*Geophys. J. R. astr. Soc.* (1986) **87**, 123–141

# Choice of norm for the density distribution of the Earth

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Accepted 1986 February 12. Received 1986 January 27; in original form 1985 June 18

**Summary**. The determination of the density distribution of the Earth from gravity data is called the inverse gravimetric problem. A unique solution to this problem may be obtained by introducing *a priori* data concerning the covariance of density anomalies. This is equivalent to requiring the density to fulfil a minimum norm condition. The generally used norm is the one equal to the integral of the square of the density distribution ($L^2$-norm), the use of which implies that blocks of constant density are uncorrelated. It is shown that for harmonic anomalous density distributions this leads to an external gravity field with a power spectrum (degree-variances) which tends too slowly to zero, i.e. implying gravity anomalies much less correlated than actually observed. It is proposed to use a stronger norm, equal to the integral of the square sum of the derivatives of the density distribution. As a consequence of this, base functions which are constant within blocks, are no longer a natural choice when solving the inverse gravimetric problem. Instead a block with a linearly varying density may be used. A formula for the potential of such a block is derived.

**Key words**: minimum norm, inversion, mixed collocation

## 1 Introduction

The problem of finding the density distribution of the Earth ($\rho$) from gravity data is called the inverse gravimetric problem. It is well known that this problem has no unique solution. On the other hand, we frequently need a solution – or several solutions – in order to study possible density distributions consistent with the known external gravity field or other data.

What we mean is that in principle it is not possible from purely gravimetric information to achieve a knowledge of the inner density $\rho$ going beyond the claim that $\rho$ has to belong to some equivalence class. The situation can be improved in no other way than by introducing more physical information. This notwithstanding it is sometimes desirable to have an element of the mentioned equivalence class, with nice functional properties, maybe incorporating some statistical information consistent with 'real world' data. We shall address the problem of determining this representer of the equivalence class or of an approximation to it (when we have only a finite set of data) as an 'estimation problem', in agreement with the statistical terminology which allows for biased estimates: consequently our 'estimates' may happen to be very far away from the true density, though satisfying the required mathematical properties.

Suppose we have given a reference density model, for example where the density depends only on the distance $r$ from the Earth's centre. The differences between this model and the actual density may be called the anomalous density. We will denote this by $\rho$ or $\delta\rho$, when it is important to distinguish between the total density and the 'density contrast'. The potential of the total density is denoted $V$ and the potential of the anomalous density is the anomalous potential, $T = \delta v$.

The observations $(y_i)$ we have will be related to $T$ or to $\rho$ through linear functionals, $L_i$, $i = 1, \ldots, n$, $y_i = L_i(\rho)$ or $y_i = L_i(T)$. A model for the density $\tilde{\rho}$ may be obtained in one of two different ways, either by supposing $\tilde{\rho}$ to be an element of a linear vector space with dimension $(m)$ smaller than $n$ or as an element of a space of dimension larger than $n$. In the first case a solution is found using a least squares principle, and in the second case by requiring the solution to have the least possible norm and being in agreement with the observations. In the first case a system of linear equations with $m$ unknowns has to be solved in order to find $\tilde{\rho}$. In the second case a subspace of dimension $n$ will be implicity given, and $\tilde{\rho}$ is found as an element of this space by solving a linear system of equations with $n$ unknowns.

This, however, requires that the norm is derived from an inner product. If this is not the case (e.g. when the supremum norm $L_\infty$ is used, cf. Backus 1971; Parker 1975), a solution may be found using linear programming techniques.

We will here only regard the situation where the norm is derived from an inner product and $n < m$. Here $m$ may be infinite. If $m$ is finite, then a solution is generally found in practice using the so-called Lanczos inverse or singular value decomposition. It is well known that this solution fulfils a minimum norm condition. If the solution space is spanned by $m$ orthonormal base functions, $f_i$, then

$$\tilde{\rho} = \sum_{j=1}^{m} x_j f_j$$

and

$$\|\tilde{\rho}\|^2 = \sum_{j=1}^{m} x_j^2$$

is minimum.

If the $L^2$ norm is used, then

$$\|\rho\|^2 = \int \rho^2 \, dB,$$

and a set of indicator functions $I_j(P)$ for non-overlapping blocks of volume $v_j$ will be orthogonal, i.e. the functions

$$\frac{1}{\sqrt{v_j}} I_j(P)$$

are orthonormal. A finite number of these functions span a reproducing kernel Hilbert space, with reproducing kernel

$$K(P, Q) = \sum_{j=1}^{m} I_j(P) I_j(Q)/v_j.$$

This kernel may also be interpreted as the covariance function of the density because all reproducing kernels are simultaneously covariance functions for the stochastic process spanning the Hilbert space (see Parzen 1959). It then expresses that density values within the same block are correlated with correlation coefficient equal to 1, and the densities in two different blocks are uncorrelated.

In a reproducing kernel Hilbert space, the collocation method may be used to find an approximation agreeing with given data. The solution is identical to the one obtained using $K(P, Q)$ as a covariance function and it is also identical to the one obtained using Lanczos' inverse (see Appendix). So even if some geophysicists do not think they use *a priori* information using the $L^2$-norm (see the discussion in Jackson 1979), in fact they do implicitly. This may be a good starting point for asking, which norm should we then use? Here it may be helpful to go to the general continuous case, $m \to \infty$.

When the blocks become smaller and smaller, but still fill the same set $B$, the covariance function behaves in the limit as one associated with a noisy stochastic process. We will in the following show that this is a process too noisy for practical use, considering the power spectrum of the anomalous gravity potential, $T$. Mathematical preparations for this analysis are given in Section 2 and the result in Section 3.

Consequently we look for stronger norms, involving derivatives of the density. Reproducing kernels for such norms may be expressed explicitly for the spherical part of the Earth. This is treated in Section 4.

Since these norms involve the derivatives of density anomalies, we cannot use the indicator functions $I_i(P)$ as basic building blocks when $m$ is finite. It is necessary to use blocks where, for example, the density may vary linearly with respect to the coordinates. An expression for the potential of such a linearly varying density is given in Section 5.

