Technical Report

TR-2005-010

Chromatic numbers and homomorphisms of large girth hypergraphs

by

Dwight Duffus, Vojtech Rodl, Bill Sands, Norbert Sauer

Mathematics and Computer Science

EMORY UNIVERSITY

Chromatic numbers and homomorphisms of large girth hypergraphs

Dwight Duffus and Vojtěch Rödl
Emory University
Mathematics & Computer Science Department
Atlanta GA USA 30322
dwight@mathcs.emory.edu, rodl@mathcs.emory.edu

Bill Sands and Norbert Sauer
University of Calgary
Mathematics & Statistics Department
2500 University Dr. NW.
Calgary AB Canada T2N1N4
sands@math.ucalgary.ca, nsauer@math.ucalgary.ca

June 18, 2005

Abstract

We consider the problem of determining the minimum chromatic number of graphs and hypergraphs of large girth which cannot be mapped under a homomorphism to a specified graph or hypergraph. More generally, we are interested in large girth hypergraphs that do not admit a vertex partition of specified size such that the subhypergraphs induced by the partition blocks have a homomorphism to a given hypergraph. In the process, a general probabilistic construction of large girth hypergraphs is obtained, and general definitions of chromatic number and homomorphisms are considered.

1 Introduction

This paper is motivated by a problem of N. Sauer and R. Winkler. Given a graph $\mathrm{G} = (V, E)$, find the smallest chromatic number of a triangle-free graph H so that after removing any induced subgraph $\mathrm{H}[V']$ which has a homomorphism to G the remaining graph $\mathrm{H}[V - V']$ still does not have a homomorphism to G. We provide an answer to this [see Theorems 1.3, 1.4 and 5.4] and in the process are led to two observations. First, our investigations are related to a recent paper by J. Nešetřil and X. Zhu [7] that connects the well-known results of P. Erdős [2] on the existence of graphs with large girth and high chromatic number to the study of homomorphisms. Second, our methods both use hypergraph constructions and yield results for hypergraph versions of the problem. This leads to consideration of how best to define chromatic numbers and homomorphisms of hypergraphs.

Graph homomorphisms have emerged as a fruitful tool within graph theory. In fact, there is a new book by Nešetřil and P. Hell [4] devoted to this subject that highlights diverse applications. However, homomorphisms of hypergraphs, and their relationship to possible definitions of chromatic number, have not been studied extensively. One of the objectives of this work is to examine these notions in the hypergraph setting.

Let us start off by stating the Nešetřil-Zhu result, showing how it is related to the Sauer-Winkler problem, and seeing that the most familiar or obvious ways to define chromatic number and homomorphisms for hypergraphs lead to very different results for the graph and hypergraph cases.

We first consider a simple version of the question of Sauer and Winkler. Given a graph A, what is the minimum chromatic number $\psi(\mathrm{A})$ of a graph G such that there is no homomorphism from G to A? It is easy to see that $\psi(\mathrm{A}) = \omega(\mathrm{A}) + 1$, where $\omega(\mathrm{A}) = \omega$ is the clique number of A. The complete graph on $\omega + 1$ vertices, $\mathrm{K}_{\omega+1}$, has no homomorphism into A, since the image of a clique under a graph homomorphism is a clique of the same size. On the other hand, any graph G of chromatic number ω has a homomorphism, defined by an ω-colouring of G, into a clique of A. This easy observation can be strengthened considerably with the Nešetřil-Zhu theorem, alluded to above.

Theorem 1.1 (J. Nešetřil and X. Zhu, [7]). *For every graph* H *and for all positive integers n and l there exists a graph* G *with the following properties:*

1. $\mathrm{girth}(\mathrm{G}) > l$.
2. *For every graph* H' *with at most n vertices, there exists a homomorphism* $g : \mathrm{G} \to \mathrm{H}'$ *if and only if there exists a homomorphism* $f : \mathrm{H} \to \mathrm{H}'$.

In Section 5 we show how this result gives the following.

Theorem 1.2. *Let* A *be a graph and let $l \geq 2$ be an integer. Then there is a graph* G *with* $\mathrm{girth}(\mathrm{G}) > l$ *and* $\chi(\mathrm{G}) = \psi(\mathrm{A})$ *which does not have a homomorphism to* A.

Now, let us generalize ψ to capture the Sauer-Winkler problem. Say that a graph G is *s-partition homomorphic* to a graph A if there exists a partition $V(\mathrm{G}) = V_1 \cup V_2 \cup$

$\ldots \cup V_s$ such that for all i there is a homomorphism of $\mathrm{G}[V_i]$ to A. Let $\psi(\mathrm{A}, s)$ be the least chromatic number of a graph which is not s-partition homomorphic to A. Then $\psi(\mathrm{A}, 1) = \psi(\mathrm{A})$ and $\psi(\mathrm{A}, 2)$ is the least chromatic number of a graph $\mathrm{G} = (V, E)$ such that whenever an induced subgraph $\mathrm{G}[V']$ has a homomorphism to A, then $\mathrm{G}[V - V']$ does not — the original problem.

Without a girth restriction, just about the same reasoning as in the case $s = 1$ shows that $\psi(\mathrm{A}, s) = s \cdot \omega(\mathrm{A}) + 1$. But we are also able to extend the result, just as in Theorem 1.2.

Theorem 1.3. *Let* A *be a graph and let* l, s *be integers with* $l \geq 2$. *Then there is a graph* G *with* $\mathrm{girth}(\mathrm{G}) > l$ *and* $\chi(\mathrm{G}) = \psi(\mathrm{A}, s)$ *which is not* s*-homomorphic to* A.

One way to prove this is based on a hypergraph packing construction, a probabilistic construction of a large girth hypergraph [with additional properties], and Theorem 1.1. This argument is outlined in Section 5. In the process, we obtained an extension of Theorem 1.3 from the graph case to the k-uniform hypergraph case. Before stating this, here is some language needed for hypergraphs.

A *hypergraph* H is a pair (V, E) where E is a family of subsets of the *vertex set* V. To avoid trivialities, assume that the members of E, that is, the *edges* of H all have size at least 2. Call a hypergraph *simple* if edges intersect in at most one vertex. Call $\mathrm{H} = (V, E)$ a k*-uniform* hypergraph if E is a family of k-element subsets of V; thus, a usual (simple, undirected) graph is a 2-uniform hypergraph.

Given a hypergraph H, we often let $V(\mathrm{H})$ denote the set of vertices of H and $E(\mathrm{H})$ stand for the set of edges of H. Given $S \subseteq V(\mathrm{H})$, $\mathrm{H}[S]$ is the hypergraph with vertex set S and edges $\mathfrak{P}(S) \cap E(\mathrm{H})$, where $\mathfrak{P}(S)$ is the power set of S. Call $\mathrm{H}[S]$ the *subhypergraph of* H *induced by* S.

An r*-colouring* of H is a map with domain $V(\mathrm{H})$, range within an r-element set, and which is not constant on any edge of H. The *(weak) chromatic number* of H, denoted by $\chi(\mathrm{H})$, is the least r for which H has an r-colouring. Note that $\chi(\mathrm{H})$ exists since every edge has at least two elements.

Given k-uniform hypergraphs F_1 and F_2, a mapping $f : V(\mathrm{F}_1) \mapsto V(\mathrm{F}_2)$ is a *homomorphism* of F_1 to F_2 if $\{f(x) : x \in e\} \in E(\mathrm{F}_2)$ for every $e \in E(\mathrm{F}_1)$. For example, if G is a graph, then $\chi(\mathrm{G}) \leq r$ if and only if the graph G has a homomorphism to K_r.

