

**CHOW RING OF PROJECTIVE NON-SINGULAR
TORUS EMBEDDING**

BY

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Let X be an n -dimensional torus embedding over an algebraically closed field k , and let S be the corresponding finite rational polyhedral partial (f.r.p.p.) decomposition of $N_R \simeq R^n$ which will be also called the (*conical*) *complex* (see [7], Chapter I, § 1,2, for definitions). Our aim is to describe the Chow ring and the l -adic cohomology of X in terms of the complex S in the case of a projective non-singular X . Some results for the case of a complete X are also given.

1. A ring associated with the conical complex. Let us fix an f.r.p.p. decomposition S of R^n such that the corresponding variety $X = X_S$ is complete and non-singular. Let a_1, a_2, \dots, a_r be the one-dimensional faces of S (the letter a will be reserved for faces of dimension 1, and ω for faces of dimension n). Each semigroup $a_j \cap Z^n$ is spanned by an element $(a_{1j}, a_{2j}, \dots, a_{nj})$ of Z^n . We attach to S the ring

$$(*) \quad Z[U_1, U_2, \dots, U_r]/I,$$

where U_j are variables corresponding to a_j , I is the homogeneous ideal generated by linear forms $\sum_{j=1}^r a_{ij} U_j$ ($i = 1, 2, \dots, n$) and monomials $U_{j_1} U_{j_2} \dots U_{j_s}$ such that $(a_{j_1}, a_{j_2}, \dots, a_{j_s})$ are minimal sequences spanning no face in S .

The ring $(*)$ will be denoted by $Z[U]/I$. It has a gradation such that the cosets of U_j are of degree 1. The Theorem from Section 2 states that $Z[U]/I$ is just the Chow ring of X .

Let us describe some properties of $Z[U]/I$.

For two cones $\sigma, \tau \in S$ we write $\sigma < \tau$ if σ is a face of τ , and we write $\sigma \text{ non-} < \tau$ if it is not. To each $\sigma \in S$, different from $\{0\}$, there corresponds a monomial of the form

$$P_\sigma = U_{j_1} U_{j_2} \dots U_{j_s},$$

Definition. Let $\omega \in S$, $\dim \omega = n$, and $\sigma \prec \omega$. Let us set

$$C_{\sigma, \omega} = \{\tau \in S : \sigma \prec \tau \prec \omega\}.$$

By *cells* in S we mean the subsets of S of the form $C_{\sigma, \omega}$ for some σ and ω . A *filtration* in S is a sequence of subcomplexes

$$(F): S = S_m \supset S_{m-1} \supset \dots \supset S_0 = \emptyset$$

such that

- (1) S_j are closed, i.e. $(\sigma \in S_j, \tau \prec \sigma) \Rightarrow (\tau \in S_j)$,
- (2) for each j , $S_{j+1} - S_j$ is a cell.

Let a cellular decomposition of S be given, that is to say a set $\{C_1, C_2, \dots, C_m\}$ of disjoint cells such that $S = \bigcup C_j$. The decomposition is said to be *filtrable* if there exists a filtration (F) of S such that $S_{j+1} - S_j$ is a cell of that decomposition for all j .

PROPOSITION 1. *Let a filtration (F) of S be given and let $S_j - S_{j-1} = C_{\sigma_j, \omega_j}$, where $\dim \omega_j = n$ and $\sigma_j \prec \omega_j$ for $j = 1, 2, \dots, m$. Then the ring $Z[U]/I$, attached to S by $(*)$, is generated, as a Z -module, by $p_{\sigma_1}, p_{\sigma_2}, \dots, p_{\sigma_m}$ (in fact, those elements form a basis of $Z[U]/I$, as it will follow from the proof of the Theorem in Section 2).*

Proof. By Lemma 2 it suffices to show that for each $\sigma \in S$ the element p_σ is of the form $\sum_k d_k p_{\sigma_k}$, where d_k are integers. Suppose that this