We have in this introduction supposed that the reader is familiar with concepts like Hilbert spaces and linear functionals. In the following we will furthermore use the concept of a Sobolev space $H^{i,j}(B)$, generally with $j = 2$. $B$ is a (bounded) set in $R^n$, the superscript $i$ indicating that the derivatives up to and inclusive order $i$ are elements of $L^j(B)$. A subscript 0 indicates that the functions and their derivatives up to and inclusive of order $i$ are zero.

The superscript $i$ does not need to be an integer. A non-integer value derivative is obtained using the Fourier transform or the Fourier transform multiplied by an appropriate constant per frequency.

## 2 General solution of the inverse problem

Suppose $S$ is the surface of the Earth, $\Omega$ the set outside the Earth and $B$ the set inside the Earth. The inverse gravimetric problem then consists, for a given $V$ on $S$ (or in $\Omega$ via the Dirichlet problem), to find $\rho$ in $B$ so that

$$V(P) = \int_B \frac{\rho(Q)}{r_{PQ}} dB_Q, \qquad \text{for } P \in S. \tag{2.1}$$

$r_{PQ}$ is the distance from $P$ to $Q$. Inverse gravimetric problems which presupose the use of other data types may be converted to this by first constructing an approximation to $V$ (using, e.g. least squares collocation).

Before going into mathematical details it might be useful to recall the definition of Sobolef spaces. We define $H^{s,p}$ ($s$ an integer) as the space of functions $h$ having distributional derivatives up to order $s$ belonging to $L^p$, $h \in H^{s,p} \leftrightarrow \{h, \partial h, \ldots, \partial^s h \in L^p\}$: the norm in these spaces is given by

$$\|h\|_{s,p}^p = \sum_0^s{}_k \sum_{|\alpha|=k} \|\partial^\alpha h\|_{L^p}^p$$

($\alpha$ = multi-index of the derivative of order $|\alpha|$). In particular when $p = 2$ we define the Hilbert spaces $H^{1+s,2}(B)$ $(B \in R^n)$ for any real $0 < s < 1$ (in this case we shall skip the exponent 2) as those subspaces of $H^{1,2}(B)$ for which the functionals

$$\|\partial^\alpha h\|_s = \int_B \int_B \frac{|\partial^\alpha h(x) - \partial^\alpha h(y)|}{|x-y|^{n+2s}} \, dB_x \, dB_y$$

($\partial^\alpha h(x)$ = derivative of $h$ corresponding to a multi-index $\alpha$ of order $|\alpha| = 1$) happen to be bounded. The norm in such spaces is defined as

$$\|h\|_{1+s}^2 = \|h\|_1^2 + \sum_{|\alpha|=1} \|\partial^\alpha h\|_s^2.$$

The same definition carries over to subsets of manifolds of dimension $n$.

*Theorem 1.* If $S$ is endowed with the continuous differentiable normal $\mathbf{n}$, and if $V \in H^{3/2}(S)$, then there exists at least one $\tilde{\rho} \in L^2(B)$ such that (2.1) is satisfied (see Sansò 1980, appendix 3).

*Theorem 2.* If (2.1) has one solution $\tilde{\rho} \in L^2(B)$, and if $S$ has a continuous normal $\mathbf{n}$, then the class of all the solutions of (2.1) in $L^2(B)$ can be represented as

$$\rho = \tilde{\rho} + \Delta h \qquad (\Delta \text{ is the Laplacian}), \tag{2.2}$$

where

$$h \in H_0^{2,2}(B)$$

(i.e. $h \in H^{2,2}(B)$ and $h|_S = 0$, $(\partial h/\partial n)|_S = 0$). In other words $\Delta h$ $[h \in H_0^{2,2}(B)]$ represents the class of $L^2$ distributions generating a null external field.

(It is easy to prove that $\rho = \Delta h$ generates no external field using Green's third identity with $P$ in $\Omega$.

$$4\pi\, V(P) = \int_B \frac{\Delta h(Q)}{r_{PQ}} \, dB = -\int_S \left[\frac{\partial h(Q)}{\partial n} \frac{1}{r_{PQ}} - h(Q) \frac{\partial}{\partial n} \frac{1}{r_{PQ}}\right] dS \equiv 0,$$

i.e. $V \equiv 0$ on $\Omega$ and whence on $S$.)

Given the non-uniqueness of the solution (2.2) we have to define some extra criterion in order to fix one solution.

For purely geodetic purposes it seems that this criterion could be completely arbitrary, as the goal is a good interpolation of the external gravity field. Nevertheless knowledge of the internal density as derived from gravity data is of interest in itself, and even from the geodetic point of view the use of qualitatively reasonable densities has an impact on the behaviour of the approximating field at the boundary. The behaviour of the field generated by a bounded density is different from that generated by an $L^2$ density and from the behaviour of the field of a single layer, especially near a rough boundary.

Typical criteria are:

the so-called minimum energy solution;
some minimum norm solution;
the restriction of the solution space,

or combinations of these criteria as, e.g. introduced in 'mixed collocation' (see Sansò & Tscherning 1982).

In the following section we will discuss these possibilities in more detail.

## 3 Uniqueness criteria

### 3.1 MINIMUM ENERGY SOLUTION

The idea is to minimize the energy from

$$E = \int_B V(B)\, d\mu_P \tag{3.1}$$

associated with the field generated by some measure $d\mu_P$ with support in the closure of $B$,

$$V(P) = \int_B \frac{1}{r_{PQ}}\, d\mu(Q), \tag{3.2}$$

under the constraint that $V(P)$ agrees with the given function $\bar{v}(P)$ on $S$. Thus we are led to the minimum principle, with Lagrange multiplier $\lambda(P)$,

$$\text{Min} \left[\!\left[ \frac{1}{2} \int_B \int_B \frac{1}{r_{PQ}}\, d\mu(P)\, d\mu(Q) + \int_S \lambda(P) \left[ \int_B \frac{d\mu(Q)}{r_{PQ}} - \bar{v}(P) \right] dS_P \right]\!\right], \tag{3.3}$$

This gives ($\delta\mu$: small change in $\mu$):

$$\int_{\bar{B}} \left[ \int_{\bar{B}} \frac{d\mu(P)}{r_{PQ}} + \int_S \frac{\lambda(P)}{r_{PQ}}\, dS_P \right] d\delta\mu(Q) = 0 \qquad \text{for all } \delta\mu,$$

so that

$$\int_{\bar{B}} \frac{d\mu(P)}{r_{PQ}} + \int_S \frac{\lambda(P)}{r_{PQ}}\, dS_P = 0 \qquad \text{for } Q \in \bar{B},$$

and

$$V(Q) = \int \frac{d\mu(P)}{r_{PQ}} = -\int_S \frac{\lambda(P)}{r_{PQ}}\, dS_P. \tag{3.4}$$

Hence $V(Q)$ in $\bar{B}$ is the potential of a single layer with density $\lambda(P)$ (i.e. it is harmonic in $B$), and since $d\mu$ has to have support in $\bar{B}$ and $\mu = 0$ in $B^0$, $V$ is the potential of a single layer everywhere.

