

# Chordal Graphs and Semidefinite Optimization

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## Abstract

Chordal graphs play a central role in techniques for exploiting sparsity in large semidefinite optimization problems and in related convex optimization problems involving sparse positive semidefinite matrices. Chordal graph properties are also fundamental to several classical results in combinatorial optimization, linear algebra, statistics, signal processing, machine learning, and nonlinear optimization. This survey covers the theory and applications of chordal graphs, with an emphasis on algorithms developed in the literature on sparse Cholesky factorization. These algorithms are formulated as recursions on elimination trees, supernodal elimination trees, or clique trees associated with the graph. The best known example is the multifrontal Cholesky factorization algorithm, but similar algorithms can be formulated for a variety of related problems, including the computation of the partial inverse of a sparse positive definite matrix, positive semidefinite and Euclidean distance matrix completion problems, and the evaluation of gradients and Hessians of logarithmic barriers for cones of sparse positive semidefinite matrices and their dual cones. The purpose of the survey is to show how these techniques can be applied in algorithms for sparse semidefinite optimization, and to point out the connections with related topics outside semidefinite optimization, such as probabilistic networks, matrix completion problems, and partial separability in nonlinear optimization.

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# 1

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## Introduction

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This survey gives an introduction to techniques from graph and sparse matrix theory that are important in semidefinite optimization. The results from graph theory we discuss are related to *chordal graphs* and *graph elimination*.

A *chordal graph* is an undirected graph with the property that every cycle of length greater than three has a chord (an edge between nonconsecutive vertices in the cycle). Chordal graphs have attracted interest in graph theory because several combinatorial optimization problems that are very difficult in general turn out to be easy for chordal graphs and solvable by simple greedy algorithms. Examples are the graph coloring problem and the problem of finding the largest clique in a graph. Chordal graphs have been studied extensively since the 1950s and their history shares some key events with the history of semidefinite optimization. In particular, it was Shannon's 1956 paper [203] that led Berge to the definition of perfect graphs, of which chordal graphs are an important subclass [28], and Lovász to one of the most famous early applications of semidefinite optimization [158].

Chordal graphs in applications often result from *graph elimination*, a process that converts a general undirected graph into a chordal graph

by adding edges. Graph elimination visits the vertices of the graph in a certain order, called the *elimination order*. When vertex  $v$  is visited, edges are added between the vertices that are adjacent to  $v$ , follow  $v$  in the elimination order, and are not yet mutually adjacent. If no edges are added during graph elimination the elimination order is called a *perfect elimination order*. It has been known since the 1960s that chordal graphs are exactly the graphs for which a perfect elimination order exists. A variety of algorithms based on different forms of ‘variable elimination’ can be described and analyzed via graph elimination. Examples include the solution of sparse linear equations (Gauss elimination), dynamic programming (eliminating optimization variables by optimizing over them), and marginalization of probability distributions (eliminating variables by summation or integration). Variable elimination is a natural approach in many applications and this partly explains the diversity of the disciplines in which chordal graphs have been studied.

The first part of this survey (Chapters 2–7) covers the basic theory of chordal graphs and graph elimination, with an emphasis on tree data structures developed in sparse matrix theory (elimination trees and supernodal elimination trees) and efficient analysis algorithms based on elimination trees.

The second part (Chapters 8–11) describes applications of chordal graphs to sparse matrices. The sparsity pattern of a symmetric sparse matrix can be represented by an undirected graph and graph elimination describes the fill-in during Cholesky factorization of a sparse positive definite matrix. Hence, the sparsity pattern of a Cholesky factor is chordal, and positive definite matrices with chordal sparsity patterns can be factored with zero fill-in. This fact underlies several classical decomposition results, discovered in the 1980s, that characterize sparse positive semidefinite matrices, sparse matrices with a positive semidefinite completion, and sparse matrices with a Euclidean distance matrix completion, when the sparsity pattern is chordal. We give an overview of these chordal decompositions and completion problems. We also present practical algorithms for solving them and several related problems, including computing the partial inverse of a sparse positive definite matrix, the inverse factorization of the

maximum determinant positive definite completion, and calculating derivatives of logarithmic barriers for cones of sparse symmetric matrices. We refer to these algorithms as *multifrontal* algorithms because they use similar recursions on elimination trees as the multifrontal Cholesky factorization algorithm. A library with implementations of most of the algorithms described in these chapters can be found at <http://cvxopt.github.io/chompack> [13].

In the last three chapters (Chapters 12–14) we discuss applications of chordal sparse matrix techniques in continuous optimization. The main focus is on decomposition results that exploit *partial separability* in nonlinear and conic optimization problems, and on techniques that exploit sparsity in interior-point methods for semidefinite optimization.

A summary of the notation used in the paper can be found on page 173.

We assume the reader is familiar with semidefinite optimization and its applications. Introductions and surveys of semidefinite optimization can be found in several books and articles, including [15, 223, 235, 24, 217, 226, 43]. On the other hand we do not assume any background in graph or sparse matrix theory. The main purpose of this survey is to give an extended introduction to the theory of chordal graphs and graph elimination, and the algorithms and elimination tree structures developed in the literature on sparse Cholesky factorization, and to describe the different ways in which these techniques can be applied in semidefinite and conic optimization algorithms. In addition, we aim to show the connections with related topics outside semidefinite optimization, such as probabilistic networks, matrix completion problems, and partial separability in nonlinear optimization.

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