Chordal Editing is Fixed-Parameter Tractable*

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Abstract

Graph modification problems are typically asked as follows: is there a set of k operations that transforms a given graph to have a certain property. The most commonly considered operations include vertex deletion, edge deletion, and edge addition; for the same property, one can define significantly different versions by allowing different operations. We study a very general graph modification problem which allows all three types of operations: given a graph G and integers k_1, k_2, and k_3, the CHORDAL EDITING problem asks if G can be transformed into a chordal graph by at most k_1 vertex deletions, k_2 edge deletions, and k_3 edge additions. Clearly, this problem generalizes both CHORDAL VERTEX/EDGE DELETION and CHORDAL COMPLETION (also known as MINIMUM FILL-IN). Our main result is an algorithm for CHORDAL EDITING in time $2^{O(k \log k)} \cdot n^{O(1)}$, where $k := k_1 + k_2 + k_3$; therefore, the problem is fixed-parameter tractable parameterized by the total number of allowed operations. Our algorithm is both more efficient and conceptually simpler than the previously known algorithm for the special case CHORDAL DELETION.

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1 Introduction

A graph is chordal if it contains no hole, that is, an induced cycle of at least four vertices. After more than half century of intensive investigation, the properties and the recognition of chordal graphs are well understood. Their natural structure earns them wide applications, some of which might not seem to be related to graphs at first sight. During the study of Gaussian elimination on sparse positive definite matrices, Rose [15, 16] formulated the CHORDAL COMPLETION problem, which asks for the existence of a set of at most k edges whose insertion makes a graph chordal, and showed that it is equivalent to MINIMUM FILL-IN. Balas and Yu [1] proposed a heuristics algorithm for the maximum clique problem by first finding a maximum (spanning) chordal subgraph. This is equivalent to the CHORDAL EDGE DELETION problem, which asks for the existence of a set of at most k edges whose deletion makes a graph chordal. Dearing et al. [5] observed that a maximum chordal subgraph can also be used to find maximum independent set and sparse matrix completion. This observation turns out to be archetypal: many NP-hard problems (coloring, maximum clique, etc.) are known to be solvable in polynomial time when restricted to chordal graphs, and hence admit a similar heuristics algorithm. Cai [3] considered the parameterized complexity of coloring

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problems on graphs close to certain graph classes. In particular, he asked as an open question on the graphs that can be made chordal by the deletion of k vertices or edges, of which the edge version was resolved by Marx [11] affirmatively. It should be noted that such a coloring algorithm needs first the set of k vertices or edges. For chordal graphs, to find them is equivalent to solving the CHORDAL VERTEX/EDGE DELETION problem. Though with slightly different purpose, the inspiration behind [1, 5] and [3] are exactly the same.

All the three problems, unfortunately but understandably, are NP-hard [18, 13, 10]. Therefore, early work of Cai [2] and Kaplan et al. [8] focused on their parameterized complexity. They proved that that the CHORDAL COMPLETION problem can be solved in time $4^k \cdot n^{O(1)}$, implying that it is fixed-parameter tractable (FPT). Marx [12] showed that the complementary deletion problems, both edge and vertex versions, are also FPT. Recently, Fomin and Villanger [7] gave an algorithm for CHORDAL COMPLETION with running time $k^{O(\sqrt{k})} \cdot n^{O(1)}$, that is, with subexponential dependence on k.

The three operations can be combined, and then the question becomes: can a graph be made chordal by deleting at most k_1 vertices and k_2 edges and adding at most k_3 edges. This leads us to the CHORDAL EDITING problem, which generalizes all three aforementioned problems in a natural way. Note that chordal graphs are hereditary, hence it does not make sense to add new vertices. The budgets for different operations are not transferable, as otherwise it degenerates to CHORDAL VERTEX DELETION. Our main result establishes the fixed-parameter tractability of CHORDAL EDITING parameterized by $k := k_1 + k_2 + k_3$.

▶ **Theorem 1.1 (Main result).** *There is a $2^{O(k \log k)} \cdot n^{O(1)}$ time algorithm for deciding, given an n-vertex graph G, whether there are a set V_- of at most k_1 vertices, a set E_- of at most k_2 edges, and a set E_+ of at most k_3 non-edges, such that the deletion of V_- and E_- and the addition of E_+ make G a chordal graph.*

As a corollary, we get a new FPT algorithm for the special case CHORDAL DELETION; our algorithm is far simpler and faster than the algorithm of [12].

Related work. Observing that a large hole cannot be fixed by the insertion of a small number of edges, it is easy to devise a bounded search tree algorithm for the CHORDAL COMPLETION problem [8, 2]. No such simple argument works for the deletion versions: the removal of a single vertex/edge suffices to break a hole of an arbitrary length. The way Marx [12] showed that this problem is FPT is to (1) prove that if the graph contains a large clique, then we can identify an irrelevant vertex whose deletion does not change the problem; and (2) observe that if the graph has no large cliques, then it has bounded treewidth, so the problem can be solved by standard techniques, such as the application of Courcelle's Theorem. In contrast, our algorithm uses simple reductions and structural properties, which reveal a better understanding of the CHORDAL VERTEX DELETION problem, and easily extend to the more general CHORDAL EDITING problem.

We remark that there were formulations that consider both edge operations, e.g., the CLUSTER EDITING problem [4], as well as the many problems studied by Natanzon et al. [13]. Their objective is to minimize the total number of edge operations, i.e., $k_2 + k_3$ in our notation, which is slightly different from them. As a matter of fact, our problem formulation is more general: if we can solve the version where the edge additions and edge deletions are bounded separately, then we can try every combination of k_2 and k_3 where $k_2 + k_3$ satisfies the given bound.