The density $\lambda(P)$ is determined using the condition

$$V(Q) = -\int_S \frac{\lambda(P)}{r_{PQ}}\, dS_P = \bar{v}(Q),$$

i.e. by solving the Dirichlet problem for $v^+$ and $v^-$ in $B$ and in $\Omega$ $(= B^c)$ with boundary values

$\bar{v}(P)$ on $S$ and then putting

$$\lambda(P) = \frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n}.$$

The minimum energy problem is a slight generalization of the so-called equilibrium distribution of a charge $e$ on a conductor $B$. The difference is that for electrical charges the potential is repulsive and not attractive as for masses. Indeed this has no impact on the form of the functional to be minimized, but it has a strong impact on its sign – in fact this is reversed.

In geodesy we define the force $\mathbf{F} = \nabla V$, but in physics the definition $\mathbf{F} = -\nabla V$ is used. So the potential energy of a mass distribution is in the physical sense negative. So what we have found is in fact a 'maximum energy' distribution. This can also be characterized by the following physical reasoning: a charge distribution of minimum energy on $S$ is in equilibrium because it cannot be changed without supplying work from the exterior, whereas if a mass is in any way distributed on $S$ it can supply work to the exterior by moving to an inner surface $S'$. In conclusion: the so-called 'minimum energy' solution has no physical meaning.

### 3.2 THE $L^2$ MINIMUM NORM SOLUTION

We first regard the $L^2$-norm, which has been discussed already in Krarup (1978) and Sansò (1980). We want to find a solution to (2.1) so that

$$\int \rho^2 \, dB = \min \tag{3.5}$$

$$\int \frac{\rho(Q)}{r_{PQ}} \, dB_Q = \bar{v}(P).$$

In variational form

$$\delta \left[\!\left[ \frac{1}{2} \int_B \rho^2 \, dB + \int \lambda(P) \left[ \bar{v}(P) - \int \frac{\rho(P)}{r_{PQ}} \, dB_Q \right] dS_P \right]\!\right] = 0$$

so that

$$\rho(Q) = \int \frac{\lambda(P)}{r_{PQ}} \, dS_P \tag{3.6}$$

which implies that

$$\Delta\rho = 0 \qquad \text{in } B.$$

The solution of the problem is obtained by solving an integral equation of the first kind, namely by putting (3.6) in (3.5)

$$\left.\begin{aligned} &\int G(P, P') \, \lambda(P) \, dS_P = \bar{v}(P') \\ &G(P, P') = \int \frac{1}{r_{PQ}} \frac{1}{r_{P'Q}} \, dB_Q \end{aligned}\right\} . \tag{3.7}$$

An explicit solution is easily obtained for the sphere. Suppose, with $Y_{nm}$ the surface spherical functions, and $\sigma_Q$ the projection of $Q$ on the unit sphere

$$\bar{v}(Q) = \sum_n \sum_m \bar{v}_{nm} \, Y_{nm}(\sigma_Q)$$

$$\rho(P) = \Sigma \, \Sigma \, \rho_{nm} \left(\frac{r}{R}\right)^n Y_{nm}(\rho_P) \qquad (\Delta\rho = 0).$$

Since

$$\frac{1}{r_{PQ}} = \frac{1}{r_Q} \Sigma \left(\frac{r_Q}{R}\right)^{n+1} P_n \,(\cos\phi)$$

with $P$ on the sphere and $Q$ inside, then

$$\frac{1}{r_{PQ}} = \frac{1}{R} \Sigma\Sigma \left(\frac{r_Q}{R}\right)^n (2n+1)^{-1} \, Y_{nm}(\sigma_P) \, Y_{nm}(\sigma_Q)$$

and

$$\begin{aligned}\int \rho \frac{1}{r_{PQ}} \, dB_Q &= \frac{1}{R} \Sigma\Sigma \frac{Y_{nm}(\sigma_P)}{(2n+1)R^n} \int_0^R r_Q^{n+2} \iint \rho(P) \, Y_{nm}(\sigma_Q) \, dr_Q \, d\sigma_Q \\ &= \Sigma\Sigma \frac{Y_{nm}(\sigma_P)}{(2n+1)R^{n+1}} \frac{\rho_{nm}}{R^n} \int_0^R r_Q^{2n+2} \, dr_Q \\ &= \Sigma\Sigma \, Y_{nm}(\sigma_P) \frac{\rho_{nm} \, R^2}{(2n+1)\,(2n+3)} = \Sigma\Sigma \bar{v}_{nm} \, Y_{nm}(\sigma_P).\end{aligned}$$

Hence

$$\rho_{nm} = \frac{(2n+1)\,(2n+3)}{R^2} \bar{v}_{nm}. \tag{3.8}$$

The condition $\rho \in L^2(B)$ implies

$$\begin{aligned}\|\rho\|^2 &= \int_B \rho^2 r^2 \, dr \, d\sigma \\ &= \Sigma\Sigma\rho_{nm}^2 \frac{R^3}{(2n+3)} < \infty\end{aligned}$$

or

$$R \, \Sigma \, (2n+1)^2 \, (2n+3) \, \bar{v}_{nm}^2 < \infty,$$

i.e. $\bar{v}_P \in H^{3/2}(S)$ (cf. Theorem 1).

Since it is somewhat problematic to use harmonic densities, we will now explicitly work with density anomalies.

