, B'$$

Exercise 3) One more characterization of transitive rationalizability (Sen, 1971)

Prove that S is transitively rationalizable if and only if it satisfied Chernoff's condition and the following condition:

Sen: $\{B \subset B' \text{ and } S(B') \cap S(B) \neq \emptyset\} \implies S(B) \subset S(B') \quad \text{all } B, B'$

Exercise 4) Characterization of subrationalizability (Deb, 1983)

Prove that S is subrationalizable if and only if it satisfies:

$$\text{for all } B \text{ there is } a \in S(B) : \{B' \subset B \text{ and } a \in B'\} \Rightarrow \{a \in S(B')\}, \text{ all } B, B' \quad (17)$$

Give an example of a choice function satisfying Aizerman and Expansion, yet not subrationalizable.

Hints for the Proofs

- 2) Apply the axiom with $B \subset B'$ to deduce Chernoff. Also, Expansion is an immediate consequence.
- 3) From Sen and Chernoff deduce NIIA: since $S(B') \cap B \subset S(B)$, if $S(B') \cap B \neq \emptyset$, then $S(B') \cap S(B) \neq \emptyset$ implying $S(B) \subset S(B')$, hence $S(B) \subset S(B') \cap B$.
- 4) Construct the sequence a_1, \dots, a_g in such a way that $a_j \in S(A \setminus \{a_1, \dots, a_{j-1}\})$ satisfies the implication in (17). Example:

$$A = \{a, b, c\} \quad S(abc) = \{ab\}, \quad S(ac) = \{c\}, \quad S(ab) = \{a\}, \quad S(bc) = \{b\}.$$

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