Our techniques. As a standard opening step, we use the *iterative compression* method introduced by Reed et al. [14] and concentrate on the compression problem, where we are equipped with a hole cover M. The subgraph $G - M$ is chordal and hence admits a clique tree decomposition. First, we break every short hole by simple branching. The main technical idea appears in the way we break long holes. We use the clique decomposition to show that the shortest hole H can be decomposed into a bounded number of segments, where the internal vertices of each segment, as well as the part of the graph "close" to them behave in a well-structured and simple way with respect to their interaction with M. To break H, we have to break some of the segments, and the properties of the segments allow us to show that we need to consider only a bounded number of canonical separators breaking these segements. Therefore, we can branch on chosing one of these canonical separators and break the hole using it, resulting in an FPT algorithm.

Notation. All graphs discussed in this paper shall always be undirected and simple. The length $|H|$ of a hole H is defined to be the number of edges in it; note that $|H| = |V(H)|$. If a pair of vertices is adjacent, we say $u \sim v$. By $v \sim X$ we mean v is adjacent to at least one vertex of the set X. Two vertex sets X and Y are completely connected if $x \sim y$ for each pair of $x \in X$ and $y \in Y$. A vertex is *simplicial* if $N(v)$ induce a clique. The notation $N_U(v)$ stands for the neighbors of v in the set U, i.e., $N_U(v) = N(v) \cap U$, regardless of whether $v \in U$ or not. We use $N_H(v)$ as a shorthand for $N_{V(H)}(v)$.

A set S of vertices *separates* x and y, and is called an *(x, y)-separator* if there is no (x, y)-path in the subgraph $G - S$; it is *minimal* if no proper subset of S separates x and y. A graph is chordal if and only if every minimal separator in it induces a clique [6].

Let $\mathcal{T}$ be a tree whose nodes, called *bags*, correspond to the maximal cliques of a graph G. With the customary abuse of notation, the same symbol K is used for a bag in $\mathcal{T}$ and its corresponding maximal clique of G. Let $\mathcal{T}(x)$ denote the subgraph of $\mathcal{T}$ induced by all bags containing x. The tree $\mathcal{T}$ is a *clique tree* of G if for any vertex $x \in V(G)$, the subgraph $\mathcal{T}(x)$ is connected. It is known that the intersection of any pair of adjacent bags K_i and K_j of $\mathcal{T}$ makes a minimal separator; in particular, it is a separator for any pair of vertices $x \in K_i \setminus K_j$ and $y \in K_j \setminus K_i$. A vertex is simplicial if and only if it belongs to exactly one maximal clique; thus, any non-simplicial vertex appears in some minimal separator(s) [9].

2 Outline of the algorithm

A subset $V_- \subseteq V(G)$ is called a *hole cover* of G if its deletion makes G chordal. We say that (V_-, E_-, E_+), where $V_- \subseteq V(G)$ and $E_- \subseteq E(G)$ and $E_+ \subseteq V(G)^2 \setminus E(G)$, is a *chordal editing set* of G if the deletion of V_- and E_- and the addition of E_+, applied successively, make G chordal. Its *size* is defined to be the 3-tuple $(|V_-|, |E_-|, |E_+|)$, and we say that it is *smaller* than (k_1, k_2, k_3) if all of $|V_-| \leq k_1$ and $|E_-| \leq k_2$ and $|E_+| \leq k_3$ hold true and at least one inequality is strict. Note that since chordal graphs are hereditary, it does not make sense to add new vertices. The main problem studied in the paper is formally defined as follows.

CHORDAL EDITING $(G, (k_1, k_2, k_3))$	
Input:	A graph G and three nonnegative integers k_1, k_2, and k_3.
Task:	Either construct a chordal editing set (V_-, E_-, E_+) of G that has size at most (k_1, k_2, k_3), or report that no such a set exists.

One might be tempted to define the editing problem by imposing a combined quota on the total number of operations, i.e., a single parameter $k = k_1 + k_2 + k_3$, instead of three separate parameters. However, this formulation is computationally equivalent to CHORDAL DELETION in a trivial sense, as vertex deletions are clearly preferable to both edge operations.

0. **return** if G is chordal or one of k_1, k_2, and k_3 becomes negative;
1. find a shortest hole H;
2. **if** H is shorter than $k+4$ **then** guess a way to fix it; **goto** 0.
3. **else** decompose H into $O(k^3)$ segments; guess a segment and break it;
4. **goto** 0.

Figure 1 Outline of our algorithm for CHORDAL EDITING COMPRESSION.

We use the technique of *iterative compression*: we define and solve a compression version of the problem first and argue that this implies the fixed-parameter tractability of the original problem. In the compression problem a hole cover M of bounded size is given in the input, making the problem somewhat easier: as $G - M$ is chordal, we have useful structural information about the graph. Note that the definition below has a slightly technical (but standard) additional condition, i.e., we are not allowed to delete a vertex in M.

CHORDAL EDITING COMPRESSION $(G, M, (k_1, k_2, k_3))$

Input: A graph G, three nonnegative integers k_1, k_2, and k_3, and a hole cover M of G whose size is at most $k_1 + k_2 + k_3 + 1$.

Task: Either construct a chordal editing set (V_-, E_-, E_+) of G such that its size is at most (k_1, k_2, k_3) and V_- is disjoint from M, or report that no such a set exists.

The set M is called the *modulator* of this instance. We use $k := k_1 + k_2 + k_3$ to denote the total numbers of operations.

We sketch how the technique of iterative compression can be applied to use an algorithm for CHORDAL EDITING COMPRESSION to solve CHORDAL EDITING.