We will be concerned with hypergraphs of large girth. A *circuit of length* c in a hypergraph $\mathrm{H} = (V, E)$ is a sequence $e_1, e_2, \ldots, e_c$ of distinct members of E and distinct elements $v_1, v_2, \ldots, v_c$ of V such that

$$e_i \cap e_{i+1} = \{v_i\} \ (i = 1, 2, \ldots, c-1), \text{ and } e_1 \cap e_c = \{v_c\}.$$

Thus, if H is k-uniform then each circuit of length c corresponds to a family of distinct k-sets whose union contains at most $c(k-1)$ elements of V. Conversely, it is well-known that given a family of c members of E with union of size at most $c(k-1)$ some subfamily constitutes a circuit of length at most c. [See, for instance, [3]].

The k-uniform hypergraph F is *s-partition homomorphic* to the k-uniform hypergraph A if there exists a partition $V(\mathrm{F}) = V_1 \cup V_2 \cup \ldots \cup V_s$ such that for all i there is a homomorphism of $\mathrm{F}[V_i]$ to A. Let $\psi(\mathrm{A}, s)$ be the least chromatic number of a k-uniform hypergraph which is not s-partition homomorphic to A.

We can now state the result for k-uniform hypergraphs.

Theorem 1.4. *Let* A *be a k-uniform hypergraph and l, s be positive integers. Then*

$$\psi(\mathrm{A}, s) = \begin{cases} s \cdot \omega(\mathrm{A}) + 1 & \text{if A is a graph,} \\ s + 1 & \text{if } k \geq 3. \end{cases}$$

For every $l \geq 2$ there is a k-uniform hypergraph F *with* $\mathrm{girth}(\mathrm{F}) > l$ *and* $\chi(\mathrm{F}) = \psi(\mathrm{A}, s)$ *which is not s-partition homomorphic to* A.

The difference between the graph and hypergraph results is due to the fact that the usual definitions of both chromatic number and homomorphism are not broad enough in the hypergraph setting.

We present an approach in the following three sections that provides more general definitions of chromatic number and homomorphism, a main result that subsumes both preceding theorems [Theorem 2.1], a quite general probabilistic hypergraph construction [Lemma 2.2], and proofs of the theorem and lemma.

In Section 5, we present an alternative proof of Theorem 1.2 based on the Nešetřil-Zhu result. We also introduce the hypergraph packing construction and, with it, lift the results for $s = 1$ to the general setting and a proof of Theorem 1.4. The second version of Theorem 1.4, Theorem 5.4, is somewhat sharper. The reader can find the required terminology in Section 5, independent of the intervening sections.

In Section 6, we close with a constructive approach to some special cases.

2 New Definitions, the Main Theorem and the Probabilistic Lemma

In Theorem 1.4, we see the difference between the graph case and the k-uniform hypergraph case, for $k \geq 3$. In order to create a more general result that subsumes both cases, we amend the definitions of homomorphism and chromatic number to better suit the hypergraph setting.

In the sequel, all hypergraphs are finite and k-uniform, $k \geq 2$. Let $1 < p, q \leq k$ and let $\mathrm{H}_i = (V_i, E_i)$, $i = 0, 1$, be hypergraphs.

(1) A *p-homomorphism* f of H_0 to H_1 is a map of V_0 to V_1 such that for all $e_0 \in E_0$ there exists $e_1 \in E_1$ such that

$$f(e_0) \subseteq e_1, \text{ and } |f(e_0)| \geq p \ ;$$

write $\mathrm{H}_0 \xrightarrow{p} \mathrm{H}_1$ if there exists a p-homomorphism of H_0 to H_1, and $\mathrm{H}_0 \stackrel{p}{\nrightarrow} \mathrm{H}_1$ if not.

(2) $\chi^{(q)}(\mathrm{H}_0) = \min\{\chi \mid \text{there exists } c : V_0 \to [\chi], \text{ for all } e \in E_0 \ |c[e]| \geq q\}$

(3) $\psi_{p,q}(\mathrm{H}_1) = \min\{\chi^{(q)}(\mathrm{H}_0) \mid \mathrm{H}_0 \stackrel{p}{\nrightarrow} \mathrm{H}_1\}$

Our results hold in the more general case of s-partitions.

(4) Say that H_0 is *s-partition p-homomorphic* to H_1 if there is a partition $X_1 \cup X_2 \cup \ldots \cup X_s$ of the vertices of H_0 such that for all i, $\mathrm{H}_0[X_i] \xrightarrow{p} \mathrm{H}_1$.

(5) Let $\psi_{p,q,s}(\mathrm{H}_1)$ be the minimum $\chi^{(q)}(\mathrm{H}_0)$ such that H_0 is not s-partition p-homomorphic to H_1.

Here is the main result. For a k-uniform hypergraph $\mathrm{H} = (V, E)$, we let the *clique number* $\omega(\mathrm{H})$ be the maximum integer ω such that there is an ω-subset V' of V all of whose k-subsets of V' are members of E.

Theorem 2.1. *Let k, p, q, s and l be positive integers and let A be a k-uniform hypergraph.*

(1) *For $1 < p \leq q \leq k$, let $q = d(p-1) + r$, where $0 < r \leq p - 1$, let $\omega = \omega(\mathrm{A})$ and set $t = d\omega + r - 1$. Then $\psi_{p,q,s}(\mathrm{A}) = st + 1$.*

(2) *For $1 < q < p \leq k$, $\psi_{p,q,s}(\mathrm{A}) = s(q-1) + 1$.*

Moreover, there is a k-uniform hypergraph H with $\mathrm{girth}(\mathrm{H}) > l$ such that $\chi^{(q)}(\mathrm{H}) = \psi_{p,q,s}(\mathrm{A})$ and H not s-partition p-homomorphic to A.

This theorem contains the results in the introduction. The usual definition of [weak] chromatic number has $q = 2$ and the definition of homomorphism used in Section 1 sets $p = k$. The graph case corresponds to $p = q = 2$, so Theorem 2.1 states that $\psi(\mathrm{A}, s) = st+1 = s\omega+1$, as given in Theorem 1.4. For $k \geq 3$, we have $q = 2 < 3 \leq k = p$, so the second part of the theorem yields $\psi(\mathrm{A}, s) = s + 1$, also as in Theorem 1.4.

The proof of Theorem 2.1 is given in Section 3. It depends upon a general probabilistic lemma which provides all the "constructions" we require. We state this now, but delay the proof until Section 4.

Call a sequence $\vec{k} = (k_1, k_2, \ldots, k_c)$ of positive integers a *k-sequence* if $\sum_{\alpha=1}^{c} k_\alpha = k$.