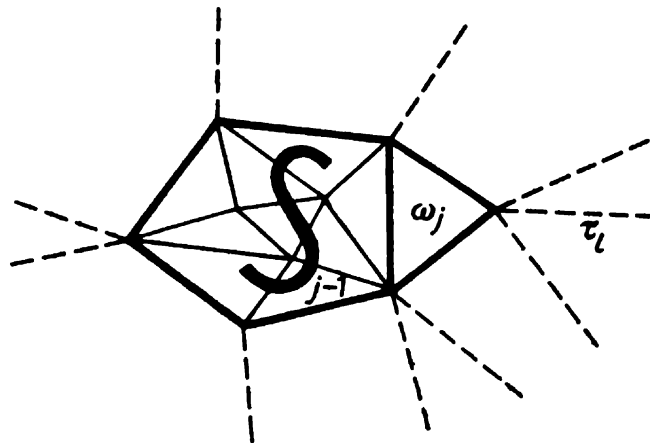


Fig. 1. The spherical complex corresponding to the 3-dimensional conical complex S_j

conclusion holds for all $\sigma \in S - S_j$ for some fixed j . We prove that it also does for $\sigma \in S - S_{j-1}$. Let $\sigma \in S_j - S_{j-1}$. Then, applying Lemma 1 to the triple $\sigma_j \prec \sigma \prec \omega_j$, we have

$$p_\sigma = \sum o_l p_{\tau_l},$$

where $\sigma_j \prec \tau_l$, and hence $\tau_l \notin S_{j-1}$. Otherwise, we have $\tau_l \text{ non-} \prec \omega_j$, and hence $\tau_l \notin S_j - S_{j-1}$ (see Fig. 1) for all l . Then $\tau_l \notin S_j$ and p_{τ_l} is a suitable

combination for all l . Then so is p_σ and the proof follows by the "decreasing induction" on j .

Remark. We compare cellular decompositions of S and those of X_S . Since

$$X_S = \bigcup_{\sigma \in S} O^\sigma,$$

for an arbitrary subset $S' \subset S$ we can put

$$X_{S'} = \bigcup_{\sigma \in S'} O$$

(warning: the affine torus embedding X_σ corresponding to a cone σ is not equal to $X_{\{\sigma\}}$). Those are locally closed T -invariant subschemes of X . In particular, let $C = C_{\sigma, \omega}$ be a cell in S . It follows easily from the non-singularity of X that X_C is an affine space. Then each cellular decomposition of S corresponds to a cellular decomposition of X .

A cellular decomposition of X ,

$$X = \bigcup_{i=1}^n W_i,$$

is said to be *filtrable* if there exists a sequence

$$(\tilde{F}): X = X_0 \supset X_1 \supset \dots \supset X_m = \emptyset$$

of closed subschemes of X such that $X_j - X_{j+1}$ is a cell of the decomposition for $j = 0, 1, \dots, m-1$ (cf. Definition 2 of [2]). Clearly, a decomposition of S is filtrable iff the corresponding decomposition of $X = X_S$ is filtrable. More precisely, to a filtration (F) of S there corresponds a filtration (\tilde{F}) of X such that

$$X_j = X_{(S-S_j)} = X - X_{S_j},$$

and then

$$X_j - X_{j+1} = X_{(S_{j+1}-S_j)}.$$

2. Chow ring of a projective toroidal embedding. Let us study the relation between the ring $Z[U]/I$ attached to S and the ring $C_{\text{rat}}^*(X_S)$ of classes of cycles on X_S , i.e. the Chow ring of X . To each face $\sigma \in S$ the orbit $O^\sigma \subset X$ corresponds (see [7]). Its closure is a cycle, and we set

$$\text{cl}(\omega) := \overline{\text{cl}_{\text{rat}}(O^\omega)} \in C_{\text{rat}}^*(X).$$

The invariant divisors \bar{O}^{α_j} will be denoted by D_j for $j = 1, 2, \dots, r$. Consider the homomorphism

$$\chi: Z[U_1, U_2, \dots, U_r] \rightarrow C_{\text{rat}}^*(X), \quad U_j \mapsto \text{cl}(\alpha_j) = \text{cl}(D_j).$$