We assume we know not only the external potential, but also the average density at all radial distances. We also suppose $S$ is a sphere of radius $R$($\sigma$, the unit sphere).

$$\bar{\rho}(r) = \int \rho(r, \sigma) \, d\sigma. \tag{3.9}$$

We may therefore go to variations

$$\delta\rho = \rho - \bar{\rho} \tag{3.10}$$

$$\delta\bar{v} = \bar{v}(P) - \frac{\bar{\mu}}{R} \left(\text{with } \bar{\mu} = \int_0^R \bar{\rho}\, r^2\, dr\right), \qquad P \in S. \tag{3.11}$$

Note that we have the consistency condition

$$\delta\bar{v}\, d\sigma = \delta\bar{v}_{00} = 0. \tag{3.12}$$

Consequently we have the problem

$$\int \delta\rho^2\, dB_Q = \min \tag{3.13}$$

$$\int \frac{\delta\rho(Q)}{r_{PQ}}\, dB_Q = \delta\bar{v}(P), \qquad P \in S \tag{3.14}$$

$$\int \delta\rho\,(r, \sigma)\, d\sigma = 0, \qquad \forall\, r \leqslant R. \tag{3.15}$$

Equation (3.15) is fulfilled by setting

$$\left.\begin{aligned} &\int \delta\rho \quad = \Sigma\, \delta\rho_{nm}(r)\, Y_{nm}(\sigma) \\ &\delta\rho_{00}(r) \equiv 0 \qquad 0 \leqslant r \leqslant R \end{aligned}\right\}. \tag{3.16}$$

Then from (3.13), (3.14) we derive for all $\delta\rho$ of the form (3.16)

$$\frac{1}{2} \int \delta\rho^2\, dB_Q + \int \lambda(P) \int \frac{\delta\rho(Q)}{r_{PQ}}\, dB_Q\, dS_P = \min$$

which implies

$$\delta\rho(Q) + \int \frac{\lambda(P)}{r_{PQ}}\, dS_P = \mu\, Y_{00}$$

(which we could avoid by stipulating $\lambda_{00} = 0$) so that

$\Delta\delta\rho = 0.$

From this we have for $\delta\rho$ the model expression

$$\delta\rho = \Sigma\Sigma\, \delta\rho_{nm} \left(\frac{r}{R}\right)^n Y_{nm}(\sigma)$$

and (3.8) holds for the variations

$$\delta\rho_{nm} = \frac{(2n+1)(2n+3)}{R^2}\, \delta\bar{v}_{nm}.$$

A trivial criticism against the $L^2$-norm approach is that by generating a harmonic density anomaly $\delta\rho$ we get a function attaining extremal values at the boundary. This property is not acceptable for the total density $\rho$; it might be reasonable for the density anomaly $\delta\rho$ since in the interior of the Earth with the increase of the pressure we expect a higher homogeniety of the masses. Nevertheless, the criticism is still valid since if we go to the discrete version of the normal equations (3.7), namely (cf. Sansò & Tscherning 1982)

$$\left.\begin{aligned} &\Sigma\, \lambda_i G(P_i, P_k) = \delta\bar{v}(P_k) \\ &\qquad \delta\rho = \Sigma\, \lambda_i \frac{1}{r_{P_iP}}, \qquad \delta v = \Sigma\, \lambda_i\, G(P_i, P) \end{aligned}\right\} \tag{3.17}$$

(here we use only the evaluation functionals as examples) we accordingly find a density which becomes unbounded at the measure-points $P_i$ and this is an odd feature.

Fortunately the behaviour of $\delta\rho$ near the boundary or at the origin can be partly controlled by introducing a suitable weight function $f(r)$ in the minimum principle, according to

$$\int_B f(r)\,(\delta\rho)^2\, dB = \min,$$

(see Tscherning & Sünkel 1981).

## 3.3 STATISTICAL INTERPRETATION

The minimum principle (3.13) with side conditions (3.14) and (3.15) has a statistcal interpretation. First note that, by defining the average on the sphere as $M[\cdot]$, we have

$$\bar{\rho}(r) = M[\rho(r, \sigma)|_r],$$

and we see that the function minimized is

$$\begin{aligned} V &= \int_r r^2 \int \delta\rho^2\, d\sigma\, dr \\ &= \int r^2\, M[(\rho - \bar{\rho})^2\,|_r]\, dr. \end{aligned}$$

This can be interpreted as a weighted (by the factor $r^2$) sum of variances on $\rho$ on each sphere of radius $r$. We stress that, due to the form of the above 'least squares' principle, an hypothesis of uncorrelation between $\rho(r, \sigma)$, $\rho(r', \sigma')$ (white noise!) is implicitly assumed. It is because of the weakness of this topology that the drawback explained in (3.17) appears.

We can find the covariance function of $\delta\rho$, since it is identical to the reproducing kernel of the space of harmonic functions in $L^2(B)$. The internal solid spherical harmonics are orthogonal in $L^2(B)$, but not normalized:

$$\begin{aligned} \int_B Y_{nm}(\sigma)^2 \left(\frac{r}{R}\right)^{2n} r^2\, dr\, d\sigma &= \int_0^R r^2 \left(\frac{r}{R}\right)^{2n} dr \\ &= R^3/(2n+3). \end{aligned}$$

The reproducing kernel is then

$$K_0(P, Q) = \frac{1}{R^3}\, \Sigma\Sigma\, (2n+3) \left(\frac{r_P r_Q}{R^2}\right)^n Y_{nm}(\sigma_P)\, Y_{nm}(\sigma_Q). \tag{3.18a}$$

This function is bounded for $r_P, r_Q < R$ and unbounded when $r_P = r_Q = R$. As our $\delta\rho$ has zero component $\delta\rho_{00} = 0$, we can ignore the summation for the zero degree in (3.18a).

Corresponding to this we get the kernel $G(P, Q)$ for the potential, namely

$$G(P, Q) = R \sum\sum \frac{1}{(2n+1)^2 (2n+3)} Y_{nm}(\sigma_P) Y_{nm}(\sigma_Q) \left( \frac{R^2}{r_P r_Q} \right)^{n+1} \tag{3.18b}$$

$$= \sum_{n=1}^{\infty} \frac{R}{(2n+1)(2n+3)} \left( \frac{R^2}{r_P r_Q} \right)^{n+1} P_n (\cos \psi_{PQ}).$$

This covariance function has degree variances

$$\sigma_n = \frac{R}{(2n+1)(2n+3)}$$

tending to zero like $n^{-2}$. A statistical analysis of the Earth's gravity field (see Forsberg 1984; Schwarz 1985), shows that $\sigma_n$ has to tend to zero like $n^{-3}$ or $n^{-4}$. (The famous Kaula's rule corresponds to $n^{-3}$.) So in order to have a more regular kernel for the potential, we must have a more regular one for $\delta\rho$. This is achieved by strengthening the topology.