Lemma 2.2. *Given $\epsilon > 0$, positive integers m, k, l, h, and k-sequences*

$$\vec{k}^{(\alpha)} = (k_1^{(\alpha)}, k_2^{(\alpha)}, \ldots, k_{c_\alpha}^{(\alpha)}),\ \alpha = 1, 2, \ldots, h,$$

there exists n_0 such that for all $n \geq n_0$, there is a k-uniform hypergraph H *with vertex set*

$$V(\mathrm{H}) = \bigcup_{i=1}^{m+1} V_i,\ |V_i| = n\ (i = 1, 2, \ldots, m + 1)$$

such that

(1) *for all $e \in E(\mathrm{H})$ there exist α, a k-sequence $\vec{k}^{(\alpha)}$, and a sequence $1 \leq j_1 < j_2 < \ldots < j_{c_\alpha} \leq m + 1$ such that for all $\beta = 1, 2, \ldots, c_\alpha$, $|e \cap V_{j_\beta}| = k_\beta^{(\alpha)}$;*

and for each $\alpha = 1, 2, \ldots, h$,

(2) *for all $\beta = 1, 2, \ldots, c_\alpha$, for all $1 \leq j_1 < j_2 < \ldots < j_{c_\alpha} \leq m+1$ and for all $V_{j_\beta}' \subseteq V_{j_\beta}$ with $|V_{j_\beta}'| \geq \epsilon n$, there exists $e \in E(\mathrm{H})$ such that $|e \cap V_{j_\beta}'| = k_\beta^{(\alpha)}$; and*

(3) girth(H) $> l$.

3 The Proof of the Main Theorem

Proof of Theorem 2.1 (1)

Let $\mathrm{A} = (V, E)$, and integers k, p, q, s, l and ω be as in the theorem statement, and, also as above, let $q = d(p - 1) + r$, where $0 < r \leq p - 1$, and $t = d\omega + r - 1$.

We first prove that $\psi_{p,q,s}(\mathrm{A}) > st$ by showing if H is any k-uniform hypergraph with $\chi^{(q)}(\mathrm{H}) = st$ then there is a partition $V(\mathrm{H}) = Y_1 \cup Y_2 \cup \ldots \cup Y_s$ such that for all i,

$\mathrm{H}^{(i)} := \mathrm{H}[Y_i] \xrightarrow{p} A$. To this end, let $c : V(\mathrm{H}) \to [st]$ be a q-colouring of H with classes $C_1, C_2, \ldots, C_{st}$. That is, for all $e \in E(\mathrm{H})$, $e \cap C_j \neq \emptyset$ for at least q distinct j's.

Group these st sets into s, say,

$$Y_i = C_{(i-1)t+1} \cup \ldots \cup C_{it},\ (i = 1, 2, \ldots, s).$$

We show that this is the required s-partition of H.

Let $W = \{w_1, w_2, \ldots, w_\omega\} \subseteq V$ induce an ω-clique in A. We argue that there is a p-homomorphism f of $\mathrm{H}^{(i)}$ to A with range contained in W. Because the vertex set of $\mathrm{H}^{(i)}$ is the union of t classes C_j and $t = d\omega + r - 1$, we can group the vertices of $\mathrm{H}^{(i)}$ into disjoint subsets $D_1, D_2, \ldots, D_\omega$ as follows:

each of $D_1, \ldots, D_{r-1}$ is the union of $d+1$ distinct C_j's,

each of $D_r, \ldots, D_\omega$ is the union of d distinct C_j's,

and each C_j, $j = (i-1)t+1, \ldots, it$, is contained in one D_m. This is possible since $(r-1)(d+1) + (\omega - r + 1)d = d\omega + r - 1 = t$.

Define $f : V(\mathrm{H}^{(i)}) \to W$ by $f(v) = w_m$ for all $v \in D_m$, $m = 1, 2, \ldots, \omega$. Proving that f is a p-homomorphism requires that for all $e \in E(\mathrm{H}^{(i)})$, $f[e]$ is contained in an edge of A and $|f[e]| \geq p$. The former is true because $|f[e]| \leq k$ and W induces a clique in A. If the latter fails then there are $m_1, \ldots, m_{p-1}$ such that

$$f[e] \subseteq \{w_{m_1}, \ldots, w_{m_{p-1}}\},$$

$$e \subseteq D_{m_1} \cup \ldots \cup D_{m_{p-1}},\ \text{and therefore}$$

$$C_{j_1} \cup \ldots \cup C_{j_q} \subseteq D_{m_1} \cup \ldots \cup D_{m_{p-1}},$$

where $e \cap C_{j_u} \neq \emptyset$ for $u = 1, \ldots, q$. This implies that at most $p-1$ D_m's contain at least q C_j's. This is impossible because $p-1$ of the D_m's contain at most

$$(r-1)(d+1) + (p-r)d = (p-1)d + r - 1 = q - 1$$

distinct C_j's. Thus f is a p-homomorphism.

In order to prove that $\psi_{p,q,s}(\mathrm{A}) \leq st+1$, we use Lemma 2.2 to provide an example of a k-uniform hypergraph $\mathrm{H}_{st+1} = (V(\mathrm{H}_{st+1}), E(\mathrm{H}_{st+1}))$ such that $\chi^{(q)}(\mathrm{H}_{st+1}) = st+1$ and for all partitions $V(\mathrm{H}_{st+1}) = X_1 \cup \ldots \cup X_s$ there is some i such that $\mathrm{H}^{(i)} := \mathrm{H}_{st+1}[X_i]$ is not p-homomorphic to A. Moreover, this hypergraph also has girth greater than any specified l. To use Lemma 2.2, we also have to specify parameters m and h, the value of ϵ and the k-sequences $\vec{k}^{(\alpha)} = (k_1^{(\alpha)}, k_2^{(\alpha)}, \ldots, k_{c_\alpha}^{(\alpha)})$, $\alpha = 1, 2, \ldots, h$.

Let $m = st$, $h = 2$, $\epsilon = 1/(st|V|)$ and specify the two k-sequences to be

$$\vec{k}^{(1)} = (1, 1, \dots, 1), \text{ and } \vec{k}^{(2)} = (k - q + 1, 1, \dots, 1).$$

The lemma gives an integer n and a k-uniform hypergraph $\mathrm{H}_{st+1} = (V(\mathrm{H}_{st+1}), E(\mathrm{H}_{st+1}))$ such that

(i) $V(\mathrm{H}_{st+1}) = V_1 \cup V_2 \cup \ldots \cup V_{st+1}$, with $|V_i| = n$ for all i;

(ii) for $\alpha = q, k$, for all $1 \le j_1 < j_2 < \ldots < j_\alpha \le st + 1$, for all $V'_{j_\beta} \subseteq V_{j_\beta}$ with $|V'_{j_\beta}| \ge \epsilon n$ for $\beta = 1, 2, \dots, \alpha$, there exists $e \in E(\mathrm{H}_{st+1})$ such that $e \subseteq \bigcup_{\beta=1}^{\alpha} V'_{j_\beta}$;

(iii) for all $e \in E(\mathrm{H}_{st+1})$, $e \cap V_j \ne \emptyset$ for exactly q or k distinct indices j; and,

(iv) $\mathrm{girth}(\mathrm{H}_{st+1}) > l$.

Observe that $\chi^{(q)}(\mathrm{H}_{st+1}) \le st+1$ because, by (iii), the classes V_i provide a q-colouring.

Suppose that $V(\mathrm{H}_{st+1}) = C_1 \cup C_2 \cup \ldots \cup C_{st}$ is a partition. For each $j = 1, 2, \dots, st+1$ choose $i(j)$ from $i = 1, 2, \dots, st$ such that $|V_j \cap C_{i(j)}|$ is maximum among all $|V_j \cap C_i|$. Then $|V_j \cap C_{i(j)}| \ge n/st > \epsilon n$ and, by the pigeonhole principle, $i(1) = i(2)$, say. By (ii), with $\alpha = q$ and $\vec{k}^{(2)}$, there exists $e \in E(\mathrm{H}_{st+1})$ with

$$e \subseteq \bigcup_{j=1}^{q} V_j \cap C_{i(j)} \subseteq C_{i(1)} \cup C_{i(3)} \cup \ldots \cup C_{i(q)}.$$

Thus, no partition of $V(\mathrm{H}_{st+1})$ into st or fewer parts can provide a q-colouring of H_{st+1}.