The linear generators of the ideal I are in $\ker \chi$. Indeed, let us consider the form $(c_1, c_2, \dots, c_n) \mapsto c_i$ on Z^n (this last group is identified with $\text{Hom}(G_m, T)$). Let \mathcal{X}_i denote the character on T corresponding to this

form (see [7]). For the rational function x_i on X and the subvariety D_j ,

$$\text{val}_{D_j}(x_i) = a_{ij}.$$

Hence $\sum_j a_{ij}D_j$ is a principal divisor for $i = 1, 2, \dots, n$.

Clearly, monomial generators of I are also in $\ker \chi$, and hence χ can be factorized by a homomorphism h defined on $Z[U]/I$.

Let l be a number prime to $\text{char}(k)$, and $H^*(X, Q_l)$ the algebra of l -adic cohomology. Consider the following diagram (see [8], the symbol X is omitted):

$$(D) \quad \begin{array}{ccccccc} Z[U]/I & \xrightarrow{h} & C_{\text{rat}}^* & \xrightarrow{g} & C_{\text{alg}}^* & \xrightarrow{f} & C_{\tau}^* \\ \downarrow i & & \downarrow & & \downarrow & & \searrow \\ (Z[U]/I) \otimes_{\mathbb{Z}} Q_l & \xrightarrow{h'} & C_{\text{rat}}^* \otimes Q_l & \xrightarrow{g'} & C_{\text{alg}}^* \otimes Q_l & \xrightarrow{f'} & C_{\tau}^* \otimes Q_l \xrightarrow{e} H^*(X, Q_l) \end{array}$$

The homomorphism e doubles the gradation and all remaining homomorphisms of the diagram are compatible with the gradation.

THEOREM. *Let X be a non-singular projective torus embedding, S the corresponding conical complex in R^n and $Z[U]/I$ the graded ring $(*)$ attached to S . Then the diagram (D) induces isomorphisms*

$$\begin{aligned} Z[U]/I &\simeq C_{\text{rat}}^*(X) \simeq C_{\text{alg}}^*(X), \\ Z[U]/I \otimes_{\mathbb{Z}} Q_l &\underset{\simeq}{\overset{\simeq}{\simeq}} H^*(X, Q_l). \end{aligned}$$

For each q the \mathbb{Z} -module $C_{\text{rat}}^q(X)$ is free. For i odd we have $H^i(X, Q_l) = 0$.

Proof. Using Theorem 2 of [3] we shall show that S has a filtration. Let us choose an element

$$a \in \mathbb{Z}^n - \bigcup_{\substack{\sigma \in S \\ \dim \sigma < n}} \sigma$$

and consider the action of G_m on X given by the homomorphism $\lambda_a: G_m \rightarrow T$. Fixed points x_1, x_2, \dots, x_m with respect to this action are the same as fixed points by T , so they correspond to n -dimensional faces $\omega_1, \omega_2, \dots, \omega_m$ belonging to S . Consider the unstable decomposition of X (see [1] and [2]),

$$X = \bigcup_i W^u(x_i),$$

where $W^u(x_i)$ are affine spaces ⁽¹⁾.

It corresponds to the cellular decomposition of S (see the Remark) of the form

$$S = \bigcup_i C_{\sigma_i, \omega_i}$$

for some faces σ_i of ω_i (see [6] for details). By Theorem 2 of [3] the first decomposition is filtrable, then so is the second one.

⁽¹⁾ The sets $W^u(x_i)$ coincide with the cells defined by Ehlers in [4].