### 3.4 USE OF THE $H^{1,2}$-NORM

We may require a linear combination of the density and of its derivatives to be minimized. We will therefore require that $\delta\rho$ is differentiable. Then

$$\int_B [a(\delta\rho)^2 + b|\nabla\delta\rho|^2]\, dB = \min \tag{3.19}$$

$$\int \frac{\delta\rho}{r_{PQ}}\, dB_Q = \delta\bar{v}(P) \qquad P \in S, \tag{3.20}$$

$$\int \delta\rho(r, \sigma)\, dr = 0 \qquad r < R, \tag{3.21}$$

where $a$ and $b$ are weight functions to be chosen at will. However we shall assume that $a$ and $b$ depend only on $r$. In this case the general variational equation for (3.19)–(3.21) is

$$a\, \delta\rho - \nabla(b \nabla \delta\rho) = \int \frac{\lambda(Q)}{r_{PQ}}\, dS_Q + \eta(r) \tag{3.22}$$

where the two Lagrange multipliers are to be determined by the side conditions (3.20), (3.21) and $\delta\rho$ has to satisfy the Neumann condition

$$b \left.\frac{\partial(\delta\rho)}{\partial n}\right|_S = 0 \quad \text{or} \quad \left.\frac{\partial(\delta\rho)}{\partial n}\right|_S = 0. \tag{3.23}$$

We shall not analyse (3.22) directly, because there is an argument against the use of (3.19) as it is.

The layered structure of the Earth's interior makes radial differentiability unlikely. Even if the average radial model $\bar{\rho}(r)$ already accounts for most of the radial discontinuity there is no reason why a density anomaly should propagate across the boundary of such discontinuities. We therefore drop the radial derivative part of the gradient, and assume from now on that $B$ is a sphere. Then we get a modified minimum principle

$$\int_B [a\delta\rho^2 + b\,|\nabla_t\delta\rho|^2]\,dB = \min \tag{3.24}$$

where only the horizontal gradient

$$\nabla_t\delta\rho = \frac{1}{r}\,\nabla_\sigma\delta\rho \qquad \left(\nabla_\sigma = \begin{Bmatrix} \dfrac{\partial}{\partial\varphi} \\ \dfrac{1}{\cos\varphi}\,\dfrac{\partial}{\partial\lambda} \end{Bmatrix}\right) \tag{3.25}$$

enters.

This approach can have a kind of physical interpretation if we refer to a model of the earth behaving like a viscous fluid and disregarding the centrifugal force: in this case in fact a radially layered density model should constitute an equilibrium figure and we may expect that any lateral density variation generates forces tending to restore such equilibrium. These forces, if they are thought of as generated by some transport phenomenon, can be expected to be more intense where the lateral density variation is steeper, so that we could imagine a real situation where in mean square values the average density contrasts and its horizontal gradient are small.

Coming back to the mathematical treatment, if we now write the variational equation for the functional (3.24) we get

$$a\,\delta\rho - \frac{b}{r^2}\,\Delta_\sigma\delta\rho = \int \frac{\lambda(Q)}{r_{PQ}}\,dS_Q + \eta(r), \tag{3.26}$$

where $\Delta_\sigma$ is the Laplace-Beltrami operator.

Integrating (3.26) over $d\sigma$ and taking into account that from (3.21)

$$\int \delta\rho\,d\sigma = 0$$

and from Green's identity applied to the sphere of radius $r$,

$$\int \Delta_\sigma\delta\rho\,d\sigma = \int \delta\rho\,\Delta_\sigma(1)\,d\sigma = 0$$

we get

$$0 = \lambda_{00} + \eta(r)$$

or

$$\eta(r) = -\lambda_{00}.$$

Subsequently (3.26) can be written as

$$\left.\begin{aligned} a\,\delta\rho - \frac{b}{r^2}\,\Delta_\sigma\delta\rho &= \int \frac{\bar{\lambda}(Q)}{r_{PQ}}\,dS \\ \bar{\lambda}(Q) &= \lambda(Q) - \lambda_{00} \end{aligned}\right\} \tag{3.27}$$

where $\bar{\lambda}(Q)$ should be determined from conditions (3.20).

Instead of going through a direct determination, we prefer first to fix $a$ and $b$. A particularly simple choice is

$$a = \text{constant}, \quad b = 4ar^2 \tag{3.28}$$

where the reason for factor 4 will become clear later.

Under these circumstances we have

$$a(I - 4\Delta_\sigma)\delta\rho = \int \frac{\bar{\lambda}(Q)}{r_{PQ}}\, dS_Q. \tag{3.29}$$

Since on the other hand

$$\Delta(I - 4\Delta_\sigma) = (I - 4\Delta_\sigma)\Delta,$$

we see that

$$(I - 4\Delta_\sigma)\Delta\delta\rho = 0. \tag{3.30}$$

As, moreover (in the sense of positive operators)

$$(I - 4\Delta_\sigma) > 1,$$

(3.30) gives

$$\Delta\delta\rho = 0. \tag{3.31}$$

Thus the choice of (3.28) gives us a solution $\delta\rho$ which is again harmonic, and since it must satisfy

$$\int_B \frac{\delta\rho(Q)}{r_{PQ}}\, dB_Q = \delta\bar{v}(P) \quad \text{given on } S,$$

$\delta\rho$ is uniquely determined and coinciding with the solution found by the $L^2$-norm.

It might seem at this point that the use of the norm (3.24) is not a useful approach, but this is not so!

The solution $\delta\rho$ when $\delta\bar{v}(P)$ is given over the complete spherical surface $S$ is the same as before, but we are making the hypothesis that $\delta\bar{v}$ is so regular that $\delta\rho$ belongs to a restricted subspace of $L^2(B)$, namely to the space of harmonic functions for which (2.24) is bounded.