Let $v_1, v_2, \ldots, v_n$ be an arbitrary ordering of $V(G)$, and let G^i be the graph induced by $\{v_1, \ldots, v_i\}$. We try to find a chordal editing set of size (k_1, k_2, k_3) for each G^i. Assume that we have obtained a solution (V_-^i, E_-^i, E_+^i) for G^i, then we can make a hole cover X^i of G^i by taking V_-^i, and an arbitrary endvertex from each edge in $E_-^i \cup E_+^i$. Clearly, $X^i \cup \{v_{i+1}\}$ is a hole cover of G^{i+1}. By guessing the (possibly empty) set X_-^i of vertices of a hypothetical solution that is in $X^i \cup \{v_{i+1}\}$ and deleting them from G^{i+1}, we make an instance of CHORDAL EDITING COMPRESSION where the graph is $G^{i+1} - X_-^i$, the modulator is $M^{i+1} := X^i \cup \{v_{i+1}\} \setminus X_-^i$, and the parameters are $(k_1 - |X_-^i|, k_2, k_3)$. Then the compression algorithm for CHORDAL EDITING COMPRESSION can be used to find a chordal editing set disjoint from M^{i+1} for $G^{i+1} - X_-^i$. If the answer is "NO," then we can conclude that the original instance is also "NO." Otherwise the obtained solution, together with X_-^i, gives the solution $(V_-^{i+1}, E_-^{i+1}, E_+^{i+1})$ for G^{i+1}. We proceed to G^{i+2}, until we reach G^n which is G. Hence the original problem is solved with at most n calls of the algorithm for CHORDAL EDITING COMPRESSION.

The main part of this paper will be focused on an algorithm for CHORDAL EDITING COMPRESSION. Its outline is described in Figure 1. We will endeavor to prove

▶ **Theorem 2.1.** CHORDAL EDITING COMPRESSION *is solvable in time* $2^{O(k \log k)} \cdot n^{O(1)}$.

Steps 1 and 2 are straightforward: we can find a shortest hole H in polynomial time, and if $|H| \leq k + 3$, then there are only $O(k^2)$ ways to fix it. To fix a hole of length $|H| \geq k + 4 > k_3 + 3$, we need to delete at least one vertex or edge from it. As we shall see in Section 3, such a hole can be divided into a bounded number of "segments" and the deletions have to "break" at least one of the segments (i.e., delete one vertex or edge from it). In our case, breaking a segment means a strange mixed form of separation: we have to separate two vertices by removing both edges and vertices. We study this notion of mixed separation on chordal graphs in Section 4. Finally, we show in Section 5 that there is a bounded number of canonical ways of breaking a segment and we may branch on choosing one segment and one of the canonical ways of breaking it. This completes the proofs of Theorem 2.1 and 1.1.

3 Segments

We shall define a hierarchy of vertex sets V_0, V_1, and V_2. Each set is a subset of the preceding one, and all of them induce chordal subgraphs. Let A denote the set of common neighbors of the shortest hole H found in step 1, and define $A_M = A \cap M$ and $A_0 = A \setminus M$. We can assume that A induces a clique: if two vertices $x, y \in A$ are nonadjacent, then together with the two nonadjacent vertices v_1 and v_3 of H, they form a 4-hole (xv_1yv_3x). The following observation follows from the fact that H is the shortest hole of G.

▶ **Proposition 3.1.** A vertex not in A is adjacent to at most three vertices of H and these vertices have to be consecutive in H.

The first set is defined by $V_0 = V(G) \setminus (M \cup A)$. Note that $\{M, V_0, A_0\}$ partitions $V(G)$, and H is disjoint from A_0. Since $|H| \geq k + 4 > |M|$, the hole H intersects both M and V_0. Every component of $H - M$ is an induced path of G_0, and there are at most $|M|$ such paths. Observing $|M| = O(k)$, to decompose H into $O(k^3)$ segments as claimed, it suffices to divide each of these paths into $O(k^2)$ parts. Let P denote such a path $(v_1 v_2 \ldots v_p)$. To avoid triviality, we may assume $p > 3$; as a result and by Proposition 3.1, the distance between v_1 and v_p in G_0 is at least 3. A further consequence is $v_1 \not\sim v_p$.

Let G_0 denote the chordal subgraph $G[V_0]$, and let $\mathcal{T}$ be a fixed clique tree for G_0. We take the unique path of bags $\mathcal{P} = (K_1, \ldots, K_q)$ that connects the disjoint subtrees $\mathcal{T}(v_1)$ and $\mathcal{T}(v_p)$ in $\mathcal{T}$, where $K_1 \in \mathcal{T}(v_1)$ and $K_q \in \mathcal{T}(v_p)$. The condition $p > 3$ implies that $q > 2$. The removal of K_1 and K_q will separate $\mathcal{T}$ into a set of subtrees, one of which contains all K_ℓ with $1 < \ell < q$; let $\mathcal{T}_1$ denote this nonempty subtree. The second set, V_1, is defined to be the union of all bags in $\mathcal{T}_1$ and $\{v_1, v_p\}$. By definition and observing that V_1 fully contains P, it induces a connected subgraph.

We then focus on bags in $\mathcal{P}$ and their union. (One may have judiciously observed that vertices in bags of $\mathcal{P}$ induce an interval graph.) From the definition of clique tree, we can infer that v_1 and v_p appear only in K_1 and K_q respectively, while every internal vertex of P appears in more than one bags of $\mathcal{P}$. For every i with $1 \leq i \leq p$, we denote by $\texttt{first}(i)$ (resp., $\texttt{last}(i)$) the smallest (resp., largest) index ℓ such that $1 \leq \ell \leq q$ and $v_i \in K_\ell$, e.g., $\texttt{first}(1) = \texttt{last}(1) = \texttt{first}(2) = 1$ and $\texttt{last}(p-1) = \texttt{first}(p) = \texttt{last}(p) = q$. As P is an induced path, for each i with $1 < i < p$, we have

$$\texttt{first}(i) \leq \texttt{last}(i-1) < \texttt{first}(i+1) \leq \texttt{last}(i). \tag{1}$$

For $1 \leq \ell < q$, we define $S_\ell := K_\ell \cap K_{\ell+1}$. For any pair of nonadjacent vertices v_i, v_j in P, (i.e., $1 \leq i < i+1 < j \leq p$,) all minimal (v_i, v_j)-separators are then $\{S_\ell \mid \texttt{last}(i) \leq \ell < \texttt{first}(j)\}$.