Let us fix a filtration (\tilde{F}) of X (and the corresponding filtration (F) of S). We begin with proving that the homomorphism h of the diagram (D) is surjective. By Proposition 7 of [5] (p. 4-31), applied to the filtration (F) , the Z -module $C_{\text{rat}}^*(X)$ is spanned by classes of cycles $\bar{X}_{\sigma_{\alpha_i, \omega_i}}$, $i = 1, 2, \dots, m$. Those cycles are nothing else but $\bigcap_{j: \alpha_j < \alpha_i} D_j$. Moreover, the divisors D_j intersect transversally. Indeed, by non-singularity of X , those divisors are locally of the form

$$D_j \simeq k^1 \times k^1 \times \dots \times 0_j \times \dots \times k^1.$$

Then

$$\text{cl}(\bar{X}_{\sigma_{\alpha_i, \omega_i}}) = \prod_{j: \alpha_j < \alpha_i} \text{cl} D_j = \prod_{j: \alpha_j < \alpha_i} h(u_j) = h(p_{\alpha_i}),$$

and hence $C_{\text{rat}}^*(X)$ is generated by $h(p_{\alpha_i})$, $i = 1, 2, \dots, m$.

Since X has a cellular decomposition, by [8] the homomorphism e of (D) is an isomorphism. Hence all horizontal arrows in (D) are surjective. It follows from Proposition 1, applied to the considered filtration of S , that the Q_i -space $(Z[U]/I) \otimes_Z Q_i$ is spanned by $p_{\alpha_1} \otimes 1, p_{\alpha_2} \otimes 1, \dots, p_{\alpha_m} \otimes 1$, and m is the number of cells of a decomposition of S . On the other hand, $\dim H^*(X, Q_i)$ is equal to the number of cells of a decomposition of X (it follows, e.g., from [2]), and hence equal to m . It is also equal to the Euler-Poincaré characteristic of X , since the cohomology in odd dimensions is null. It follows that the composed epimorphism of linear spaces

$$(Z[U]/I) \otimes Q_i \rightarrow H^*(X, Q_i)$$

is bijective. Since the elements $p_{\alpha_i} \otimes 1$ form a basis of $(Z[U]/I) \otimes Q_i$, p_{α_i} are independent, and hence they form a basis of the Z -module $Z[U]/I$.

Thus f', g', h' are bijective, t is injective, and hence f, g, h are bijective.

3. Non-projective case. Let now S be such a conical complex that X is non-singular and complete, not necessarily projective. If S has any filtration, then the assertion of the Theorem remains true for X . Unfortunately, the decompositions of X , resulting from the G_m -actions described above, need not to be filtrable (see the Example in [6]; besides, there exists a filtrable decomposition for this example). We are not able to prove that every "non-singular complete" complex S has a filtration (P 1183). Nevertheless, we have a provisional

PROPOSITION 2. *If $\dim S \leq 3$, then S has a filtration*

The proof is immediate for dimensions 1 and 2. Instead of a 3-dimensional conical complex in R^3 we consider the corresponding 2-dimensional spherical complex \tilde{E} , in other words — a triangulation of the 2-dimensional sphere. One should show that there exists a closed filtration

$$E = E_m \supset E_{m-1} \supset \dots \supset E_0 = \emptyset$$

such that, for $i = 0, 1, \dots, m-1$, $E_{i+1} - E_i$ has one of the following four forms (see Fig. 2):

- (I) $\{\omega\} = C_{\omega, \omega}$,
 - (II) $\{\omega, \rho\} = C_{\rho, \omega}$,
 - (III) $\{\omega, \rho, \tau, \alpha\} = C_{\alpha, \omega}$,
 - (IV) $\{\omega, \rho, \sigma, \tau, \alpha, \beta, \gamma\} = C_{0, \omega}$
- (0 is the origin, hence it has no image in the figure).