With the choice (3.28), the norm (3.24) is easily computed for a harmonic function, so that

$$\begin{aligned}\|\delta\rho\|_1^2 &= a \int_B [\delta\rho + 4|\Delta_\sigma\delta\rho|^2]\, dB \\ &= a \int_0^R \int_\sigma [\delta\rho^2 - 4\delta\rho\Delta_\sigma\delta\rho]\, d\sigma\, r^2\, dr \\ &= a R^3 \sum_{\substack{n=0\\ m=0}} \frac{(2n+1)^2}{2n+3}\, \delta\rho_{nm}^2\end{aligned} \tag{3.32}$$

($\delta\rho_{00} = 0$ as a consequence of (3.27)).

From (3.32) we see that the space with norm $\|\ \|_1$ has the reproducing kernel

$$K_1(P, Q) = \frac{1}{R^3} \sum_{n=0} \frac{2n+3}{(2n+1)^2} \left(\frac{r_P r_Q}{R^2}\right)^n Y_{nm}(\sigma_P) Y_{nm}(\sigma_Q) \tag{3.33}$$

which is a well behaving even at the boundary, contrary to $K_0(P, Q)$ which was the kernel corresponding to the $L^2$-norm.

As a consequence of the regularity of $K_1$ we have in the case of discrete data a more regular estimate of $\delta\hat{\rho}$. We can see this at least in the case of the evaluation functionals (operating on $\delta v$)

$$\min \|\delta\hat{\rho}\|_1^2 \tag{3.34}$$

$$L_i(\delta\hat{\rho}) = \int \frac{1}{r_{P_iQ}} \delta\hat{\rho}(Q)\, dB_Q = \delta\bar{v}(P_i), \qquad i = 1, \ldots, n. \tag{3.35}$$

We find $\delta\hat{\rho}$ as a linear combination of the Riez representers of the functionals $L_i$ which are obtained using (3.33) and (3.8) (and $r_{P_i} = R$)

$$K_1(L_i, Q) = \frac{1}{R} \sum_{n=0} \frac{1}{(2n+1)^3} Y_{nm}(\sigma_{P_i}) Y_{nm}(\sigma_Q) \left(\frac{r_Q}{R}\right)^n.$$

Since these functions are bounded, $\delta\hat{\rho}$ is bounded. Nevertheless, if at $P_i$ we had measured the gravity disturbance, we would have

$$K_1(\delta g_{P_i}, Q) = \frac{1}{R} \sum_{n=0} \frac{n+1}{R} \frac{1}{(2n+1)^3} Y_{nm}(\sigma_{P_i}) Y_{nm}(\sigma_Q) \left(\frac{r_Q}{R}\right)^n,$$

and this is not now bounded on $S$.

Note, finally, that to $K_1(P, Q)$ corresponds a reproducing kernel for the exterior potential, $G(P, Q)$. On $S$ it becomes, using (3.8) twice

$$G_1(P', Q') = R \sum_{n=0} \frac{1}{(2n+1)^4 (2n+3)} Y_{nm}(\sigma_P) Y_{nm}(\sigma_{Q'}).$$

This entails $\delta v \in H^{5/2}(S)$ as was to be expected.

## 4 General principles

At this point it would have been possible to go to higher-order norms, constructed with differential operators, which in some cases would have involved additional side conditions (see e.g. Skorvanek 1981). Instead we will summarize what has been done in more general terms in a few principles.

(1) We decide to work with harmonic density anomalies since these correspond to reasonable minimum principles with suitable choices of the free parameters. In this way the solution $\delta\rho$ for a given field $\delta v$ on $S$ is fixed and unique; indeed $\delta\rho$ will satisfy a certain minimum principle with a norm of order $n$ ($n$ denotes the number of derivatives of $\delta\rho$ implied) if $\delta v$ is accordingly suitably smooth. The fact that $\delta\rho$ is unique does not, however, fix the estimate $\delta\hat{\rho}$ when a discrete (finite) number of observations is given. In fact we can have a family of $\delta\hat{\rho}$ according to the norms chosen.

(2) We classify the norms for the estimates $\delta\hat{\rho}$ according to the reproducing kernels of the corresponding spaces We assume that these have the form

$$K_k(P, Q) = \Sigma F_k(n) Y_{nm}(\sigma_P) Y_{nm}(\sigma_Q) \left(\frac{r_P r_Q}{R^2}\right)^n, \tag{4.1}$$

where $F_k(n)$ is a positive, never zero, rational function of $n$ satisfying

$$F_k(n) \underset{n\to\infty}{\sim} n^{-k}. \tag{4.2}$$

To the reproducing kernel (4.1) for $\delta\rho$ corresponds a reproducing kernel $G_k(P, Q)$ for $\delta v$, namely

$$G_k(P, Q) = R^4 \ \Sigma \ \frac{F_k(n)}{(2n+3)^2\,(2n+1)^2} \ Y_{nm}(\sigma_P)\,Y_{nm}(\sigma_Q). \tag{4.3}$$

Moreover, always fixing $K_k(P, Q)$, we fix the representer of the evaluation of the anomalous potential $\delta v$ at the point $P_i$, namely

$$\begin{aligned} \delta v(P_i) &= \int \frac{1}{r_{P_iQ}}\,\delta\rho\, dB_Q = R^2 \ \Sigma \ \frac{\delta\rho_{nm}}{(2n+3)\,(2n+1)} \ Y_{nm}(\sigma_{P_i}) \\ &= \langle g_{P_i}, \delta\rho \rangle_k \qquad (r_{P_i} = R \Leftrightarrow P_i \in S), \end{aligned} \tag{4.4}$$

where

$$g_{P_i}(Q) = \Sigma \ \frac{F_k(n)}{(2n+3)\,(2n+1)} \ Y_{nm}(\sigma_{P_i})\,Y_{nm}(\sigma_Q) \left(\frac{r_Q}{R}\right)^n. \tag{4.5}$$

All the other functionals on $\delta v$, $L_i(\delta v)$, have representers $L_{P_i}(g_{P_i})$, since

$$L_{P_i}\delta u = \langle L_{P_i}\,g_{P_i}, \delta\rho \rangle_k. \tag{4.6}$$

However for some functionals (e.g. $(\partial/\partial r)|_{P_i}$) it might be necessary to continue analytically $g_{P_i}$ in $r_{P_i} > R$ and then restore $r_{P_i} = R$.