The third set, V_2, is defined to be the union of vertices in all induced (v_1, v_p)-paths in G_0. Since a vertex x is an internal vertex of an induced (v_1, v_p)-path of G_0 if and only if it is in some minimal (v_1, v_p)-separator of G_0, we have (noting $q > 2$)

▶ **Proposition 3.2.** A vertex is in $V_2 \setminus \{v_1, v_p\}$ if and only if it appears in more than one bags of $\mathcal{P}$. Moreover, $V_2 \setminus \{v_1, v_p\} \subseteq \bigcup_{1<\ell<q} K_\ell$.

The definition of V_0 and G_0 depend upon the hole H, while the definition of V_1 and V_2 depend upon both the hole H and the path P. In this paper, we are always concerned with a particular path of a particular hole, which will be specified before the usage of V_1 and V_2.

The set $V_0 \setminus V_1$ is easily understood, and we now consider $V_1 \setminus V_2$. Given a pair of nonadjacent vertices $x, y \in V_2$, we say that x lies to the *left* (resp., *right*) of y if the bags of $\mathcal{P}$ containing x have smaller (resp., greater) indices than those containing y. If an induced path of $G[V_2]$ consists of three or more vertices, then its endvertices are nonadjacent and have a left-right relation. This relation can be extended to all pairs of consecutive (and adjacent) vertices x, y in this path, the one with smaller distance to the left endvertex of the path is said *to the left of the other*. It is easy to verify that these two definitions are compatible.

▶ **Lemma 3.3.** *For any component C of the subgraph induced by $V_1 \setminus V_2$, the set $N_{V_0}(C)$ induces a clique and there exists ℓ such that $1 < \ell < q$ and $N_{V_0}(C) \subseteq K_\ell$.*

Proof. Consider a vertex $x \in C$, which is different from v_1 and v_p. Since $x \in V_1$, it appears in some bag of $\mathcal{T}_1$. Recall that the only bag of $\mathcal{T}_1$ that is adjacent to K_1 is K_2. Thus if $x \in K_1$, then it has to be in K_2 as well, which is impossible as $x \notin V_2$ (Proposition 3.2). Therefore, $x \notin K_1$; for the same reason, $x \notin K_q$. As a result, $N_{V_0}(x) \subseteq V_1$, and then $N_{V_0}(C) \subseteq V_2$.

It now suffices to show that $N_{V_0}(C)$ induces a clique. Suppose that, for contradiction, there is a pair of nonadjacent vertices $x, y \in N_{V_0}(C)$. We can find an induced (v_1, v_p)-path P' through x and y; without loss of generality, let x lie to the left of y, i.e., $P' =(v_1 \cdots x \cdots y \cdots v_p)$. Let x' and y' be the first and last vertices in P' that are adjacent to C, and $(x'P''y')$ be an induced path with all internal vertices from C. Note that x' either is x or lies to the left of x in P' and y' either is y or lies to the right of y, which imply $x' \not\sim y'$. Thus $(v_1 \cdots x'P''y' \cdots v_p)$ is an induced (v_1, v_p)-path through C, which is impossible. This completes the proof. ◀

Such a component C is called a *branch* of P, and we say that it is *near to* $v_i \in P$ if there is an ℓ with $\texttt{first}(i) \leq \ell \leq \texttt{last}(i)$ satisfying the condition of Lemma 3.3. Since a component C is near to $v_i \in P$ if and only if $N_{V_0}(C) \subseteq N[v_i]$, and applying Proposition 3.1 on any vertex in $N_{V_0}(C)$, we conclude that a branch is near to at most three vertices of P. If a hole passes through C, then C has to be adjacent to M: by Lemma 3.3, $N_{V_0}(C)$ is a clique, thus a hole cannot enter and leave C both via $N_{V_0}(C)$. The converse is not necessarily true: some branch that is adjacent to M might still be disjoint from all holes, e.g., if $N(C)$ is a clique. This observation inspires us to generalize the definition of simplicial vertices to sets of vertices.

▶ **Definition 3.4.** A set X of vertices is called *simplicial in a graph G* if $N[X]$ induces a chordal subgraph of G and $N(X)$ induces a clique of G.

It is easy to verify that a simplicial set of vertices is disjoint from all holes. This suggests that simplicial sets are irrelevant to CHORDAL EDITING problem and we may never want to add/delete edges incident to a vertex in a simplicial set. However, this is not true in general, and we may need to add/delete such edges if $N(X)$ was modified. As characterized by the following lemma, this is the only reason for touching X in the solution: set X will only concern us after $N(X)$ has been changed. We say that a chordal editing set (V_-, E_-, E_+) *edits* a set $X \subset V(G)$ of vertices if either V_- contains a vertex of X or $E_- \cup E_+$ contains an edge with at least one endpoint in X. We use a classic result of Dirac [6] stating that the graph obtained by identifying two cliques of the same size from two chordal graphs is also chordal.

▶ **Lemma 3.5.** *A minimal chordal editing set edits a simplicial set U only if it removes at least one edge induced by $N(U)$.*

Proof. Let (V_-, E_-, E_+) be a minimal editing set of G such that E_- does not contain any edge induced by $N(U)$. We restrict the editing set to the subgraph $G - U$, i.e., we consider the set $(V_- \setminus U, E_- \setminus (U \times V(G)), E_+ \setminus (U \times V(G)))$, and let G' be the graph obtained by applying it to G. Clearly $G' - U = G - U$ is chordal, where $N(U) \setminus V_-$ induces a clique. Also chordal is the subgraph of G' induced by $N[U] \setminus V_-$. Both of them contain the clique $N(U) \setminus V_-$. Since G' can be obtained from them by identifying $N(U) \setminus V_-$, it is also chordal. Then by the minimality of (V_-, E_-, E_+), it must be the same as $(V_- \setminus U, E_- \setminus (U \times V(G)), E_+ \setminus (U \times V(G)))$, and this proves this lemma. ◀

Now we are ready to define segments of P, which are delimited by some special vertices called junctions. By definition, a branch is simplicial in G_0, but unnecessarily in G. We say that a vertex $w \notin K$ is *adjacent* to a bag K if w is adjacent to at least one vertex in K.