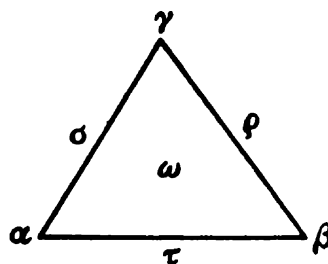


Fig. 2

Moreover, $E_m - E_{m-1}$ has to be of the form (I), $E_1 - E_0$ — of the form (IV), and the remaining $S_{i+1} - S_i$ — of the form (II) or (III). We choose an arbitrary simplex $\omega \in Q$ of dimension 2 and set

$$E_{m-1} = E - \{\omega\}.$$

E_{m-1} may be regarded as a triangulation of a plane set, homeomorphic to the closed disc D of dimension 2. It remains to find a filtration for E_{m-1} . Such a filtration can easily be constructed by induction, with the help of the following

LEMMA 3. *Given a curvilinear triangulation Q of D , containing $m \geq 2$ simplices of dimension 2, there exist at least 2 closed subcomplexes Q' and Q'' of Q such that $Q - Q'$ and $Q - Q''$ are of the form (II) or (III) and both topological spaces $|Q'|$ and $|Q''|$ are homeomorphic to D .*

Proof follows by induction on m . The assertion is obvious for $m = 2$. Suppose that it is true for all m less than some k and consider a triangulation

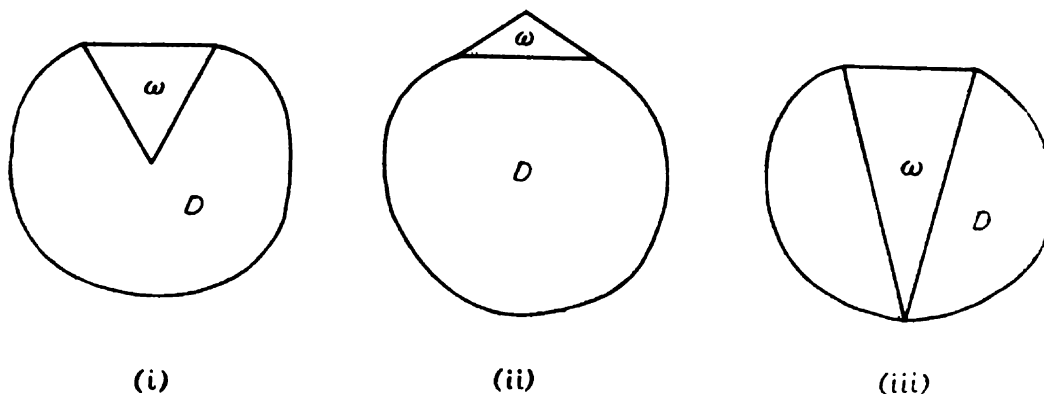


Fig. 3

of D containing k triangles. Let $\omega \in Q$ be such a triangle that at least one edge of ω is contained in the boundary of D . There are three possibilities (Fig. 3).

It suffices to consider the case of triangulation Q containing a triangle ω in position (iii). Removing the closed triangle ω from D , we have a decomposition into connected components:

$$D - |\bar{\omega}| = U_1 \cup U_2.$$

Let \bar{U}_i denote the closure of U_i in D , and P_i the restriction of the triangulation Q to \bar{U}_i for $i = 1, 2$. The triangulation P_i either contains only one triangle or satisfies the assumption of the lemma with $m < k$. In both cases there exists at least one closed subcomplex P'_i in P_i such that

$P_i - P'_i$ is of the form (II) or (III);

$|P'_i|$ is homeomorphic to D ;

no edge of ω belongs to $P_i - P'_i$.

In order the last condition to be satisfied we have to prove the existence of two subcomplexes in Q .

Indeed, the subcomplexes

$$Q' = Q - (P_1 - P'_1) \quad \text{and} \quad Q'' = Q - (P_2 - P'_2)$$

of Q satisfy the assertion of the lemma. Thus Proposition 2 is proved.