We have shown that $\chi^{(q)}(\mathrm{H}_{st+1}) = st + 1$.

We now prove that H_{st+1} is not s-partition p-homomorphic to A. Let us argue that for an arbitrary partition $V(\mathrm{H}_{st+1}) = X_1 \cup X_2 \cup \ldots \cup X_s$, and $\mathrm{H}^{(i)} = \mathrm{H}[X_i]$ $(i = 1, 2, \dots, s)$, there is some i such that $\mathrm{H}^{(i)}$ is not p-homomorphic to A. For each $j = 1, 2, \dots, st+1$ choose $i(j)$ from $i = 1, 2, \dots, s$ such that $|V_j \cap X_{i(j)}|$ is maximum among all $|V_j \cap X_i|$. In particular, $|V_j \cap X_{i(j)}| \ge n/s$. By the pigeonhole principle, we may assume, without loss of generality, that

$$X_{i(1)} = X_{i(2)} = \cdots = X_{i(t+1)} = X.$$

Let $B_j = V_j \cap X$, $j = 1, 2, \dots, t+1$, and let $\mathrm{H}' = \mathrm{H}_{st+1}[B_1 \cup B_2 \cup \ldots \cup B_{t+1}]$. We shall show that H' is not p-homomorphic to A.

Assume to the contrary that f is a p-homomorphism of H′ to A. For $j = 1, 2, \ldots, t+1$, choose $a_j \in V(\mathrm{A})$ such that $|f^{-1}(a_j) \cap B_j|$ is maximum among all $|f^{-1}(a) \cap B_j|$, $a \in V$. With $U_j = f^{-1}(a_j) \cap B_j$, we see that

$$|U_j| \geq \frac{|B_j|}{|A|} \geq \frac{n/s}{|A|} \geq \epsilon n, \ j = 1, 2, \ldots, t+1.$$

Let $a_1, a_2, \ldots, a_v$ be the distinct elements among $a_1, a_2, \ldots a_{t+1}$.

Case (1): $v \geq k$. We claim that for each k-set $\{i_1, i_2, \ldots, i_k\} \subseteq [v]$, $\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} \in E = E(\mathrm{A})$. From (ii), using $\alpha = k$ and $\vec{k}^{(1)}$, there is $e \in E(\mathrm{H}_{st+1})$ such that $e \subseteq \bigcup_{\beta=1}^{k} U_{i_\beta}$. It is clear that $e \in E(\mathrm{H}')$. Thus,

$$f[e] = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} \text{ and } f[e] \subseteq e' \in E,$$

so $e' = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} \in E$. Thus, $\{a_1, a_2, \ldots, a_v\}$ induces a clique in A, so $v \leq \omega$.

For $i = 1, 2, \ldots, v$, let β_i be the number of j's such that $f[U_j] = a_i$, and assume that $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_v$. We now consider the size of $\beta = \beta_1 + \beta_2 + \cdots + \beta_{p-1}$. If $\beta \geq q$ then there are distinct indices $j_1, j_2, \ldots, j_q$ such that each $f[U_{j_i}] \subseteq \{a_1, a_2, \ldots, a_{p-1}\}$. By (ii), with $\alpha = k$ and $\vec{k}^{(2)}$, $\bigcup_{i=1}^{q} U_{j_i}$ contains an edge of H_{st+1} and, hence, of H′. This contradicts the assumption that f is a p-homomorphism. On the other hand, if

$$\beta \leq q - 1 = d(p-1) + r - 1,$$

then, using the fact that $\beta/(p-1) \geq \beta_p \geq \ldots \geq \beta_v$, we have

$$\frac{1}{p-1}\left(d(p-1) + r - 1\right) \geq \beta_p \geq \ldots \geq \beta_v.$$

Since each β_i is an integer and $r \leq p - 1$, this implies that $d \geq \beta_p \geq \ldots \geq \beta_v$. This gives a contradiction as follows:

$$\begin{aligned} t + 1 &= d\omega + r \\ &= \beta_1 + \beta_2 + \ldots + \beta_v \\ &\leq d(p-1) + r - 1 + (v - p + 1)d \\ &\leq d\omega + r - 1 \\ &= t \end{aligned} \tag{3.1}$$

Again, we see that the assumption that f is a p-homomorphism leads to a contradiction.

Case (2): $v < k$. Define the β's in exactly the same way and argue in the same manner. The fact $v < k \leq \omega$ allows us to obtain a contradiction just as in equation (3.1) above.

Proof of Theorem 2.1 (2)

Recall that $\mathrm{A} = (V, E)$ is a k-uniform hypergraph, and that p, q are integers satisfying $1 < q < p \leq k$. We wish to show that $\psi_{p,q,s}(\mathrm{A}) = s(q-1)+1$.

To see that $\psi_{p,q,s}(\mathrm{A}) > s(q-1)$, we let H be a k-uniform hypergraph with $\chi^{(q)}(\mathrm{H}) = s(q-1)$ and assume that $C_1, C_2, \ldots, C_{s(q-1)}$ are classes of a q-colouring of H. Define

$$X_j = C_{(j-1)(q-1)+1} \cup C_{(j-1)(q-1)+2} \cup \ldots \cup C_{j(q-1)},\ j = 1, 2, \ldots, s.$$

Since each edge of H must intersect at least q C_j's, each of the subhypergraphs $\mathrm{H}[X_i]$ is empty. Hence, we have p-homomorphisms to A, trivially. This shows that $\psi_{p,q,s}(\mathrm{A}) \geq s(q-1)+1$.

To see the opposite inequality, we invoke the probabilistic lemma to obtain the required k-uniform hypergraph with girth greater than a specified l. Again, we must give m and h, the value of ϵ and the k-sequences $\vec{k}^{(\alpha)} = (k_1^{(\alpha)}, k_2^{(\alpha)}, \ldots, k_{c_\alpha}^{(\alpha)})$, $\alpha = 1, 2, \ldots, h$.

Let $m = s(q-1)$, $h = 1$, $\epsilon = 1/(s(q-1)|V|)$, and $\vec{k}^{(1)} = (k-q+1, 1, \ldots, 1)$. The lemma gives an integer n and a k-uniform hypergraph $\mathrm{H}_{s(q-1)+1} = (V(\mathrm{H}_{s(q-1)+1}), E(\mathrm{H}_{s(q-1)+1}))$ such that

(i) $V(\mathrm{H}_{s(q-1)+1}) = V_1 \cup V_2 \cup \ldots \cup V_{s(q-1)+1}$, with $|V_i| = n$ for all i;

(ii) for all $1 \leq j_1 < j_2 < \ldots < j_q \leq s(q-1)+1$, for all $V'_{j_\beta} \subseteq V_{j_\beta}$ with $|V'_{j_\beta}| \geq \epsilon n$ for $\beta = 1, 2, \ldots, q$, there exists $e \in E(\mathrm{H}_{s(q-1)+1})$ such that $e \subseteq \bigcup_{\beta=1}^{q} V'_{j_\beta}$;

(iii) for all $e \in E(\mathrm{H}_{s(q-1)+1})$, $e \cap V_j \neq \emptyset$ for exactly q indices j; and,

(iv) $\mathrm{girth}(\mathrm{H}_{s(q-1)+1}) > l$.