▶ **Definition 3.6** (Segment). A vertex $v \in P$ is called a *junction (of P)* if *(1)* some bag K that contains v is adjacent to $M \setminus A_M$; *(2)* some branch near to v is adjacent to $M \setminus A_M$; *(3)* some branch near to v is not simplicial in G; or *(4)* $N_{V_2}(v)$ is not completely connected to A. A sub-path $(v_s \cdots v_t)$ of P is called a *segment*, denoted by $[v_s, v_t]$, if v_s and v_t are the only junctions in it.

We point out that the four types are not exclusive, and one junction might be in more than one types. For a junction v of type (1) or (2), we say that the vertex in $M \setminus A_M$ used in its definition *witnesses* it.

▶ Remark. Informally speaking, for a junction v of type (1) or (2), there is a connection from v to $M \setminus A_M$ that is *local* to v in some sense; for a junction v of type (3) or (4), there is a hole near to v, and its disposal might interfere with that of H. If another hole H' intersects a segment $[v_s, v_t]$, then H' has to go through the whole segment, or more specifically, it necessarily enters and exits the segment via $N[v_s]$ and $N[v_t]$, respectively.

The definition of junction and segment extends to all paths of $H - M$. In polynomial time, we can construct V_0 for H and V_1, V_2 for each path P of $H - M$, from which all junctions of H can be identified. For each path of $H - M$, the endvertices are adjacent to $M \setminus A_M$, hence junctions. As a result, every vertex in $V(H) \setminus M$ is contained in some segment, and in each path of $H - M$, the number of segments is the number of junctions minus one.

We are now ready for the main result of this section that gives a cubic bound on the number of segments of H. It should be noted the constants—both the exponent and the coefficient—in the following statement are not tight, and the current values simplify the argument significantly. Recall that a vertex not in A sees at most three vertices in H, and they have to be consecutive.

▶ **Theorem 3.7.** *If H contains more than $|M| \cdot (12k^2 + 92k + 82)$ segments, then we can either find a vertex that has to be in V_-, or return "NO."*

Proof. We show that H contains at most $|M| \cdot (12k^2 + 92k + 82)$ junctions. Recall that there are at most $|M|$ paths in $H - M$. To obtain a contradiction, we suppose that some path P of $H - M$ contains $12k^2 + 92k + 82$ junctions. Let us first attend to junctions of type (1) in P.

▶ **Claim 1.** Each $w \in M \setminus A_M$ witness at most 15 junctions of type (1).

Proof. Suppose, for contradiction, that 15 vertices in H appears in some bag adjacent to w; let X be this set of vertices. Assume first that X is consecutive. At most 3 of them are adjacent to w, and they are consecutive in H. Thus, we can always pick 6 consecutive vertices from X that are disjoint from $N_H(w)$; let them be $\{v_i, \ldots, v_{i+5}\}$. By definition, there are two vertices $u_1, u_2 \in V_0 \cap N(w)$ such that $u_1 \sim v_i$ and $u_2 \sim v_{i+5}$. It is easy to verify that $u_2 \not\sim v_{i+2}$ and $u_1 \not\sim v_{i+3}$ and $u_1 \not\sim u_2$. Therefore, we can find an induced (u_1, u_2)-path with all interval vertices from $\{v_i, \ldots, v_{i+5}\}$. The length of this path is at least 3, and hence it makes a hole with w of length at most 9. Assume now that X is not consecutive in P, then we can pick a pair of nonadjacent vertices v_i, v_j from X such that the $v_\ell \notin X$ for every $i < \ell < j$. There are two vertices $u_1, u_2 \in V_0 \cap N(w)$ such that $u_1 \sim v_i$ and $u_2 \sim v_j$. It is easy to verify that $(wu_1v_i \cdots v_ju_2w)$ is a hole. By assumption that $|X| \geq 15$, we have $j - i \leq |H| - 13$. In either case, we end with a hole strictly shorter than H. The contradictions prove this claim. ⌟

▶ **Claim 2.** If some vertex $w \in M \setminus A_M$ witnesses $5k + 80$ junctions of the first two types in P, then we can return "NO."

Proof. Let X be this set of junctions, we order them according to their indices in P and group each consecutive five from the beginning. We omit groups that contain junctions of type (1) witnessed by w, and in each remaining group, we pair the second and last vertices in it. According to Claim 1, we end with at least $k + 1$ pairs, which we denote by (v_{ℓ_1}, v_{r_1}), $\cdots$, $(v_{\ell_{k+1}}, v_{r_{k+1}})$, $\cdots$.

For each pair (v_{ℓ_j}, v_{r_j}), where $1 \leq j \leq k + 1$, we construct a hole H_j as follows. By definition, there is a branch C_{ℓ_j} (resp., C_{r_j}) whose neighborhood in H is a proper subset of $\{v_{\ell_j-1}, v_{\ell_j}, v_{\ell_j+1}\}$ (resp., $\{v_{r_j-1}, v_{r_j}, v_{r_j+1}\}$). By the selection of the pair v_{ℓ_j} and v_{r_j} (two vertices of X have been skipped in between), they are nonadjacent, and $r_j - \ell_j > 2$. Therefore, C_{ℓ_j} and C_{r_j} are distinct and necessarily nonadjacent. Since C_{ℓ_j} induces a connected subgraph and is adjacent to both w and $\{v_{\ell_j-1}, v_{\ell_j}, v_{\ell_j+1}\}$, we can find an induced (w, v_{ℓ_j+1})-path P_{ℓ_j} with all internal vertices from $C_{\ell_j} \cup \{v_{\ell_j-1}, v_{\ell_j}\}$. Likewise, we can obtain an induced (w, v_{r_j-1})-path P_{r_j} with all internal vertices from $C_{r_j-1} \cup \{v_{r_j}, v_{r_j+1}\}$. These two paths P_{ℓ_j} and P_{r_j}, together with $(v_{\ell_j+1} \ldots v_{r_j-1})$, make the hole H_j: we have $\ell_j + 1 < r_j - 1$; for each $\ell_j + 1 \leq s \leq r_j - 1$, $v_s \not\sim w$; and for each $\ell_j + 1 < s < r_j - 1$, $v_s \not\sim C_{\ell_j}, C_{r_j}$. This hole goes through w. This way we can construct $k + 1$ holes, and it can be easily verified that they intersect only in w. Since we are not allowed to delete w, we cannot fix all these holes by at most k operations. Thus we can return "NO." ⌟