(3) The sought (collocation) solution is given by

$$\delta\hat{\rho} = \Sigma\lambda_i(L_i g_{P_i}), \tag{4.7}$$

where the constants $\lambda_i$ are determined from

$$\{\langle L_i g_{P_i},\ L_j g_{P_j}\rangle_k\}\ \{\lambda_j\} = \{L_i\,\delta v(P_i)\}, \tag{4.8}$$

and $\langle L_i\, g_{P_i}, L_j g_{P_j}\rangle_k = K_k(L_i, L_j)$. Such expressions may in many cases be calculated using closed expressions (see Tscherning 1976).

## 5 Mixed collocation and restriction of the solution space

The general principles discussed in Section 4 can only be applied on a spherical earth model. However they may be combined with the method of restricting the solution space, which is the only technique widely used in practice to invert gravimetric data. This combined technique is called 'mixed collocation' (see Sansò & Tscherning 1982).

The general idea of mixed collocation is to combine the 'internal collocation' method using a Hilbert space of density functions with the classical 'external collocation' for the anomalous potential using a 'spherical' covariance function. The difficulty is to apply the minimum norm principle to a body like the Earth of a quite complicated form. This refers not only to the shape of the topographic masses, but it might also include all the masses between the topographic surface and the Moho.

For this reason it is convenient to treat the outer layers in discrete form, i.e. to restrict the solution space for the contribution of these masses to the outer potential. However, we keep the analytic representation inside some sphere within the earth ( a Bjerhammar sphere).

Outside this sphere we may use and combine the potential generated by a simple density distribution in a simple geometrical figure, such as rectangular boxes. Previously, mainly boxes with constant density have been used, but it has been proposed also to use densities varying linearly with the height (see, e.g. Murthy & Rao 1979).

If we want to use one of the higher-order norms for the density inside the Bjerhammar sphere, we should also use the same type of norm for these boxes. We must therefore also use density distributions at least varying linearly with the (local) coordinates $(x, y, z)$, because otherwise the derivatives will not give any contribution to the norm. We will therefore derive the expression for the potential of a density function depending linearly on one of the coordinates $(z)$.

To this aim we can use an indefinite integral of $1/r$ (see MacMillan 1958, section 4.3),

$$\int \frac{1}{\sqrt{x^2+y^2+z^2}}\, dx\, dy\, dz = f_0(x, y, z), \tag{5.1}$$

i.e. a particular solution of the equation

$$\frac{\partial^3 f_0}{\partial x\, \partial y\, \partial z} = \frac{1}{\sqrt{x^2+y^2+z^2}}. \tag{5.2}$$

One such particular integral (as a matter of fact the only one symmetrical with respect to $(x, y)$, $(x, z)$ and $(y, z)$) is

$$f_0(x, y, z) = xy \log(z + r) + xz \log(y + r) + yz \log(x + r)$$
$$- \frac{x^2}{2} \operatorname{arctg} \frac{yz}{xr} - \frac{y^2}{2} \operatorname{arctg} \frac{xz}{yr} - \frac{z^2}{2} \operatorname{arctg} \frac{xy}{zr} \tag{5.3}$$

with $r^2 = x^2 + y^2 + z^2$.

The solution can be used to compute the definite integral

$$\int_{z^-}^{z^+} \int_{y^-}^{y^+} \int_{z^-}^{x^+} \frac{dx_0\, dy_0\, dz_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} = \Pi_0(x, y, z) \tag{5.4}$$

by evaluating the function $f_0(x - x_0, y - y_0, z - z_0)$ with respect to $x_0, y_0, z_0$ between the given limits.

Now we want to generalize (5.3), (5.4) slightly in order to compute

$$\int_{z^-}^{z^+} \int_{y^-}^{y^+} \int_{x^-}^{x^+} \frac{(\rho_0 - z_0\rho_1)\, dx_0\, dy_0\, dz_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$

$$= \int \frac{(\rho_0 + \rho_1 z) + \rho_1(z - z_0)}{\sqrt{(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2}}\, dV_0$$

$$= (\rho_0 + \rho_1 z)\, \Pi_0(x, y, z) + \rho_1\, \Pi_1(x, y, z). \tag{5.5}$$

It remains to compute $\Pi_1$ and to this aim we must evaluate the indefinite integral

$$\iiint \frac{dx\, dy\, z\, dz}{\sqrt{x^2+y^2+z^2}} = f_1(x, y, z).$$

This is easily done, noting that

$$\iint \frac{dx\,dy}{\sqrt{x^2+y^2+z^2}} = \frac{\partial f_0(x,y,z)}{\partial z},$$

so that

$$f_1(x,y,z) = \int \frac{\partial f_0}{\partial x}\, z\, dz$$

$$= {}^1\!/_3\, r\,xy - \frac{z^3}{3}\operatorname{arctg}\frac{xy}{zr} + \frac{r^2(x+y)}{4} - \frac{x^2}{3}\log(r-y) - \frac{y^3}{3}\log(r-x)$$

$$+ \frac{x(3z^2-x^2)}{6}\log(y+r) + \frac{y(3z^2-y^2)}{6}\log(x+r). \tag{5.6}$$

The approximation $\tilde{T}$ to the anomalous potential will then consist of two parts

$$\tilde{T} = \tilde{T}_t + \tilde{T}_S, \tag{5.7}$$

where $\tilde{T}_t$ is the topographic part and $\tilde{T}_S$ the spherical part. $\tilde{T}_t$ is modelled in blocks $B_k$ (with index $k$) so that

$$\tilde{T}_t(P) = \sum_k \left[\!\left[ \rho_{0k}\, \Pi_{0k}(P) + \sum_{j=1}^{3} \rho_{jk}\,[x_j\, \Pi_{0k}(P) + \Pi_{jk}(P)] \right]\!\right] \tag{5.8}$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$ and $\Pi_{jk}$ is the potential of the density with $x_j$ varying linearly.

$T_S$ belongs to a space with given reproducing kernel $K(P, Q)$.