Observe that $\chi^{(q)}(\mathrm{H}_{s(q-1)+1}) \leq s(q-1)+1$ because, by (iii), the classes V_i provide a q-colouring. The argument that $\chi^{(q)}(\mathrm{H}_{s(q-1)+1}) > s(q-1)$ is almost exactly the same as that showing $\chi^{(q)}(\mathrm{H}_{st+1}) > st$ in the previous part, and so we omit it.

Let us show that for every s-partition of $V(\mathrm{H}_{s(q-1)+1})$ there is some part that induces a subhypergraph of $\mathrm{H}_{s(q-1)+1}$ with no p-homomorphism to A. Suppose that $V(\mathrm{H}_{s(q-1)+1}) = X_1 \cup X_2 \cup \ldots \cup X_s$ is any partition. Use the pigeonhole principle, just as in the previous part, to show that there are q distinct indices $j_1, j_2, \ldots, j_q$ in $[s(q-1)+1]$ and some $X = X_j$ such that

$$U_i = V_{j_i} \cap X \text{ satisfies } |U_i| \geq \frac{n}{s} > \epsilon n \quad i = 1, 2, \ldots, q.$$

Let $\mathrm{H}' = \mathrm{H}_{s(q-1)+1}[X]$ and let's consider any map f of H' to A. We choose $a_i \in V$ so that $|f^{-1}(a_1) \cap U_i|$ is maximum among all $|f^{-1}(a) \cap U_i|$, $a \in V$, for $i = 1, 2, \ldots, q$. Because $|f^{-1}(a_i) \cap U_i| \geq n/(s|A|)$, (ii) yields $e \in E(\mathrm{H}')$ such that

$$e \subseteq \bigcup_{i=1}^{q} f^{-1}(a_i) \cap U_i, \text{ implying } f[e] \subseteq \{a_1, a_2, \ldots, a_q\}.$$

Since $q < p$, f is not a p-homomorphism. Hence, $\mathrm{H}_{s(q-1)+1}$ is not s-partition p-homomorphic to A.

4 The Proof of the Probabilistic Lemma

We shall construct H by deleting edges from a union $\bigcup_{\alpha=1}^{h} \mathrm{H}^{(\alpha)}$, where each $\mathrm{H}^{(\alpha)}$ corresponds to some k-sequence $\vec{k}^{(\alpha)}$, $\alpha = 1, 2, \ldots, h$. Our probabilistic proof is a modification of the well-known argument due to Erdős [2] and Erdős and Hajnal [3].

First, here are the two basic results from probability that we shall need. Following notation in [5], $\mathbf{X} \in \mathrm{Bi}(r, p)$ means that $\mathbf{X}$ is a random variable with binomial distribution, the sum of r independent Bernoulli random variables. The statement of Chernoff's Inequality is based on Theorem 2.1 in [5].

Markov's Inequality : for $\mathbf{X} \geq 0$ and $t > 0$,

$$\mathbb{P}(\mathbf{X} \geq t) \leq \frac{\mathbb{E}(\mathbf{X})}{t} \ .$$

Chernoff's Inequality : for $\mathbf{X} \in \mathrm{Bi}(r, p)$ and $0 \leq \delta \leq 1$,

$$\mathbb{P}(\mathbf{X} \leq \delta r p) \leq \exp\left(-\frac{(1-\delta)^2}{2} r p\right) \ .$$

Proof of Lemma 2.2

Let $V_1, V_2, \ldots, V_{m+1}$ be disjoint sets each of size n, with n sufficiently large. To construct $\mathrm{H}^{(\alpha)}$, $\alpha = 1, 2, \ldots, h$, we shall select k-element sets, each with probability $p = (\log n)/n^{k-1}$ from the set

$$\bigcup_{j_1 < \ldots < j_{c_\alpha}} \left\{ \bigcup_{i=1}^{c_\alpha} \binom{V_{j_i}}{k_i^{(\alpha)}} : \ 1 \leq j_1 < j_2 < \ldots < j_{c_\alpha} \leq m+1 \right\}.$$

Thus, there are

$$\binom{t+1}{c_\alpha}\prod_{i=1}^{c_\alpha}\binom{n}{k_i^{(\alpha)}}$$

independent trials in total.

Since our eventual hypergraph H is obtained by deleting edges from the union of the $\mathrm{H}^{(\alpha)}$'s, we see that **(1)** of the lemma is immediate.

Second, we consider property **(2)** of the lemma. For an arbitrary $1 \le j_1 < j_2 < \ldots < j_{c_\alpha} \le m+1$, choose a sequence of c_α sets $V'_{j_\beta} \subseteq V_{j_\beta}$, with $|V'_{j_\beta}| \ge \epsilon n$. We call such a sequence *large*. Let $\mathbf{X}(V'_{j_1}, V'_{j_2}, \ldots, V'_{j_{c_\alpha}})$ denote the random variable counting the number of k-sets e satisfying

$$|e \cap V'_{j_\beta}| = k_\beta, \ \beta = 1, 2, \ldots, c_\alpha. \tag{4.1}$$

Then $\mathbf{X} \in \mathrm{Bi}(r,p)$ where

$$r = \prod_{\beta=1}^{c_\alpha}\binom{|V_{j_\beta}|}{k_\beta} \ \text{ and } \ p = \frac{\log n}{n^{k-1}}.$$

Thus,

$$\mathbb{E}(\mathbf{X}) = rp \ge \prod_{\beta=1}^{c_\alpha}\left(\frac{\epsilon n}{k_\beta}\right)^{k_\beta}\frac{\log n}{n^{k-1}} \ge \left(\frac{\epsilon}{k}\right)^k n\log n.$$

Apply Chernoff's inequality with $\delta = 1/2$ to obtain

$$\mathbb{P}\left(\mathbf{X} \le \frac{1}{2}\left(\frac{\epsilon}{k}\right)^k n\log n\right) \le \exp\left(-\frac{1}{8}\left(\frac{\epsilon}{k}\right)^k n\log n\right).$$

The number of sequences $(V'_{j_1}, V'_{j_2}\ldots, V'_{j_{c_\alpha}})$ with $1 \le j_1 < j_2 < \ldots < j_{c_\alpha}$ is less than $\binom{m+1}{c_\alpha}2^{c_\alpha n}$ and thus the probability that there is a large sequence which contains fewer than $\frac{1}{2}\left(\frac{\epsilon}{k}\right)^k n\log n$ satisfying (4.1) is at most

$$\binom{m+1}{c_\alpha}2^{c_\alpha n}\exp\left(-\frac{1}{8}\left(\frac{\epsilon}{k}\right)^k n\log n\right) \tag{4.2}$$

which tends to 0 as $n \to \infty$.

We now want to estimate the number of circuits of length at most r in the hypergraph $\mathrm{H} = \bigcup_{\alpha=1}^{h} \mathrm{H}^{(\alpha)}$. First, recall that a circuit of length $j \geq 2$ corresponds to a family of distinct k-sets $e_1, e_2, \ldots, e_j$ such that $|\bigcup_{i=1}^{j} e_i| \leq j(k-1)$ and, conversely, a family satisfying these conditions contains a circuit of length at most j. The number of such families is less than

$$\binom{(m+1)n}{j(k-1)} \binom{j(k-1)}{k}^j \leq c_j(m,k) n^{j(k-1)} ,$$

where $c_j(m,k) = c_j$ depends only on j, k, m.