If Claim 2 applies, then we are already done; otherwise, there are at most $|M| \cdot (5k + 80)$ junctions of the first two types. We proceed by considering the set B of junctions that are only of type (3) or (4) but not of the first two types. Its number is at least

$$(12k^2 + 92k + 82) - (5k + 80) \cdot |M| \geq 7k^2 + 7k + 1.$$

We order B according to their indices in P, and let b_i denote the index of the ith vertex of B in P. For each $0 \leq i \leq k(k + 1)$, we use the $(7i + 3)$th vertex of B to construct a hole H_i. Then we argue that this collection of holes either allows us to identify a vertex that has to be in the solution, or conclude infeasibility.

The first case is when there is a pair of nonadjacent vertices $x \in N_{V_2}(v_{b_{7i+3}})$ and $y \in A$. In this case we can assume that x is adjacent to neither $v_{b_{7i+1}}$ nor $v_{b_{7i+5}}$; otherwise $(xv_{b_{7i+1}}yv_{b_{7i+3}}x)$ or $(xv_{b_{7i+3}}yv_{b_{7i+5}}x)$ is a 4-hole, which contradicts the fact that H is the shortest. In other words, x only appears in some bag between $K_{\texttt{last}(b_{7i+1})}$ and $K_{\texttt{first}(b_{7i+5})}$; on the other hand, by definition of V_2, it appears in at least two of these bags. There is thus an induced $(v_{b_{7i+1}}, v_{b_{7i+5}})$-path P_i via x in $G[V_2]$. Starting from x, we traverse P_i to the left until the first vertex x_1 that is adjacent to y; the existence of such a vertex is ensured by the fact that $y \sim v_{b_{7i+1}}$. Similarly, we find the first neighbor x_2 of y in P_i to the right of x. Then the sub-path of P_i between x_1 and x_2, together with y, gives the hole H_i. By construction, no vertex of $H_i - y$ is adjacent to $v_{b_{7i}}$ or $v_{b_{7i+6}}$.

In the other case, some branch C_i near to $v_{b_{7i+3}}$ is not simplicial in G. By definition, either the subgraph induced by $N(C_i)$ is not a clique, or the subgraph induced by $N[C_i]$ is not chordal. Since $v_{b_{7i+3}}$ does not satisfy the conditions of type (1) and (2), $N(C_i) \cap M \subseteq A_M$, i.e., $N(C_i) \setminus V_0 \subseteq A$. On the other hand, according to Lemma 3.3, $N(C_i) \cap V_0$ induces a clique. Therefore, there must be a pair of nonadjacent vertices $x \in N(C_i) \cap V_0$ and $y \in A_M$. As C_i is near to $v_{b_{7i+3}}$, it must hold that $x \in N(v_{b_{7i+3}})$; this has already been discussed in the previous case. Suppose now that $N(C_i)$ induces a clique and there is a hole H_i in $N[C_i]$. We have seen that $N[C_i] \cap M = A_M$, thus this hole H_i intersects A_M; let w be a vertex in $V(H_i) \cap A_M$. If H_i is disjoint from A_0, then no vertex in $H_i \setminus M$ can be adjacent to $v_{b_{7i}}$ or $v_{b_{7i+5}}$. Otherwise, it contains some vertex $u \in A_0$; noting that A induces a clique, $H_i \cap A = \{u, w\}$. Moreover, $N(C_i) \cap V_2$ is in the neighborhood of $v_{b_{7i+3}}$ and therefore $N(C_i) \cap V_2$ and $N(C_j) \cap V_2$ are disjoint for $i \neq j$: the existence of a vertex $x \in V_2$ adjacent to both C_i and C_j would contradict Proposition 3.1 (noting that the distance of $v_{b_{7i+3}}$ and $v_{b_{7j+3}}$ is greater than 2 on the hole H).

In sum, we have a set $\mathcal{H}$ of at least $k(k+1)+1$ distinct holes such that (1) each hole in $\mathcal{H}$ contains at most one vertex of A_0, and (2) the intersection of any pair of them is in A. Recall that each hole has length at least $k+4$, hence cannot be fixed by edge additions only. If there is a $u \in A_0$ contained in at least $k+1$ holes of $\mathcal{H}$, then we have to put u into V_-; otherwise we have to delete distinct elements (edges or vertices) to break different holes, which is impossible. Now assume that no such a vertex u exists, then there must be $k+1$ holes that intersect only in M, which allow us to return "NO." ◀

4 Mixed separators in chordal graphs

Given a pair of nonadjacent vertices x, y of a graph, we say that a pair of vertex set V_S and edge set E_S is a *mixed (x, y)-separator* if the deletion of V_S and E_S leaves x and y in two different components; its size is defined to be $(|V_S|, |E_S|)$. A mixed (x, y)-separator is *inclusive-wise minimal* if there exists no other mixed (x, y)-separator (V'_S, E'_S) such that $V'_S \subseteq V_S$ and $E'_S \subseteq E_S$ and at least one containment is proper.

▶ **Lemma 4.1.** *Let x and y be a pair of nonadjacent vertices in a chordal graph F. For any pair of nonnegative integers (a, b), we can find a mixed (x, y)-separator of size at most (a, b) or asserts its nonexistence in time $3^{a+b+1} \cdot |V(F)|^{O(1)}$.*

Another interpretation of this lemma is

▶ **Corollary 4.2.** *Let x and y be a pair of nonadjacent vertices in a chordal graph F. For any nonnegative integer $a \leq k_1$, in time $3^{k_1+k_2+1} \cdot |V(F)|^{O(1)}$ we can find the minimum number b such that $b \leq k_2$ and there is a mixed (x, y)-separator of size (a, b) or assert that there is no mixed (x, y)-separator of size (a, k_2).*

5 Proof of Theorem 2.1

We are now ready to put everything together and finish the analysis of the algorithm. We say that a chordal editing set is minimum if there exists no chordal editing set with a smaller size. Note that a segment is contained in a unique path of $H - M$, which determines V_1 and V_2.