The variational principle giving the solution is then

$$L_i T(P_i) = \sum_k \left[\!\left[ \rho_{0k}\, L_i\, \Pi_{0k}(P_i) + \Sigma \rho_{jk}\, L_i\, [x_j\, I_{0k}(P_i) + \Pi_{jk}(P_i)] \right]\!\right] + L_i T_S(P_i) \tag{5.9}$$

(observation equations)

$$\sum_k \left[ A_k \rho_{0k}^2 + \sum_j (B_{jk}\, \rho_{jk}^2 + 2C_{jk}\, \rho_{0k}\, \rho_{jk}) \right] + \| T_S \|_H^2 = \min \tag{5.10}$$

where the weights $A_k$, $B_{jk}$, $C_{jk}$ can be chosen as

$$A_k = \int_{B_k} dB, \quad B_{jk} = \int_{B_k} x_j^2\, dB, \quad C_{jk} = \int_{B_k} x_j\, dB,$$

so that the first sum in (5.10) is the $L^2$-norm of the density $\rho_{0k} + \Sigma \rho_{jk} x_j$ in the block $B_k$. Constants corresponding to norms involving the derivatives with respect to $x_j$ are easily computed.

## 6 Conclusions

We have here seen how minimum principles different from the $L^2$-norm can be introduced. The corresponding norms should involve the square sum of derivatives in order to produce a covariance function which is realistic. As a consequence of this, density models with boxes of constant density should not be used exclusively. Linearly varying densities may be introducted, the potential of which are not more complicated than that of a usual rectangular box.

Using the minimum principle (norm) (5.10) the densities in the individual blocks will still be uncorrelated, but the correlation of the densities within a block will vary. Correlations between blocks may be imposed, e.g. by letting the linear part extend to all neighbouring blocks.

The effect of using such density models on the solution of inverse gravimetric problems remains to be seen. It should, however, give a possibility for a more simple and realistic density modelling, since the density frequently is not exactly horizontally stratified, but may have a considerable slope.

The density models may be evaluated using seismic data. These should be used in order to select the most realistic norm. This, on the other hand, could be used to find a norm, which the Earth's density minimalizes. This again should give us information about elastic properties fulfilled by the Earth – if any are fulfilled.

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## Appendix: identity between collocation solution and solution obtained using Lanczos inverse (generalized inverse)

Suppose

$$\delta\rho = \sum_{j=1}^{m} x_j I_j,$$

where $I_j$ is the indicator function for the $j$th block, which we suppose has unit volume. Let the observations $y_i$ be related to the density through linear functionals $L_i$,

$$\{y_i\} = \sum_{j=1}^{m} x_j L_i(I_j).$$

We put $L_i(I_j) = A_{ij}$, and we will suppose that the matrix $A = \{A_{ij}\}$ has full rank, $n$. Let $\lambda_j^2$ be the eigenvalues and $v_j$ the eigenvectors for the $m \times m$ matrix $A^{\mathrm{T}}A$. Suppose they are arranged so that the $n$ first are the ones different from zero. They will be the eigenvalues for $AA^{\mathrm{T}}$, which has eigenvectors $u_i$. Then

$$A^{\mathrm{T}}Av_j = \lambda_j^2 v_j, \qquad j = 1, \ldots, m$$

$$AA^{\mathrm{T}}u_i = \lambda_i^2 u_i, \qquad i = 1, \ldots, n.$$

Let the matrices $U$ and $V$ be formed with the vectors $u_i$ and $v_j$ as columns, respectively, and let $\Lambda$ be the $n \times n$ matrix with the $\lambda_i$ in the diagonal. Then

$$A = U\Lambda V^{\mathrm{T}}$$

and

$$AA^{\mathrm{T}} = U\Lambda^2 U^{\mathrm{T}}.$$

The Lanczos inverse is then

$$H_{\mathrm{L}} = V\Lambda^{-1} U^{\mathrm{T}}.$$

Hence

$$\{x_j\} = H_{\mathrm{L}} \{y_i\},$$

and

$$\delta\tilde{\rho}_{\mathrm{L}} = \{y_i\}^{\mathrm{T}} H_{\mathrm{L}}^{\mathrm{T}}\{I_j\}.$$

The collocation solution is obtained using the reproducing kernel

$$K(P, Q) = \sum_{j=1}^{m} I_j(P) I_j(Q) \equiv \operatorname{cov} [(\delta\rho(P), \quad \delta\rho(Q)].$$

The solution is, cf. (4.7)

$$\delta\tilde{\rho}_{\mathrm{C}}(Q) = \sum_{i=1}^{n} b_i \operatorname{cov} [y_i, \delta\rho(Q)] = \sum_{i=1}^{n} b_i \sum_{j=1}^{m} L_i(I_j) I_j(Q) = \sum_{i=1}^{n} b_i \sum_{j=1}^{m} A_{ij} I_j$$

$$\left[ = \sum_{j=1}^{m} x_j I_j(Q) \right],$$

where $\{b_i\}$ is determined from the collocation condition

$$y_j = L_j(\delta\rho) = L_j(\delta\tilde{\rho}_{\mathrm{C}}) = \sum_{k=1}^{n} b_k \operatorname{cov}(y_i, y_k)$$

or

$$\{b_i\} = [\mathrm{cov}\,(y_i, y_k)]^{-1}\,\{y_k\} = \left\{\left[\sum_{j=1}^{m} L_i(I_j)L_k(I_j)\right]\right\}^{-1}\{y_k\}$$

$$= (\{A_{ij}\}\ \{A_{kj}\}^{\mathrm{T}})^{-1}\,\{y_k\}.$$

To show that $\delta\tilde{\rho}_{\mathrm{C}} = \delta\tilde{\rho}_{\mathrm{L}}$, we must prove

$$H_{\mathrm{L}}^{\mathrm{T}} = (AA^{\mathrm{T}})^{-1}A.$$

But

$$A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1} = V\Lambda U^{\mathrm{T}}(U\Lambda^2U^{\mathrm{T}})^{-1} = V\Lambda U^{\mathrm{T}}U\Lambda^{-2}U^{\mathrm{T}} = V\Lambda^{-1}U^{\mathrm{T}} = H_{\mathrm{L}}.$$