Let $\mathbf{Y}_j$ be the random variable counting the number of j-circuits $j = 1, 2, \ldots, l$. Then

$$\mathbb{E}(\mathbf{Y}_j) \leq c_j n^{j(k-1)} p^j = c_j \log^j n .$$

Let $\mathbf{Y} = \sum_{j=2}^{l} \mathbf{Y}_j$. Then $\mathbb{E}(\mathbf{Y}) \leq \sum_{j=2}^{l} c_j \log^j n < c_0 \log^l n$. Apply Markov's inequality to $\mathbb{E}(\mathbf{Y})$:

$$\mathbb{P}(\mathbf{Y} > 2c_0 \log^l n) \leq \frac{\mathbb{E}(\mathbf{Y})}{2c_0 \log^l n} = \frac{1}{2} . \tag{4.3}$$

Summarizing (4.2) and (4.3), we infer that for $n \geq n(m,k,l,\epsilon)$, there is a k-uniform hypergraph H_n on the vertex set $\bigcup_{i=1}^{m+1} V_i$ such that

(i) for all $\alpha = 1, 2, \ldots, h$, $1 \leq j_1 < j_2 < \ldots < j_{c_\alpha} \leq m+1$ and large sequences $(V'_{j_1}, V'_{j_2}, \ldots, V'_{j_{c_\alpha}})$ there exists $e \in \mathrm{H}^{(\alpha)}$ such that $|e \cap V'_{j_\beta}| = k_\beta$, for $\beta = 1, 2, \ldots, c_\alpha$; and,

(ii) the number of circuits of length at most l is at most $2c_0 \log^l n$.

Delete one edge from each such circuit, resulting in a k-uniform hypergraph H of girth greater than l in which each large family contains at least one edge of H. Thus, we have item **(3)** of the lemma.

This completes the proof of Lemma 2.2.

5 A Packing Construction

An alternative approach to Theorem 1.4 uses Theorem 1.1 and the well-known result of Erdős to establish the theorem for $s = 1$ [see Lemma 5.1]. Then a packing construction for hypergraphs [Lemma 5.2] and a probabilistic construction [Lemma 5.3] can be used to both generalize from the graph case to hypergraphs and extend the result to s-partitions. This argument may be of independent interest, so we present it in this section.

We revert to the usual meanings of homomorphism, colouring, and chromatic number for graphs and hypergraphs, as used in Section 1. That is, a homomorphism of graphs is a 2-homomorphism, of k-uniform hypergraphs, a k-homomorphism, a colouring c of a hypergraph H has $|c[e]| \geq 2$, for all edges e of H, and chromatic number for hypergraphs is the [weak] chromatic number.

First, here is Theorem 1.4 in the case $s = 1$, proved using using probabilistic results from [2] and [7].

Lemma 5.1. *Let* A *be a k-uniform hypergraph and let $l \geq 2$. Then there is a k-uniform hypergraph* C *with* $\mathrm{girth}(\mathrm{C}) > l$ *and* $\chi(\mathrm{C}) = \psi(\mathrm{A})$ *such that* $C \nrightarrow A$.

Proof. Let A be a graph with $\omega(\mathrm{A}) = \omega$ and let $l \geq 2$. As noted in the introduction, it is straightforward to argue that

$$\psi(\mathrm{A}) = \begin{cases} \omega(\mathrm{A}) + 1 & \text{if } k = 2, \\ 2 & \text{if } k \geq 3, \end{cases}$$

if there is no restriction on girth.

Let C be a graph with $\mathrm{girth}(\mathrm{C}) > l$, obtained from Theorem 1.1 in the case that $\mathrm{H} = \mathrm{K}_{\omega+1}$ and $n = \max\{|V(\mathrm{A})|, \omega + 1\}$. Since $\mathrm{K}_{\omega+1} \nrightarrow \mathrm{A}$, we have that $\mathrm{C} \nrightarrow \mathrm{A}$. On the other hand, $\mathrm{C} \rightarrow \mathrm{K}_{\omega+1}$, and thus $\chi(\mathrm{C}) \leq \omega + 1$. Of course, $\mathrm{C} \nrightarrow \mathrm{K}_\omega$ because $\mathrm{K}_{\omega+1} \nrightarrow \mathrm{K}_\omega$. Thus, $\chi(\mathrm{G}) = \omega + 1$. This finishes the argument in the case of graphs.

Let $k \geq 3$ and let A be a k-uniform hypergraph. Given l, we construct a k-uniform hypergraph C such that $\chi(\mathrm{C}) = 2$, $\mathrm{girth}(\mathrm{C}) > l$, and $\mathrm{C} \nrightarrow \mathrm{A}$.

For a k-uniform hypergraph H, we denote by $Gr(\mathrm{H})$ the graph on vertex set $V(\mathrm{H})$, with edge set all pairs $\{u, v\}$ such that u and v belong to an edge of H. [This is a special case of the construction below and is called the *2-section* of H in [1].] Note that if f is a homomorphism of H to a k-uniform hypergraph L then f is a graph homomorphism of $Gr(\mathrm{H})$ to $Gr(\mathrm{L})$.

Let $\chi(Gr(\mathrm{A})) = c$. By the well-known result in [2], there exists a graph G such that $\chi(\mathrm{G}) \geq c+1$ and girth(G) $> l$. Let W be a set of size $(k-2)|E(\mathrm{G})|$ which is disjoint from $V(\mathrm{G})$. We associate with every edge e of G a subset $\delta(e)$ of W with $|\delta(e)| = k-2$ and $\delta(e) \cap \delta(f) = \emptyset$ for any two different edges e and f of G.

Let the k-uniform hypergraph C have $V(\mathrm{C}) = V(\mathrm{G}) \cup W$ and the set of edges

$$\{e \cup \delta(e) : e \text{ is an edge of G}\}.$$

Since $k \geq 3$ the partition $V(\mathrm{G}) \cup W$ is a 2-colouring of C. And, $\mathrm{F} \nrightarrow \mathrm{A}$ because $\chi(Gr(\mathrm{F})) \geq c+1$, while $\chi(Gr(\mathrm{A})) = c$. It is easy to see that girth(C) $> l$. □

In the rest of this section, we outline the alternative approach to Theorem 1.4 for arbitrary s. This requires a few new terms.

Let A be a k-uniform hypergraph and let H be a simple $|V(\mathrm{A})|$-uniform hypergraph. We let $\mathcal{H}(\mathrm{A})$ denote the set of all k-uniform hypergraphs obtained by inserting a copy of A in each edge of H. More formally, $\widehat{\mathrm{H}} \in \mathcal{H}(\mathrm{A})$ if and only if $V(\widehat{\mathrm{H}}) = V(\mathrm{H})$, for all $e \in E(\mathrm{H})$, $\widehat{\mathrm{H}}[e]$ is isomorphic to A, and for all $\widehat{e} \in E(\widehat{\mathrm{H}})$ there is some $e \in E(\mathrm{H})$ such that $\widehat{e} \subseteq e$. Note that

$$|\mathcal{H}(\mathrm{A})| = \left(\frac{|V(\mathrm{A})|!}{|\mathrm{Aut}(\mathrm{A})|}\right)^{|E(\mathrm{H})|}.$$

Refer to any member of $\mathcal{H}(\mathrm{A})$ as an A-*packing* of H.