Proof of Theorem 2.1. Let (V_-^*, E_-^*, E_+^*) be a minimum chordal editing set of G of size no more than (k_1, k_2, k_3). We start from a closer look at how it breaks H; by Theorem 3.7, we may assume that H contains $O(k^3)$ segments. There are three options for breaking H. In the first case, V_-^* contains some junction, or E_-^* contains some edge of H that is in $M \times V_0$. In this case, we can branch on including one of these vertices or edges into the solution; there are $O(k^3)$ of them. Otherwise, we need to delete an internal vertex or edge from some segment. Let $d = 2k+4$. In the second case, there is either (1) some i with $s < i \leq s+d$ such that $v_i \in V_-^*$ or $v_{i-1}v_i \in E_-^*$; or (2) some j with $t-d \leq j < t$ such that $v_j \in V_-^*$ or $v_jv_{j+1} \in E_-^*$. In particular, if the segment to be broken satisfies $t - s \leq 2d$, then we must be in this case. If one of the two aforementioned cases is correct, then we can identify one vertex or edge of the solution by branching. In total, there are $O(k^4)$ branches we need to try.

Henceforth, we assume that none of these two cases holds. We still have to delete at least one vertex or edge from H; this vertex or edge must belong to some segment $[v_s, v_t]$ with $t - s > 2d$. For such a segment, we consider V_1 and V_2 corresponding to it. For any pair of indices i, j with $s \leq i < i+3 \leq j \leq t$, we use $U_{[i,j]}$ to denote the union of the set of bags in the nonempty subtree of $\mathcal{T} - \{K_{\texttt{last}(i)}, K_{\texttt{first}(j)}\}$ that contains $\{K_{\texttt{last}(i)+1}, \ldots, K_{\texttt{first}(j)-1}\}$ as well as $\{v_i, v_j\}$. Let $G_{[i,j]}$ be the subgraph induced by $U_{[i,j]}$.

▶ **Claim 3.** There must be some segment $[v_s, v_t]$ such that vertices v_{s+d} and v_{t-d} are disconnected in $G_{[s,t]} - V_-^* - E_-^*$.

Proof. We prove by contradiction. For a segment $[v_s, v_t]$ with $t - s \leq 2d$, the path $(v_s \cdots v_t)$ remains intact in $G - V_-^* - E_-^*$. Thus it suffices to consider segments $[v_s, v_t]$ with $t - s > 2d$. Let $s' = s + d$ and $t' = t - d$. For such a segment, we can find an induced $(v_s,\, v_t)$-path $P_{[s,t]}$ in $G_{[s,t]} - V_-^* - E_-^*$, which is also an (unnecessarily induced) path of G. This path has to visit every bag K_ℓ with $\texttt{last}(s) \leq \ell \leq \texttt{first}(t)$. In other words, in the original graph G, the path $P_{[s,t]}$ intersects every $N[v_i]$ with $s < i \leq s'$. Since we delete at most $k_2 \leq k$ edges each of which is adjacent to a single vertex in the sub-path $(v_s \cdots v_{s'})$, and $(d - k_2) \geq k+4$, there must be a vertex $v_{s''}$ with $s'' \geq s + k + 4$ that is not incident to any edge in E_-^*. This vertex is either in or adjacent to $P_{[s,t]}$ in $G_{[s,t]} - V_-^* - E_-^*$. Likewise, we can find a vertex $v_{t''}$ with $t'' \leq t - k - 4$ that is in or adjacent to $P_{[s,t]}$ in $G_{[s,t]} - V_-^* - E_-^*$. We now change the path into $(v_s \cdots v_{s''} P' v_{t''} \cdots v_t)$, where P' is an induced $(v_{s''},\, v_{t''})$-path with all internal vertices from $P_{[s,t]}$.

Let s''' with $s \leq s''' \leq s''$ be the smallest index such that $v_{s'''}$ is adjacent to P'. We argue that $s''' \geq s'' - 2$. Otherwise, the neighbor x of $v_{s'''}$ in P' (noting that it is not in A) is to the left of $v_{s''}$. Any path from $v_{s'''}$ to v_t in G_0 has to visit $N[v_{s''}]$. Since no edge incident to $v_{s''}$ is deleted, the path P' has a chord, which is impossible. Similarly, let t''' with $t'' \leq t''' \leq t$ be the greatest such that $v_{t'''}$ is adjacent to P', and we have $t''' \leq t'' + 2$. We can take an induced $(v_{s'''},\, v_{t'''})$-path of $G - V_-^* - E_-^*$ with all internal vertices from P', and extend it by including $(v_s \cdots v_{s'''})$ and $(v_{t'''} \cdots v_t)$ to make a chordless $(v_s,\, v_t)$-path $P'_{[s,t]}$ in $G - V_-^* - E_-^*$. The length of this path is at least $2(k+4-2) \geq 2k_3 + 4$.

Therefore, for each segment $[v_s, v_t]$ of H, we have obtained an induced $(v_s,\, v_t)$-path $P'_{[s,t]}$ in $G - V_-^* - E_-^*$. Concatenating all these paths, as well as edges of H in $M \times V(G)$, we get a cycle C. To verify that C is a hole, it suffices to verify that the internal vertices of $P'_{[s,t]}$ is disjoint and nonadjacent to other parts of C. On the one hand, no internal vertex of $P'_{[s,t]}$ is adjacent to $M \setminus A_M$ by definition (C is disjoint from A). On the other hand, all internal vertices of $P'_{[s,t]}$ appear in the subtree that contains $K_{\texttt{last}(s+4)}$ in $\mathcal{T} - \{K_{\texttt{last}(s+3)}, K_{\texttt{first}(t-3)}\}$, while no vertex in the $(v_t,\, v_s)$-path in C does. This verifies that C is a hole of $G - V_-^* - E_-^*$. Since the length of C is longer than $2k_3 + 4$, there must be a hole after the addition of E_+^*, which contains at most k_3 edges. This contradiction proves the claim. ◀

In other words, (V_-^*, E_-^*) contains some inclusive-wise minimal mixed $(\{v_s, \ldots, v_{s'}\}, \{v_{t'}, \ldots, v_t\})$-separator (V_S^*, E_S^*) in $G_{[s,t]}$. The resulting graph obtained by deleting (V_S^*, E_S^*) from $G_{[s,t]}$ is characterized by the following claim.