COROLLARY. *The Theorem of Section 2 holds if the assumption of projectivity of X is replaced by: "X is complete of dimension less than or equal to 3".*

4. Ehlers' formula for Betti numbers of X_S .

PROPOSITION 3. *Let S be such that $X = X_S$ is a non-singular complete variety, and let l be an integer prime to $\text{char}(k)$. Let $b_i = \dim H^i(X, Q_l)$. Then $b_i = 0$ for i even.*

Let P_X be the Poincaré polynomial of X such that

$$P_X(U) = \sum_i b_{2i} U^i,$$

let d_i be the number of faces of codimension i in S , and D_S the polynomial

$$D_S(U) = \sum_i d_i U^i.$$

Then $P_X(U) = D_S(U-1)$, and

$$b_{2i} = \sum_{j=1}^n (-1)^{j-i} \binom{j}{i} d_j.$$

This formula was obtained by Ehlers in [4]. We shall show how it follows from the cellular decomposition of X .

Let us consider a cellular decomposition

$$S = \bigcup_j C_{\sigma_j, \omega_j}$$

and the corresponding decomposition

$$X = \bigcup_j W_j$$

(indeed, they exist; for example, this can be the unstable decomposition described in the proof of the Theorem in Section 2). By [2], we have

$$b_{2i} = \#\{j: \dim W_j = i\}.$$

Since the subscheme W_j corresponds to C_{σ_j, ω_j} and $\dim W_j = \text{codim } \sigma_j$, Betti numbers of X can be expressed by

$$b_{2i} = \#\{j: \text{codim } \sigma_j = i\}.$$

Hence both polynomials D_S and P_X are given in terms of the complex S .

Now we present one of the ways of deriving the relation between the both polynomials, which gives the interpretation of D_S .

Consider the complex vector space $N \otimes_{\mathbb{Z}} \mathbb{C}$ (where N is the group of one-parameter subgroups of the torus T) and its standard immersion into complex projective n -dimensional space P . The Chow ring of P is $\mathbb{Z}[U]/U^{n+1}$, the coset u of U being the class of a hyperplane. Let us denote by $\text{Cl}(\lambda)$, where λ is a subset of P , the class of the cycle $\bar{\lambda}$ (Zariski's closure) in the Chow ring of P . If S' is a subset of the complex S , we write

$$\text{Cl}(S') = \sum_{\omega \in S'} \text{Cl}(\omega).$$

With this notation we can describe the polynomial D_S as

$$D_S(u) = \text{Cl}(S).$$

This equality determines $D_S(u)$ uniquely, since $\deg D_S = n$.

We now find the above-mentioned class using the decomposition of S . Let us begin with a single cell $C_{\sigma, \omega} \subset S$ and let r be the codimension of σ . The cone ω has n faces of dimension $n-1$. Let V_1, V_2, \dots, V_n be the linear hulls of those faces in N_R , and H_i the open half-space spanned by V_i and ω for $i = 1, 2, \dots, n$. Then

$$\omega = \bigcap_{i=1}^n (V_i \cup H_i).$$

Since σ is a face of ω , we can assume that

$$\sigma = \left(\bigcap_{i=1}^r V_i \right) \cap \omega.$$

The (not necessarily compact) polyhedron $|C_{\sigma, \omega}|$ the faces of which are precisely the elements of $C_{\sigma, \omega}$ is of the form

$$|C_{\sigma, \omega}| = \bigcap_{i=1}^r (V_i \cup H_i) \cap \bigcap_{i=r+1}^n H_i.$$

Since $\text{Cl}(V_i) = u$ and $\text{Cl}(H_i) = 1$, it follows easily from the transversality of V_i 's that

$$\text{Cl}(C_{\sigma, \omega}) = \prod_{i=1}^r (\text{Cl}(V_i) + \text{Cl}(H_i)) \prod_{i=r+1}^n \text{Cl}(H_i) = (u+1)^r.$$

For the whole complex S we have

$$\text{Cl}(S) = \sum_j \text{Cl}(C_{\sigma_j, \omega_j}) = \sum_j (u+1)^{\text{codim } \sigma_j} = \sum_{i=1}^n b_{2i}(u+1)^i = P_{\mathbf{x}}(u+1),$$

and the formula is proved.

Added in proof. When the paper was in print, Danilov [9] proved by other methods that the Theorem holds for an arbitrary non-singular complete torus embedding.

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