Let A be a hypergraph and s be a nonnegative integer. Let

$$\pi(\mathrm{A}, s+1) = \min\{\chi(\widehat{\mathrm{H}}) \mid \chi(\mathrm{H}) = s+1,\ \widehat{\mathrm{H}} \in \mathcal{H}(\mathrm{A})\}.$$

with the minimum taken over all $(s+1)$-chromatic $|V(\mathrm{A})|$-uniform hypergraphs H and all $\widehat{\mathrm{H}} \in \mathcal{H}(\mathrm{A})$.

Lemma 5.2. *Let* A *be a hypergraph and let* s *be a nonnegative integer. Then*

$$\pi(\mathrm{A}, s+1) \geq (\chi(\mathrm{A}) - 1)s + 1.$$

Proof. Let H be any simple $|V(A)|$-uniform hypergraph of chromatic number $s+1$ and let ϕ be any mapping of $V(\mathrm{H})$ to $X \times Y$ where $|X| = \chi(\mathrm{A}) - 1$ and $|Y| = s$. For each $v \in V(\mathrm{H})$ let $\hat{\phi}(v) = y$ where $\phi(v) = (x, y)$. Since $|Y| < \chi(\mathrm{H})$, there is some $e \in E(\mathrm{H})$ such that $\hat{\phi}|e$ is constant. Now let $\widehat{\mathrm{H}} \in \mathcal{H}(\mathrm{A})$. Since $|X| < \chi(\mathrm{A})$ and $\widehat{\mathrm{H}}[e]$ is isomorphic to A, there is some $e' \in E(\mathrm{A})$ such that $e' \subseteq e$ and $\phi|e'$ is constant. Thus, ϕ cannot be a colouring of $\widehat{\mathrm{H}}$. Hence, $\pi(\mathrm{A}, s+1) \geq (\chi(\mathrm{A}) - 1)s + 1$. □

Lemma 5.3. *let* A *be a k-uniform hypergraph and let s and l be positive integers. Then there is a* $|V(\mathrm{A})|$*-uniform hypergraph* H *such that* $\chi(\mathrm{H}) = s + 1$*,* $\mathrm{girth}(\mathrm{H}) > l$*, and an* A*-packing* $\widehat{\mathrm{H}}$ *of* H *such that*

$$\chi(\widehat{\mathrm{H}}) = (\chi(\mathrm{A}) - 1)s + 1.$$

Consequently,

$$\chi(\widehat{\mathrm{H}}) = \pi(\mathrm{A}, s + 1) = (\chi(\mathrm{A}) - 1)s + 1.$$

Proof. We are given k, s and l and the k-uniform hypergraph A. Let $|V(\mathrm{A})| = a$, $\chi = \chi(\mathrm{A})$, and $V(\mathrm{A}) = W_1 \cup W_2 \cup \ldots \cup W_\chi$ be a partition induced by a χ-colouring of A, with $|W_i| = a_i$ for $i = 1, 2, \ldots, \chi$.

We apply Lemma 2.2 with $k = a$, $h = 1$, $c_1 = \chi$, $\epsilon = 1/s$, the a-sequence $\vec{a} = (a_1, a_2, \ldots, a_\chi)$, and $m = (\chi - 1)s$. This yields an a-uniform H satisfying **(1), (2)** and **(3)** of the lemma. Define a k-uniform hypergraph $\widehat{\mathrm{H}}$ on $V(\mathrm{H})$ as follows: for each $e \in E(\mathrm{H})$ insert a copy of A in e by identifying W_β with $e \cap V_{j_\beta}$, where $|e \cap V_{j_\beta}| = a_\beta$ for $\beta = 1, 2, \ldots, \chi$, as guaranteed by **(1)**. It is clear that $\widehat{\mathrm{H}} \in \mathcal{H}(\mathrm{A})$.

We know from Lemma 5.2 that $\chi(\widehat{\mathrm{H}}) \geq m+1$. The upper bound is immediate from the construction of $\widehat{\mathrm{H}}$, so we see that $\chi(\widehat{\mathrm{H}}) = m + 1$. It remains to prove that $\chi(\mathrm{H}) = s + 1$.

To see that $\chi(\mathrm{H}) \leq s + 1$, recall that $V(\mathrm{H}) = \bigcup_{i=1}^{m+1} V_i$. Create a new partition of $V(\mathrm{H})$ with parts U_j, $j = 1, 2, \ldots, s + 1$, by letting each U_j be the union of $\chi - 1$ distinct V_i's, for $j = 1, 2, \ldots, s$, and $U_{s+1} = V_{m+1}$. By **(1)** of Lemma 2.2, each edge of H intersects exactly χ V_i's, so the partition by U_j, $j = 1, 2, \ldots, s + 1$ provides an $(s + 1)$-colouring of H.

To prove that $\chi(\mathrm{H}) > s$, suppose that $V(\mathrm{H}) = X_1 \cup X_2 \cup \ldots \cup X_s$ is a partition. For each $i = 1, 2, \ldots, m+1 = (\chi-1)s+1$ there is some $j(i) \in [s]$ such that $|V_i \cap X_{j(i)}| > n/s = \epsilon n$. There is some index j_0 such that $j_0 = j(i)$ for at least χ distinct i's. By **(2)**, X_{j_0} contains an edge of H, so the partition by X_j, $j = 1, 2, \ldots, s$ cannot give an $(s - 1)$-colouring of H. □

We can now state and prove a somewhat sharpened version of Theorem 1.4.

Theorem 5.4. *Let* A *be a k-uniform hypergraph and let s and l be positive integers. Then there exist a k-uniform hypergraph* C *and a* $|V(\mathrm{C})|$*-uniform hypergraph* H*, both of girth greater than l, such that* $\chi(\mathrm{C}) = \psi(\mathrm{A})$*,* $\mathrm{C} \nrightarrow \mathrm{A}$*,* $\chi(\mathrm{H}) = s + 1$ *and such that some* $\widehat{\mathrm{H}} \in \mathcal{H}(\mathrm{C})$ *satisfies:*

(1) $\widehat{\mathrm{H}}$ *is not s-partition homomorphic to* A*;*

(2) $\chi(\widehat{\mathrm{H}}) = \psi(\mathrm{A}, s) = s(\psi(A) - 1) + 1$*; and*

(3) $\mathrm{girth}(\widehat{\mathrm{H}}) > l$.

Just as in Theorem 1.4, this gives the value of $\psi(\mathrm{A}, s)$, once we recall that $\psi(\mathrm{A}) = \omega(\mathrm{A}) + 1$ for graphs and $\psi(\mathrm{A}) = 2$ for nontrivial hypergraphs, and shows that there is a large girth hypergraph realizing $\psi(\mathrm{A}, s)$.

Proof. Let A, s and l be as in the preceding statement. Apply Lemma 5.1 to A and l to obtain a k-uniform C such that $\mathrm{girth}(\mathrm{C}) > l$, $\chi(\mathrm{C}) = \psi(\mathrm{A})$, and $\mathrm{C} \nrightarrow \mathrm{A}$.

Apply Lemma 5.3 to C, $s+1$ and l to obtain a $|V(C)|$-uniform hypergraph H such that $\mathrm{girth}(\mathrm{H}) > l$, and $\chi(\mathrm{H}) = s+1$, and, for some $\widehat{\mathrm{H}} \in \mathcal{H}(\mathrm{C})$

$$\chi(\widehat{\mathrm{H}}) = \pi(\mathrm{C}, s+1) = (\chi(\mathrm{C}) - 1)s + 1. \tag{5.1}$$

For any partition of the vertices of $V(\widehat{\mathrm{H}}) = V(\mathrm{H})$ into s parts, some part, say W, contains an edge of H. Then $\widehat{\mathrm{H}}[W]$ contains a copy of A and, thus, has no homomorphism to A. Therefore, $\widehat{\mathrm{H}}$ is not s-partition homomorphic to A, proving **(1)**.