▶ **Claim 4.** Let (V_S, E_S) be an inclusive-wise minimal mixed $(v_{s'}, v_{t'})$-separator in $G[U_{[s',t']}]$. For any pair of indices s'', t'' with $s \leq s'' \leq s' < t' \leq t'' \leq t$, both $X \setminus \{v_{s''}\}$ and $Y \setminus \{v_{t''}\}$ are simplicial in $G' = G - V_S - E_S$, where X and Y be the components of $G_{[s'',t'']} - V_S - E_S$ containing $v_{s''}$ and $v_{t''}$, respectively.

Proof. It is easy to verify that $N_{G'}(X \setminus \{v_{s''}\}) \subseteq (K_{\texttt{last}(s'')} \cap V_2) \cup A$. The set $K_{\texttt{last}(s'')} \cap V_2$ is completely connected to A; otherwise $s'' + 1$ is a junction, which is impossible. Let $X' = N_{G'}[X \setminus \{v_{s''}\}]$; a vertex in X' is either in V_2, some branch, or A. We now verify that X' induces a chordal subgraph of G', which means that $X \setminus \{v_{s''}\}$ is simplicial in G'. Since (V_S, E_S) is inclusive-wise minimal, no edge in E_S is induced by X or Y. As a result, for every branch C near to some vertex v_i with $s < i < t$, $C \cap X'$ is simplicial. On the other hand, by definition of segments, $V_2 \cap X'$ is completely connected to A. Therefore, $G'[X']$ is chordal. A symmetric argument applies to $Y \setminus \{v_{t''}\}$. ⌟

We consider the subgraph obtained from G by deleting (V_S^*, E_S^*), i.e., $G' = G - V_S^* - E_S^*$. Note that $(V_-^* \setminus V_S^*, E_-^* \setminus E_S^*, E_+^*)$ is a minimum chordal editing set of G'.

▶ **Claim 5.** For any mixed $(\{v_s, \ldots, v_{s'}\}, \{v_{t'}, \ldots, v_t\})$-separator (V_S^*, E_S^*) of size at most $(|V_S^*|, |E_S^*|)$ in $G_{[s,t]}$, substituting (V_S, E_S) for (V_S^*, E_S^*) in (V_-^*, E_-^*, E_+^*) gives another minimum editing set to G.

Proof. We first argue the existence of some vertex $v_{s''}$ with $s \leq s'' \leq s'$ such that E_- contains no edge induced by $K_{\texttt{last}(s'')}$. For each s'' with $s \leq s'' \leq s'$, since $\texttt{last}(s'') \geq \texttt{first}(s'' + 1)$ and every vertex in them is adjacent to at most 3 vertices of H (Proposition 3.1), bags $K_{\texttt{last}(s'')}$ and $K_{\texttt{last}(s''+2)}$ are disjoint. In particular, an edge cannot be induced by both $K_{\texttt{last}(s'')}$ and $K_{\texttt{last}(s''+2)}$. Suppose that E_- contains an edge induced by $K_{\texttt{last}(s'')}$ for each s'' with $s \leq s'' < s'$, then we must have $|E_-| > (s' - s)/2 \geq k_2$, which is impossible. Likewise, we have some vertex $v_{t''}$ with $t' \leq t'' \leq t$ such that E_- contains no edge induced by $K_{\texttt{last}(t'')}$. By Claim 4, it follows that every vertex of $U_{[s'',t'']}$ is in a simplicial set of $G - V_S^* - E_S^*$. Since $(V_-^* \setminus V_S^*, E_-^* \setminus E_S^*, E_+^*)$ is a minimum chordal editing set to $G - V_S^* - E_S^*$, we have by Lemma 3.5 that $(V_-^* \setminus V_S^*, E_-^* \setminus E_S^*, E_+^*)$ does not edit any vertex of $U_{[s'',t'']}$.

Suppose that there is a hole C in the graph obtained by applying $((V_-^* \setminus V_S^*) \cup V_S, (E_-^* \setminus E_S^*) \cup E_S, E_+^*)$ to G. By construction, C contains a vertex of $U_{[s',t']} \subseteq U_{[s'',t'']}$. However, by Claim 4, every vertex of $U_{[s'',t'']}$ is in some simplicial set of $G - V_S - E_S$ and, as $(V_-^* \setminus V_S^*, E_-^* \setminus E_S^*, E_+^*)$ does not edit $U_{[s'',t'']}$, every such vertex is in a simplical set after applying $((V_-^* \setminus V_S^*) \cup V_S, (E_-^* \setminus E_S^*) \cup E_S, E_+^*)$ to G. Thus no vertex of $U_{[s'',t'']}$ is on a hole, a contradiction. ⌟

For any segment $[v_s, v_t]$, we can use Corollary 4.2 to find all possible sizes of minimum mixed $(\{v_s, \ldots, v_{s'}\}, \{v_{t'}, \ldots, v_t\})$-separator. There are at most k_1 of them. By Claim 5, one of them can be used to compose a minimum chordal editing set. In each iteration, we branch into $O(k^4)$ instances to break a hole, and in each branch decreases k by at least 1. The runtime is thus $O(k)^{4k} \cdot n^{O(1)} = 2^{O(k \log k)} \cdot n^{O(1)}$. This completes the proof. ◀

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