The proof of **(2)** will be complete once we show that $\psi(\mathrm{A}, s) = \pi(\mathrm{C}, s+1)$. Let us see that

$$\psi(\mathrm{A}, s) \leq \pi(\mathrm{C}, s+1). \tag{5.2}$$

Given any simple $(s+1)$-chromatic, $|V(\mathrm{C})|$-uniform hypergraph F and any $\widehat{\mathrm{F}} \in \mathcal{F}(\mathrm{C})$, a partition of $\widehat{\mathrm{F}}$ into s parts must result in a part which contains an edge of F and, hence, a copy of C. Then the subhypergraph of $\widehat{\mathrm{F}}$ induced on that part has no homomorphism to A, so $\widehat{\mathrm{F}}$ is not s-partition homomorphic to A and $\psi(\mathrm{A}, s) \leq \chi(\widehat{\mathrm{F}})$. Since F and $\widehat{\mathrm{F}}$were chosen arbitrarily, $\psi(\mathrm{A}, s) \leq \pi(\mathrm{C}, s+1)$.

In order to prove the opposite inequality, we need that

$$s(\psi(\mathrm{A}) - 1) + 1 \leq \psi(\mathrm{A}, s). \tag{5.3}$$

To prove it, let F be any k-uniform hypergraph with $\chi(\mathrm{F}) \leq s(\psi(\mathrm{A}) - 1)$. Let V_i ($i = 1, 2, \ldots, s(\psi(\mathrm{A}) - 1)$) be the classes of a colouring of F. Now let W_j ($j = 1, 2, \ldots, s$) be pairwise disjoint, with each W_j the union of $\psi(\mathrm{A}) - 1$ distinct V_i's. Since each of the s induced subhypergraphs $\mathrm{F}[W_j]$ has a $(\psi(\mathrm{A})-1)$-colouring, defined by the V_i's contained in W_j, for $j = 1, 2, \ldots, s$, $\mathrm{F}[W_j] \to \mathrm{A}$. Thus, F is s-partition homomorphic to A. This proves that $s(\psi(\mathrm{A}) - 1) + 1 \leq \psi(\mathrm{A}, s)$.

We now conclude that $\psi(\mathrm{A}, s) = \pi(\mathrm{C}, s+1)$ from (5.4) below, which follows from (5.3), (5.2), (5.1), and the fact that $\chi(\mathrm{C}) = \psi(\mathrm{A})$:

$$\begin{aligned} s(\psi(\mathrm{A}) - 1) + 1 \leq \psi(\mathrm{A}, s) \leq \pi(\mathrm{C}, s+1) \leq \\ \leq s(\chi(\mathrm{C}) - 1) + 1 \leq s(\psi(\mathrm{A}) - 1) + 1. \end{aligned} \tag{5.4}$$

Finally, **(3)**, regarding the girth of a packing, is an immediate consequence of the following observation: $\text{girth}(\widehat{\text{H}}) \geq \min\{\text{girth}(\text{H}), \text{girth}(\text{C})\}$. □

6 Explicit Packings in Specific Cases

Given s and a k-uniform hypergraph A, Theorem 5.4 produces an $(s+1)$-chromatic, large girth $|V(\text{A})|$-uniform hypergraph H such that some A-packing $\widehat{\text{H}}$ of H has chromatic number equal to the minimum possible, $\pi(\text{A}, s+1) = (\chi(\text{A}) - 1)s + 1$.

The methods used are probabilistic, so there is no explicit description of $\widehat{\text{H}}$. We present an explicit construction of large girth hypergraphs for $\text{A} = \text{K}_k$. Here is a restatement of the objective: for integers k, s and l, explicitly construct a k-uniform hypergraph H of girth greater than l, weak chromatic number $s+1$, and strong chromatic number $(k-1)s+1$.

The construction is based on a general method created by Nešetřil and Rödl [6]. The following is not explicitly stated in [6] but is a consequence of the proof of the main theorem of that paper.

Lemma 6.1 (The Girth Machine). *Let* L *be a* k*-uniform hypergraph with* $V(\text{L}) = \{v_1, v_2, \ldots, v_t\}$*, weak chromatic number* χ_w*, and strong chromatic number* χ_s*. Then there exists a* k*-uniform hypergraph* $\widetilde{\text{L}}$ *with* $V(\widetilde{\text{L}}) = V_1 \cup V_2 \cup \ldots \cup V_t$*, weak chromatic number* χ_w*, strong chromatic number* χ_s*, and* $\text{girth}(\widetilde{\text{L}}) > l$*. Moreover, the mapping* $V_i \to v_i$ *is a homomorphism of* $\widetilde{\text{L}}$ *to* L*.*

Apply the Girth Machine to the k-uniform hypergraph $L = \text{K}^{(k)}_{(k-1)s+1}$, that is, the hypergraph on vertex set $[(k-1)s+1]$ with edge set all k-element subsets of $[(k-1)s+1]$. The weak chromatic number of $\text{K}^{(k)}_{(k-1)s+1}$ is $s+1$, by a pigeonhole argument, and the strong chromatic number is the size of the vertex set, $(k-1)s+1$. Therefore, $\widetilde{\text{L}}$ has the desired properties. Unfortunately, the cardinality of $V(\widetilde{\text{L}})$ is a tower function of considerable height.

We end with a concrete, small construction that handles the case $k = l = s = 3$.

The set of vertices of H is the set $S := \{1, 2, 3, 4, 5\}$ together with the 10 2-element subsets of S, so $|V(\text{H})| = 15$. The set of hyperedges of H is the set

$$\{\{x, y, \{x, y\}\} : x, y \in S\} \cup \{\{\{x, y\}, \{y, z\}, \{z, x\}\} : x, y, z \in S\}.$$

To see that $\chi(\text{H}) \geq 3$, assume for a contradiction that H has a colouring γ with colours a and b. Then three of the elements in S receive the same colour. We may assume, without loss of generality, that the elements 1, 2 and 3 all receive colour a. Then the three 2-element subsets $\{1,2\},\{2,3\},\{3,1\}$ must be coloured with b, which is a contradiction.

Note that the strong chromatic number of H is equal to the total chromatic number of the complete graph K_5, which is 5. (The set {1, {2,5}, {3,4}} is a colour class and one obtains the others by rotation.)

References

[1] C. Berge, *Hypergraphs*, North-Holland Mathematical Library, **45**, Elsevier Science Publishers B.V. (1989).

[2] P. Erdős, Graph theory and probability, *Canad. J. Math.* **11** (1959) 34–38.

[3] P. Erdős and A. Hajnal, On chromatic number of graphs and setsystems, *Acta. Math. Acad. Sci. Hung.* **17** (1966) 61–99.

[4] P. Hell and J. Nešetřil, *Graphs and Homomorphisms*, Clarendon Press, Oxford (2004).

[5] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*, Wiley, New York (2000).

[6] J. Nešetřil and V. Rödl, A short proof of the existence of highly chromatic hypergraphs without short cycles, *J Combinatorial Theory B* **27** (1979) 225–227.

[7] J. Nešetřil and X. Zhu, On sparse graphs with given colorings and homomorphisms, *J Combinatorial Theory B* **90** (2004) 161–172.